SYMPLECTIC COHOMOLOGY FOR STABLE FILLINGS

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Abstract. We discuss a generalisation of symplectic cohomology for symplectic manifolds which weakly fill their contact boundary and satisfy an additional stability condition. Furthermore, we develop a geometric setting for proving maximum principles for Floer trajectories, and prove a Moser-type result for weak fillings. This is a preliminary version of the paper.

1. Introduction

Symplectic cohomology is a generalisation of Hamiltonian Floer homology, which has proved to be useful in symplectic topology, for example in the study of symplectic fillings or Stein manifolds. A natural class of manifolds for which the invariant is defined, are *Liouville domains*, i.e. manifolds equipped with an exact symplectic form satisfying certain convexity properties at the boundary. We refer the reader to [6] for a survey on symplectic cohomology for Liouville domains.

Symplectic cohomology can be defined for the broader class of symplectic manifolds with contact type boundary. Such manifolds are equipped with a symplectic form that no longer needs to be exact, apart from a neighbourhood of the boundary, where it satisfies the same convexity conditions as Liouville manifolds. In other words, they *strongly fill* their contact boundary. Symplectic cohomology for strong fillings was used in [5] to study Lagrangian submanifolds of certain Liouville domains by considering non-exact perturbations of the original, exact symplectic structure. These perturbations preserved the exactness of the symplectic form at the boundary. Deformations that are not exact at the boundary would result in symplectic manifolds that fail to satisfy the strong filling condition. The question whether symplectic cohomology can still be defined for such manifolds was the main motivation for this project. Such a generalisation could potentially be helpful in understanding deformations of Liouville domains that are not exact at the boundary, such as the ones discussed in [3].

Recently, the notion of a *weak symplectic filling* in higher dimensions was introduced in [4]. Contrary to the notion of a strong filling, it is defined by an open condition, which means that a small deformation of a weak filling is again a weak filling. Furthermore, a symplectic manifold weakly filling its boundary admits a tamed almost complex structure making the boundary pseudoconvex. This suggests that, as in the case of strong fillings, there could be a maximum principle preventing solutions of the Floer equation from escaping to infinity, which is one of the main tools required to extend the classical definition of Floer homology to non-compact cases.

We show that such a maximum principle can indeed be established, and hence symplectic cohomology defined, under the additional assumption that the weak filling is *stable*, i.e. induces a stable Hamiltonian structure at the boundary. Specifically, we prove the following

Theorem A. *Let (M,ω) be a stable symplectic filling of a contact manifold (V,λ). If bubbling can be eliminated (for instance, if M is monotone or $c_1(M)=0$) and all the Reeb orbits of (V,λ) are non-degenerate, then symplectic cohomology $SH^*(M,\omega)$ is well-defined.*

Failed attempts at proving the maximum principle for Floer trajctories without the stability condition led us to the following question.

Question. Let (M,ω) be a symplectic manifold equipped with a real-valued function $r: M \to \mathbb{R}$, a Hamiltonian $H: M \to \mathbb{R}$, and an almost complex structure J tamed by ω. Under what assumptions on r, H, and J does the function $r \circ u$ satisfy a maximum principle[1] for any Floer trajectory $u: \mathbb{R} \times S^1 \to M$? What if we assume that both H and J depend on parameters $(s,t) \in \mathbb{R} \times S^1$ and ask the same question for (s,t)-dependent Floer trajectories?

We present a partial answer to this problem. The main idea is to interpret Floer trajectories in M as pseudoholomorphic sections of the trivial fibration $\mathbb{R}\times S^1 \times M \to \mathbb{R}\times S^1$ equipped with a certain almost complex structure. Such an interpretation was previously considered in [2]. Subsequently, for an almost complex manifold (M,J) we introduce the class of *functions satisfying an elliptic inequality* (which can be seen as a generalisation of plurisubharmonic functions) such that if $f: M \to \mathbb{R}$ satisfies an elliptic inequality, then a maximum principle holds for $f \circ u$ for any J-holomorphic curve $u: \Sigma \to M$. Then the following holds.

Theorem B. *Let $M, \omega, J, H,$ and r be as before. Let ϕ^t be the Hamiltonian flow of H and $\mathcal{L}_H$ the Lie derivative along ϕ^t. Denote by $\xi = \{dr = d^J r = 0\}$ the distribution of complex tangencies of r, and by $\mathcal{I}_\xi = \langle dr, d^J r\rangle$ the ideal of differential forms on M vanishing on ξ. Assume that*

(1) *r satisfies an elliptic inequality,*
(2) *r is ϕ^t-invariant,*
(3) *$\mathcal{L}_H \mathcal{I}_\xi \subset \mathcal{I}_\xi$.*

Then the function $r \circ u$ satisfies a maximum principle for any solution of the Floer equation $u: \mathbb{R} \times S^1 \to M$ corresponding to (M,ω,H,J).

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[1] By a maximum principle for a function $f: \Sigma \to \mathbb{R}$ on a manifold Σ we understand that $f\big|_K$ attains its maximum at the boundary of K for any compact subset $K \subset \Sigma$.

Condition (3) can be interpreted as infinitesimal invariance of ξ under the Hamiltonian flow and in the case of weak fillings is related to the stability condition.

We end with a Moser-type stability result for deformations of weak fillings.

Theorem C. *Let (M, Ω_0) be a weak filling of a contact manifold $(V, \xi = \ker\lambda)$. Denote by $\widehat{M} = M \cup (\mathbb{R}\times V)$ its cylindrical completion. Let $\{\Omega_s\}_{s\in[0,1]}$ be a smooth family of symplectic forms on $\widehat{M}$ whose cohomology class $[\Omega_s]$ in $H^2(\widehat{M}, \mathbb{R})$ does not depend on s, and which on the collar $\mathbb{R}\times V$ has the form*

$$\Omega_s = \omega_s + d(r\lambda)$$

for a family $\{\omega_s\}_{s\in[0,1]}$ of two-forms on V. Then Ω_0 and Ω_1 are isotopic.

We conclude that every weak filling whose symplectic form has rational cohomology class at the boundary, is symplectomorphic to a stable filling after attaching a cylindrical end.

The paper is organised as follows. In section 2, we present basic definitions and properties of weak and stable fillings, based mostly on [4]. Then we recall the construction of symplectic cohomology, with main references being [6] and [5], discuss how it can be generalised to stable fillings, and prove Theorem A. Section 3 is devoted to our attempt to find a geometric way of thinking about maximum principles for Floer trajectories. We show how they can be seen as pseudoholomorphic curves in a larger symplectic manifold and use this interpretation to show Theorem B. In section 4, we prove Theorem C.

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2. Weak fillings and symplectic cohomology

2.1. **Weak and stable symplectic fillings.** We begin with a brief discussion of weak symplectic fillings of contact manifolds in higher dimensions. Our approach is based on [4], where all the details and proofs can be found.

Remark. We assume all manifolds to be oriented and all contact structures to be co-oriented. Whenever we write $\partial M = V$, we mean the equality as oriented manifolds, where the boundary ∂M is equipped with the natural orientation induced from M. If η is an n-form on an n-dimensional manifold M, by $\eta > 0$ we mean that η is a positive volume form on M. For simplicity, we denote $\omega_V := \omega\big|_V = i^*\omega$, whenever ω is a differential form on M and $i : V = \partial M \to M$ denotes the inclusion, and $\omega_\xi = \omega\big|_\xi$ for a subbundle $\xi \subset TM$.

