SYMPLECTIC FILLABILITY

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Contents

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Introduction

Many contact manifolds arise naturally as boundaries of symplectic manifolds. Typical examples are odd-dimensional spheres and more generally, boundaries of pseudoconvex domains in $\mathbb{C}^n$ and unit sphere bundles in cotangent bundles. This leads to the notion of a *symplectic filling.* Let (M, ξ) be a closed $(2n-1)$-dimensional contact manifold. A symplectic filling of (M, ξ) is a compact $2n$-dimensional symplectic manifold (W, ω) such that $\partial W = M$ and the symplectic and contact structures on the boundary are compatible in a certain way. If a contact manifold admits a symplectic filling, we call it *symplectically fillable.*

We can ask the following questions:

(1) Which closed contact manifolds are symplectically fillable?
(2) If a contact manifold admits a symplectic filling, is it in some sense unique?

In this essay, I consider the first question. It turns out to be closely related to other problems in contact topology. For example, consider the three-dimensional case. Note that by a theorem of Rokhlin, any closed oriented three-manifold bounds an oriented four-manifold, so there are no purely topological obstructions to symplectic fillability. However, it turns out that there are plenty of contact manifolds that do not admit a symplectic filling.

Theorem. *Any closed three-manifold carries a contact structure that is not symplectically fillable.*

This theorem is an easy corollary of other, more fundamental results. Recall that a major discovery in three-dimensional contact topology was the observation of Eliashberg that contact structures are divided into two very different classes: the overtwisted and the tight ones. The former are flexible, meaning that two overtwisted contact structures are isotopic if and only if they are homotopic as plane fields. Furthermore, they can be classified by means of homotopy theory. On the other hand, tight contact structures are not governed solely by topological data and contain geometric information. They capture the difference between the smooth and the contact realms and hence are of great interest to contact topologists.

Definition. Let (M, ξ) be a three-dimensional contact manifold. Let $D \subset M$ be an embedded disc and $\mathcal{F} = TD \cap \xi$ its characteristic foliation. We call D an *overtwisted disc (with singular boundary)* if the leaves of $\mathcal{F}$ are radial lines and the singular set of $\mathcal{F}$ consists of the origin and the boundary ∂D.

Definition. A three-dimensional contact manifold is *overtwisted* if it contains an overtwisted disc. Otherwise it is called *tight.*

Remark. Usually one assumes the boundary of an overtwisted disc to be a leaf rather than a singular set of the characteristic foliation. Here, however, we will use the above definition. An overtwisted disc with singular boundary gives rise to a standard one after a small perturbation.

A relationship between overtwistedness and symplectic fillability is the content of the following beautiful and difficult result.

Theorem (Eliashberg-Gromov)**.** *If a closed contact three-manifold is symplectically fillable, then it is tight.*

In other words, the existence of an overtwisted disc obstructs symplectic fillability. It is known, however, that this is not the only obstruction, as there are examples of tight contact three-manifolds that are not symplectically fillable [8]. Still, the Eliashberg-Gromov theorem is a result of great importance, as it provides a criterion for non-fillability as well as many examples of tight contact structures.

A contact structure on a three-manifold can be modified by a so-called Lutz twist to obtain a new contact structure that is overtwisted and homotopic to the original one through plane fields. By Martinet's theorem, every closed three-manifold admits a contact structure. Applying a Lutz twist, we can make it overtwisted and hence non-fillable by the Eliashberg-Gromov theorem. In this way, we obtain the theorem mentioned at the beginning.

The purpose of this essay is to present the proof of an analogous result for higher-dimensional contact manifolds.

Theorem (Niederkrüger-van Koert [20]). *Every closed contact manifold carries a contact structure that does not admit any semi-positive weak symplectic filling.*

The first section explains the notion of a semi-positive weak filling. The proof of the theorem has two steps. The first one is a higher-dimensional generalisation of the Eliashberg-Gromov theorem.

Theorem (Niederkrüger [18]). *A closed PS-overtwisted contact manifold does not admit any semi-positive weak symplectic filling.*

In analogy to dimension three, we call a contact manifold *PS-overtwisted* if it contains a certain submanifold with boundary, called *bordered Legendrian open book*, whose characteristic foliation is of a particular form. Starting from overtwisted three-dimensional manifolds, Presas [21] gave the first examples of PS-overtwisted manifolds in all dimensions. Using contact surgery techniques, Niederkrüger and van Koert obtained from these examples PS-overtwisted contact structures on all odd-dimensional spheres. Consequently, the contact connected sum operation allows us to make any contact manifold PS-overtwisted, and hence non-fillable, without changing its diffeomorphism type.

In the essay, I present the details of both steps: the non-fillability theorem and the construction of PS-overtwisted spheres. I discuss the former in section 1, based mostly on [18] and [19], and the latter in section 2, to which the main reference is [20]. The construction of PS-overtwisted spheres presented here differs from the original one.

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Notation and conventions. Unless stated otherwise, symplectic and contact manifolds are assumed to be compact and equipped with the orientation compatible with a given symplectic, respectively contact structure. We consider only co-oriented contact structures and assume all contact forms to be positive. If W and M are oriented manifolds, then the equality $M = \partial W$ should be understood as an equality of oriented manifolds, where ∂W is equipped with the boundary orientation induced from W.

Whenever $A \subset W$ is a submanifold and α is a differential form on W, by $\alpha\big|_A$ we mean the differential form $i^*\alpha$ on A, where $i\colon A \hookrightarrow W$ is the inclusion of A.

1. PS-overtwistedness and fillability

The overtwisted-tight dichotomy does not have an immediate generalisation to higher dimensions. At the moment, it is not clear what the correct notion of overtwistedness in higher dimensions should be. Recent progress by various authors (see in particular [1], [17], and [18]) shows that the notion of PS-overtwistedness, introduced by Niederkrüger, is a plausible generalisation[1].

In this section, we will discuss PS-overtwisted contact manifolds and the result that motivated their definition, namely the theorem of Niederkrüger that a PS-overtwisted contact manifold is not symplectically fillable. It is a generalisation of the Eliashberg-Gromov theorem in dimension three. The proof is based on similar ideas and goes back to Gromov, who introduced the technique of filling with pseudoholomorphic discs.

First, we briefly discuss symplectic fillings in higher dimensions and specify what we mean by symplectically fillable. Next, we introduce bordered Legendrian open book, particular submanifolds which generalise overtwisted discs to higher dimensions. In analogy to the three-dimensional case, a contact manifold is called PS-overtwisted if it contains such a submanifold. Finally, we state the main result and discuss its proof, which is based on the analysis of the moduli space of pseudoholomorphic discs in a symplectic manifold with contact boundary.

Our goal is to present the main ideas of the proof rather than to give a comprehensive treatment. Therefore, we sometimes skip certain technical details, referring to the original papers [18] and [19], on which this exposition is almost entirely based. In several places of this section we use some standard facts about pseudoconvexity and pseudoholomorphic curves, which for the convenience of the reader are gathered in appendix A.

1.1. Symplectic fillings.

Let (M,ξ) be a $(2n-1)$-dimensional contact manifold. A *symplectic filling* of (M,ξ) is a compact $2n$-dimensional symplectic manifold (W,ω) with boundary $\partial W = M$ such that the symplectic and contact structures on the boundary are compatible in a certain way. There are various compatibility conditions that one can impose and we list those that are most commonly used.

Definition 1.1. (W,ω) is a *strong symplectic filling* if on a neighbourhood U of the boundary $\partial W = M$ we have $\omega = d\alpha$ for a one-form α on U such that the restriction $\alpha\big|_M$ is a contact form on (M,ξ). If U can be chosen to be the whole manifold W, meaning in particular that ω is globally exact, we call (W,ω) an *exact filling.*

Definition 1.2. (W,ω) is a *weak symplectic filling* if for every contact form α on (M,ξ) the two-form $(\omega + d\alpha)$ restricts to a symplectic structure on the bundle ξ.

We immediately see that

$$\{\text{exact fillings}\} \subset \{\text{strong fillings}\} \subset \{\text{weak fillings}\}$$

and the inclusions are strict. Indeed, since the definition of a weak filling is given by an open condition, one can always perturb the symplectic form of a strong filling so that it is no longer exact at the boundary, but still weakly fills it. Similarly, any exact filling can be perturbed far from the boundary to a non-exact strong filling. What is more interesting, and much more difficult to prove, is that there exist contact manifolds that are weakly but not strongly fillable, and ones that are strongly but not exactly fillable (see [6] and [12], and [15] for higher-dimensions).

[1] In fact, just a few days before submitting this essay the preprint [2] appeared, in which the authors propose what seems to be the right notion of overtwistedness in higher dimensions. They also show that every overtwisted contact manifold is PS-overtwisted, and hence all the results we mentioned, including non-fillability, still hold.

The definition of a strong filling can be restated as follows. Recall that a vector field X on a symplectic manifold (W, ω) is called a *Liouville vector field* if $\mathcal{L}_X \omega = \omega$, where $\mathcal{L}_X$ denotes the Lie derivative along X. A symplectic manifold (W, ω) is a strong filling of (M, ξ) if and only if in a neighbourhood of the boundary $\partial W = M$ there exists a Liouville vector field X positively transverse to M such that $(\iota_X \omega)\big|_M$ is a contact form on (M, ξ). The filling is exact if and only if such a vector field exists globally. That this definition is equivalent to the previous one is an easy consequence of Cartan's formula.

Example 1.3. The standard symplectic ball (D^{2n}, ω_0) is an exact filling of the standard contact sphere (S^{2n-1}, ξ_0). Consider the vector field

$$X = \frac{1}{2} r \partial_r = \frac{1}{2} \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i}) .$$

It is positively transverse to the sphere S^{2n-1} and satisfies the Liouville condition. Moreover,

$$\alpha = \iota_X \omega_0 = \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$$

is the standard contact form on (S^{2n-1}, ξ_0).

Note that D^{2n} is a pseudoconvex domain in $\mathbb{C}^n$ (for the defintion of pseudoconvexity and related notions see appendix A). Generally, there is a close connection between symplectic fillings and pseudoconvexity, which allows us to effectively apply techniques of pseudoholomorphic curves. We will explore this relation in the next few paragraphs. The first observation is that certain pseudoconvex domains in complex manifolds give rise to symplectic fillings.

Lemma 1.4. *Let (X, J) be a complex manifold and $\phi\colon X \to [0, \infty)$ a proper J-convex function. Let $M = \phi^{-1}(c)$ be a regular level set and $\xi = TM \cap J(TM)$ its distribution of complex tangencies. Denote $W = \phi^{-1}([0, c])$ and $\omega_\phi = -dd^J \phi$. Then (M, ξ) is a contact manifold and (W, ω_ϕ) its exact filling.*

Remark 1.5. In this case, (W, J) is called a *Stein filling* of (M, ξ). Every Stein filling is exact, but the converse is not true. There exist contact manifolds that are exactly but not Stein fillable [4].

Proof. First of all, W is indeed a compact codimension zero submanifold with smooth boundary M, because the map ϕ is proper and c is its regular value. The tangent space $TM \subset TX$ is given by the equation $\{d\phi = 0\}$ and the distribution of complex tangencies ξ by the additional condition $\{d^J \phi = 0\}$. Thus, $d^J \phi\big|_M$ is a defining one-form for ξ. Since ϕ is J-convex, the Levi form $\omega_\phi = -dd^J \phi$ is a symplectic form compatible with J and it is non-degenerate on ξ. This proves that ξ is a contact structure on M and (W, ω_ϕ) is its exact filling. □

Note that in the proof we did not actually use the integrability of J. We could have as well assumed (X, J) to be an almost complex manifold. On the other hand, the boundary of any strong symplectic filling can be made pseudoconvex with respect to a compatible almost complex structure.

Proposition 1.6. *If (W, ω) is a strong symplectic filling of a contact manifold (M, ξ), then there exists an ω-compatible almost complex struture J on W such that $\xi = TM \cap J(TM)$ and M is J-convex.*

Proof. Let X be a Liouville vector field on a neighbourhood of M, positively transverse to M and such that $\alpha = (\iota_X \omega)\big|_M$ is a contact form. Let R be the Reeb vector field on M corresponding to α. Then we have the decomposition

$TW\big|_M = \langle X\rangle \oplus \langle R\rangle \oplus \xi$. Choose a bundle complex structure on ξ compatible with $\omega\big|_\xi$ and extend it to $TW\big|_M$ by setting $JX = R$. This way we obtain an almost complex structure on $TW\big|_M$ compatible with ω. By definition, it satisfies $\xi = TM \cap J(TM)$. Since the bundle of ω-compatible almost complex structures has contractible fibre, we can extend J from the boundary to the whole manifold W, obtaining the desired almost complex structure. □

If we consider almost complex structures that are only tamed rather than compatible with the symplectic structure, a similar result holds for weak fillings. Moreover, it turns out that the weak filling condition is equivalent to the existence of a tamed almost complex structure making the boundary pseudoconvex. The proof is rather involved and can be found in [15].

Theorem 1.7. *Let (M,ξ) be a contact manifold and (W,ω) a symplectic manifold with boundary $\partial W = M$. Then (W,ω) is a weak symplectic filling of (M,ξ) if and only if there exists an ω-tamed almost complex structure J on W such that $\xi = TM \cap J(TM)$ and M is J-convex.*

The following technical assumptions on a symplectic manifold are often introduced in order to have some control over the bubbling of pseudoholomorphic curves.

Definition 1.8. A $2n$-dimensional symplectic manifold (W,ω) is called

1. *symplectically aspherical* if $\langle[\omega], A\rangle = 0$ for every $A \in \pi_2(W)$,
2. *semi-positive* if for every $A \in \pi_2(W)$ such that $\langle[\omega], A\rangle \geq 0$ and $c_1(A) \geq 3-n$ we have $c_1(A) \geq 0$.

Here, $c_1(A)$ denotes the first Chern number of the complex bundle $f^*(TW,J)$ over S^2, where $f\colon S^2 \to W$ represents the class A, and J is an ω-tamed almost complex structure on W.

Note that if W is symplectically aspherical, then every pseudoholomorphic sphere $u\colon S^2 \to W$ is constant, because its energy is zero. It follows from the Stokes theorem that every exact symplectic manifold is symplectically aspherical. Clearly, every semi-positive manifold is also symplectically aspherical, but in general the latter condition is more restrictive and imposes certain topological contraints on the manifold (see for example [14])). The semi-positivity condition is more general. For example, every four- or six-dimensional symplectic manifold is semi-positive.

Since now, we will consider *semi-positive weak symplectic fillings*. As we have said before, the weak filling condition is the most general one that guarantees the pseudoconvexity of the boundary with respect to a tamed almost complex structure. On the other hand, semi-positivity allows us to control the bubbling phenomenon. One can hope to remove the semi-positivity assumption in the future, for instance by means of Hofer-Wysocki-Zehnder polyfold theory.

1.2. **Bordered Legendrian open books.** Having discussed symplectic fillings, we introduce the next *dramatis personae*: bordered Legendrian open books (abbreviated bLob). They are certain submanifolds of a contact manifold that carry a singular codimension one foliation by Legendrian submanifolds. They can be seen as higher-dimensional analogues of overtwisted discs, and indeed in dimension three a bLob is just an overtwisted disc with singular boundary. Importantly, the presence of a bLob obstructs symplectic fillability of a contact manifold. The key point is that if the boundary of a symplectic filling contains a bLob, then we can control the behaviour of pseudoholomorphic curves with boundary on the bLob.

We begin with some basic facts about Legendrian foliatons. Let (M,ξ) be a contact manifold of dimension $(2n-1)$. Let α be a contact form and $P \subset M$ a submanifold. Then $\mathcal{F} = TP \cap \xi$ is a singular codimension one distribution whose set of singular points

$$\operatorname{Sing}(\mathcal{F}) = \{p \in P \mid (\alpha_p)\big|_{T_pP} = 0\}$$

consists of the points at which P is tangent to ξ. By the Frobenius theorem, $\mathcal{F}$ gives rise to a singular foliation if and only if

$$(\alpha \wedge d\alpha)\big|_{TN} = 0.$$

Assume that this condition is satisfied. In order to exlude the degenerate case when all the points are singular, assume also that α does not identically vanish on P, i.e. P is not isotropic. In this case, $d\alpha\big|_{\mathcal{F}} = 0$ at every regular point p of the foliation, hence $\mathcal{F}_p$ is an isotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$. In particular, we have

$$\dim P - 1 = \dim \mathcal{F} \leq \frac{1}{2}\dim \xi = \frac{1}{2}(\dim M - 1) = n - 1,$$

so $\dim P \leq n$. In the case of largest possible dimension $\dim P = n$, the leaves of the foliation $\mathcal{F}$ are Legendrian submanifolds of (M, ξ).

