RESEARCH SUMMARY (EXTENDED VERSION)

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My research is in differential geometry and geometric analysis, particularly in *gauge theory*, which is concerned with partial differential equations originating from particle physics. Since the pioneering work of Atiyah, Donaldson, and Witten, gauge theory has led to significant advances in low-dimensional topology, algebraic geometry, and symplectic geometry.

My work fits into two new, rapidly developing lines of research whose goal is to extend the scope of applications of gauge theory to geometry:

(1) *Beyond Yang–Mills and Seiberg–Witten.* Most of research in mathematical gauge theory so far has focused on solutions of the Yang–Mills equations and the closely related Seiberg–Witten equations, which lead to topological invariants of 3– and 4–dimensional manifolds. Following work of Taubes, there has been recently a surge in the study of more general equations considered in physics. These include the Kapustin–Witten equations, which are expected to lead to new insights into the topology of knots and 3–manifolds, and the Vafa–Witten equations, which have deep links to modular forms and algebraic geometry.

(2) *Beyond dimensions* 3 *and* 4. Donaldson, Thomas, and Segal observed that many attractive features of Yang–Mills theory on 4–manifolds generalize to manifolds of dimensions 6, 7, 8 provided that they are equipped with Riemannian metrics with *special holonomy*, given by the Lie groups SU(3), G_2, Spin(7) respectively. These authors put forward an idea of defining invariants of such manifolds using higher-dimensional Yang–Mills theory. Such invariants would be helpful in classifying the existing millions of examples of special holonomy metrics, of which, at present, there is no systematic understanding. Special holonomy metrics, due to their relationship to supersymmetry, are of independent interest in string theory, a branch of theoretical physics.

The main difficulty in both of these research directions stems from possible degenerations of solutions to gauge-theoretic equations, which occur when their energy concentrates along a lower-dimensional set inside the manifold. For example, if X is a Riemannian 7–manifold with the holonomy group G_2, then the energy of Yang–Mills connections can concentrate along a 3–dimensional minimal submanifold $M \subset X$. This leads to an intriguing relationship between Yang–Mills theory on X and solutions of different gauge-theoretic equations on M, called *multi-monopoles* and *harmonic* $\mathbb{Z}/2$ *spinors*. Interestingly, harmonic $\mathbb{Z}/2$ spinors appear also in many other contexts in gauge theory, namely as degenerations of flat $\mathrm{SL}(2,\mathbb{C})$ connections and solutions of the Kapustin–Witten and Vafa–Witten equations.

In my thesis, I studied multi-monopoles and harmonic $\mathbb{Z}/2$ spinors on 3–manifolds. In particular, in [Doa1, Doa2], I constructed the first non-trivial examples of moduli spaces of multi-monopoles by relating them to algebraic geometry of Riemann surfaces. In joint work with Walpuski [DW1], I described a wall-crossing phenomenon for multi-monopoles which we then used to prove the existence of harmonic $\mathbb{Z}/2$ spinors on all 3–manifolds with $b_1 > 1$ [DW3]. These are so far the only general existence results for multi-monopoles and harmonic $\mathbb{Z}/2$ spinors.

Another research direction I pursued is related to manifolds with holonomy SU(3), that is: Calabi–Yau three-folds. In this case, the Donaldson–Uhlenbeck–Yau theorem establishes a correspondence between Yang–Mills connections and stable holomorphic vector bundles, building a bridge between gauge theory and algebraic geometry. Guided by this correspondence, Thomas constructed invariants of projective Calabi–Yau three-folds using methods of algebraic geometry. Walpuski and I initiated a project of defining an analogous invariant of a more general class of *symplectic Calabi–Yau three-folds* (that is: symplectic 6–manifolds with vanishing first Chern class) using pseudo-holomorphic curves and gauge theory [DW2]. Such a construction, apart from providing new ways of distinguishing symplectic 6–manifolds, should lead to a novel approach to the famous MNOP conjecture in algebraic geometry, which relates various enumerative invariants of Calabi–Yau three-folds. Our papers [DW4, DW5] make progress in this direction by establishing new results on compactness and equivariant transversality for pseudo-holomorphic curves in symplectic 6–manifolds.