Let (M, ω) be a compact symplectic manifold with boundary $\partial M = V$. Assume that ξ is a positive contact structure on V. Denote by CS_ξ the conformal class of symplectic structures on ξ induced by the contact structure (i.e. the class of $d\lambda_\xi$ for any defining one-form λ).

Definition 2.1. (M, ω) is a *weak filling* of (V, ξ) if

(1) ω_ξ is a symplectic bundle structure on ξ,
(2) $\omega_\xi + \mathrm{CS}_\xi$ is a ray of symplectic structures on ξ.

In other words, for every one-form λ defining ξ, we have that on $V = \partial M$,

$$\lambda \wedge (d\lambda + \omega_\xi)^{n-1} > 0 \qquad \text{and} \qquad \lambda \wedge \omega_\xi^{n-1} > 0.$$

Weak fillings can be characterised in terms of pseudoconvexity of their boundary. Before we state it as a theorem, let us recall the following notion:

Definition 2.2. A contact manifold (V, ξ) is called the *tamed pseudoconvex boundary* of an almost complex manifold (M, J) if $\partial M = V$ and

(1) V is J-convex, meaning that for any defining one-form λ, we have
$$d\lambda(v, Jv) > 0 \quad \text{for all nonzero } v \in \xi,$$
(2) ξ is J-invariant or, equivalently, $\xi = TV \cap J(TV)$.

Theorem 2.3. *A symplectic manifold (M, ω) with $\partial M = V$ is a weak filling of (V, ξ) if and only if it admits an almost complex structure J tamed by ω which makes (V, ξ) the tamed pseudoconvex boundary of (M, J).*

We will be interested in a particular class of weak fillings satisfying an additional stability property at the boundary.

Definition 2.4. A *stable Hamiltonian structure* on an oriented $(2n-1)$-dimensional manifold V is a pair (λ, ω), where λ is a one-form and ω is a closed two-form of maximal rank (i.e. ω^{n-1} is nowhere vanishing) such that

$$\lambda \wedge \omega^{n-1} > 0 \qquad \text{and} \qquad \ker \omega \subset \ker d\lambda.$$

Definition 2.5. A *stable filling* of a contact manifold (V, ξ) is a weak filling (M, ω) such that (ω_V, λ) is a stable Hamiltonian structure on V for some contact one-form λ.

Remark. From now on, for a given stable filling (M, ω) we will assume that a contact form λ on V has been chosen so that (ω_V, λ) is a stable Hamiltonian structure. In fact, such a contact form is unique up to multiplication by a constant. Indeed, assume that λ' is another contact form on (V, ξ) with this property. Then $\lambda' = \psi\lambda$ for a positive function $\psi : V \to \mathbb{R}$. Let $\mathcal{R}$ be the Reeb vector field of λ and $X \in \xi$. Then

$$\begin{aligned} 0 = d\lambda'(\mathcal{R}, X) &= (d\psi \wedge \lambda + \psi d\lambda)(\mathcal{R}, X) = \\ &= d\psi(\mathcal{R})\lambda(X) - d\psi(X)\lambda(\mathcal{R}) + \psi d\lambda(\mathcal{R}, X) = -d\psi(X). \end{aligned}$$

Therefore, $d\psi\big|_\xi = 0$. Since ξ is a contact structure, it follows that $d\psi = 0$ (otherwise ξ would be integrable) and $\lambda' = c\lambda$ for $c \in \mathbb{R}$.

It turns out that weak fillability and stable fillability are equivalent.

Proposition 2.6. *Any weak filling (M,ω) of a contact manifold (V,ξ) can be deformed to a stable one, i.e. there exists a smooth family $\{\omega_s\}$ of symplectic structures on M such that (M,ω_s) is a weak filling of (V,ξ) for all $s\in[0,1]$, $\omega_0=\omega$, and (M,ω_1) is a stable weak filling of (V,ξ).*

Let (M,ω) be a weak filling of $(V,\xi=\ker\lambda)$. Then there exists a tubular neighbourhood of V in M symplectomorphic to $V\times(-\epsilon,0]$ equipped with the symplectic form $\omega_V+d(r\lambda)$. Denote by $(\widehat{M},\widehat{\omega})$ the *magnetic* (or *cylindrical*) *completion* of (M,ω), i.e. the manifold obtained from (M,ω) by attaching the cylindrical end $V\times[0,\infty)$:

$$\widehat{M}:=M\cup_V(V\times[0,\infty)),$$

$$\widehat{\omega}\big|_M:=\omega,\qquad \widehat{\omega}\big|_{V\times[0,\infty)}:=\omega_V+d(r\lambda).$$

If (M,ω) is a stable weak filling, we will assume that the contact form λ is chosen in such a way that $\ker d\lambda=\ker\omega_V$.

Remark. If the cohomology class of ω_V is rational, i.e. $[\omega_V]\in H^2(V,\mathbb{Q})$, then in fact the deformation in Proposition 2.6 can be chosen so that $[\omega_s]=[\omega]$ for all $s\in[0,1]$. As we prove in Section 3, this induces an isotopy of symplectic structures on the cylindrical completion $\widehat{M}$. In particular, the cylindrical completions of (M,ω_0) and (M,ω_1) are symplectomorphic. However, in the general case, first ω needs to be perturbed so that its cohomology class is rational at the boundary and we cannot use Moser's trick.

2.2. **Symplectic cohomology.** We briefly recall the definition of symplectic cohomology, adapting it to the case of weak fillings. See [6] and [5] for details in the case of strong fillings.

Let (M,ω) be a stable filling of a contact manifold $(V,\xi=\ker\lambda)$, and $(\widehat{M},\widehat{\omega})$ its cylindrical completion. To avoid bubbling, we assume that M is monotone. Furthermore, we assume that all Reeb orbits of (V,λ) are non-degenerate. We want to construct Hamiltonian Floer homology for Hamiltonians and almost complex structures whose behaviour on the non-compact collar $V\times[0,\infty)$ is subject to certain conditions. Firstly, we want orbits of a Hamiltonian on the collar to correspond to Reeb orbits on V. Secondly, a maximum principle should hold for solutions $u:\mathbb{R}\times S^1\to\widehat{M}$ of the Floer equation

$$\partial_s u+J\left(\partial_t u-X_H\right)=0$$

that prevents them from escaping to infinity. In fact, we have to establish a maximum principle for (s,t)-dependent Floer trajectories, i.e. solutions of the equation

$$\partial_s u+J(s,t)\left(\partial_t u-X_{H(s,t)}\right)=0$$

for families of Hamiltonians and almost complex structures. We consider t-dependency, because we need to perturb H and J to achieve transversality, and s-dependency in order to define the continuation maps for different choices of Hamiltonians.

The following definitions and proofs will not differ much from the standard ones, used in the case of strong fillings. However, they will make use of the stability condition (in the sense of Definition 2.5).

Definition 2.7. We call a smooth function $H : \widehat{M} \to \mathbb{R}$ an *admissible Hamiltonian* if $H(x, r) = h(r)$ on the collar $V \times [0, \infty)$.