Definition 1.9. Let (M, ξ) be a $(2n-1)$-dimensional contact manifold and $P \subset M$ a submanifold of dimension n. If $\mathcal{F} = TP \cap \xi$ is a singular codimension one foliation on P, we call it a *Legendrian foliation.*

As we said in the previous section, if a contact manifold admits a weak symplectic filling, then it can be made into a pseudoconvex boundary with respect to a tamed almost complex structure. A particularly important feature of submanifolds with a Legendrian foliation is that in this setting they become totally real (for relevant definitions in almost complex geometry see appendix A). This will be crucial in understanding the behaviour of pseudoholomorphic curves.

Lemma 1.10. *Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ). If $P \subset M$ if a submanifold such that $\mathcal{F} = TP \cap \xi$ is a Legendrian foliation, then the set of regular points $P \setminus \operatorname{Sing}(\mathcal{F})$ is a totally real submanifold of W.*

Proof. Let $X \in TP$ be a non-zero vector. If JX also belongs to TP, then

$$X \in TP \cap J(TP) \subset TM \cap J(TM) = \xi,$$

hence both X and JX are tangent to the Legendrian foliation $\mathcal{F} = TP \cap \xi$. Let α be a contact form on M, i.e. $\xi = \ker \alpha$. Its derivative $d\alpha\big|_{\xi}$ is the Levi form of M. Since M is J-convex and $X, JX \in \xi$, we have $d\alpha(X, JX) > 0$. At the same time, the restriction $d\alpha\big|_{\mathcal{F}}$ vanishes on $P \setminus \operatorname{Sing}(\mathcal{F})$. The contradiction shows that there is no non-zero vector tangent to $TP \cap J(TP)$ at points of $P \setminus \operatorname{Sing}(\mathcal{F})$, thus the latter is totally real. □

As a corollary, we obtain some information about pseudoholomorphic curves with boundary on $P \setminus \operatorname{Sing}(\mathcal{F})$.

Lemma 1.11. *Let (W, J) be an almost complex manifold with J-convex boundary (M, ξ). Let $P \subset M$ be a submanifold such that $\mathcal{F} = TP \cap \xi$ is a Legendrian foliation and $u\colon (\Sigma, \partial\Sigma) \to (W, P \setminus \operatorname{Sing}(\mathcal{F}))$ a J-holomorphic curve. If u is not constant, then it does not touch M at any interior point and the restriction $u\big|_{\partial\Sigma}$ is positively transverse to $\mathcal{F}$.*

Proof. By Proposition A.14, there exists a smooth function $f\colon W \to [-\infty, 0]$ such that $M = f^{-1}(0)$, df does not vanish on M and f is J-convex on a neighbourhood of M. Then the one-form $\alpha = -d^J f$ is a positive contact form on M.

The first statement follows from the maximum principle for $f \circ u$ and is the content of Proposition A.10. On the other hand, by the boundary point lemma (described in the same proposition) if $w \in T\Sigma\big|_{\partial\Sigma}$ points out of $\partial\Sigma$, then $u_* w$ is positively transverse to M. Choose a vector $v \in T(\partial\Sigma)$ that orients the boundary $\partial\Sigma$. Then $w = iv$ points inwards and we have

$$\alpha(u_* v) = -d^J f(u_* v) = -df(Ju_* v) = -df(u_*(w)) = -d(f \circ u)(w) > 0,$$

which shows that the boundary map $u\big|_{\partial\Sigma} \colon \partial D^2 \to P \backslash \operatorname{Sing} \mathcal{F}$ is positively transverse to ξ, hence also to the leaves of $\mathcal{F}$. □

Submanifolds with Legendrian foliations exist and determine their tubular neighbourhood uniquely up to a contactomorphism. Before stating the relevant theorems, we need one more definition.

Definition 1.12. A codimension one singular foliation $\mathcal{F}$ on a manifold P with boundary is called *neat* if it satisfies the following two conditions:

(1) $\mathcal{F}$ admits a *regular equation*, i.e. it is given by the kernel of a global one-form λ such that $d\lambda \neq 0$ at all singular points of $\mathcal{F}$.
(2) If a connected component of ∂P contains a singular point of $\mathcal{F}$, then it is tangent to $\mathcal{F}$ in a small neighbourhood of this point.

Note that every Legendrian foliation admits a regular equation. Indeed, it is defined as the zero set of $\alpha\big|_P$, where α is a contact form. If $p \in P$ is a singular point, then $T_pP \subset \xi_p$ and the condition $d\alpha_p\big|_{T_pP} = 0$ would imply that T_pP is an isotropic subspace of ξ_p, which is impossible since $\dim T_pP > \frac{1}{2}\dim \xi_p$. Hence, for a Legendrian foliation to be neat it is necessary and sufficient to satisfy the boundary condition (2).

Now we can state the theorems about the existence and tubular neighbourhood of manifolds carrying a neat Legendrian foliation.

Theorem 1.13. *Let P be a manifold with a neat singular foliation $\mathcal{F}$. There exists an open contact manifold (M, ξ) that contains P in such a way that $\mathcal{F} = TP \cap \xi$ is a Legendrian foliation.*

Theorem 1.14. *Let P be a compact manifold with a neat singular foliation $\mathcal{F}$. If P can be embedded into two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) such that $\mathcal{F}$ is a Legendrian foliation in both of them, then there exist neighbourhoods U_1 and U_2 of P in M_1 and M_2, respectively, and a contactomorphism*

$$\Psi \colon (U_1, \xi_1) \to (U_2, \xi_2)$$

restricting to the identity on P.

The proofs are complicated and we will not discuss them, referring the reader to [19]. Instead, let us focus on a particular class of submanifolds whose Legendrian foliation is related to an additional topological structure.

Definition 1.15. Let P be a compact manifold with non-empty boundary. A *relative open book* on P is a pair (B, ϑ) where

(1) B is a non-empty codimension two closed submanifold of P with trivial normal bundle,
(2) $\vartheta \colon P \backslash B \to S^1$ is a locally trivial fibration whose fibres are transverse to ∂P and which in a neighbourhood $B \times D^2$ of B is isomorphic to the fibration $B \times (D^2 \setminus \{0\}) \to S^1$ given by $(b, z) \mapsto z/|z|$.

The submanifolds B and $\vartheta^{-1}(z)$ are called, respectively, the *binding* and *page* of the relative open book.

Remark 1.16. Relative open books and their closed analogues, open books, are intimately related to contact topology. In dimension three, a celebrated theorem of Giroux establishes a bijective correspondence between contact structures up to isotopy and open book decompositions up to a certain equivalence relation. See [7] for an exposition of the topic.

Definition 1.17. Let (M, ξ) be a contact manifold. A submanifold with boundary $P \subset M$ is called a *bordered Legendrian open book* if it carries a Legendrian foliation $\mathcal{F} = TP \cap \xi$ and a relative open book (B, ϑ) such that

(1) the regular leaves of $\mathcal{F}$ are the fibres of ϑ,
(2) $\mathrm{Sing}(\mathcal{F}) = \partial P \cup B$.

A contact manifold is called *PS-overtwisted* if it contains a bLob.

Therefore, the singular set of the Legendrian foliation on a bLob consists of two components: the codimension one boundary and codimension two binding, where singularities are given by the concrete model.

Note that a three-dimensional contact manifold is PS-overtwisted if and only if it is overtwisted in the classical sense, as a bLob is just an overtwisted disc with singular boundary. In higher dimensions, one can find easily open PS-overtwisted contact manifolds thanks to Theorem 1.13. Closed example also exist, but their construction is more challenging. In fact, every contact manifold can be given a PS-overtwisted contact structure. In section 2 we will come back to this issue.

1.3. **The non-fillability theorem.** We are now in place to state the main theorem describing the relationship between PS-overtwistedness and symplectic fillability. In dimension three, it recovers the Eliashberg-Gromov theorem.

Theorem 1.18 (Niederkrüger). *A PS-overtwisted contact manifold (M, ξ) containing a* bLob *P does not admit any semi-positive weak symplectic filling (W, ω) for which the restriction $\omega\big|_P$ is exact.*

It is worth pointing out three important special cases.

Corollary 1.19. *Let (M, ξ) be a PS-overtwisted contact manifold.*

(1) *(M, ξ) is not exactly fillable.*
(2) *If $\dim M = 5$, then (M, ξ) is not strongly fillable.*
(3) *If $\dim M = 3$, then (M, ξ) is not weakly fillable.*

Proof. Any exact filing clearly satisfies the assumptions of the theorem. As regards points (2) and (3), every six- and four-dimensional symplectic manifold is semi-positive. If $\dim M = 5$, then the exactness of $\omega\big|_P$, where P is a bLob, follows from the strong filling condition. On the other hand, if $\dim M = 3$, then a bLob in M is just an overtwisted disc, hence $H^2(P, \mathbb{R}) = 0$ and $\omega\big|_P$ is automatically exact. □

Remark 1.20. In the proof we will actually use a stronger condition than the exactness of $\omega\big|_P$. We will assume that there is a neighbourhood P in W in which (W, ω) looks like an exact filling, that is, on which $\omega = d\eta$ for a one-form η that restricted to M agrees with a positive contact form. Let us say that in such a situation the symplectic filling is *exact around* P. It follows from Lemma 2.10 in [15] that if $\omega\big|_P$ is exact, then the filling can be deformed so that it is exact around P. Hence, we do not need to add the stronger assumption to Theorem 1.18.

The remaining part of the section is devoted to a discussion of the proof of Theorem 1.18. Although our exposition is far from being comprehensive, we hope that it will introduce the reader to the main ideas and techniques. Before going into somewhat technical details, let us present a brief overview of the proof.

The proof uses the method of filling with pseudoholomorphic discs, developed by Gromov and Eliashberg. One assumes by contradiction that there exists a weak symplectic filling (W, ω) satisfying the assumptions of the theorem. Then W is equipped with a certain ω-tamed almost complex structure J making the boundary M pseudoconvex. Then, away from the binding B and the boundary ∂P, the bLob P is a totally real submanifold.

The next step is to study the moduli space of J-holomorphic discs

$$u\colon (D^2, \partial D^2) \to (W, P)$$

with a marked point $z \in \partial D^2$ at the boundary. We identify two pairs (u, z) and (u', z') if they differ by a Möbius transformation mapping z' to z. For reasons that will become clear later, we denote the moduli space of such maps by $\partial\mathcal{M}$. It is equipped with the evaluation map

$$\begin{aligned} \mathrm{ev}\colon \partial\mathcal{M} &\to P, \\ [u, z] &\mapsto u(z). \end{aligned}$$

Using transversality techniques, one proves that the almost complex structure J can be chosen so that $\partial\mathcal{M}$ is a non-compact smooth manifold of dimension $n = \dim \partial\mathcal{M} = \dim P$. Furthermore, by adding all possible limits of sequences of J-holomorphic discs in $\partial\mathcal{M}$, one can define the Gromov compactification $\partial\overline{\mathcal{M}}$. For the compactness theorem to hold it is crucial that the bLob P is totally real with respect to J and that the filling is exact around P.

Under the additional assumption that the symplectic filling (W, ω) is symplectically aspherical, the compactified moduli space is a closed smooth manifold of the particularly simple form

$$\partial\overline{\mathcal{M}} = \partial\mathcal{M} \cup B.$$

Here we identify the binding B with the set of maps $u\colon D^2 \to W$ mapping the whole disc to a single point in B.

Around B, the compactified moduli space $\partial\overline{\mathcal{M}}$ has a neighbourhood $\mathcal{A}$ consisting of an explicit family of J-holomorphic discs, called the Bishop family. The Bishop family $\mathcal{A}$ has the following two properties:

(1) the evaluation map maps $\mathcal{A}$ diffeomorphically onto a neighbourhood $U \cong D^2 \times B$ of the binding B in P,
(2) there are no other J-holomorphic discs in $\partial\overline{\mathcal{M}}$ passing through U.

In particular, it follows that the restriction

$$\mathrm{ev}\big|_{\partial\mathcal{A}}\colon \partial\mathcal{A} \to P \setminus B$$

maps the fundamental class $[\partial A]$ to the generator $[\partial D^2 \times B]$ of the homology group $H_{n-1}(P \setminus B, \mathbb{Z}_2)$. On the other hand, we have the map

$$\mathrm{ev}\big|_{\partial\overline{\mathcal{M}}\setminus\mathcal{A}}\colon \partial\overline{\mathcal{M}} \setminus \mathcal{A} \to P \setminus B.$$

It is well-defined, because apart from the Bishop family no J-holomorphic discs can come close to the binding. Since $\partial\overline{\mathcal{M}} \setminus \mathcal{A}$ is a smooth n-dimensional manifold with boundary $\partial\mathcal{A}$, it follows that $\mathrm{ev}_*[\partial A]$ must be trivial in $H_{n-1}(P \setminus B, \mathbb{Z}_2)$. The contradiction finishes the proof.

If the symplectic filling (W, ω) is only semi-positive rather than symplectically aspherical, the compactified moduli space does no longer have such a simple form, as it contains also bubble curves. However, the semi-posivitity condition gives us some control over the bubbles and a more subtle argument can be used to find the desired contradiction.

In the next parts of this section, we present some of the details of the above proof. First, we describe a model neighbourhood of the binding and show that every pseudoholomorphic curve intersecting this neighbourhood must belong to the

Bishop family. Then in a similar way we investigate what happens around the boundary of the bLob, and prove that no pseudoholomorphic curve can come close to it. It is an important step in understanding the compactification of the moduli space. Finally, we discuss shortly the construction and compactification of the moduli space and present the proof again, this time with more details.

1.4. **The Bishop family.** Assume that (W, ω) is a weak filling of a contact manifold (M, ξ) containing a bLob P. We also assume the filling to be exact around P. Here we describe a model neighbourhood U of the binding $B \subset P$ in W. Then we show that if J is an almost complex structure of a particular form on U, then all J-holomorphic discs partially contained in U and with boundary on P are in fact contained in U and belong to an explicit family of discs, called the *Bishop family.*

First we should understand well the the simplest, four-dimensional case. Let $B^4 \subset \mathbb{C}^2$ be the four-dimensional ball and $S^3 \subset \mathbb{C}^2$ the unit three-sphere equipped with the standard contact form

$$\alpha_0 = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2,$$

where $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ are real coordinates on $\mathbb{C}^2$. Consider the embedding of the unit disc $\Phi\colon D^2 \hookrightarrow S^3$ given by

$$\Phi(z) = \left(z, \sqrt{1 - |z|^2}\right).$$

Denote by N the image of Φ. We easily check that $\Phi^*\alpha_0 = x dy - dx$, hence the characteristic foliation of N is the radial foliation with an elliptic singular point at the origin $\Phi(0) = (0, 1)$.

Note that N is not an overtwisted disc, because its boundary is not singular. Indeed (S^3, α_0) does not contain any such disc because it is symplectically fillable. However, the neighbourhood of the elliptic point $(0, 1)$ in B^3 defined by

$$U = B^3 \cap \{x_2 \geq 1 - \delta\}$$

for δ small enough, provides a model neighbourhood of the singular point of an overtwisted disc in any symplectic filling.

Lemma 1.21. *Let (W, ω) be a weak filling of a three-dimensional contact manifold (M, ξ). If $D \subset M$ is an overtwisted disc and the filling is exact around D, then the singular origin of D has a neighbourhood $U_W \subset W$ symplectomorphic to the model U described above, with $D \cap U_W$ corresponding to $N \cap U$.*

Proof. Theorem 1.14 guarantees that a neighbourood of p in M is contactomorphic to $N \cap U \subset S^3$. Since the symplectic filling is exact around the overtwisted disc, it follows (see Lemma 2.6 in [15]) that p has a neighbourhood in W symplectomorphic to the symplectisation of $N \cap U$, which is symplectomorphic to U itself, because B^4 is an exact filling of S^3. □

We want to study pseudoholomorphic discs in W with boundary on the overtwisted disc D. In the model described above we are able to write down explicitely all holomorphic discs staying close to the singular point.