1. Multi-monopoles and harmonic $\mathbb{Z}/2$ spinors. The notion of a harmonic spinor is central to Atiyah–Singer index theory. A related notion is that of a *harmonic* $\mathbb{Z}/2$ *spinor*: a two-valued harmonic spinor on a Riemannian spin manifold M which vanishes along a codimension two set $Z \subset M$ and changes sign when we go around a small loop linking Z. We say that such a spinor is *singular* if Z is non-empty. A basic example is the function $z \mapsto \sqrt{z}$ on $M = \mathbb{C}$, with $Z = \{0\}$. Harmonic $\mathbb{Z}/2$ spinors were introduced by Taubes who discovered that they appear in compactifications of the moduli spaces of flat $\mathrm{SL}(2,\mathbb{C})$ connections on 3–manifolds and solutions to the Kapustin–Witten and Vafa–Witten equations [Tau1, Tau2, Tau3]. Harmonic $\mathbb{Z}/2$ spinors appear also as rescaling limits of sequences of Yang–Mills connections on G_2–manifolds whose energy concentrates along a 3–dimensional submanifold [Wal]. In both of these contexts, it is crucial to study the existence problem for harmonic $\mathbb{Z}/2$ spinors.

Problem 1. Given M of dimension 3 or 4, is there a singular harmonic $\mathbb{Z}/2$ spinor on M?

Very little is known about this problem; even in the simplest case $M = \mathbb{R}^3$ it is unknown if there is a harmonic $\mathbb{Z}/2$ spinor with Z non-empty and compact. So far, the only examples of harmonic $\mathbb{Z}/2$ spinors were constructed in special cases using complex geometry. Walpuski and I remedied this situation by proving that 3–manifolds abound with harmonic $\mathbb{Z}/2$ spinors. Motivated by applications in gauge theory, we considered a twist of the Dirac equation by a connection ∇ on a fixed $\mathrm{SU}(2)$–bundle on M. As a result, the existence of harmonic $\mathbb{Z}/2$ spinors depends not only on the choice of a Riemannian metric on M but also on the choice of ∇.

Theorem 2 ([DW3])**.** *For every compact spin 3–manifold M with $b_1(M) > 1$ there are a Riemannian metric g and a connection ∇ with respect to which there exists a singular harmonic $\mathbb{Z}/2$ spinor on M. In fact, there are uncountably many such (g, ∇).*

This result opened up two new research directions. First, its proof suggests a way of constructing explicit examples of harmonic $\mathbb{Z}/2$ spinors using gluing methods (discussed in Section 3). Second, it proves the first part of the following conjecture in the case $b_1(M) > 1$.

Conjecture 3 (Donaldson)**.** *Let $\mathscr{P}$ be the space of all metrics g and connections ∇ on M, and let $\mathscr{P}_{\mathbb{Z}/2} \subset \mathscr{P}$ be the subspace of those (g, ∇) with respect to which there exists a singular harmonic $\mathbb{Z}/2$ spinor. The subspace $\mathscr{P}_{\mathbb{Z}/2}$ is non-empty and has codimension one in $\mathscr{P}$.*

The proof of Theorem 2 suggests that the second part of the conjecture is also true; it uses a cohomology class of *degree one* in $\mathscr{P}$, which can be informally thought of as the Poincaré dual of $\mathscr{P}_{\mathbb{Z}/2}$ in $\mathscr{P}$. The construction of this class exploits a relationship between harmonic $\mathbb{Z}/2$ spinors and *multi-monopoles*: solutions of a generalization of the Seiberg–Witten equation. Haydys and

Walpuski showed that a sequence of multi-monopoles either converges to another multi-monopole or degenerates to a harmonic $\mathbb{Z}/2$ spinor [HW]. Walpuski and I studied the converse problem.

Problem 4. Given a harmonic $\mathbb{Z}/2$ spinor on a 3–manifold M, is there a sequence of multi-monopoles which degenerates to it in the way described by the compactness theorem [HW]?