Recall that the Hamiltonian vector field X_H of a function H is defined by

$$\widehat{\omega}(\cdot, X_H) = dH.$$

Lemma 2.8. *Let $H : \widehat{M} \to \mathbb{R}$ be an admissible Hamiltonian and $\mathcal{R}$ the Reeb vector field of (V, λ). Then $X_H = h'(r)\mathcal{R}$ on the collar.*

Proof. We want to show that for any vector field X on $V \times [0, \infty)$ we have

$$\widehat{\omega}(X, h'(r)\mathcal{R}) = dH(V).$$

On the collar $\widehat{\omega} = \omega_V + d(r\lambda)$ and $dH = h'(r)dr$ and we want to prove that

$$\omega_V(X, h'(r)\mathcal{R}) + dr \wedge \lambda(X, h'(r)\mathcal{R}) + rd\lambda(X, h'(r)\mathcal{R}) = h'(r)dr(X),$$

which, since $\mathcal{R}$ belongs to $\ker d\lambda = \ker \omega_V$, simplifies to

$$dr \wedge \lambda(X, h'(r)\mathcal{R}) = h'(r)dr(X).$$

The last equality follows immediately from $dr(\mathcal{R}) = 0$ and $\lambda(\mathcal{R}) = 1$. Note that in the proof we have used the stability condition. □

As in the standard case, the lemma implies that any non-constant one-periodic orbit $x(t)$ of X_H which intersects the collar, is contained in the level-set $V \times \{r\}$ and corresponds to a Reeb orbit $z(t) = x(t/T)$ with period $T = h'(r)$.

Definition 2.9. An almost complex structure J on $\widehat{M}$ is *admissible* if

(1) J is tamed by ω on M,
(2) On the collar $V \times [0, \infty)$,
  - $J\partial_r = \mathcal{R}$ and $J\mathcal{R} = -\partial_r$,
  - J preserves ξ and its restriction to ξ is tamed by ω_ξ and $d\lambda$.

By Theorem 2.3, there always exists an admissible almost complex structure on $\widehat{M}$. We just need choose any almost complex structure on M making (V, ξ) into a tamed pseudoconvex boundary and extend it to the collar $V \times [0, \infty)$ in an r-independent way. Furthermore, the space of admissible almost complex structures is contractible (see appendix in [4]).

Lemma 2.10. *An admissible almost complex structure is $\widehat{\omega}$-tamed.*

Proof. We want to show $\widehat{\omega}(X, JX) > 0$ for any non-zero vector $X \in T\widehat{M}$. Since $\widehat{\omega}$ restricts to ω on W, which tames J, it is enough to check the condition on the collar $V \times [0, \infty)$, where we have the decomposition

$$T\widehat{M} = \xi \oplus \langle \mathcal{R} \rangle \oplus \langle \partial_r \rangle.$$

Let us consider three basic cases. If $X \in \xi$, then

$$\widehat{\omega}(X, JX) = \omega_V(X, JX) + dr \wedge \lambda(X, JX) + rd\lambda(X, JX) > 0,$$

because $dr(JX) = dr(X) = 0$ and both ω_V and $d\lambda$ tame J on ξ.

Assume now that $X = \mathcal{R}$. Then $J\mathcal{R} = -\partial_r$ and we easily calculate

$$\widehat{\omega}(\mathcal{R}, J\mathcal{R}) = -\omega_V(\mathcal{R}, \partial_r) - dr \wedge \lambda(\mathcal{R}, \partial_r) - rd\lambda(\mathcal{R}, \partial_r) = \lambda(\mathcal{R})dr(\partial_r) = 1,$$

because $\mathcal{R}$ belongs to $\ker d\lambda = \ker \omega_V$ and $\lambda(\mathcal{R}) = 1$.

Finally, for $v = \partial_r$ we have

$$\widehat{\omega}(\partial_r, J\partial_r) = \widehat{\omega}(\partial_r, \mathcal{R}) = \widehat{\omega}(\mathcal{R}, -\partial_r) = 1.$$

In general, for $X = a\mathcal{R} + b\partial_r + Y$ with $Y \in \xi$, we have

$$\begin{aligned}\widehat{\omega}(X, JX) &= \widehat{\omega}(a\mathcal{R} + b\partial_r + Y, b\mathcal{R} - a\partial_r + JY) = \\ &= a^2\widehat{\omega}(\mathcal{R}, J\mathcal{R}) + b^2\widehat{\omega}(\partial_r, J\partial_r) + \widehat{\omega}(Y, JY) \geq 0\end{aligned}$$

and equality holds if and only if $a = b = 0$ and $Y = 0$. Note that we have used the stability condition to get $\widehat{\omega}(\mathcal{R}, Y) = 0$ for any $Y \in \xi$. □

For admissible Hamiltonians and almost complex structures, there is a maximum principle which guarantees a C^0-bound for Floer trajectories. The proof is essentially the same as for strong fillings.

Proposition 2.11. *If H is an admissible Hamiltonian, and J an admissible almost complex structure, then for any smooth map $u : \Omega \to V \times [0, \infty)$ defined on a compact set $\Omega \subset \mathbb{R} \times S^1$ and satisfying the Floer equation*

$$\partial_s u + J(\partial_t u - X_H) = 0,$$

the maxima of the function $r \circ u$ are attained at the boundary $\partial\Omega$.

Proof. We can write $u(s, t) = (v(s, t), f(s, t))$, where v takes values in V and $f = r \circ u$. The Floer equation, projected into the components of the decomposition $T(V \times [0, \infty)) = \xi \oplus \langle \mathcal{R} \rangle \oplus \langle \partial_r \rangle$ gives us the following equations:

$$\begin{aligned}\pi_\xi(\partial_s v) + J\pi_\xi(\partial_t v) &= 0,\\ \lambda(\partial_s v) + \partial_t f &= 0,\\ \partial_s f - \lambda(\partial_t v) + h' \circ f &= 0,\end{aligned}$$

where π_ξ denotes the projection onto ξ. Here we use that for v tangent to V, the projection of v on $\langle \mathcal{R} \rangle$ is $\lambda(v)\mathcal{R}$ and that $X_H = h'(r)\mathcal{R}$.

Using the equations and the fact that $J\big|_\xi$ is tamed by $d\lambda$, we obtain

$$\begin{aligned}0 \leq d\lambda(\pi_\xi(\partial_s v), J\pi_\xi(\partial_s v)) &= d\lambda(\partial_s v, \partial_t v) = \\ &= \partial_s[\lambda(\partial_t v)] - \partial_t[\lambda(\partial_s v)] = \left(\partial_s^2 + \partial_t^2\right) f + \left(h'' \circ f\right)\partial_s f\end{aligned}$$

Therefore, $f = r \circ u$ satisfies the inequality $Lf \geq 0$ for the second order elliptic operator $L = \partial_s^2 + \partial_t^2 + h''(f)\partial_s$ and the general theory of elliptic PDE establishes the maximum principle for f. □

We want to generalise the above maximum principle to the case when $H = H(s,t)$ and $J = J(s,t)$ depends on parameters (s,t). Since t-dependency does not change anything in the proof, we will omit it for simplicity. Let us introduce the following defition.