Definition 1.22. The *Bishop family* is the family of holomorphic discs given by

$$u_t\colon (D^2, \partial D^2) \to (U, N \cap U)$$
$$u_t(z) = (\sqrt{1 - t^2} z, t)$$

for $t \in [1 - \delta, 1)$.

Proposition 1.23. *Let $u\colon (\Sigma, \partial\Sigma) \to (U, N \cap U)$ be a holomorphic curve in U with boundary on N. If u is injective at a boundary point, then $\Sigma = D^2$ and $u = u_t \circ \phi$ for a Möbius transformation $\phi\colon D^2 \to D^2$ and $t \in [1 - \delta, 1)$.*

Proof. Consider the projection $y_2\colon U \to \mathbb{R}$ on the coordinate axis $y_2 = \mathrm{Im}(z_2)$ for $(z_1, z_2) \in U$. Since the surface Σ is compact and the function $y_2 \circ u$ is harmonic, it attains its maximal and minimal values at the boundary $\partial\Sigma$. However, the image $u(\Sigma)$ is contained in N, where the function y_2 vanishes. Hence, $y_2 \circ u = 0$ on Σ. Consequently, the Cauchy-Riemann equations imply that the holomorphic function $z_2 \circ u$ is constant on Σ, and therefore the image $u(\Sigma)$ is contained in a subset $L_t \subset U$ of the form

$$L_t = \{(z_1, z_2) \in U \mid z_2 = t\} = \left\{(z_1, t) \in \mathbb{C}^2 \mid |z_1| \leq \sqrt{1-t^2}\right\},$$

where $t \in \mathbb{R}$ is a fixed real number. Note that L_t is a complex submanifold of C^2 homeomorphic to a disc, and its boundary ∂L_t is a simple closed loop in N around the elliptic singularity $(0, 1)$.

Define the holomorphic map

$$\phi = u_t^{-1} \circ u\colon (\Sigma, \partial\Sigma) \to (D^2, \partial D^2),$$

where $u_t\colon (D^2, \partial D^2) \to (L_t, \partial L_t)$ is a Bishop disc. It is well-defined, since the image of u is contained in L_t. We want to show that ϕ is a biholomorphism, and hence u and u_t differ only by a Möbius transformation.

Consider the boundary map $\phi\big|_{\partial\Sigma}\colon \partial\Sigma \to \partial D^2$. It follows from Lemma 1.10 that $u\big|_{\partial\Sigma}$, and hence also $\phi\big|_{\partial\Sigma}$, has a nowhere-vanishing derivative and the restriction of ϕ to every boundary component of $\partial\Sigma$ is a covering of ∂D^2. Since ϕ is also injective at a boundary point, the boundary has one connected component and the covering map is a diffeomorphism. Moreover, it is an orientation-preserving diffeomorphism, because holomorphic maps preserve orientation.

We have the following commutative diagram of cohomology groups:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\partial\Sigma) & \longrightarrow & H^2(\Sigma, \partial\Sigma) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \phi^*|_{\partial\Sigma}} & & \uparrow{\scriptstyle \phi^*} & & \\ 0 & \longrightarrow & H^1(\partial D^2) & \longrightarrow & H^2(D^2, \partial D^2) & \longrightarrow & 0 \end{array}$$

Both horizontal arrows come from the long exact sequences of pairs and under the identification of all four cohomology groups with $\mathbb{Z}$ they correspond to identities. Since the boundary map $\phi\big|_{\partial\Sigma}$ is an orientation-preserving diffeomorphism, also the left vertical arrow is the identity, and so is the right one. In other words, the map $\phi\colon \Sigma \to D^2$ has degree 1, and since it is a holomorphic map of Riemann surfaces, it is a biholomorphism. Thus, $u = u_t \circ \phi$ for a Möbius transformation $\phi\colon D^2 \to D^2$. $\square$

Next we prove that in fact all pseudoholomorphic curves that intersect U are completely contained in U and, therefore, belong to the Bishop family up to a reparametrisation.

Proposition 1.24. *Let $u\colon (\Sigma, \partial\Sigma) \to (B^4, N)$ be a holomorphic curve. If the intersection $u(\Sigma) \cap U$ is non-empty, then $u(\Sigma)$ is contained in U.*

Proof. Assume that u is non-constant, as otherwise the statement is trivially true. The region $U = B^4 \cap \{x_2 \geq 1 - \delta\}$ has the boundary ∂U which is a manifold with corners consisting of two smooth pieces:

$$\partial U = \partial_+ U \cup \partial_- U,$$
$$\partial_+ U = S^3 \cap \{x_2 \geq 1 - \delta\},$$
$$\partial_- U = B^4 \cap \{x_2 = 1 - \delta\}.$$

We want to show that the coordinate function $x_2 = \mathrm{Re}(z_2)$ is constant on the image of u. This implies that the holomorphic curve cannot escape from U, since otherwise it would have to cross the bottom boundary $\partial_- U$.

After a small change of δ we may assume the map u to be transverse to $\partial_- U$. Since it is automatically transverse to the top boundary $\partial_+ U$ by the boundary point lemma (see Proposition A.10), the preimage $\Omega = u^{-1}(U)$ is a closed region of Σ, whose boundary $\partial\Omega$ is a manifold with corners consisting of two smooth pieces:

$$\begin{aligned}\partial\Omega &= \partial_+\Omega \cup \partial_-\Omega,\\ \partial_+\Omega &= u^{-1}(\partial_+ U),\\ \partial_-\Omega &= u^{-1}(\partial_- U).\end{aligned}$$

Without loss of generality, assume that Ω is connected (otherwise repeat the following reasoning for each connected component). Consider the coordinate map $x_2 = \mathrm{Re}(z_2)\colon U \to \mathbb{R}$. It is harmonic and hence the composition $x_2 \circ u\colon \Omega \to \mathbb{R}$ attains its maximum at the boundary $\partial\Omega$. As it attains its minimum $x_2 = 1-\delta$ on the bottom boundary $\partial_-\Omega$, the maximum must be attained at a point $z_0 \in \partial_+\Omega$. Let v be a non-zero vector tangent to $\partial_+\Omega$ at z_0. The point z_0 is a maximum of $x_2 \circ u$ on Ω, and hence also on the smooth manifold $\partial_+\Omega$, hence we have

$$d(x_2 \circ u)(v) = 0.$$

Consider now the imaginary coordinate $y_2 = \mathrm{Im}(z_2)$. As u maps the boundary $\partial_+\Omega$ to N, where y_2 vanishes, we also have

$$d(y_2 \circ u)(v) = 0.$$

The above equalities and the Cauchy-Riemann equation for the holomorphic function $z_2 \circ u$ imply that $d(x_2 \circ u) = 0$ at z_0. It shows that $x_2 \circ u$ is constant, as otherwise it would be positively transverse to $\partial_+\Omega$ at z_0 by Proposition A.10. □

The proposition holds for any four-dimensional almost complex manifold that contains the model described above for the parameter δ small enough. The next theorem summarises what we have proved so far.

Theorem 1.25. *Let (W, ω) be a weak symplectic filling of a three-dimensional contact manifold (M, ξ). Assume that M contains an overtwisted disc D and the filling is exact around D. Then there exists an ω-tamed almost complex structure J on W and a neighbourhood $U \subset W$ of the origin $p \in D$ such that*

1. *(M, ξ) is J-convex,*
2. *if a J-holomorphic curve $u\colon (\Sigma, \partial\Sigma) \to (W, D)$ is somewhere injective at the boundary and intersects U, then it belongs to the Bishop family around p, up to a Möbius reparametrisation.*

Proof. By Lemma 1.21, there exists a neighbourhood $U \subset W$ of p symplectomorphic to the model we have discussed. It is equipped with the natural complex structure such that Propositions 1.23 and 1.24 hold. Since the space of ω-tamed almost complex structures making the boundary (M, ξ) pseudoconvex is contractible (see Theorem 1.7), we can extend this structure to the whole symplectic filling. □

Now we want to describe an analogous higher-dimensional model. It is more complicated than the four-dimensional one, but the key point is that it contains the latter as a Cartesian component. This allows us to prove easily a higher-dimensional analogue of Theorem 1.25.

Assume that (M, ξ) is a PS-overtwisted contact manifold with a weak filling (W, ω). Let B be the binding of a bLob in M and assume that the filling is exact around the bLob. We will describe a model neighbourhood of B in W.

Equip the cotangent bundle T^*B with an almost complex structure J_B making the zero section $B \subset T^*LB$ a totally real submanifold. By Proposition A.14 from appendix A, there exists a smooth function $f\colon T^*B \to [0, \infty)$ such that $B = f^{-1}(0)$, df does not vanish on B, and f is J-convex on a neighbourhood of B. Without loss of generality, assume that f is J-convex on $f^{-1}[0,1]$. Consider now the manifold $\mathbb{C}^2 \times T^*B$ with the product almost complex structure $J = i \oplus J_B$, and the function $F\colon \mathbb{C}^2 \times T^*B \to [0,\infty)$ given by

$$F(z_1, z_2, q, p) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + f(q,p).$$

The regular level set $M_0 = F^{-1}(1/2)$ is a contact manifold with the contact form $\alpha_0 = -d^J F\big|_M$ and the compact region $W_0 = F^{-1}[0,1/2]$ equipped with the symplectic form $\omega_F = -dd^J F$ is its exact filling (see Lemma 1.4).

Define an embedding $\phi\colon D^2 \times B \hookrightarrow M_0$ by the formula

$$\phi(z,q) = \left(z, \sqrt{1-|z|^2}, q, 0\right) \in \mathbb{C}^2 \times T^*B$$

and denote its image by P_0. Note that $P_0 = N \times B$, where $N \subset S^3$ is the disc described in the four-dimensional model and $B \subset T^*B$ is the zero section. The submanifold P_0 contains the binding B and carries a Legendrian foliation $\mathcal{F}$ defined by the one-form

$$\phi^*\alpha_0 = -\phi^*\left(d^J F\right) = xdy - ydx.$$

Finally, consider the neighbourhood of B in W_0 given by

$$U = \{\mathrm{Re}(z_2) \geq 1-\delta\} \cap W_0$$

As in the four-dimensional case, $(U \cap M_0, \alpha_0)$ provides a model neighbourhood of the binding B in the contact manifold (M,ξ), and (U, ω_F) is a model neighbourhood of B in the symplectic filling (W,ω).

Lemma 1.26. *Let (W,ω) be a weak filling of a contact manifold (M,ξ). If $P \subset M$ is a* bLob *and the filling is exact around P, then its binding B has a neighbourhood $U_W \subset W$ symplectomorphic to the model U described above, with $P \cap U_W$ corresponding to $P_0 \cap U$.*

In this model neighbourhood, we have the Bishop family of J-holomorphic discs:

$$u_{t,q}\colon (D^2, \partial D^2) \to (U, P_0 \cap U),$$
$$u_{t,q}(z) = \left(\sqrt{1-t^2}\,z, t, q, 0\right),$$

for $t \in [1-\delta, 1)$ and $q \in B$. The main result states that these are the only pseudoholmorphic curves close to the binding.

Theorem 1.27. *Let (W,ω) be a weak symplectic filling of a contact manifold (M,ξ). Assume that M contains an* bLob *P and the filling is exact around P. Then there exists an ω-tamed almost complex structure J on W and a neighbourhood $U \subset W$ of the binding $B \subset P$ such that*

1. *(M,ξ) is J-convex,*
2. *if a J-holomorphic curve $u\colon (\Sigma, \partial\Sigma) \to (W,P)$ is somewhere injective at the boundary and intersects U, then it belongs to the Bishop family around B, up to a Möbius reparametrisation.*

Proof. Repeating word by word the proof of Proposition 1.24, we show that the image of u is completely contained in U. We can now decompose the curve into two pseudoholomorphic maps $u = (u_1, u_2)$, where

$$u_1\colon (\Sigma, \partial\Sigma, i) \to (\mathbb{C}^2, N, i),$$
$$u_2\colon (\Sigma, \partial\Sigma, i) \to (T^*B, B, J_B).$$

If we show that u_2 is constant, then the theorem will follow from Proposition 1.23 applied to u_1. This, however, is an immediate consequence of the fact that $B \subset T^*B$ is totally real (see Proposition A.15). □

1.5. **Pseudoholomorphic discs near the boundary.** In this section, we prove that pseudoholomorphic curves cannot come close to the boundary of a bLob P. Similarly to the previous section, we show this by finding a model neighbourhood of ∂P in the symplectic filling.

In the previous section we proved Lemma 1.26 stating that a neighbourhood of the binding of P is always isomorphic to the model we had described. The proof used Theorem 1.14, according to which a Legendrian foliation of a submanifold fully determines its contact neighbourhood. That was all we needed, because by the definition of a bLob, the Legendrian foliation around the binding is isomorphic to the radial foliation on $B \times D^2$. Around the boundary ∂P we have no such model. However, the next lemma shows that the Legendrian foliation in a neighbourhood of ∂P has a particular form and is determined by a certain foliation on ∂P. This foliation is an additional piece of data we need to consider.

Lemma 1.28. *Let P be a compact manifold with boundary, equipped with a singular codimension one foliation $\mathcal{F}$ given by a regular equation β. If the boundary ∂P is contained in* $\operatorname{Sing}(\mathcal{F})$*, then it has a collar*

$$c\colon (-\epsilon, 0] \times \partial P \hookrightarrow P$$

such that $c^*\beta = r\widehat{\beta}$, where r denotes the coordinate on $(-\epsilon, 0]$ and $\widehat{\beta}$ is a non-vanishing one-form on ∂P that defines a regular codimension one foliation on ∂P.

The notion of a regular equation and related definitions concerning foliations can be found in the subsection devoted to bordered Legendrian open books. For a proof of the lemma, see [19].

We proceed with the construction of a model neighbourhood of the boundary of a bLob in a symplectic filling. For simplicity, we again begin with the four-dimensional case. In this case any bLob is just an overtwisted disc and its boundary is a circle. Consider the complex manifold $\mathbb{C} \times \mathbb{R} \times S^1$ with the product complex structure coming from $\mathbb{C}$ and the cylinder $\mathbb{R} \times S^1 \cong \mathbb{C}^*$. Define a map $F\colon \mathbb{C} \times \mathbb{R} \times S^1 \to [0, \infty)$

$$F(z, r, s) = \frac{1}{2}\left(|z|^2 + r^2\right),$$

where $z = x + iy$ is the complex coordinate on $\mathbb{C}$ and (r, s) are cylindrical coordinates on $\mathbb{R} \times S^1$. We easily check that F is plurisubharmonic and gives rise to the symplectic form

$$-dd^{\mathbb{C}}F = 2dx \wedge dy + dr \wedge ds.$$

By Lemma 1.4, the domain $F^{-1}[0, 1/2] = B^3 \times S^1$ is an exact filling of its boundary $F^{-1}(1/2) = S^2 \times S^1$. The latter is equipped with the contact structure defined by the one-form

$$-d^{\mathbb{C}}F = xdy - ydx + rds.$$

Consider also the embedding $\Phi\colon (-\epsilon, 0] \times S^1 \hookrightarrow \mathbb{C} \times \mathbb{R} \times S^1$ given by

$$\Phi(r, s) = \left(\sqrt{1 - r^2}, r, s\right).$$

Let $N \subset \mathbb{C} \times \mathbb{R} \times S^1$ be its image. It is a two-dimensional submanifold of $S^2 \times S^1$ whose boundary is the circle

$$S = \Phi(\{0\} \times S^1) = \{1\} \times \{0\} \times S^1.$$

The contact structure on $S^2 \times S^1$ induces a singular codimension one distribution on N given by the kernel of the form rds. The circle S is its singular set. We define a closed neighbourhood of S in $B^3 \times S^1$:

$$U = \{(z, r, s) \in B^3 \times S^1 \mid x \geq 1 - \delta\}.$$

Note that for $(x, y, r) \in B^3$, the condition $x \geq 1-\delta$ implies automatically $y^2+r^2 \leq 2\delta$, so by decreasing δ we can make U an arbitrarily small neighbourhood of S in $B^3 \times S^1$.

By Lemma 1.28, in a neighbourhood of the boundary every overtwisted disc looks like $(-\epsilon, 0] \times S^1$ with the foliation given by the one-form $r ds$. Hence, using Theorem 1.14, we can prove the following result in the same way as we proved Lemma 1.26 in the previous section.