We proved that the answer is positive when $Z = \emptyset$, that is: for non-singular harmonic spinors.

Theorem 5 ([DW1, DW3])**.** *Every nowhere vanishing harmonic spinor ψ on a compact spin 3–manifold M arises as a degeneration of a family of multi-monopoles.*

As a consequence, we derived a *wall-crossing formula* describing how the signed count of multi-monopoles changes when we vary (g, ∇) in $\mathscr{P} \setminus \mathscr{P}_{\mathbb{Z}_2}$. There is a similar wall-crossing phenomenon in gauge theory on G_2–manifolds: the signed count of Yang–Mills connections can change when we deform the G_2–metric. This change occurs whenever there exists a certain 3–dimensional submanifold with a non-zero harmonic spinor [DS, Wal]. Since both the count of Yang–Mills connections and that of multi-monopoles jump due to the appearance of harmonic spinors, Haydys and Walpuski conjectured that a combination of the two numbers should be invariant under deformations of G_2–metrics. Our wall-crossing formula is the first step in making this proposal rigorous.

2. Gauge theory on Riemann surfaces. Since the proofs of Theorem 2 and Theorem 5 are non-constructive, it is desirable to find concrete examples of multi-monopoles and harmonic $\mathbb{Z}/2$ spinors. In articles [Doa1, Doa2], I constructed the first examples of multi-monopoles and harmonic $\mathbb{Z}/2$ spinors arising as their degenerations, and described their moduli spaces. The starting point was proving an isomorphism, true for every compact Riemann surface Σ, between

- the moduli space $\mathscr{M}_{\text{gauge}}$ of multi-monopoles on the 3–manifold $M = S^1 \times \Sigma$, and
- the moduli space $\mathscr{M}_{\text{alg}}$ of certain algebro-geometric data on Σ.

This is a version of a *Hitchin–Kobayashi correspondence* relating gauge theory to algebraic geometry. A related isomorphism was established in dimension 4 by Bryan and Wentworth [BW]. I used this correspondence to study the following problem, which is related to extending the wall-crossing formula mentioned above to the case of singular harmonic $\mathbb{Z}/2$ spinors.

Problem 6. Use the compactness theorem [HW] to define a compactification of the moduli space of multi-monopoles which is compatible with Fredholm deformation theory.

For example, we might look for a compact analytic space containing the moduli space, with its natural analytic structure induced from deformation theory, as a Zariski open, dense subset. There are two problems with constructing such a space. First, the convergence statement in [HW] uses a weak Sobolev topology which does not interact well with Fredholm theory. Second, there is at present no Fredholm theory describing the moduli space of harmonic $\mathbb{Z}/2$ spinors. I solved these problems in the case $M = S^1 \times \Sigma$ by defining a gauge-theoretic compactification $\bar{\mathscr{M}}_{\text{gauge}}$ and comparing it to an algebro-geometric compactification $\bar{\mathscr{M}}_{\text{alg}}$ constructed in [BW].

Theorem 7 ([Doa1, Doa2])**.** *The isomorphism of analytic spaces $\mathscr{M}_{\text{gauge}} \cong \mathscr{M}_{\text{alg}}$ extends to a homeomorphism of their compactifications $\bar{\mathscr{M}}_{\text{gauge}} \cong \bar{\mathscr{M}}_{\text{alg}}$. In particular, $\bar{\mathscr{M}}_{\text{gauge}}$ has the structure of a complex analytic space with $\mathscr{M}_{\text{gauge}}$ being a Zariski open, dense subset.*

The construction of $\bar{\mathscr{M}}_{\text{gauge}}$ uses a strengthening of the Haydys–Walpuski compactness theorem for $M = S^1 \times \Sigma$. Its proof involves a novel method which can be described as a *quantitative Hitchin–Kobayashi correspondence*. The point is to establish a priori estimates on complex gauge transformations used in the construction of the isomorphism $\mathscr{M}_{\text{gauge}} \cong \mathscr{M}_{\text{alg}}$ as we approach the ends of the moduli space. I plan to apply this method to study compactness for other gauge-theoretic equations, such as the Vafa–Witten equation, on Kähler manifolds.