Definition 2.12. An *admissible perturbation* is a family $(\omega_s, \lambda_s, H_s, J_s)$ depending smoothly on a parameter $s \in \mathbb{R}$, where

(1) (ω_s) is a family of symplectic structures on $\widehat{M}$, and $(\ker \lambda_s)$ is a family of contact structures on V, such that
$$\omega_s = (\omega_s)_V + d(r\lambda_s) \quad \text{on } V \times [0,\infty)$$
and $((\omega_s)_V, \lambda_s)$ is a stable homotopy of Hamiltonian structures,

(2) (H_s) is a monotone family of admissible Hamiltonians, that is, there exists a smooth function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that on the collar
$$H_s(x,r) = h(r,s) \quad \text{and} \quad \partial_s \partial_r h \leq 0.$$

(3) (J_s) is a family of almost complex structures on $\widehat{M}$ such that J_s preserves $\ker \lambda_s$, is tamed by $d\lambda_s$ and ω_s on $\ker \lambda_s$, and satisfies $J_s \partial_r = \mathcal{R}_s$, where $\mathcal{R}_s$ is the Reeb vector field of (V, λ_s).

If $(\omega_s, \lambda_s, H_s, J_s)$ is an admissible perturbation, then we can define the corresponding Hamiltonian vector field X_s as the s-dependent vector field on $\widehat{W}$ given by the condition $\omega(\cdot, X_s) = dH_s$. Repeating word by word the proof of Lemma 2.8, we see that on the collar $X_s = \partial_r h(s,r)\mathcal{R}_s$.

The following generalisation of Proposition 2.11 holds.

Proposition 2.13. *If (ω_s, H_s, J_s) is an admissible perturbation and X_s denotes the corresponding Hamiltonian vector field, then for any function $u : \Omega \to M \times [0,\infty)$ defined on a compact set $\Omega \subset \mathbb{R} \times S^1$ and satisfying the parametrised Floer equation*
$$\partial_s u + J_s(\partial_t u - X_s) = 0,$$
the maxima of the function $r \circ u$ are attained at the boundary $\partial\Omega$.

Proof. Denote $\xi_s = \ker \lambda_s \subset TV$. As in the proof of Proposition 2.11, we write $u(s,t) = (f(s,t), v(s,t))$ and decompose the Floer equation into three equations:
$$\pi_{\xi_s}(\partial_s v) + J_s \pi_{\xi_s}(\partial_t v) = 0,$$
$$\lambda_s(\partial_s v) + \partial_t f = 0,$$
$$\partial_s f - \lambda_s(\partial_t v) - \partial_r h(s,f) = 0.$$
Here we use in particular that $J_s \partial_r = \mathcal{R}_s$ and $J_s \xi_s = \xi_s$ for each s. Furthermore, again we can use the fact that each J_s is tamed by $d\lambda_s$. The

only difference is that this time we have to take under account that $\partial_r h$ also depends on s:

$$\begin{aligned} 0 \leq d\lambda_s\left(\pi_{\xi_s}\left(\partial_s v\right), J_s \pi_{\xi_s}\left(\partial_s v\right)\right) = d\lambda_s\left(\partial_s v, \partial_t v\right) = \\ = \partial_s\left[\lambda_s\left(\partial_t v\right)\right] - \partial_t\left[\lambda_s\left(\partial_s v\right)\right] = \\ = \left(\partial_s^2 + \partial_t^2\right) f + \partial_r^2 h(s,f)\partial_s f + \partial_s\partial_r h(s,f). \end{aligned}$$

Since we have assumed that $\partial_s \partial h \leq 0$, we see that f again satisfies the inequality $Lf \geq 0$ for the second-order elliptic operator $L = \partial_s^2 + \partial_t^2 + \partial_r^2 h(s,f)\partial_s$, and the maximum principle follows as in the previous case. □

Let H be a Hamiltonian linear at infinity, i.e. on the collar $H(x,r) = \tau r + c$ for r large enough and $\tau > 0$ which is not a period of any Reeb orbit. Then Proposition 2.11 guarantees that $SH^*(H)$ is well-defined as in the standard case. Proposition 2.13 for admissible perturbations of the form $(\omega, \lambda, H_s, J)$ allows us to define the continuation homomorphism:

$$SH^*(H_+) \to SH^*(H_-)$$

for Hamiltonians H_+, H_- linear at infinity with slopes $\tau_+ < \tau_-$. Hence, symplectic cohomology may be defined as the direct limit

$$SH^*(M,\omega) = \varinjlim SH^*(H)$$

taken over the continuation homomorphisms for Hamiltonians linear at infinity. Once we have the maximum principles, the rest is the same as in the case of manifolds with contact-type boundary (see for example [5]) and the details will added in the final version of the paper. We also plan to discuss how $SH^*(M,\omega)$ changes under deformations of ω inducing a stable homotopy of Hamiltonian structures at the boundary.

3. Maximum principle revisited

As we have seen, establishing a maximum principle preventing Floer trajectories from escaping to infinity is crucial for the definition of symplectic cohomology. The situations we have encountered so far fit into the more general framework of the question stated in the introduction. Recall

Question 3.1. Let (M,ω) be a symplectic manifold equipped with a real-valued function $r : M \to \mathbb{R}$, a Hamiltonian $H : M \to \mathbb{R}$, and an almost complex structure J tamed by ω. Under what assumptions on r, H, and J does the function $r \circ u$ satisfy a maximum principle? What if we assume that both H and J depend on parameters $(s,t) \in \mathbb{R} \times S^1$ and ask the same question for (s,t)-dependent Floer trajectories?

Example 3.2. In the definition of symplectic cohomology, we set $M = \mathbb{R} \times V$ to be the symplectisation of a contact manifold V and $r : \mathbb{R} \times V \to \mathbb{R}$ be the projection on the first coordinate. Then a maximum principle is satisfied if $H = h(r)$ depends only on r and J is an admissible almost contact structure. In order to define the continuation map, we considered the case when H and J were (s,t)-dependent.

Example 3.3. In order to prove the invariance of symplectic cohomology under symplectomorphism of contact type at infinity, one needs to establish a maximum principle for the function $\rho : \mathbb{R} \times V \to \mathbb{R}$, $\rho(r, x) = r + f(x)$, Hamiltonians of the form $H = H(\rho)$ and a suitable family of almost complex structures. For details, see for example [5].

In this section we present an attempt to develop a more general setting for proving maximum principles for Floer trajectories. Our answer is far from being complete, and, in particular, it does not include the s-dependent cases described in the above examples.

3.1. **Floer trajectories and pseudoholomorphic curves.** We begin with the interpretation of Floer trajectories in a symplectic manifold as pseudoholomorphic curves in the symplectisation of the mapping torus of the identity map. Such an interpretation is discussed in [2].

Let (M, ω) be a symplectic manifold. Denote by (s, t) the standard coordinates on the cylinder $\mathbb{R} \times S^1$. Here we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and the coordinate t is a real number mod 1. Whenever we consider a family of functions of vector fields depending on t, we will assume that it is 1-periodic. Let $J = J(s, t)$ be a smooth family of almost complex structures on M tamed by ω, and $H = H(s, t)$ a smooth family of Hamiltonians. Let X_H be the (s, t)-dependent Hamiltonian vector field corresponding to H:

$$\omega(\cdot, X_H(s, t)) = dH(s, t).$$

Consider the manifold $W = \mathbb{R} \times S^1 \times M$ equipped with the symplectic form

$$\underline{\omega} = \omega + ds \wedge dt,$$

and an almost complex structure $\underline{J}$ given by the following conditions:

(1) $\underline{J}X = J(s, t)X$ for any $X \in T_xM$ at $(s, t, x) \in W$,
(2) $\underline{J}\partial_s = X_H(s, t) + \partial_t$,
(3) $\underline{J}\partial_t = -\partial_s - J(s, t)X_H(s, t)$.