Lemma 1.29. *Let (W, ω) be a weak symplectic filling of a three-dimensional contact manifold (M, ξ). If $D \subset M$ is an overtwisted disc and the filling is exact around D, then the boundary ∂D has a neighbourhood $U_W \subset W$ symplectomorphic to the model U described above, with $D \cap U_W$ being mapped to $N \cap U$.*

It turns out that no non-constant holomorphic curve with boundary at N can come close to the circle S.

Proposition 1.30. *Let $u\colon (\Sigma, \partial\Sigma) \to (B^3 \times S^1, N)$ be a holomorphic curve. If its image $u(\Sigma)$ has a non-empty intersection with the domain U, then u is constant.*

Proof. First we prove that the image of u is contained in the slice $U \cap \{z = t\}$ for a fixed real number t. The proof is very similar to the one of Proposition 1.24. Assume that u is not constant and denote the lower and upper boundary of the domain U by

$$\partial_+ U = U \cap (S^3 \times S^1) = \left\{(z, r, s) \mid x \geq 1-\delta,\ y = 0,\ r = -\sqrt{1-|z|^2}\right\},$$
$$\partial_- U = U \cap \{x = 1-\delta\} = \left\{(z, r, s) \mid x = 1-\delta,\ |z|^2 + r^2 \leq 1\right\}.$$

By decreasing slightly the parameter δ, we may assume that the map u is transverse to $\partial_- U$. Since it is also transverse to $\partial_+ U$ by Proposition A.10, the preimage $\Omega = u^{-1}(U)$ has a piecewise smooth boundary

$$\partial\Omega = \partial_+\Omega \cup \partial_-\Omega,$$

where

$$\partial_+\Omega = u^{-1}(\partial_+ U),$$
$$\partial_-\Omega = u^{-1}(\partial_- U)$$

are smooth curves in Σ. Withous loss of generality, assume that Ω is connected. Consider coordinate function $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ on U. We want to prove that the compositions $x \circ u$ and $y \circ u$ are constant. Since both functions are harmonic, they attain their maxima and minima on the boundary $\partial\Omega$. If $y \circ u = 0$ everywhere, then it follows from the Cauchy-Riemann equations for the holomorphic function $z \circ u$ that $x \circ u$ is constant. If $y \circ u$ does not vanish identically, then it attains an extremum at a point $z_0 \in \partial_-\Omega$, since $y = 0$ on the other part of the boundary. At the same time, $x = 1-\delta$ on $\partial_-\Omega$, hence z_0 is at the same time a minimum of the function $x \circ u$. Therefore, for any vector v tangent to $\partial_-\Omega$ at z_0, we have

$$d(x \circ u)(v) = d(y \circ u)(v) = 0,$$

and the Cauchy-Riemann equations for $z \circ u$ imply that $d(x \circ u) = d(y \circ u) = 0$ at z_0. This contradicts the boundary point lemma (see Proposition A.10) and shows that either u is a constant curve or both functions $x \circ u$ and $y \circ u$ are constant.

Now let us prove that u in fact must be constant. We have just proved that $u(\Sigma)$ is contained in a subset of the form $U \cap \{z = t\}$ for a fixed real number t. In particular, at the boundary $\partial\Sigma$ we have $r \circ u = -\sqrt{1-t^2}$, because $u(\partial\Sigma) \subset N$. Since the function $r \circ u$ is harmonic on Σ, it follows that it is constant and the Cauchy-Riemann equations for the holomorphic function $(r, s) \circ u\colon \Sigma \to \mathbb{R} \times S^1$ imply that also $s \circ u$ is constant. □

This way we have obtained the following result analogous to Theorem 1.25 from the previous section.

Theorem 1.31. *Let (W, ω) be a weak symplectic filling of a three-dimensional contact manifold (M, ξ). Assume that M contains an overtwisted disc D and the filling is exact around D. Then there exists an ω-tamed almost complex structure J on W and a neighbourhood $U \subset W$ of ∂D such that (M, ξ) is J-convex and every J-holomorphic curve $u\colon (\Sigma, \partial\Sigma) \to (W, D)$ intersecting U is constant.*

We want to find an analogous model in higher dimensions. The outcome will be somewhat involved but hopefully the four-dimensional case will make it easier to understand.

If P is a bLob in a contact manifold (M, ξ) then by Lemma 1.28 its boundary ∂P has a collar of the form $(-\epsilon, 0] \times \partial P$ such that the Legendrian foliation is defined by the one-form $r\widehat{\beta}$. Here r is the coordinate on $(-\epsilon, 0]$ and $\widehat{\beta}$ is a one-form on ∂P that defines a regular foliation $\mathcal{F}_0$ on ∂P.

The relative open book structure on P induces a fibration $\partial P \to S^1$ and by the definition of a bLob, its fibres are precisely the leaves of $\mathcal{F}_0$. Let $F_0 \subset \partial P$ be such a leaf and $\psi\colon F_0 \to F_0$ the monodromy diffeomorphism. Therefore, the boundary ∂P of the bLob is diffeomorphic to the mapping torus S of ψ:

$$\partial P \cong S = (\mathbb{R} \times F_0)/\sim,$$

where the equivalence relation $\sim$ identifies a point (s, x) with $(s+1, \psi(x))$. Hence, the fibre F_0 and monodromy diffeomorphism ψ fully determine ∂P itself as well as its contact neighbourhood in (M, ξ). Let us underline again that this follows from Lemma 1.28 and Theorem 1.14. We will build a model for any given F_0 and ψ.

The diffeomorphism $\psi\colon F_0 \to F_0$ induces the symplectomorphism

$$(\psi, (d\psi^{-1})^*)\colon T^*F_0 \to T^*F_0.$$

Consider its mapping torus $(\mathbb{R} \times T^*F_0)/\sim$, where the equivalence relation identifies a point $(s, \mathbf{q}, \mathbf{p})$ with $(s+1, \psi(\mathbf{q}), (d\psi^{-1})^*\mathbf{p})$. Note that the obtained manifold naturally containes $S = (\mathbb{R} \times F_0)/\sim$ as an embedded submanifold. Furthermore, we consider the projection

$$\pi\colon \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*F_0)/\sim \to \mathbb{C} \times \mathbb{R} \times S^1,$$

$$\pi(z, r, s, \mathbf{q}, \mathbf{p}) = (z, r, s),$$

whose each fibre T^*F_0 is an exact symplectic manifold. Since the monodromy map is a symplectomorphism, we obtain a symplectic structure on the vertical bundle $\ker \pi_*$. Choose any compatible almost complex structure on this symplectic vector bundle and define J to be its extension to the whole tangent bundle of $\mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*F_0)/\sim$ given by the complex structure on $\mathbb{C}$ and condition $J\partial_r = \partial_s$. The almost complex manifold we obtain is a higher-dimensional analogue of the product $\mathbb{C} \times \mathbb{R} \times S^1$ that we considered in the four-dimensional case, and admits a J-holomorphic fibration π over the latter.

Just like in the four-dimensional case, we want to find a J-convex function on the neighbourhood of S. Such a function is given by

$$F\colon \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*F_0)/\sim \longrightarrow [0, \infty),$$

$$F(z, r, s, \mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(|z|^2 + r^2\right) + h(z, r, s, \mathbf{q}, \mathbf{p}),$$

where for fixed (z, r, s) the function h is a non-negative plurisubharmonic function on the fibre T^*F_0 vanishing on the zero section (for example, the length with respect to a compatible bundle metric). The function F gives rise to a symplectic form

$$-dd^JF = 2dx \wedge dy + dr \wedge ds - dd^Jh.$$

Define W_0 to be the region $F^{-1}([0,1/2])$ equipped with the above symplectic structure. Its boundary $M_0 = \partial W_0 = F^{-1}(1/2)$ has an induced contact form $\alpha_0 = -d^J F$ and contains the submanifold $N \subset M_0$ given by the embedding

$$(-\epsilon, 0] \times (\mathbb{R} \times F_0)/\sim \hookrightarrow \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*F_0)/\sim,$$

$$(r, s, \mathbf{q}) \mapsto \left(\sqrt{1-r^2}, r, s, \mathbf{q}, \mathbf{0}\right).$$

Recall that $S = (\mathbb{R} \times F_0)/\sim$ is the boundary of our given bLob. Note that just like in the four-dimensional case, the Legendrian foliation on N is given by the one-form rds and the boundary

$$S' = \partial N = \{1\} \times \{0\} \times S$$

is its singular set. Hence, by construction, a neighbourhood of S' in N looks like a collar of $S = \partial P$ in the bLob P. Define a closed neighbourhood of S' in W_0:

$$U = W_0 \cap \{x \geq 1 - \delta\}.$$

By Theorem 1.14, for δ small enough, the subset $U \cap M_0$ is contactomorphic to a neighbourhood of S in M, and U itself is symplectomorphic to a small neighbourhood of S in the symplectic filling. We have a higher-dimensional analogue of Lemma 1.29.

Lemma 1.32. *Let (W, ω) be a weak symplectic filling of a contact manifold (M, ξ). If $P \subset M$ is a* bLob *and the filling is exact around P, then the boundary ∂P has a neighbourhood $U_W \subset W$ symplectomorphic to the model U described above, with $P \cap U_W$ corresponding to $N \cap U$.*

We can now prove the main result of this subsection.

Theorem 1.33. *Let (W, ω) be a weak symplectic filling of a contact manifold (M, ξ). Assume that M contains a* bLob *P such that the filling is exact around P. Then there exists an ω-tamed almost complex structure J on W and a neighbourhood $U \subset W$ of ∂P such that (M, ξ) is J-convex and every J-holomorphic curve $u\colon (\Sigma, \partial\Sigma) \to (W, P)$ intersecting U is constant.*

Proof. The theorem follows easily from the corresponding four-dimensional result. By the previous lemma, we may choose a neighbourhood of ∂P symplectomorphic to the model described above. In this neighbourhood, take J to be the almost complex structure defined before.

Assume, as in the proof of Proposition 1.30, that the domain $\Omega = u^{-1}(U)$ is connected and has a piecewise smooth boundary. Consider the holomorphic map $\pi \circ u\colon (\Omega, \partial\Omega) \to \mathbb{C} \times \mathbb{R} \times S^1$, where

$$\pi\colon \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*F_0)/\sim \longrightarrow \mathbb{C} \times \mathbb{R} \times S^1,$$

is the J-holomorphic fibration described before. This map must be constant by Proposition 1.30. Hence, the image of u is contained in a fibre $E = T^*F_0$ of π. By our choice of the almost complex structure, E is an almost complex submanifold equipped with the plurisubharmonic function $F\big|_E = h\big|_E + 1$. The boundary of u is contained in N. The latter intersects E along the zero section, which is the zero set of the function h. Therefore, $F \circ u$ is a subharmonic function on Ω which is constant at the boundary, hence it is constant on Ω. Since $F \circ u$ is strictly subharmonic at any injective point of u, it follows that u is constant. $\square$

1.6. **The moduli space.** We want to say a few words about the construction and compactness of the moduli space. The topic involves some subtle analytical considerations and we are unable to present it in detail. As usually, we refer to [19] instead. A general framework used in this kind of problems is described in [16], whereas [10] discusses a compactness theorem applicable to our situation.

Throughout this subsection, (W,ω) is assumed to be a semi-positive weak symplectic filling of (M,ξ) and $P \subset M$ is a bLob, whose binding we denote by B. We assume that the filling is exact around P (see Remark 1.20).

We consider a class of certain ω-tamed almost complex structures on the symplectic filling (W,ω) that are carefully tailored to our situation.

Definition 1.34. An almost complex structure J on W is called *admissible* if:

(1) it is tamed by ω,
(2) the boundary $M = \partial W$ is J-convex,
(3) it coincides with the natural almost complex structures in the model neighbourhoods of B and ∂P described in, respectively, Lemmas 1.26 and 1.29.

Denote by $\mathcal{J}$ the space of all admissible almost complex structures. In the same way as in Theorems 1.27 and 1.33 one proves that $\mathcal{J}$ is non-empty. We equip $\mathcal{J}$ with the subspace C^∞-topology coming from the space $\Gamma(W, \mathrm{End}(TW))$ of smooth sections of the endomorphism bundle $\mathrm{End}(TW)$. Note that all the various notions of C^∞-topology coincide since W is compact. Equipped with this topology, $\mathcal{J}$ turns out to be a contractible Fréchet manifold.

For a given almost complex structure $J \in \mathcal{J}$, we will consider the moduli space of unparametrised discs with boundary on P and a marked point. First, let us consider the space of parametrised discs

$$\widetilde{\mathcal{M}}(D^2, P; J) = \left\{ u \in C^\infty(D^2, W) \;\middle|\; \begin{array}{c} \overline{\partial}_J u = 0,\ u(\partial D^2) \subset P, \\ u \text{ is somewhere injective} \end{array} \right\}.$$

Here, $\overline{\partial}_J$ is the Cauchy-Riemann operator given by

$$\overline{\partial}_J u = J(u) \circ du - du \circ i.$$

Its image lies in the space $\Omega^{0,1}(D^2, u^*TW)$ of $(0,1)$-forms with values in the pullback bundle u^*TW over D^2. By definition, a smooth map $u\colon D^2 \to W$ is J-holomorphic if and only if $\overline{\partial}_J u = 0$.

For a point $z_0 \in D^2$ we have the natural evaluation map

$$\begin{aligned} \mathrm{ev}_{z_0}\colon \widetilde{\mathcal{M}}(D^2, P; J) &\to W, \\ [u] &\mapsto u(z_0). \end{aligned}$$

Theorem 1.35 (Transversality)**.** *There exists a dense subspace $\mathcal{J}_{reg} \subset \mathcal{J}$ such that for any $J \in \mathcal{J}_{reg}$ the moduli space $\widetilde{\mathcal{M}}(D^2, P; J)$ is a disjoint union of finite-dimensional smooth manifolds and for any given point $z_0 \in D^2$ the evaluation map $\mathrm{ev}_{z_0}\colon \widetilde{\mathcal{M}}(D^2, P; J) \to W$ is smooth.*

The dimension of a connected component containing a map $u \in \widetilde{\mathcal{M}}(D^2, P; J)$ is

$$\dim_u \widetilde{\mathcal{M}}(D^2, P; J) = \frac{1}{2} \dim W + \mu(u^*TW, u^*TP),$$

where μ denotes the Maslov index.

Recall that the *Maslov index* is an integer invariant classifying up to homotopy loops of totally real subspaces in a given complex vector space. Here, the map u gives us a totally real subspace $T_{u(z)}P$ in $T_{u(z)}W$ for every $z \in \partial D^2$. All the tangent spaces to W can be identified with a single complex vector space, because the bundle u^*TW over D^2 is trivial. This way the boundary map $u\big|_{\partial D^2}$ induces a loop of totally real subspaces.

Outline of the proof. The proof relies on a combination of standard transversality techniques and some computations in our concrete model. We will not present it here, but let us discuss shortly its main ideas.

Consider the space

$$\mathcal{B} = \left\{ u \in C^\infty(D^2, W) \mid u(\partial D^2) \subset P \right\}$$

of all smooth discs in W with boundary on P. We equip $\mathcal{B}$ with C^∞-topology, which makes it a Fréchet manifold. Furthermore, the set of somewhere injective maps form an open subspace $\mathcal{B}^* \subset \mathcal{B}$. There is a Fréchet vector bundle over $\mathcal{B}$

$$\mathcal{E} \to \mathcal{B}$$

whose fibre over a map $u \in \mathcal{B}$ is given by

$$\mathcal{E}_u = \Omega^{0,1}(D^2, u^*TW).$$

The Cauchy-Riemann operator gives rise to a smooth section

$$\mathcal{S} \colon \mathcal{B} \to \mathcal{E},$$

$$\mathcal{S}(u) = (u, \overline{\partial}_J u).$$

We see that the moduli space is the zero set of this section:

$$\widetilde{\mathcal{M}}(D^2, P; J) = \{ u \in \mathcal{B}^* \mid \mathcal{S}(u) = 0 \}.$$

The idea is to equip the moduli space with a manifold structure coming from $\mathcal{B}$. If $\mathcal{S}$ is transverse to the zero section of $\mathcal{E}$, we expect $\widetilde{\mathcal{M}}(D^2, P; J)$ to be a submanifold of $\mathcal{B}$. Since the Cauchy-Riemann equations are elliptic, the differential of $\mathcal{S}$ at a point is a Fredholm operator and the dimension of $\widetilde{\mathcal{M}}(D^2, P; J)$ is finite. Around a curve u it is given by the index of the linearised Cauchy-Riemann operator, which is related to topological properties of u by the Atiyah-Singer index theorem. Transversality can be achieved by perturbing the almost complex structure in the infinite-dimensional space $\mathcal{J}$. The essential ingredient here is the Sard-Smale theorem.