Using the algebro-geometric description of the compactification, I verified, in this special setting, a version of Conjecture 3, and described generic properties of the moduli space.

Theorem 8 ([Doa2])**.** *Denote by $\mathscr{P}_\Sigma \subset \mathscr{P}$ the set of Riemannian metrics g and connections ∇ on $M = S^1 \times \Sigma$ which are pulled-back from Σ. The set $\mathscr{P}_\Sigma \cap \mathscr{P}_{\mathbb{Z}/2}$ is of complex codimension one in $\mathscr{P}_\Sigma$. Moreover, for a generic choice of (g, ∇) from $\mathscr{P}_\Sigma \setminus \mathscr{P}_{\mathbb{Z}/2}$ the corresponding moduli space of multi-monopoles $\mathscr{M}_{\text{gauge}}$ is a compact Kähler manifold (in particular, $\mathscr{M}_{\text{gauge}} = \bar{\mathscr{M}}_{\text{gauge}}$).*

I supplemented this general result with a detailed analysis of $\mathscr{M}_{\text{alg}}$ and $\bar{\mathscr{M}}_{\text{alg}}$ for Riemann surfaces of low genus, in which I explored connections of multi-monopoles with classical topics in algebraic geometry, such as moduli of stable bundles, theta divisors, and Brill–Noether theory.

3. J–holomorphic curves in 6–manifolds. In 1889, Castelnuovo proved that for every projective variety there is an upper bound on the genus of an irreducible holomorphic curve contained in that variety and representing a prescribed homology class. In joint work with Walpuski, I studied an analogous problem in almost complex geometry.

Question 9. If (X, J) is a compact almost complex manifold, is there a bound on the genus of a J–holomorphic curve in X representing a given class $A \in H_2(X, \mathbb{Z})$?

The adjunction inequality gives an affirmative answer when $\dim X = 4$. For $\dim X > 6$, a simple dimension counting argument shows that Castelnuovo's bound exists when J is *generic*. We focused on the remaining case $\dim X = 6$, in particular on those classes A for which the dimension of the moduli space of J–holomorphic maps is zero, e.g. all classes in symplectic Calabi–Yau three-folds. Our motivation for considering this case comes from our project, described below, to construct new invariants of symplectic Calabi–Yau three-folds [Doa1]. Castelnuovo's bound is also closely related to the famous Gopakumar–Vafa conjecture [IP]. Walpuski and I related this problem to the property of *super-rigidity*, which is expected to be satisfied by a generic almost complex structure.

Theorem 10. [DW5] *Suppose that $\dim X = 6$, J is a super-rigid almost complex structure, and $A \in H_2(X, \mathbb{Z})$ is such that the virtual dimension of the moduli space of J–holomorphic maps representing A is zero. Given any $\Lambda > 0$, there are only finitely many simple J–holomorphic maps representing A with energy at most Λ. In particular, if X carries a symplectic structure which tames J, then there are only finitely many simple J–holomorphic maps representing A.*

In the situation of the theorem, Gromov's compactness theorem shows that there are finitely many simple J–holomorphic maps representing A from Riemann surfaces of a *fixed genus*—it is thus not of much use for proving Theorem 10, which deals with maps from surfaces of *any genus*. Instead, we establish a compactness theorem for *J–holomorphic cycles*: formal sums of embedded J–holomorphic curves. Its proof relies on ideas of Taubes and results of geometric measure theory, in particular recent work of DeLellis, Spadaro, and Spoloar on semi-calibrated currents.

Eftekhary and Wendl made significant progress towards proving that super-rigidity is a generic property [Eft, Wen]. Their work can be seen a theory of equivariant transversality for Cauchy–Riemann operators. Inspired by Wendl's ideas, Walpuski and I developed a general theory dealing with equivariant transversality for families of elliptic operators [DW4]. We plan to apply this theory to study multiple covers of calibrated submanifolds in Riemannian manifolds of special holonomy, for instance associatives in G_2–manifolds and special Lagrangians in Calabi–Yau three-folds.

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