It is easy to check that these conditions indeed define a unique almost complex structure on W. If we complete a basis of TM to a basis of TW by the vectors $X_H + \partial_t$ and ∂_s, then $\underline{J}$ is represented by the matrix

$$\underline{J} = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & J(s,t) \end{pmatrix},$$

where $J(s, t)$ is the matrix of the almost complex structure on M.

Lemma 3.4. *The almost complex structure $\underline{J}$ is tamed by $\underline{\omega}$.*

Proof. Let X be a vector tangent to W. Decompose X according to the decomposition $TW = \langle \partial_s \rangle \oplus \langle \partial_t \rangle \oplus TM$ to get $X = a\partial_s + b\partial_t + Y$, where $Y \in TM$. By the following simple calculations:

$$\underline{\omega}(\partial_s, \underline{J}\partial_s) = (\omega + ds \wedge dt)(\partial_s, X_H + \partial_t) = ds \wedge dt(\partial_s, \partial_t) = 1,$$

$$\underline{\omega}(\partial_t, \underline{J}\partial_t) = (\omega + ds \wedge dt)\,(\partial_t, -\partial_s - JX_H) = ds \wedge dt(\partial_t, -\partial_s) = 1,$$
$$\underline{\omega}(\partial_s, Y) = \underline{\omega}(\partial_t, Y) = 0,$$

we eventually obtain

$$\underline{\omega}(X, \underline{J}X) = a^2 + b^2 + \omega(Y, J(s,t)Y) \geq 0,$$

where the equality holds if and only if $a = b = 0$ and $Y = 0$. □

Consider the (s,t)-dependent Floer equation for $u : \mathbb{R} \times S^1 \to M$:

$$\partial_s u + J(\partial_t u - X_H) = 0.$$

Note that here both J and X_H depend on (s,t), but for the sake of simplicity of the notation we will omit this dependency whenever it is obvious. We will call maps satisfying the above equations *Floer trajectories*.

The reason for considering the symplectic manifold $(W, \underline{\omega})$ is the following bijective correspondence between Floer trajectories in M and pseudoholomorphic sections of the bundle $W \to \mathbb{R} \times S^1$.

Proposition 3.5. *A differentiable function $u : \mathbb{R} \times S^1 \to M$ satisfies the Floer equation if and only if the corresponding section $v(s,t) = (s,t,u(s,t))$ of the trivial bundle $W \to \mathbb{R} \times S^1$ is $\underline{J}$-holomorphic.*

Proof. Let $v = (\alpha, \beta, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times M = W$ be a differentiable function, where $u : \mathbb{R} \times S^1 \to M$. Consider the Cauchy-Riemann equation for $(W, \underline{J})$:

$$\partial_s v + \underline{J}\partial_t v = 0.$$

Projected on the components of the decomposition $TW = \langle \partial_s \rangle \oplus \langle \partial_t \rangle \oplus TM$, it is equivalent to the following three equations:

$$\begin{cases} \partial_s \alpha - \partial_t \beta = 0, \\ \partial_s \beta - \partial_t \alpha = 0, \\ \partial_s u + J\partial_t u + (\partial_t \alpha) X_H - (\partial_t \beta) J X_H = 0. \end{cases}$$

If v is a section, i.e. $\alpha = \beta = \mathrm{id}$, then

$$\partial_s \alpha = \partial_t \beta = 1, \qquad \partial_t \alpha = \partial_s \beta = 0.$$

The first two equations are then trivially satisfied and the third one reads:

$$\partial_s u + J(\partial_t u - X_H) = 0.$$

Therefore, the Cauchy-Riemann equation for a section $v(s,t) = (s,t,u(s,t))$ is equivalent to the Floer equation for $u(s,t)$. □

If the Hamiltonian function H on M is only t-dependent and does not depend on s, we can consider another almost complex structure $\widehat{J}$ on $W = \mathbb{R} \times S^1 \times M$ which is equivalent to $\underline{J}$. We define $\widehat{J}$ as follows:

$$\widehat{J}\partial_s = \partial_t, \qquad \widehat{J}\partial_t = -\partial_s,$$
$$\widehat{J}X = (\phi^{-t})_* J(t) (\phi^t)_* X \quad \text{for } X \in T_xM \subset T_{(s,t,x)}W.$$

We easily check that $\widehat{J}^2 = -\mathrm{id}$. Intuitively, the almost complex structure $\widehat{J}$ can be seen as the product almost complex structure on $\mathbb{R} \times S^1 \times M$, where the almost complex structure on the M factor is conjugated by the Hamiltonian flow of H.

Recall that we call a time-dependent vector field X_t on a manifold M *complete* if its flow $\phi^t : M \to M$ given by the differential equation

$$\frac{d\phi^t}{dt} = X_t$$

exists for all times t and therefore, is a one-parameter group of diffeomorphisms of M.

Lemma 3.6. *If $H = H(t)$ is a time-dependent function whose Hamiltonian flow is complete, then the map $\Psi : W \to W$ given by*

$$\Psi(s,t,x) = (s,t,\phi^t(x)).$$

is a pseudoholomorphic diffeomorphism between $(W, \widehat{J})$ and $(W, \underline{J})$.

Proof. From the completness of the Hamiltonian flow it follows that the map is a diffeomorphism. We just have to check that it preserves the almost complex structure (here $X \in TM$):

$$\Psi_* \widehat{J} \partial_s = \Psi_* \partial_t = \partial_t + X_H = \underline{J} \partial_s = \underline{J} \Psi_* \partial_s,$$
$$\Psi_* \widehat{J} \partial_t = -\Psi_*(\partial_s) = -\partial_s = \underline{J}(\partial_t + X_H) = \underline{J} \Psi_* \partial_t,$$
$$\Psi_* \widehat{J} X = \phi^t_* \phi^{-t}_* J(t) \phi^t_* X = J(t) \phi^t_* X = \underline{J} \Psi_* X.$$

□

3.2. **The maximum principle.** Assume that $H = H(t)$ is a time-dependent Hamiltonian on (M, ω). By Proposition 3.5 and Lemma 3.6, solutions of the Floer equation for (M, ω, J, H) correspond bijectively to pseudoholomorphic sections of $(W, \widehat{J})$ considered as a trivial bundle over $\mathbb{R} \times S^1$. Explicitely, the correspondence is given by

$$\big(u : \mathbb{R} \times S^1 \to M\big) \mapsto \big(\widehat{u} : \mathbb{R} \times S^1 \to W\big)$$
$$\widehat{u}(s,t) = \big(s, t, \phi^{-t}(u(s,t))\big),$$

where ϕ^t denotes the Hamiltonian flow of H.