There are several technical difficulties that one has to overcome in order to make this approach work. For example one needs to work with Banach rather than Fréchet manifolds in order to apply the inverse function and Sard-Smale theorems. For that, we have to consider maps and almost complex structures of regularity lower than C^∞, and establish an elliptic regularity result that would allow us to come back to the smooth realm. For more details we refer to [16], where the authors prove that transversality indeed can be achieved at every somewhere-injective disc for a generic choice of an almost complex structure.

However, in our situation we encounter an additional problem. Namely, we cannot consider arbitrary perturbations of almost complex structures, because we have assumed elements of $\mathcal{J}$ to agree with a specific almost complex structure on neighbourhoods of the binding and boundary of the bLob. This problem can fixed by showing that transversality of the Cauchy-Riemann operator is automatically satisfied on the Bishop family. Since there are no other pseudoholomorphic discs coming close to the binding and no discs at all close to the boundary of the bLob, this is sufficient. We refer to section III.1.3. in [19] for details. □

Let $\widetilde{\mathcal{M}}_0(D^2, P; J)$ be the connected component of $\widetilde{\mathcal{M}}_0(D^2, P; J)$ containing the Bishop family at B. Consider the product $\widetilde{\mathcal{M}}_0(D^2, P; J) \times D^2$ consisting of J-holomorphic discs together with a marked point in D^2. The group $\mathrm{Aut}(D^2)$ of Möbius transformations preserving the disc acts on $\widetilde{\mathcal{M}}_0(D^2, P; J) \times D^2$ by $(u, z) \mapsto (u \circ \phi, \phi^{-1}(z))$ for $\phi \in \mathrm{Aut}(D^2)$.

Consider the moduli space of unparametrised J-holomorphic discs with boundary on P and a marked point, defined as the quotient

$$\mathcal{M} = \left(\widetilde{\mathcal{M}}_0(D^2, P; J) \times D^2\right) / \mathrm{Aut}(D^2).$$

It is equipped with the evaluation map

$$\begin{aligned} \mathrm{ev}\colon \mathcal{M} &\to W, \\ [u, z] &\mapsto u(z). \end{aligned}$$

It is not difficult to see that the action of $\mathrm{Aut}(D^2)$ on $\widetilde{\mathcal{M}}_0(D^2, P; J) \times D^2$ is smooth, proper, and free. Its restriction to $\dim \widetilde{\mathcal{M}}_0(D^2, P; J) \times \partial D^2$ also have these properties. Since $\dim \mathrm{Aut}(D^2) = 3$, the expected dimension of the quotient equals $\widetilde{\mathcal{M}}_0(D^2, P; J) + 2 - 3 = \dim P + 1$. Therefore, we have the following result.

Proposition 1.36. *The moduli space $\mathcal{M}$ is a non-compact smooth manifold with boundary. Its dimension is given by*

$$\dim \mathcal{M} = \frac{1}{2} \dim W + 1 = \dim P + 1.$$

Furthermore, the evaluation map $\mathrm{ev}\colon \mathcal{M} \to W$ *is smooth.*

The boundary of $\mathcal{M}$ is the quotient

$$\partial\mathcal{M} = \left(\widetilde{\mathcal{M}}_0(D^2, P; J) \times \partial D^2\right) / \mathrm{Aut}(D^2)$$

consisting of equivalences classes $[u, z]$ with $z \in \partial D^2$. Note that the image of $\partial\mathcal{M}$ under the evaluation map is contained in P.

By definition, for any choice of an admissible almost complex structure $J \in \mathcal{J}_{reg}$, the binding B and boundary ∂P of the bLob have neighbourhoods biholomorphic to the models described in the two previous subsections. This gives us some information about the local structure of $\mathcal{M}$.

From Theorem 1.33 it follows that no non-constant J-holomorphic discs can come close to ∂P. In particular, there is a neighbourhood of ∂P in W, whose preimage under the evaluation map $\mathrm{ev}\colon \mathcal{M} \to W$ is empty. On the other hand, the binding B has a neighbourhood isomorphic to the model U described in Lemma 1.26. Recall briefly that

$$U = \left\{(z_1, z_2, \mathbf{q}, \mathbf{p}) \in \mathbb{C}^2 \times T^*B \;\middle|\; \mathrm{Re}(z_2) > 1 - \delta\right\} \cap F^{-1}([0, 1/2]),$$

where the function $F\colon \mathbb{C}^2 \times T^*B \to [0, \infty)$ is of the form

$$F(z_1, z_2, \mathbf{q}, \mathbf{p}) = \frac{1}{2}(|z_1|^2 + |z_2|^2) + f(\mathbf{q}, \mathbf{p}).$$

Consider the subset $Q \subset U$ given by

$$Q = U \cap \{\mathrm{Im}(z_2) = 0\}.$$

It is a submanifold of dimension $\dim Q = \dim P + 1$ and it intersects the boundary $\partial U = U \cap M$ along the subset $P_0 \cong D^2 \times B$, which was introduced as the model neighbourhood of B in the bLob.

Theorem 1.27 shows that Q is foliated by the Bishop family of J-holomorphic discs. In other words, for every point $p \in Q \setminus B$, there exists a unique, up to a Möbius reparametrisation, non-constant J-holomorphic disc $u\colon (D^2, \partial D^2) \to (W, P)$ passing through p. Bishop discs are injective along the boundary, so that they define elements of $\mathcal{M}$. Therefore, there is an open subset $V = \mathrm{ev}^{-1}(U)$ of the moduli space $\mathcal{M}$ such that the evaluation map maps V diffeomorphically onto $Q \setminus B$. We have to remove B from Q as the only J-holomorphic discs passing through the points of the binding are constant maps, which are not elements of the moduli space $\mathcal{M}$ (we considered only somewhere injective maps). Similarly, at the boundary $\partial\mathcal{M}$ we have the open subset $V \cap \partial\mathcal{M}$ that is mapped by the evaluation map diffeomorphically onto $P_0 \setminus B$.

Now we would like to know something about the global structure of $\mathcal{M}$. The moduli space can be compactified by adding the limits of sequences of maps in $\mathcal{M}$. The Gromov compactness theorem tells us what all such possible limits look like.

Theorem 1.37 (Gromov compactness)**.** *Let (W, ω) be a compact symplectic manifold equipped with an almost complex structure J tamed by ω. Let $L \subset W$ be a compact totally real submanifold. Let*

$$u_k \colon (D^2, \partial D^2) \to (W, L)$$

be a sequence of J-holomorphic maps whose energy

$$E(u_k) = \int_{D^2} u_k^* \omega$$

is bounded from above by a constant $C > 0$.

Then there exists a subsequence (u_{k_l}) that converges in the sense of Gromov to a stable J-holomorphic map v_∞ of genus zero with connected boundary. Furthermore, the relative homology class of discs is preserved under the Gromov convergence, that is, if $[u_k] = x$ in $H_2(W, L)$ for all k and a fixed homology class x, then also $[v_\infty] = x$.

For the definitions of the Gromov convergence and stable maps, as well as the proof of the theorem, see [10]. Let us just recall that in this case a stable J-holomorphic map of genus zero consists of a finite collection (more specifically, a tree) of J-holomorphic discs with boundary on L and J-holomorphic spheres. If we consider a sequence of discs with marked points, the limit stable map will also have a marked point on one of the disc or sphere components. In particular, the evaluation map is still defined for the limit stable maps. Note also that in our case P is actually not totally real at the points of the binding B and boundary ∂P, but this is not a problem, because we control J-holomorphic curves around B and ∂P regardless of the theorem.

Lemma 1.38 (Energy bound)**.** *There exists a constant $C > 0$ such that the energy of any J-holomorphic disc $u \in \mathcal{M}$ is bounded from above by C:*

$$E(u) = \int_{D^2} u^* \omega \le C.$$

Proof. Here we use the fact that the symplectic filling is exact around the bLob (see Remark 1.20). Let α be a contact form on (M, ξ) such that $\omega = d\alpha$ along P.

An element of $\mathcal{M}$ is represented by a J-holomorphic map $u \colon (D^2, \partial D^2) \to (W, P)$ that can be connected by a smooth family of J-holomorphic discs with a constant map $u_0(z) = b_0$ for some $b_0 \in B$. In other words, we have a smooth map $U \colon D^2 \times [0,1] \to W$ such that for each $t \in (0,1]$, the restriction $u_t = (\cdot, t)$ is a non-constant J-holomorphic map mapping ∂D^2 to P, and $u_1 = u$. We have

$$\partial(D^2 \times [0,1]) = \bigl(\partial D^2 \times [0,1]\bigr) \cup \bigl(D^2 \times \{0,1\}\bigr)$$

and by the Stokes theorem,

$$0 = \int_{D^2 \times [0,1]} U^*(d\omega) = \int_{\partial D^2 \times [0,1]} u_t^* \omega + \int_{D^2} u_1^* \omega - \int_{D^2} u_0^* \omega.$$

The last integral obviously vanishes since u_0 is a constant map, and using the Stokes theorem again we obtain the following formula for the energy:

$$E(u) = -\int_{\partial D^2 \times [0,1]} u_t^* \omega = -\int_{\partial D^2 \times [0,1]} u_t^*(d\alpha) = \int_{\partial D^2} u^* \alpha.$$

The change of sign in the last equality comes from the fact that $\partial D^2 \times [0,1]$ and $D^2 \times \{1\}$ induce the opposite boundary orientations on $\partial D^2 \times \{1\}$.

Let $\vartheta\colon P \setminus B \to S^1$ be the fibration defining the relative open book structure on P. The one-forms $d\vartheta$ and $\alpha\big|_{TP}$ are both positive defining forms of the Legendrian foliation, hence there exists a smooth function $f\colon P \to [0,\infty)$ such that

$$\alpha\big|_{TP} = f d\vartheta.$$

Since the binding B and boundary ∂P are singular sets of the Legendrian foliation, f vanishes on both of them, and hence it is bounded on P. Let

$$C = 2\pi \sup_{x\in P} |f(x)|.$$

The map u is not constant, so by Lemma 1.11 the boundary map $u\big|_{\partial D^2}$ is positively transverse to the Legendrian foliation. In particular, $u\big|_{\partial D^2}$ intersects every leaf exactly once and we have

$$E(u) = \int_{\partial D^2} u^*(f d\vartheta) \le \sup_{x\in P} |f(x)| \int_{\partial D^2} u^*(d\vartheta) = 2\pi \sup_{x\in P} |f(x)| = C.$$

□

The global energy bound allows us to apply the Gromov compactness theorem and see the possible limits of sequences of J-holomorphic discs. More complicated stable maps are excluded by the geometry of our problem and we end up with the following list of cases.

Proposition 1.39. *If (u_k) is a sequence of J-holomorphic discs in $\mathcal{M}$, then it has a subsequence converging in the sense of Gromov to one of the following:*

(1) *a J-holomorphic disc $u_\infty \in \mathcal{M}$,*
(2) *a constant disc $u_\infty(z) = b_0$ for some $b_0 \in B$,*
(3) *a stable map consisting of a single non-constant J-holomorphic disc*

$$u_\infty\colon (D^2, \partial D^2) \to (W, P)$$

and a finite tree of non-constant J-holomorphic spheres.

Proof. Assume first that (u_k) contains a subsequence of discs coming arbitrarily close to the binding B. Such discs have to belong to the Bishop family and using the compactness of B we can choose a subsequence converging to a point $b_0 \in B$.

On the other hand, none of the discs u_k can come close to the boundary ∂P of the bLob. Therefore, assume that all u_k stay outside some open subset $V \subset P$ containing B and ∂P. Since $L = P \setminus V$ is a compact totally real submanifold of W, the assumptions of Theorem 1.37 are satisfied and (u_k) contains a subsequence converging in the sense of Gromov to a stable J-holomorphic map v.

We just need to show that v has only one disc component. Note first that every disc $u \in \mathcal{M}$ is homotopic to a fixed Bishop disc u_0, which yields the equality $[u_k] = [u_0]$ in homology $H_2(W, L)$. In particular, under the boundary map

$$\partial_*\colon H_2(W, L) \to H_1(L)$$

the homology class $[u_k]$ is mapped to the homology class of the boundary of u_0, which is a generator of $H_1(L)$ going once around the binding B. Geometrically, it is equivalent to the content of Lemma 1.11 which states that the boundary of a non-constant J-holomorphic disc is positively transverse to the Legendrian foliation.

The relative homology class is preserved by Gromov convergence, hence $[v] = [u_0]$ in $H_2(W, L)$. However, if v contained $k > 1$ distinct non-constant J-holomorphic discs, then we would have $\partial_*[v] = k\partial_*[u_0]$, because spherical bubbles do not contribute to the image of $[v]$ under ∂_*. Geometrically, it would mean that the boundary of

v intersect each leaf of the Legendrian foliation multiple times. This contradiction shows that v contains only one disc component. □

In particular, if (W, ω) is symplectically aspherical, the third possibility cannot occur, as there are no non-constant J-holomorphic spheres in W.

Corollary 1.40. *If (W, ω) is symplectically aspherical, then the moduli space $\mathcal{M}$ admits a compactification $\overline{\mathcal{M}} = \mathcal{M} \cup B$ which is an smooth oriented compact manifold with boundary $\partial\overline{\mathcal{M}} = \partial\mathcal{M} \cup B$. Here, the binding B is identified with the set of constant maps with values in B.*

Unfortunately, in general spherical bubbles can appear and the compactification does not have such a simple description. However, it turns out that the image of the bubbles under the evaluation map has at least codimension two in W.

Proposition 1.41. *The Gromov compactification $\overline{\mathcal{M}}$ consists of B and the set of bubble curves $\mathcal{C} \subset \overline{\mathcal{M}}$. There exist finitely many smooth manifolds $X_1, \ldots, X_N$ and smooth maps $f_i \colon X_i \to W$ such that $\dim X_i \le \dim \mathcal{M} - 2$ for all i, and*

$$\operatorname{ev}(\mathcal{C}) \subset \bigcup_{i=1}^{N} f_i(X_i).$$

An analogous statement holds for the boundary $\partial\overline{\mathcal{M}}$, i.e. the image of bubble curves in $\partial\overline{\mathcal{M}}$ under the evaluation map has codimension at least two.

We refer to [19] for the proof. It is based on the semi-positivity of the symplectic filling and a detailed analysis of the bubbling phenomena.

1.7. **Proof of the non-fillability theorem.** Let (M, ξ) be a contact manifold containing a bLob P. We want to prove that M does not admit any semi-positive weak symplectic filling (W, ω) which is exact around P. We will assume that there is such a filling and derive a contradiction.

Choose a regular admissible almost complex structure, that is, an element $J \in \mathcal{J}_{reg}$. Consider the moduli space $\mathcal{M}$ of unparametrised J-holomorphic discs with boundary on P and a marked point. Denote by $\overline{\mathcal{M}}$ its Gromov compactification. It is equipped with the evaluation map $\operatorname{ev} \colon \overline{\mathcal{M}} \to W$.

Let us first discuss the simpler case when (M, ω) is symplectically aspherical. Then, by Corollary 1.40 the boundary moduli space $\partial\overline{\mathcal{M}} = \partial\mathcal{M} \cup B$ is a closed smooth manifold of dimension $n = \dim P$. Furthermore, there is a neighbourhood of B in $\partial\overline{\mathcal{M}}$

$$\mathcal{A} = (V \cap \partial M) \cup B$$

that is mapped by the evaluation map diffeomorphically onto a neighbourood $U \cap P_0$ of the binding B in P (the notation that we use here was introduced in the previous subsection). This neighbourhood is diffeomorphic to $D^2 \times B$. Under this identification, the Bishop discs in $\mathcal{A}$ are defined by

$$u_{z,b} \colon D^2 \to D^2 \times B,$$
$$w \mapsto (zw, b),$$

and the evaluation map is simply given by $u_{z,b} \mapsto (z, b)$.