Assume that a real-valued function $r : M \to \mathbb{R}$ is given. It induces the function $\widehat{r} : W \to \mathbb{R}$ defined by

$$\widehat{r}(s,t,x) = r(\phi^t(x)).$$

In other words, $\widehat{r} = (\pi \circ \Psi)^* r$, where $\Psi : (W, \widehat{J}) \to (W, \underline{J})$ is the pseudoholomorphic diffeomorphism from Lemma 3.6 and $\pi : W \to M$ is the projection $\mathbb{R} \times S^1 \times M \to M$.

Let $u : \mathbb{R} \times S^1 \to M$ be a Floer trajectory in M and $\widehat{u} : \mathbb{R} \times S^1 \to W$ the corresponding $\widehat{J}$-holomorphic section of $W \to \mathbb{R} \times S^1$. Then $\pi \circ \Psi \circ \widehat{u} = u$ and hence for any function $r : M \to \mathbb{R}$ we have $\widehat{r} \circ \widehat{u} = r \circ u$. Therefore, proving that the function $r \circ u$ does not attain a local maximum for any

Floer trajectory u in M is equivalent to proving that the function $\widehat{r}\circ v$ does not attain a local maximum for any $\widehat{J}$-holomorphic section $v : \mathbb{R}\times S^1 \to W$.

The second problem has the advantage of dealing with pseudoholomorphic maps rather than more complicated solutions to the Floer equation. For pseudoholomorphic curves we have the following well-known lemma:

Lemma 3.7. *Let (X, I) be an almost complex manifold and $f : X \to \mathbb{R}$ a weakly I-convex function. Then for any I-holomorphic curve $v : (\Sigma, i) \to (X, I)$ the function $f\circ v$ either is constant or does not attain a local maximum in the interior of Σ.*

However, typically the function $\widehat{r}$ on $(W, \widehat{J})$ is not be weakly $\widehat{J}$-convex even if the original function $r : M \to \mathbb{R}$ is J-convex[2]. We can replace J-convexity by a more general condition that also guarantees a maximum principle for restrictions to pseudoholomorphic curves.

Definition 3.8. Let (X, I) be an almost complex manifold. We will say that a real-valued function $f : X \to \mathbb{R}$ *satisfies an elliptic inequality* if there exist real one-forms α and β such that for all $V \in TX$

$$\left(-dd^{\mathbb{C}} f + \alpha\wedge df + \beta\wedge d^{\mathbb{C}} f\right)(V, IV) \geq 0.$$

Note that for $\alpha = \beta = 0$ this is the condition for f to be weakly I-convex.

Lemma 3.7 easily generalises to the following result:

Lemma 3.9. *Let (X, I) be an almost complex manifold. If a function $f : X \to \mathbb{R}$ satisfies an elliptic inequality, then for any pseudoholomorphic curve $v : (\Sigma, i) \to (X, I)$ the function $f\circ v : \Sigma \to \mathbb{R}$ either is constant or it does not attain a local maximum in the interior of Σ.*

Proof. Let $v : (\Sigma, i) \to (X, I)$ be an I-holomorphic curve. Denote by $z = x + iy$ a complex chart on an open subset $U \subset \Sigma$. Plugging $V = \partial_x v$ and $JV = \partial_y v$ into the elliptic inequality for f, we get

$$\Delta(f\circ v) + (\alpha(\partial_x v) - \beta(\partial_y v))\,\partial_y(f\circ v) - (\alpha(\partial_y v) + \beta(\partial_x v))\,\partial_x(f\circ v) \geq 0.$$

Hence, the function $f\circ v : U \to \mathbb{R}$ satisfies an inequality of the form $L(f\circ v) \geq 0$ for an elliptic operator $L = \Delta + A\partial_x + B\partial_y$. From a general theory (see for instance [1]) it follows that if $f\circ v$ is not constant, then it cannot attain a local maximum in U. □

We will show that if a function r on M satisfies an elliptic inequality and is in a certain way compatible with the Hamiltonian dynamics, then the induced function $\widehat{r}$ on W also satisfies an elliptic inequality.

Before stating the result, let us introduce some notation. A time-dependent Hamiltonian $H = H(t)$ induces a time-dependent Hamiltonian vector field X_t and the flow of diffeomorphisms ϕ^t given by

$$\frac{d\phi^t}{dt} = X_t.$$

[2] In fact, it does not even need to have weakly convex level sets.

Let $\mathcal{L}_H$ denote the Lie derivative corresponding to this flow. Note that it is a derivation of the ring of differential forms $\Omega^*(M)$. Recall that for a differential form α the following formula holds):

$$\frac{d}{dt}\left((\phi^t)^*\alpha\right) = (\phi^t)^*\mathcal{L}_H\alpha.$$

Recall that for a function $r : M \to \mathbb{R}$ on an almost complex manifold (M, J) we define its complex differential $d^{\mathbb{C}}r = dr \circ J$ and the distribution of complex tangencies $\xi = \{dr = d^{\mathbb{C}}r = 0\} \subset TM$. Denote by $\mathcal{I}_\xi \lhd \Omega^*(M)$ the ideal of differential forms vanishing on ξ, i.e. generated by the forms dr and $d^{\mathbb{C}}r$.

Theorem 3.10. *Let $H = H(t)$ be a time-dependent Hamiltonian and J an almost complex structure on M. Let $r : M \to \mathbb{R}$ be a smooth function and ξ its distribution of complex tangencies. Assume that*

(1) *r satisfies an elliptic inequality,*
(2) *r is invariant with respect to the Hamiltonian flow of H,*
(3) $\mathcal{L}_H\mathcal{I}_\xi \subset \mathcal{I}_\xi$.

Then the induced function $\widehat{r} : W \to \mathbb{R}$ also satisfies an elliptic inequality.

Proof. Let $\widehat{r} : W \to \mathbb{R}$ be the function on $W = \mathbb{R} \times S^1 \times M$ induced by $r : M \to \mathbb{R}$. Recall that

$$\widehat{r}(s, t, x) = r(\phi^t(x)).$$

In order to prove that $\widehat{r}$ satisfies an elliptic inequality we will compute its Levi form $-dd^{\mathbb{C}}\widehat{r}$ with respect to the almost complex structure $\widehat{J}$. We have

$$d\widehat{r} = \frac{d(r \circ \phi^t)}{dt}dt + (\phi^t)^*dr,$$

$$d^{\mathbb{C}}\widehat{r} = \frac{d(r \circ \phi^t)}{dt}ds + (\phi^t)^*d^Jr.$$

Here $d^Jr = dr \circ J$ denotes the complex differential of r on M with respect to the almost complex structure J. Differentiating the second equation on W, we obtain

$$dd^{\mathbb{C}}\widehat{r} =$$

$$= \frac{d^2(r \circ \phi^t)}{dt^2}dt \wedge ds + d\left(\frac{d(r \circ \phi^t)}{dt}\right) \wedge ds + \frac{d}{dt}[(\phi^t)^*d^Jr] \wedge dt + (\phi^t)^*dd^Jr =$$

$$= \frac{d^2(r \circ \phi^t)}{dt^2}dt \wedge ds + \left((\phi^t)^*\mathcal{L}_H dr\right) \wedge ds + \left((\phi^t)^*\mathcal{L}_H d^Jr\right) \wedge dt + (\phi^t)^*dd^Jr.$$

Since r is ϕ^t-invariant, we end up with

$$d\widehat{r} = (\phi^t)^*dr, \quad d^{\mathbb{C}}\widehat{r} = (\phi^t)^*d^Jr,$$

$$dd^{\mathbb{C}}\widehat{r} = \left((\phi^t)^*\mathcal{L}_H d^Jr\right) \wedge dt + (\phi^t)^*dd^Jr.$$

By condition (3) of the hypothesis, we have $\mathcal{L}_H d^Jr \in \mathcal{I}_\xi$. Hence, there exist smooth functions $f, g : M \to \mathbb{R}$ such that

$$\mathcal{L}_H d^Jr = f dr + g d^Jr.$$

Furthermore, since r satisfies an elliptic inequality, there exist one-forms α and β on M such that for all $V \in TM$ we have

$$\left(-dd^J r + dr \wedge \alpha + d^J r \wedge \beta\right)(V, JV) \geq 0.$$

For a vector $X \in TW$ denote by $\pi_* X \in TM$ its image under the projection $\pi : \mathbb{R} \times S^1 \times M \to M$. Then, we have

$$\begin{aligned}\left(-dd^{\mathbb{C}}\widehat{r} + d\widehat{r} \wedge (f dt + (\phi^t)^*\alpha) + d^{\mathbb{C}}\widehat{r} \wedge (g dt + (\phi^t)^*\beta)\right)(X, \widehat{J}X) = \\ = (\phi^t)^*\left(-dd^J r + dr \wedge \alpha + d^J r \wedge \beta\right)(\pi_* X, (\phi^{-t})_* J (\phi^t)_* \pi_* X) = \\ = \left(-dd^J r + dr \wedge \alpha + d^J r \wedge \beta\right)((\phi^t)_* \pi_* X, J(\phi^t)_* \pi_* X) \geq 0.\end{aligned}$$

Therefore, if we introduce the following one-forms on W:

$$\widehat{\alpha} = f dt + (\phi^t)^*\alpha,$$
$$\widehat{\beta} = g dt + (\phi^t)^*\beta,$$

then the function $\widehat{r} : W \to \mathbb{R}$ satisfies the elliptic inequality

$$\left(-dd^{\mathbb{C}}\widehat{r} + d\widehat{r} \wedge \widehat{\alpha} + d^{\mathbb{C}}\widehat{r} \wedge \widehat{\beta}\right)(X, \widehat{J}X) \geq 0.$$

□

Remark. We might also consider the case when the almost complex structure $J = J(t)$ is time-dependent. Then the proposition still holds, except that we have to replace the standard Lie derivative with its time-dependent analogue. For a time-dependent differential form $\alpha = \alpha_t$ we define

$$\mathcal{L}^t_H \alpha = \mathcal{L}_H \alpha_t + \frac{d\alpha_t}{dt}.$$

This defines a derivation of the ring of time-dependent differential forms $\Omega^{t,*}(M)$. The complex differential $d^{J_t} r$, and hence the complex tangency distribution $\xi = \xi_t$ also depend on time, and we consider the ideal

$$\mathcal{I}^t_\xi = \langle dr, d^{J_t} r \rangle \triangleleft \Omega^{t,*}(M)$$

of time-dependent forms vanishing on ξ_t. In this case, condition (3) reads

$$\mathcal{L}^t \mathcal{I}^t_\xi \subset \mathcal{I}^t_\xi$$

and the proposition can be proved in the same way as above.

In order to understand conditions (2) and (3) better, let us consider function satisfying a very particular elliptic inequality, namely those with vanishing Levi form. In this case, invariance conditions appear in a natural way. Let us begin with the following lemma:

Lemma 3.11. *Let $f : X \to \mathbb{R}$ be a real-valued function on an almost complex manifold (X, I). The following conditions are equivalent:*

1. *f is locally the real part of an I-holomorphic function.*
2. *The Levi form of f vanishes.*
3. *The distribution of complex tangencies of f is integrable.*

Proof. The equivalence of (2) and (3) follows immediately from the Frobenius theorem. If f is the real part of an I-holomorphic function $h = f + ig$, then the Cauchy-Riemann equation reads $d^{\mathbb{C}} f = -dg$, hence $dd^{\mathbb{C}} f = -d^2 g = 0$, which proves the implication from (1) to (2). On the other hand, if $dd^{\mathbb{C}} f = 0$, then by Poincaré lemma, locally there exists a function g such that $d^{\mathbb{C}} f = -dg$ and hence the function $h = f + ig$ is locally well-defined and I-holomorphic. □

We might ask the following question: if $r : M \to \mathbb{R}$ is a function with vanishing Levi form, when the induced function $\widehat{r} : W \to \mathbb{R}$ also a function with vanishing Levi form? Note that this is a special case of the problem we have considered so far, as functions with vanishing Levi form clearly satisfy a maximum principle, because their restriction to any I-holomorphic curve is harmonic. The following proposition provides us with an answer to the question.

Proposition 3.12. *Let $U \subset M$ be an open subset. If $h : U \to \mathbb{C}$ is J-holomorphic, then $\widehat{h} : \mathbb{R} \times S^1 \times U \to \mathbb{C}$ is $\widehat{J}$-holomorphic if and only if h is invariant with respect to the Hamilotnian flow of H.*

Remark. Note that in this case we have $\mathcal{L}_H dr = \mathcal{L}_H d^{\mathbb{C}} r = 0$.

Proof. Let $h = f + ig$. By the Cauchy-Riemann equations, $d^J f = -dg$. The induced function $\widehat{h}$ is $\widehat{J}$-holomorphic if and only if we have $d^{\widehat{J}} \widehat{f} = -d\widehat{g}$. We easily compute

$$d\widehat{g} = \frac{d(g \circ \phi^t)}{dt} dt + (\phi^t)^* dg,$$

$$d^{\widehat{J}} \widehat{f} = \frac{d(f \circ \phi^t)}{dt} ds + (\phi^t)^* d^J f.$$

Now it is clear that the Cauchy-Riemann equations for $\widehat{h}$ are satisfied if and only if both f and g are ϕ^t-invariant. □

Therefore, in the special case when $r : M \to \mathbb{R}$ is locally given as the real part of a J-holomorphic function $h : M \to \mathbb{C}$, for the induced function $\widehat{r}$ on W to be also of this form it is necessary and sufficient that both real and imaginary part of h are invariant with respect to the Hamiltonian flow ϕ^t. In the more general case, the complex tangency distribution of r is non-integrable, and although we can still require r to be ϕ^t-invariant, there is no corresponding imaginary condition, as the form $d^{\mathbb{C}} r$ is not even locally the differential of any function. We can still, however, demand at least the *infinitesimal* ϕ^t-invariance of the complex tangency distribution, and this is how we should intuitively understand condition (3) from Theorem 3.10.

3.3. **Examples.** Theorem 3.10 may be applied to reprove maximum principles in the situations we have discussed before.