Thus, the evaluation map restricted to $\partial\mathcal{A}$

$$\operatorname{ev}\big|_{\partial\mathcal{A}} \colon \partial\mathcal{A} \to P \setminus B$$

represents a non-trivial homology class $[\partial D^2 \times B] \in H_{n-1}(P \setminus B, \mathbb{Z}_2)$. On the other hand, this map can be extended to

$$\operatorname{ev}\big|_{\partial\overline{\mathcal{M}} \setminus \mathcal{A}} \colon \partial\overline{\mathcal{M}} \setminus \mathcal{A} \to P \setminus B.$$

Indeed, the evaluation map maps $\partial\overline{\mathcal{M}}$ to P, because the J-holomorphic discs that we consider have boundary on P. The image misses B, since the only non-constant J-holomorphic discs that can come close to the binding belong to the Bishop family contained in $\mathcal{A}$. Since $\overline{\mathcal{M}} \setminus \mathcal{A}$ is a smooth compact manifold with boundary $\partial\mathcal{A}$, this shows that the map

$$\mathrm{ev}\big|_{\partial\mathcal{A}}\colon \partial\mathcal{A} \to P \setminus B$$

must represent zero in the homology group $H_{n-1}(P \setminus B, \mathbb{Z}_2)$. We have obtained the desired contradiction.

If (W, ω) is only semi-positive rather than symplectically aspherical, then $\partial\overline{\mathcal{M}} \setminus \mathcal{A}$ is not a smooth manifold. However, it follows from Proposition 1.41 that the map $\mathrm{ev}\big|_{\partial\overline{\mathcal{M}}\setminus\mathcal{A}}\colon \partial\overline{\mathcal{M}} \setminus \mathcal{A} \to P \setminus B$ still defines a so-called pseudocycle in the sense of McDuff-Salamon [16]. This way we could obtain a contradiction as in the previous case. Instead, let us present the following *ad hoc* argument that does not make use of the theory of pseudocycles.

Assume that we can find a smooth path $\gamma\colon [0,1] \to P$ such that

(1) $\gamma(0) \in B$, $\gamma(1) \in \partial P$, and $\gamma\left((0,1)\right) \subset P \setminus (B \cup \partial P)$,
(2) γ is transverse to the evaluation map $\mathrm{ev}\colon \partial\mathcal{M} \to P$,
(3) its image $\gamma([0,1])$ is disjoint from the image $\mathrm{ev}(\mathcal{C})$ of bubble curves $\mathcal{C} \subset \partial\overline{\mathcal{M}}$.

Because $\dim \partial\mathcal{M} = \dim P$, condition (2) implies that the preimage of $\gamma([0,1])$ under the evaluation map is a smooth one-dimensional submanifold of $\partial\mathcal{M}$. Let Γ be its connected component that contains the Bishop discs intersecting $\gamma([0,\epsilon]))$ for small $\epsilon > 0$. Note that Γ is compact after adding to it the constant disc $u_0(z) = \gamma(0)$. Indeed, any sequence of points of Γ has a Gromov convergent subsequence. This subsequence cannot converge to a bubble curve because of condition (3). Hence, by Proposition 1.39 the limit is either an element of Γ or the degenerate disc u_0.

Therefore, $\Gamma \cup \{u_0\}$ is a compact connected one-dimensional manifold. By assumption (1), it cannot be a circle, thus is is diffeomorphic to a closed interval whose endpoints correspond to u_0 and some element $[u,z] \in \partial\mathcal{M}$. Because a non-constant J-holomorphic disc cannot come close to the boundary ∂P, we see that $\mathrm{ev}([u,z]) = z$ is contained in $P \setminus (B \cup \partial P)$. Thus, $\mathrm{ev}\colon \partial\mathcal{M} \to P$ is transverse to γ at $[u,z]$ and it follows that Γ can be extended through $[u,z]$. It means that $[u,z]$ cannot be an endpoint of Γ and we obtain a contradiction.

It remains to prove that a path $\gamma\colon [0,1] \to P$ with properties $(1) - (3)$ indeed exists. This follows essentially from Proposition 1.41, because we can perturb any curve to miss a given subset of codimension at least two. Note however, that solely by perturbing γ we cannot arrange it to miss $\mathrm{ev}(\mathcal{C})$ and be transverse to the evaluation map at the same time. However, this can be achieved by first choosing γ and then perturbing the almost complex structure J within $\mathcal{J}_{reg}$ to make the evaluation map transverse to γ. Details are described in [19].

2. Construction of PS-overtwisted structures

Recall that a contact manifold is called *PS-overtwisted* if it contains an embedded bLob. In the previous section we discussed the relationship between PS-overtwistedness and symplectic fillability established by Niederküger [18], which was the main motivation for introducing the notion of a bLob. At the beginning it was not clear that compact PS-overtwisted contact manifolds exist apart from dimension three. First examples in an arbitrary dimension were constructed by Presas [21] by considering connected sums of contact fibrations. Subsequently, Niederkrüger and van Koert [20] managed to surger down these examples to get PS-overtwisted structures on all odd-dimensional spheres. As a corollary they obtained the beautiful theorem that any contact manifold admits a PS-overtwisted, and hence non-fillable, contact structure.

This section is devoted to a detailed discussion of a construction of PS-overtwisted spheres. Following advice of Klaus Niederkrüger, we apply a more general Presas gluing than the one used in [20]. In this approach, we are able to control easily the diffeomorphism type of surgered manifolds by keeping track of normal framings. Consequently, we avoid problems with possible exotic differential structures or knotted spheres that had to be dealt with in the original construction.

We begin with a brief introduction to contact surgery and gluing techniques. Although we skip some of the details, we do present proofs of Propositions 2.7 and 2.15, which deal with contact surgery and fibre connected sum in the presence of a contact submanifold. We could not find Proposition 2.7 in the existing literature, and the proof of Proposition 2.15 is based on a similar result from [20].

2.1. **Contact surgery.** The technique of surgery, well-known in differential topology, has been successfully adapted to contact geometry providing an important tool of constructing and modifying contact manifolds. Here we recall basic definitions and briefly discuss those aspects of surgery that will be useful for our purposes. For a comprehensive treatment of topological surgery we refer the reader to [13]. A detailed discussion of the contact case can be found in [11].

First, let us recall the definition of *topological surgery.*

Definition 2.1. Let M be a smooth n-dimensional manifold and $S^k \hookrightarrow M$ a framed embedding of a sphere. By this, we mean an embedding together with a trivialisation of the normal bundle (*normal framing*), or equivalently, an embedding $S^k \times D^{n-k} \hookrightarrow M$. Then we say that the manifold

$$(M - S^k \times \mathring{D}^{n-k}) \cup_{S^k \times S^{n-k-1}} (D^{k+1} \times S^{n-k-1})$$

is obtained by the *surgery on M along S^k*. Here the two manifolds are glued along the natural identification of the boundaries:

$$\partial(M - S^k \times \mathring{D}^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1}).$$

For convenience, we will denote the result of the surgery on M along an embedded sphere $S \subset M$ by M_S. This is not a standard notation.

The diffeomorphism type of M_{S^k} depends only on the isotopy class of the embedding $S^k \hookrightarrow M$ and the homotopy class of the normal framing. Equivalently, it is determined by the isotopy class of the tubular neighbourhood embedding $S^k \times D^{n-k} \hookrightarrow M$.

As an example, let us discuss surgeries on S^n along lower-dimensional spheres. This will be important for the construction of PS-overtwisted contact structures.

Example 2.2. For $k < n$, consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$ as the subset of $\mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$ satisfying $\|x\|^2 + \|y\|^2 = 1$ for $(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$. This induces the decomposition

$$S^n = (D^{k+1} \times S^{n-k-1}) \cup (S^k \times D^{n-k}),$$

where the subsets

$$D^{k+1} \times S^{n-k-1} = \left\{(x,y) \in S^n \mid \|y\|^2 \leq 1/2\right\},$$
$$S^k \times D^{n-k} = \left\{(x,y) \in S^n \mid \|x\|^2 \leq 1/2\right\}$$

are glued along their common boundary

$$S^k \times S^{n-k-1} = \left\{(x,y) \in S^n \mid \|x\|^2 \geq 1/2, \|y\|^2 \geq 1/2\right\}.$$

The surgery on S^n along $S^k \times \{0\}$ with respect to the framing induced by the embedding $S^k \times D^{n-k} \subset S^n$ yields the manifold

$$(D^{k+1} \times S^{n-k-1}) \cup (D^{k+1} \times S^{n-k-1}) = S^{k+1} \times S^{n-k-1},$$

since the gluing map is given by the identity on the boundary $S^k \times S^{n-k-1}$.

Note that this process can be reversed by sugery on $\{pt\} \times S^{n-k-1} \subset S^{k+1} \times S^{n-k-1}$. Specifically, the surgery defined by the embedding $D^{k+1} \times S^{n-k-1} \subset S^{k+1} \times S^{n-k-1}$, where $D^{k+1} \subset S^{k+1}$ is the northern hemisphere, is diffeomorphic to S^n. We will be particularly interested in the case $k = n-2$, when by the surgery on $S^{n-1} \times S^1$ along S^1 one obtains S^n.

Let (M, ξ) be a contact manifold with a contact form α. In the realm of contact topology, one can still perform a surgery along an embedded sphere $S \subset M$ under the additional assumption that S is isotropic, i.e. α vanishes on S. Similarly to the topological case, apart from an isotropic embedding, one needs to choose a trivialisation of a certain vector bundle over S. However, this time we have to keep track of an additional geometric structure.

Recall that the restriction $d\alpha\big|_\xi$ equips the contact distribution ξ with a symplectic structure whose conformal class does not depend on the choice of the contact form α. Below, the symbol $\perp$ stands for the orthogonal complement in ξ with respect to the form $d\alpha$.

Definition 2.3. Let $S \subset M$ be an isotropic submanifold. i.e. $\alpha\big|_S = 0$. Then TS is an isotropic subbundle of the symplectic bundle $(\xi, d\alpha)$ and we define the *conformal symplectic normal bundle* of S in M as

$$\mathrm{CSN}_M(S) = TS^{\perp}/TS$$

equipped with the conformal symplectic structure induced by $d\alpha$.

Here is the corresponding definition for contact submanifolds.

Definition 2.4. Let $N \subset M$ be a contact submanifold with the contact structure $\xi' = \xi \cap TN$. The *conformal symplectic normal bundle* of N in M is the bundle

$$\mathrm{CSN}_M(N) = (\xi')^{\perp}$$

equipped with the conformal symplectic structure induced by $d\alpha$,

Note that for a contact submanifold $N \subset M$, we have

$$TM = TN \oplus \mathrm{CSN}_M(N).$$

Thus, the conformal symplectic normal bundle of N is isomorphic to its normal bundle, and a symplectic trivialisation of the former is in particular a normal framing of N. Similarly, if $S^k \hookrightarrow M$ is an isotropic embedding of a sphere, then a trivialisation of its conformal symplectic normal bundle induces the so-called *natural framing* of the normal bundle of S^k. Indeed, there is a bundle isomorphism

$$\mathcal{N}_M(S^k) \cong \epsilon \oplus TS^k \oplus \mathrm{CSN}_M(S^k),$$

where $\epsilon \to S^k$ denotes the trivial line bundle. The component $\epsilon \oplus TS$ has the standard trivialisation coming from the inclusion $S^k \subset \mathbb{R}^{k+1}$, hence a trivialisation of $\mathrm{CSN}_M(S^k)$ induces a normal framing of S^k. See see section 6.2 in [11] for details.

The following theorem is the main result in contact surgery.

Theorem 2.5 (Contact surgery theorem)**.** *Let S be an isotropic sphere in a contact manifold (M, ξ). Assume that a symplectic trivialisation of $\mathrm{CSN}_M(S)$ is given. Then the manifold M_S obtained from the surgery on M along S with respect to the natural framing carries a contact structure that coincides with ξ away from the surgery region.*

In fact, one can prove that the contact manifolds M and M_S are symplectically cobordant. Although will not give the full proof of this theorem (it can be found in [11]), it is important for our purpose to discuss briefly how the contact structure on M_S is constructed. This description is usually referred to as Weinstein's picture of contact surgery.

Let (M, ξ) be a $(2n-1)$-dimensional contact manifold, and S^{k-1} an isotropic sphere in M with trivial conformal symplectic normal bundle. The sphere S^{k-1} has a neighbourhood in M diffeomorphic to $S^{k-1} \times D^{2n-k}$ which we want to replace by $D^k \times S^{2n-k-1}$ in a way that would give us a natural contact structure on the resulting manifold. This can be done as follows. We realise $S^{k-1} \times D^{2n-k}$ and $D^k \times S^{2n-k-1}$ as open neighbourhoods of spheres $S^{k-1} \subset A$ and $S^{2n-k-1} \subset B$, where A and B are certain contact hypersurfaces in $\mathbb{R}^{2n}$ diffeomorphic to $\mathbb{R}^{2n-1}$. Moreover, $S^{k-1} \subset A$ is isotropic and has trivial conformal symplectic normal bundle. Then we construct a Liouville vector field Y on $\mathbb{R}^{2n}$ transverse to both A and B. Its flow enables us to identify the collars of $S^{k-1} \subset A$ and $S^{2n-k-1} \subset B$ through a contactomorphism and to perform a surgery on A along S^{k-1} by gluing the contact structures coming from A and B. Finally, by the isotropic neighbourhood theorem we identify the neighbourhoods of S^{k-1} in A and the original manifold M, so that this local model describes the surgery on M.

Here are the somewhat tedious details. Consider $\mathbb{R}^{2n}$ with coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

We look at $\mathbb{R}^{2n}$ as the product $\mathbb{R}^k \times \mathbb{R}^{2n-k}$ with coordinates $(q_1, \ldots, q_k)$ and $(q_{k+1}, \ldots, q_n, p_1, \ldots, p_n)$ respectively. Let Y be the Liouville vector field defined by

$$Y = \sum_{i=1}^{k} \left(-q_i \partial_{q_i} + 2p_i \partial_{p_i}\right) + \frac{1}{2} \sum_{i=k+1}^{n} \left(q_i \partial_{q_i} + p_i \partial_{p_i}\right).$$

One verifies easily that Y is the gradient vector field of the function

$$g(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{k} \left(-\frac{1}{2} q_i^2 + p_i^2\right) + \frac{1}{4} \sum_{i=k+1}^{n} \left(q_i^2 + p_i^2\right).$$

In particular, Y is transverse to the level sets $A = g^{-1}(-1)$ and $B = g^{-1}(1)$. Hence, the one-form

$$\alpha_0 = \iota_Y \omega_0 = \sum_{i=1}^{k} \left(q_i dp_i + 2p_i dq_i\right) + \frac{1}{2} \sum_{j=k+1}^{n} \left(-q_i dp_i + p_i dq_i\right)$$

induces contact structures on A and B. It is an easy check that these contact hypersurfaces are both diffeomorphic to $\mathbb{R}^{2n}$ and contain spheres

$$S_A^{k-1} = \left\{ \sum_{i=1}^{k} q_i^2 = 2,\ q_{k+1} = \cdots = q_n = p_1 \cdots = p_n = 0 \right\} \subset A,$$

$$S_B^{2n-k-1} = \left\{ q_1 = \cdots = q_k = 0,\ \sum_{i=1}^{k} p_i^2 + \frac{1}{4} \sum_{i=k+1}^{n} \left(q_i^2 + p_i^2\right) = 1, \right\} \subset B.$$

Note that $S_A^{k-1} \subset A$ is isotropic and has trivial conformal symplectic normal bundle. In particular, by the contact neighbourhood theorem, we may identify its neighbourhood with a neighbourhood of a given isotropic sphere S^{k-1} in M. Furthermore, the flow of the Liouville vector field Y in the region $g^{-1}([-1,1])$ has exactly one stationary point $(0,0)$, whose stable and unstable manifolds are the linear subspaces, respectively, $\{(q_1,\ldots,q_k) = 0\}$ and $\{(q_{k+1},\ldots,q_n,p_1,\ldots,p_n) = 0\}$. Outside these subspaces, the flow of Y yields a contactomorphism of open sets

$$A \setminus \nu(S^{k-1}) \cong B \setminus \nu(S_B^{2n-k-1}),$$

where ν denotes a closed tubular neighbourhood. This contactomorphism enables us to identify the collars of $S_A^{k-1} \times D^{2n-k}$ in A and $D^k \times S_B^{2n-k-1}$ in B and hence cut out the former and glue in the latter, obtaining a contact structure on the resulting manifold. Since we have identified $S_A^{k-1} \times D^{2n-k}$ with a tubular neighbourhood of S^{k-1} in M, this corresponds to a contact structure on the manifold resulting from the surgery on M along S^{k-1}.

A surgery can be performed relative to a submanifold. In the topological setting, if $N \subset M$ is a submanifold and $S \subset N$ an embedded sphere, then normal framings of $S \subset N$ and $N \subset M$ induce a normal framing of $S \subset M$, and it is not difficult to prove that the corresponding surgery M_S contains N_S as a submanifold with trivial normal bundle.

Example 2.6. Let $M_1,\ldots,M_k$ be a collection of submanifolds of a manifold M. Assume that they have trivial normal bundles and intersect transversely. For every subset of indices $\{i_1,\ldots,i_r\} \subset \{1,\ldots,k\}$ we consider

$$M_{i_1,\ldots,i_r} = M_{i_1} \cap \cdots \cap M_{i_r}.$$

Assume that the lowest dimensional intersection $M_{1,\ldots,k}$ has positive dimension and contains an embedded sphere S with trivial normal bundle. Then one can perform a surgery on M along S in such a way that it induces a surgery on each of the submanifolds $M_{i_1,\ldots,i_r}$. This can be done as follows. Choose normal framings of the submanifolds $M_1,\ldots,M_k$. They induce normal framings of each $M_{i_1,\ldots,i_r}$ by the decomposition

$$\mathcal{N}_M(M_{i_1,\ldots,i_r}) = \mathcal{N}_M(M_{i_1}) \oplus \cdots \oplus \mathcal{N}_M(M_{i_r}).$$

Moreover, if we choose a normal framing of S in the lowest dimensional intersection $M_{1,\ldots,k}$, then for each permutation of indices $\{i_1,\ldots,i_k\} = \{1,\ldots,k\}$ we obtain a trivialisation of the normal bundle of S in M since

$$\begin{aligned}
\mathcal{N}_M(S) &= \mathcal{N}_{M_{i_1,\ldots,i_k}}(S) \oplus \mathcal{N}_{M_{i_1,\ldots,i_{k-1}}}(M_{i_1,\ldots,i_k}) \oplus \cdots \oplus \mathcal{N}_M(M_{i_1}) = \\
&= \mathcal{N}_{M_{1,\ldots,k}}(S) \oplus \mathcal{N}_M(M_{i_k}) \oplus \cdots \oplus \mathcal{N}_M(M_{i_1}) = \\
&= \mathcal{N}_{M_{1,\ldots,k}}(S) \oplus \mathcal{N}_M(M_1) \oplus \cdots \oplus \mathcal{N}_M(M_k).
\end{aligned}$$

The latter equality shows that the framing does not depend on the choice of the permutation. In particular, for any subset of indices $\{i_1,\ldots,i_r\} \subset \{1,\ldots,k\}$ the normal framing of S in M is the sum of the normal framing of S in $M_{i_1,\ldots,i_r}$ and the trivialisation of the normal bundle of the latter. Therefore, the result of the surgery on M along S with respect to this framing naturally contains the surgery on $M_{i_1,\dots,i_r}$ along S as a submanifold.

We will be particularly interested in the case when $M = S^n \times S^1$ is the product of the sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x_1|^2 + \cdots + |x_{n+1}|^2 = 1\}$$

with the circle, and $M_1, \dots, M_{n-1}$ are the codimension one submanifolds given by

$$M_i = \{x \in S^n \mid x_i = 0\} \times S^1.$$

Each of them is diffeomorphic to $S^{n-1} \times S^1$ in an obvious way. They intersect transversely, and for each subset of indices $\{i_1, \dots, i_r\} \subset \{1, \dots, n-2\}$ the submanifold

$$M_{i_1,\dots,i_r} = M_{i_1} \cap \cdots \cap M_{i_r}$$

is naturally identified with $S^{n-r} \times S^1$. The lowest dimensional intersection $M_{1,\dots,n-1}$ is diffeomorphic to $S^1 \times S^1$ and contains an embedded circle $\Gamma = \{*\} \times S^1$, which generates the fundamental group of each $M_{i_1,\dots,i_r}$.

Using the technique discussed above, we can perform the surgery on M along Γ so that it induces the surgery on each of the submanifolds $M_{i_1,\dots,i_r}$. The result is diffeomorphic to the sphere S^{n+1} (see Example 2.2) and turns $M_{i_1,\dots,i_r}$ into the standard codimension r sphere in S^{n+1} given by the equations $x_1 = x_2 = \cdots = x_r = 0$.

The following deals with a similar situation in the contact setting.

Proposition 2.7. *Let (M, ξ) be a contact manifold, $N \subset M$ a contact submanifold with trivial normal bundle, and $S \subset N$ an isotropic sphere with trivial conformal symplectic normal bundle. Then*

(1) $\mathrm{CSN}_M(S) = \mathrm{CSN}_M(N) \oplus \mathrm{CSN}_N(S)$,
(2) *N_S is a contact submanifold of M_S.*

Here we assume that trivialisations of $\mathrm{CSN}_M(N)$ *and* $\mathrm{CSN}_N(S)$ *are given, inducing a trivialisation of* $\mathrm{CSN}_M(S)$, *and surgeries on M and N are performed with respect to the corresponding framings.*

Proof. The first statement is true for arbitrary embeddings $S \subset N \subset M$, where N is a contact submanifold and S is isotropic, regardless of the triviality of normal bundles. Let α be a contact form for ξ and $\xi' = \xi \cap TN$ denote the contact distribution on N. By definition, we have $TS \subset \xi' \subset \xi$, hence $(\xi')^\perp \subset TS^\perp$ and the decomposition $\xi = \xi' \oplus (\xi')^\perp$ yields

$$TS^\perp = (TS^\perp \cap \xi) \oplus (\xi')^\perp = TS^{\perp_{\xi'}} \oplus (\xi')^\perp,$$

where $TS^{\perp_{\xi'}}$ denotes the $d\alpha$-orthogonal complement of TS in ξ'. Thus,

$$\begin{aligned} \mathrm{CSN}_M(S) = TS^\perp/TS = (TS^{\perp_{\xi'}}/TS) \oplus (\xi')^\perp = \\ = \mathrm{CSN}_N(S) \oplus \mathrm{CSN}_M(N). \end{aligned}$$

In order to prove the second part, we need to recall Weinstein's picture of contact surgery. Assume that $\dim M = 2m-1$, $\dim N = 2n-1$, and $\dim S = k$ with $k < n < m$. Consider the contact hypersurfaces $A \cong \mathbb{R}^{2m-1}$ and $B \cong \mathbb{R}^{2m-1}$ in $(\mathbb{R}^{2m}, \omega_0)$ described previously. Then the subset

$$A' = A \cap \{q_{n+1} = \cdots = q_m = p_{n+1} = \cdots = q_m = 0\}$$

is a contact submanifold of A of dimension $2n-1$ with trivial normal bundle. Hence, the pair $A' \subset A$ is a local model for the neighbourhood of N in M. Furthermore, we have $S_A^{k-1} \subset A'$ and a neighbourhood of S in N is modelled on this embedding. The resulting product tubular neighbourhood of S in M corresponds to the trivialisation of $\mathrm{CSN}_M(S)$ induced from the given trivialisations of $\mathrm{CSN}_N(S)$ and $\mathrm{CSN}_M(N)$.

In the local description constructed above, Weinstein's picture of surgery on N along S is contained in Weinstein's picture of surgery on M along S through the natural embedding $\mathbb{R}^{2n} \subset \mathbb{R}^{2m}$ given by

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (q_1, \ldots, q_n, \underbrace{0, \ldots, 0}_{m-n}, p_1, \ldots, p_n, \underbrace{0, \ldots, 0}_{m-n}).$$

Specifically, consider the ingredients of Weinstein's picture of surgery on N: the Liouville vector field Y' on $\mathbb{R}^{2n}$, the function g', and contact hypersurfaces $A', B' \subset \mathbb{R}^{2n}$. By Y, g, A, B we denote the analogous objects on $\mathbb{R}^{2m}$ corresponding to the surgery on M. Then we have

$$Y' = Y\big|_{\mathbb{R}^{2n}}, \quad g' = g\big|_{\mathbb{R}^{2n}},$$
$$A' = A \cap \mathbb{R}^{2n}, \quad B' = B \cap \mathbb{R}^{2n}.$$

Therefore, performing the surgery on M along S induces the surgery on N along S and N_S is a submanifold of M_S. Recall that N_S and M_S consist of pieces of, respecitvely, A', B' and A, B, glued together along contactomorphism. Since A' is a contact submanifold of A, and B' is a contact submanifold of B, it follows that N_S is a contact submanifold of M_S. □

2.2. **Fibre connected sum and Presas gluing.** The connected sum operation can be defined in contact category, allowing us to construct a new contact manifold

$$M_1 \# M_2 = (M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\}) / \sim$$

out of contact manifolds M_1 and M_2 of the same dimension, by gluing them along punctured neighbourhoods of points $p_1 \in M_1$ and $p_2 \in M_2$. The *fibre connected sum* is a generalisation of this construction which enables us to glue two contact manifolds along a contact submanifold of codimension two.

Let us again begin with the topological definition.

Definition 2.8. Let M_1 and M_2 be oriented n-dimensional manifolds and N a closed oriented manifold of dimension $k < n$. Assume that two embeddings $N \hookrightarrow M_1$ and $N \hookrightarrow M_2$ are given together with a fibre orientation-reversing isomorphism of their normal bundles

$$\begin{CD} \mathcal{N}_{M_1}(N) @>{\Phi}>> \mathcal{N}_{M_2}(N) \\ @VVV @VVV \\ N @>{\mathrm{id}}>> N. \end{CD}$$

We identify the images of embeddings with N itself and their tubular neighbourhoods with the normal bundles, in which N is embedded as the zero section. Choose an auxilliary orientation-reversing diffeomorphism $\alpha\colon (0,\infty) \to (0,\infty)$ and bundle metrics on the normal bundles making Φ into an isometry. We define the *fibre connected sum* of M_1 and M_2 along N as the quotient

$$M_1 \#_N M_2 = (M_1 \setminus N) \cup (M_2 \setminus N) / \sim,$$

where the identification $\sim$ is given by the orientation-preserving diffeomorphism

$$\mathcal{N}_{M_1}(N) \setminus N \to \mathcal{N}_{M_2}(N) \setminus N$$
$$\mathbf{v} \mapsto \alpha(\|\mathbf{v}\|) \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The diffeomorphism type of $M_1 \#_N M_2$ depends on the isotopy class of embeddings $N \hookrightarrow M_1$ and $N \hookrightarrow M_2$, as well as the homotopy class of their normal framings.

Example 2.9. A fibre connected sum of two manifolds M and N along a point is just the connected sum $M\#N$.

Example 2.10. Let $S^k \subset S^n$ be a standard embedding. Assume that an embedding $S^k \subset M^n$ is given together with a trivialisation of the normal bundle. Then the fibre connected sum $M\#_{S^k}S^n$ is diffeomorphic to the result of the surgery along $S^k \subset M$. In particular, $S^n\#_{S^k}S^n$ is the surgery on S^n along S^k, which by Example 2.2 is diffeomorphic to $S^{k+1} \times S^{n-k-1}$.

Similarly to the surgery along spheres, in the contact case, the fibre connected sum along a codimension two submanifold can be performed so that it induces a contact structure on the resulting manifold.

Theorem 2.11 (Contact fibre connected sum)**.** *Let (M_1, ξ_1), (M_2, ξ_2), and (N, ξ') be cooriented contact manifolds such that $\dim M_1 = \dim M_2$ and $\dim N = \dim M_1 - 2$. Let $N \hookrightarrow M_1$ and $N \hookrightarrow M_2$ be contact embeddings that respect the coorientations, and $\Phi\colon \mathrm{CSN}_{M_1}(N) \to \mathrm{CSN}_{M_2}(N)$ a fibre-orientation-reversing isomorphism of the conformal symplectic normal normal bundles. Then the fibre connected sum $M_1\#_N M_2$ admits a contact structure that coincides with ξ_1 and ξ_2 outside tubular neighbourhoods of N in M_1 and M_2, respectively.*

As we will need the theorem only in the situation when the normal bundle is trivial, we refer to [11] for the proof in the general case. What is important to us, is the following observation due to Presas that allows us to construct higher dimensional PS-overtwisted manifold with the fibre connected sum.

Theorem 2.12 (Presas gluing)**.** *Let (M_1, ξ_1) and (M_2, ξ_2) be contact manifolds of the same dimension that both contain a PS-overtwisted contact submanifold (N, ξ') of codimension two with trivial normal bundle. Then the fibre connected sum $M_1\#_N M_2$ is a PS-overtwisted manifold. Specifically, if P is a* bLob *in N, then $M_1\#_N M_2$ contains a* bLob *diffeomorphic to $S^1 \times P$.*

Proof. By the contact neighbourhood theorem, the submanifold N has neighbourhoods $U_1 \subset M_1$ and $U_2 \subset M_2$ contactomorphic to

$$(D^2 \times N, \alpha_N + r^2 d\phi),$$

where $D^2 = \{r^2 < \epsilon\}$ is an open disc with radial coordinates (r, ϕ), and α_N is a contact form for (N, ξ'). Denote by s the new coordinate $s = r^2$. Note that (s, ϕ) are coordinates only on $D^2 \setminus \{0\}$. After removing N, the open sets $U_1 \setminus N$ and $U_2 \setminus N$ are contactomorphic to

$$\left((0, \epsilon) \times S^1 \times N, \alpha_N + s d\phi\right).$$

We can attach the negative collar $(-\epsilon, 0] \times S^1 \times N$ to $M_1 \setminus N$ and $M_2 \setminus N$ through the obvious identifications. This operation does not change the contactomorphism type by the Gray stability theorem. Now we can glue together the extended manifolds we have obtained via the contactomorphism

$$(-\epsilon, \epsilon) \times S^1 \times N \to (-\epsilon, \epsilon) \times S^1 \times N,$$
$$(s, \phi, p) \mapsto (-s, -\phi, p).$$

By the construction, the resulting manifold is contactomorphic to the fibre connected sum $M_1\#_N M_2$.

Let $P \subset N$ be a bLob. Let $B \subset P$ be its binding and $\vartheta\colon P \setminus B \to S^1$ the fibration defining the relative open book structure. The open set $(-\epsilon, \epsilon) \times S^1 \times N$ contains the submanifold $\widehat{P} = \{0\} \times S^1 \times P$. We can equip it with the relative open book structure with binding $\{0\} \times S^1 \times B$ and fibration $\widehat{\vartheta}\colon \widehat{P} \to S^1$ given by

$$(\phi, x) \mapsto \vartheta(x).$$

The contact form on $M_1 \#_N M_2$ restricts to α_N on $\widehat{P}$. Hence, $\widehat{P}$ is equipped with a Legendrian foliation whose leaves are the pages of the open book. Furthermore, from the corresponding statements for P it follows that the boundary of $\widehat{P}$ consists of singular points and around the binding $\widehat{P}$ the foliation looks like the radial foliation on $D^2 \times S^1 \times B$. This proves that $\widehat{P}$ is a bLob in $M_1 \#_N M_2$.

□

The theorem allows us to construct PS-overtwisted contact manifolds of arbitrary dimension. Recall that any three-manifold admits an overtwisted contact structure.

Example 2.13. Let M be a closed PS-overtwisted contact manifold, for instance an overtwisted three-manifold. By a result of Bourgeois [3], there exists a contact structure on the product $M \times T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two-dimensional torus, such that each fibre $M \times \{*\}$ is a contact submanifold with trivial normal bundle. If N is another contact manifold containing M as a codimension two contact submanifold with trivial normal bundle, then the result of the gluing $N \cup_M M \times T^2$ is PS-overtwisted by Theorem 2.12. This was the original result proved using different methods by Presas in [21].

In particular, we can glue two copies of $M \times T^2$ together along $M \times \{*\}$ to obtain a PS-overtwisted contact structure on $M \times \Sigma_2$, where Σ_2 is the surface of genus two. This gives us examples of closed PS-overtwisted manifolds in any dimension.

As in the case of the contact surgery (see Proposition 2.7), we want to be able to perform fibre connected sum in a way inducing the same operation on a given submanifold. This will be important for the inductive step in the construction of PS-overtwisted spheres. First, let us consider the following topological example.