Alternative proof of Proposition 2.11. We need to show that the function $r : V \times [0,\infty) \to [0,\infty)$ defined by the projection on the second coordinate, satisfies the conditions of Theorem 3.10. We easily compute:

$$d^{\mathbb{C}}r = -\lambda,$$
$$-dd^{\mathbb{C}}r(X,JX) = d\lambda(X,JX) \geq 0 \quad \text{for } X \in T(V \times [0,\infty)).$$

Hence, r is weakly J-convex. Furthermore, the Hamiltonian vector field $X_H = h'(r)\mathcal{R}$ is tangent to level sets $\{r = \text{const}\}$, so the Hamiltonian flow ϕ^t preserves the function r. In order to check condition (3), we calculate the Lie derivative:

$$\mathcal{L}_{X_H} d^{\mathbb{C}}r = -\mathcal{L}_{(h' \circ r)\mathcal{R}}\lambda = -(h' \circ r)\mathcal{L}_{\mathcal{R}}\lambda - \lambda(\mathcal{R})d(h' \circ r) = -(h'' \circ r)dr.$$

Therefore, all the hypotheses of Theorem 3.10 are satisfied. □

4. Moser stability for weak fillings

Assume that (M,ω) is a weak filling of a contact manifold $(V, \xi = \ker\lambda)$. Choose a closed two-form η in the cohomology class $[\omega_V] \in H^2(V)$. In [4] it is shown that the symplectic structure ω on M can be deformed through weak fillings to another structure ω' such that $\omega'_V = \eta$. In particular, Proposition 2.6 follows from this. Here we show that after attaching cylindrical ends such a deformation yields in fact an isotopy of symplectic structures. In particular, this shows that if ω_V has a rational cohomology class in $H^2(V)$, then the cylindrical completion of (M,ω) is symplectomorphic to the cylindrical completion of a stable filling (see Remark 2.1).

As always, denote by $\widehat{M} = M \cup_V (\mathbb{R} \times V)$ manifold obtained from M by attaching the cylindrical end. Let $r : \widehat{M} \to \mathbb{R}$ be a smooth function such that $r < 0$ on the interior of M and r agrees with the natural projection $\mathbb{R} \times V \to \mathbb{R}$ on the cylindrical end.

Theorem 4.1. *Let $\{\Omega_s\}_{s\in[0,1]}$ be a smooth family of symplectic forms on $\widehat{M}$ whose cohomology class $[\Omega_s] \in H^2(\widehat{M},\mathbb{R})$ does not depend on s. Assume that on the collar $\mathbb{R} \times V$ it has the form*

$$\Omega_s = \omega_s + d(r\lambda)$$

for a family $\{\omega_s\}_{s\in[0,1]}$ of two-forms on V. Then Ω_0 and Ω_1 are isotopic.

Remark. The deformations considered in [4] are of the above form.

Proof. Since the cohomology class of Ω_s does not depend on s, by the parametric version of the Poincaré lemma, there exists a family of one-forms η_s on $\widehat{M}$ such that

$$\frac{d\Omega_s}{ds} = d\eta_s.$$

Furthermore, since on the collar

$$\frac{d\Omega_s}{ds} = \frac{d\omega_s}{ds} \in \Omega^2(V,\mathbb{R})$$

and the zero section $V \to \mathbb{R} \times V$ induces an isomorphism of de Rham cohomology $H^*(\mathbb{R} \times V, \mathbb{R}) \cong H^*(V, \mathbb{R})$ commuting with the exterior derivative, we may assume that each of η_s restricts to a one-form on V on the collar $\mathbb{R} \times V$, that is to say, it does not depend on r.

We proceed as in the standard proof of Moser's lemma. Let v_s be the unique time-dependent vector field on $\widehat{M}$ such that

$$\iota_{v_s} \Omega_s + \eta_s = 0. \tag{1}$$

Consider the flow of the vector field v_s given by the differential equation

$$\frac{d\rho_s}{ds} = v_s \circ \rho_s.$$

We need to prove that ρ_s is well defined for $s \in [0, 1]$. For this, it is enough to prove that v_s is bounded with respect to an r-invariant metric.

On the collar $\mathbb{R} \times V$, we can decompose v_s according to the decomposition $T\widehat{M} = \langle \partial_r \rangle \oplus TV$ into $v_s = \phi_s \partial_r + w_s$ for a family of functions $\phi_s : \mathbb{R} \times V \to \mathbb{R}$ and a time-dependent vector field w_s on V. Then equation 1 reads:

$$\iota_{w_s} (\omega_s + r d\lambda) + \phi_s \lambda - \lambda(w_s) dr + \eta_s = 0. \tag{2}$$

Since $\omega_s, d\lambda$, and η_s are all differential forms on V, it follows that $\lambda(w_s) = 0$, and therefore w_s belongs to the contact distribution $\xi \subset TV$.

Let R by the Reeb vector field of λ on V. Introduce an auxilliary almost complex structure J on ξ compatible with $d\lambda$, and an r-invariant metric on $\mathbb{R} \times V$ such that the decomposition $T(\mathbb{R} \times V) = \langle \partial_r \rangle \oplus \langle R \rangle \oplus \xi$ is orthogonal, $\|\partial_r\| = \|R\| = 1$, and it coincides with $d\lambda(\cdot, J\cdot)$ on ξ. In particular, for every $X \in \xi$ we have $d\lambda(X, JX) = \|X\|^2$.

Evaluating both sides of equation 2 on the vector $Jw_s \in \xi$, we obtain

$$\omega_s(w_s, Jw_s) + r d\lambda(w_s, Jw_s) + \eta_s(Jw_s) = 0,$$
$$r\|w_s\|^2 = r d\lambda(w_s, Jw_s) \leq \|\eta_s\| \|w_s\| + \|\omega_s\| \|w_s\|^2.$$

Therefore, for r large enough, we have $\|w_s\| \leq \frac{C}{r}$, which proves that $\|w_s\|$ is globally bounded on $\widehat{M}$. Taking the supremum over $s \in [0, 1]$ we obtain a bound independent on s. Applying equation 2 to the Reeb vector field R, we get

$$\phi_s = -\eta_s(R) - \omega_s(w_s, R),$$
$$|\phi_s| \leq \|\eta_s\| + \|\omega_s\| \|w_s\|,$$

which proves that ϕ_s is also bounded. Therefore, there exists a constant C such that for all $s \in [0, 1]$ and $x \in \widehat{M}$ we have

$$\|v_s\|^2 = |\phi_s|^2 + \|w_s\|^2 \leq C.$$

□

Corollary 4.2. *Let (M, ω) be a weak filling of (V, ξ). If $[\omega_V] \in H^2(V, \mathbb{R})$ is rational, then there exists a symplectic structure ω' on M such that (M, ω') is a stable filling of (V, ξ), and the cylindrical completions of (M, ω) and (M, ω') are symplectomorphic.*

References

[1] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
[2] Oliver Fabert. Product operations on Floer homology and symplectic field theory. 2004. Diplomarbeit at Ludwig-Maximilians-Universität München.
[3] Richard M. Harris. Distinguishing between exotic symplectic structures. *J. Topol.*, 6(1):1–29, 2013.
[4] Patrick Massot, Klaus Niederkrüger, and Chris Wendl. Weak and strong fillability of higher dimensional contact manifolds. *Invent. Math.*, 192(2):287–373, 2013.
[5] Alexander F. Ritter. Deformations of symplectic cohomology and exact Lagrangians in ALE spaces. *Geom. Funct. Anal.*, 20(3):779–816, 2010.
[6] Paul Seidel. A biased view of symplectic cohomology. arXiv:0704.2055 [math.SG].