Example 2.14. The sphere S^{2n-1}, considered as a subset of $\mathbb{C}^n$

$$S^{2n-1} = \{z \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

contains the n codimension two spheres $S_i = S^{2n-1} \cap \{z_i = 0\}$, as well as their transverse intersections

$$S_{i_1,\ldots,i_r} = S_{i_1} \cap \cdots \cap S_{i_r} \cong S^{2n-2r-1}.$$

A discussion of normal framings similar to the one in Example 2.6 shows that the fibre connected sum $S^{2n-1} \#_{S_n} S^{2n-1} \cong S^{2n-2} \times S^1$ contains each of the gluing

$$S_{i_1,\ldots,i_r} \#_{S_{i_1,\ldots,i_r,n}} S_{i_1,\ldots,i_r}$$

as a submanifold with trivial normal bundle for any collection of indices $\{i_1, \ldots, i_r\}$ not containing n. In fact, under the identification of the ambient manifold $S^{2n-1} \#_{S_n} S^{2n-1}$ with $S^{2n-2} \times S^1$, the codimension two submanifolds $S_i \#_{S_{i,k}} S_i$ for $i = 1, \ldots, n-1$, correspond to $(S^{2n-2} \cap \{z_i = 0\}) \times S^1$.

The next result allows us deal with situations as the one discussed above in the presence of a contact structure.

Proposition 2.15. *Let M_1, M_2, and N be as in Theorem 2.11. Assume that contact submanifolds $S_1 \subset M_1$, $S_2 \subset M_2$, and $S \subset N$ with trivial normal bundles are given, so that S_1 and S_2 intersect N transversly along S in M_1 and M_2, respectively. Then the fibre connected sum $M_1 \#_N M_2$ contains $S_1 \#_S S_2$ as a contact submanifold.*

Remark 2.16. As usually, we need to carefully choose framings. One easily checks that there are natural isomorphisms of conformal symplectic bundles

$$\mathrm{CSN}_{S_1}(S) \cong \mathrm{CSN}_{M_1}(N)\big|_S,$$
$$\mathrm{CSN}_{S_2}(S) \cong \mathrm{CSN}_{M_2}(N)\big|_S.$$

Under these identifications, the isomorphism $\Phi\colon \mathrm{CSN}_{M_1}(N) \to \mathrm{CSN}_{M_2}(N)$ from Theorem 2.11 yields an isomorphism $\mathrm{CSN}_{S_1}(S) \to \mathrm{CSN}_{S_2}(S)$, and we glue S_1 and S_2 using this map.

Proof. By the contact neighbourhood theorem, the submanifold N has a neighbourhood in M_1 contactomorphic to a neighbourhood of the zero section in the conformal symplectic normal bundle $\mathrm{CSN}_{M_1}(N)$. Similarly, $S = N \cap S_1$ has a neighbourhood in S_1 contactomorphic to a neighbourhood of the zero section in $\mathrm{CSN}_{S_1}(S)$. Since there is an isomorphism $\mathrm{CSN}_{M_1}(N)\big|_S \cong \mathrm{CSN}_{S_1}(S)$, we can choose both neighbourhoods in a compatible way. In other words, there exists a neighbourhood of N in M_1 contactomorphic to $N \times D^2$ such that S_1 in this neighbourhood is given by $(N \cap S_1) \times D^2$. Analogously for S_2 in M_2.

Using such neighbourhoods, we can construct the Presas gluing $M_1 \#_N M_2$ in the way described in the proof of Theorem 2.12, so that it induces the analogous construction for the fibre connected sum $S_1 \#_S S_2$. □

2.3. **PS-overtwisted spheres.** Having discussed all the tools of contact topology that we need, we can now proceed to the construction of PS-overtwisted contact structures on spheres.

First, consider the non-standard contact structure on the unit sphere $S^{2n-1} = \{z \in \mathbb{C}^n \mid |z| = 1\}$ given by the one-form

$$\alpha_- = i \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j) - i(f d\bar{f} - \bar{f} df),$$

where $f\colon \mathbb{C}^n \to \mathbb{C}$ is the standard Morse function

$$f(z) = z_1^2 + \cdots + z_n^2.$$

Remark 2.17. The contact structure (S^{2n-1}, α_-) can be constructed from the structure of an open book with page $(T^*S^{n-1}, d\lambda_{can})$ and monodromy consisting of a left-handed Dehn twist. See Remark 8 in [20] for details.

Proposition 2.18. *For every $n \geq 1$, the sphere (S^{2n-1}, α_-) is a contact manifold. Furthermore,*

(1) *(S^3, α_-) is overtwisted.*
(2) *There are contact embeddings $(S^{2k-1}, \alpha_-) \hookrightarrow (S^{2k+1}, \alpha_-)$ given by the standard inclusions $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_{j-1}, 0, z_j, \dots, z_k)$.*

Proof. The proof that α_- is a contact form is a straightforward computation. Consider the following one-form on $\mathbb{C}^n \setminus \{0\}$:

$$\widehat{\alpha}_- = i \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j) - i \left(\frac{f}{|z|} d\left(\frac{\bar{f}}{|f|}\right) - \frac{\bar{f}}{|z|} d\left(\frac{f}{|z|}\right) \right)$$

and its differential

$$\omega_- = d\widehat{\alpha}_- = 2i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - 2i d\left(\frac{f}{|z|}\right) \wedge d\left(\frac{\bar{f}}{|z|}\right).$$

One easily checks that ω_- is a symplectic form on $\mathbb{C}^n \setminus \{0\}$ and that $\widehat{\alpha}_- = \iota_X \omega_-$ for the Liouville vector field

$$X = \frac{1}{2}(z_1, \dots, z_n).$$

In particular, since X is transverse to the unit sphere S^{2n-1} and the restriction of $\widehat{\alpha}_-$ to S^{2n-1} is equal to α_-, it follows that the latter is a contact form.

As the second statement is trivially true, we need to prove that (S^3, α_-) is overtwisted. This follows from Remark 2.17, but we can also find explicitely an overtwisted disc. Let $F \subset \mathbb{C}^2$ be the intersection of S^3 and the hyperplane $\{\mathrm{Im}(z_1) = \mathrm{Re}(z_2)\}$. It is naturally identified with the two-dimensional sphere S^2. Consider the inverse of the stereographic projection

$$\Phi\colon \mathbb{C} \to F \subset \mathbb{C}^2$$

$$\Phi(x+iy) = \frac{1}{\sqrt{2}(1+x^2+y^2)} \begin{pmatrix} (x+1)^2 + y^2 - 2 \\ 2y \\ 2y \\ (x-1)^2 + y^2 - 2 \end{pmatrix}.$$

A straightforward but tedious calculation shows that the pull-back of the contact form is given by the formula

$$\Phi^*\alpha_- = \frac{4(3r^4 - 10r^2 + 3)}{(1+r^2)^4}(ydx - xdy),$$

where $r^2 = x^2 + y^2$. We see that the characteristic foliation of the sphere F has an isolated singularity at the point corresponding to $x = y = 0$ and a singular circle $r^2 = 1/3$. Thus, the image of $\{x^2 + y^2 \leq 1/3\}$ under the map Φ is an overtwisted disc in (S^3, α_-). □

We are now ready to construct a PS-overtwisted structure on the sphere S^5. The general case will be proved by induction, so it is important to understand the first step well in order to see why we need to go through a somewhat complicated induction procedure.

Proposition 2.19. *The five-dimensional sphere S^5 carries a PS-overtwisted contact structure.*

Proof. The contact sphere (S^5, α_-) contains the overtwisted sphere (S^3, α_-) as a contact submanifold with trivial normal bundle, with the embedding $S^3 \hookrightarrow S^5$ being the standard inclusion $(z_1, z_2) \mapsto (z_1, z_2, 0)$. Consider the Presas gluing $M = S^5 \#_{S_3} S^5$ corresponding to the standard trivialisation of the normal bundle of S^3 in S^5. Topologically, the fibre connected sum with S^5 along S^3 is the same as the surgery along S^3, hence M is diffeomorphic to $S^4 \times S^1$ (see Examples 2.2 and 2.10). By Theorem 2.12, the manifold M carries a PS-overtwisted contact structure.

Let P be an overtwisted disc in S^3. Consider the open subset of M that was used in the proof of Theorem 2.12 to glue the two pieces of M:

$$\left((-\epsilon, \epsilon) \times S^1 \times S^3, \alpha_{S^3} + s d\phi\right),$$

where (s, ϕ) are coordinates on the cylinder $(-\epsilon, \epsilon) \times S^1$. Then $\{0\} \times S^1 \times P$ is a bLob in M. Choose any point $p \in S^3$ which does not belong to the overtwisted disc $P \subset S^3$ and consider the loop

$$\Gamma = \{0\} \times S^1 \times \{p\} \subset M.$$

Under the diffeomorphism $M \cong S^4 \times S^1$, it corresponds to the standard generator of the fundamental group $\{*\} \times S^1$. The loop Γ is isotropic and has trivial normal bundle isomorphic to $\langle \partial_s \rangle \oplus T_p S^3$. The conformal symplectic normal bundle of Γ corresponds in this decomposition to $\xi'_p \subset T_p S^3$, where ξ' is the contact distribution on S^3. This trivialisation of the conformal symplectic normal bundle induces the standard normal framing of $\{*\} \times S^1$ in $S^4 \times S^1$. Therefore, we can perform the contact surgery on M along Γ so that topologically it corresponds to the standard surgery on $S^4 \times S^1$ along the circle $\{*\} \times S^1$. The resulting contact manifold is diffeomorphic to the five-sphere S^5. Since the loop Γ was chosen so that it did not intersect the $S^1 \times P$, and outside the surgery region the old and new contact structures coincide, we have obtained a contact structure on S^5 that contains a bLob. □

If the resulting PS-overtwisted sphere S^5 could be embedded as an unknotted contact submanifold of a contact sphere S^7, then proceeding in the same way we could construct a PS-overtwisted contact structure on S^7. However, although we started with the contact sphere (S^5, α_-) which naturally embeds into (S^7, α_-), it is not at all clear that the manifold we obtain after performing Presas gluing and surgery again admits such a contact embedding. This is why in the general case we need to inductively construct PS-overtwisted contact spheres together with contact embeddings into higher-dimensional spheres.

Theorem 2.20. *For every $n \geq 1$, the sphere S^{2n-1} carries a PS-overtwisted contact structure.*

Proof. We will show by reverse induction on $k \leq n-2$, that there exists a contact structure on S^{2n-1} containing an embedded PS-overtwisted sphere of codimension $2k$ as a contact submanifold. In the induction step, k is decreased by 1, and finally, for $k = 0$ we obtain a PS-overtwisted contact structure on S^{2n-1} itself.

Here is the precise statement of the inductive hypothesis. There exist a contact manifold S of dimension $2n-1$ and contact submanifolds $S_1, \ldots, S_k$ of codimension two such that:

(1) There is a diffeomorphism $S \cong S^{2n-1}$ mapping S_i onto the sphere $S^{2n-1} \cap \{z_i = 0\}$ for each $i = 1, \ldots, k$.
(2) For every subset of indices $\{i_1, \ldots, i_r\} \subset \{1, \ldots, k\}$ the intersection
$$S_{i_1,\ldots,i_r} = S_{i_1} \cap \cdots \cap S_{i_r}$$
is a contact submanifold of codimension $2r$.
(3) The lowest-dimensional intersection $S_{1,\ldots,k}$ is PS-overtwisted.

We start the induction from $k = n-2$. Consider the contact manifold (S^{2n-1}, α_-) introduced before. It contains the contact spheres $S_i = S^{2n-1} \cap \{z_i = 0\}$. The family $S_1, \ldots, S_{n-2}$ satisfies the first two conditions and their lowest-dimensional intersection $S_{1,\ldots,n-2}$ is indeed an overtwisted three-sphere, as we have proved in Proposition 2.18.

Assume that the inductive hypothesis holds for some $k \leq n-2$, i.e. there is a contact $(2n-1)$-sphere S with contact submanifolds $S_1, \ldots, S_k$ satisfying conditions (1) – (3). Consider the Presas gluing of two copies of S along S_k:
$$M = S \#_{S_k} S.$$
Since the sphere S_k is unknotted in S, after the gluing we obtain a contact manifold diffeomorphic to $S^{2n-2} \times S^1$ (see Example 2.10). Furthermore, by Proposition 2.15, M contains as a contact submanifold the Presas gluing
$$M_j = S_j \#_{S_{j,k}} S_j,$$
for each $j = 1, \ldots, k-1$. More generally, for any subset of indices $\{i_1, \ldots, i_r\} \subset \{1, \ldots, k-1\}$, the result of the gluing
$$M_{i_1,\ldots,i_r} = S_{i_1,\ldots,i_r} \#_{S_{j_1,\ldots,j_r,k}} S_{i_1,\ldots,i_r}$$
is a contact submanifold of $M_{i_1,\ldots,\widehat{i_s},\ldots i_r}$.

Note that in order to perform the gluing we do not need to specify the normal framing, as it is unique in this case. Indeed, homotopy classes of normal framings of a codimension two l-dimensional sphere with oriented trivial normal bundle are in one-to-one correspondence with elements of the group $\pi_l(\mathrm{SO}(2))$, which is trivial for $l \geq 2$. Hence, the fibre connected sum of two manifolds along such a sphere is specified solely by the embeddings of the sphere. In particular, by condition (1) our gluing is topologically equivalent to the one discussed in Example 2.14. Hence, each of the fibre connected sums $M_{i_1,\ldots,i_r}$ is diffeomorphic to the standard gluing

$$S^{2n-2r-1}\#_{S^{2n-2r-2}}S^{2n-2r-1} \cong S^{2(n-r-1)} \times S^1,$$

and there exists a diffeomorphism $M \cong S^{2n-2} \times S^1$ mapping each of the submanifolds M_i diffeomorphically to the corresponding submanifold

$$S_i^{2n-4} \times S^1 = (S^{2n-2} \cap \{z_i = 0\}) \times S^1 \subset S^{2n-2} \times S^1.$$

Since the lowest dimensional sphere $S_{1,\ldots,k}$ is PS-overtwisted, Theorem 2.12 guarantees that the manifold $M_{1,\ldots,k-1} \cong S^{2(n-k)} \times S^1$ obtained from Presas gluing is also PS-overtwisted. Furthermore, as in the five-dimensional case discussed in Proposition 2.19, it contains an isotropic circle Γ of the form $\{*\} \times S^1$, which is disjoint from the given bLob. The loop Γ is contained in every higher-dimensional manifold $M_{i_1,\ldots,i_r}$ and corresponds to $\{*\} \times S^1$ under the identification $M_{i_1,\ldots,i_r} \cong S^{2(n-r-1)} \times S^1$. By Proposition 2.7, we may perform the contact surgery on M along Γ, inducing the contact surgery on every submanifold $M_{i_1,\ldots,i_r}$. We claim that a framing of the conformal symplectic normal bundle of Γ can be chosen so that topologically this surgery is equivalent to the one discussed in Example 2.6. In this case, the resulting contact manifold $S' = M_\Gamma$ is diffeomorphic to S^{2n-1} and contains a family of $k-1$ contact spheres of codimension two

$$S_1' = M_{1,\Gamma}, \quad \ldots, \quad S_{k-1}' = M_{k-1,\Gamma},$$

which satisfies conditions $(1) - (3)$. This proves the induction step.

We still need to address the problem of choosing the right framings. We want to choose them in a way allowing us to apply Proposition 2.7 and so that the result is compatible with the framings from Example 2.6. First, choose a framing of the conformal symplectic normal bundle of Γ in $M_{1,\ldots,k-1}$ so that the induced normal framing corresponds to the standard framing of $\{*\} \times S^1$ in $S^{2(n-k)} \times S^1$. This can be done in the same way as in the five-dimensional case. Then choose the normal framing of each $M_i \cong S_i^{2n-4} \times S^1$ in $M \cong S^{2n-2} \times S^1$ corresponding to the standard normal framing of S_i^{2n-4} in S^{2n-2}. These framings induce normal framings of each submanifold $M_{i_1,\ldots,i_r}$ in M thanks to the decomposition

$$\mathrm{CSN}_M(M_{i_1,\ldots,i_r}) = \mathrm{CSN}_M(M_{i_1}) \oplus \cdots \oplus \mathrm{CSN}_M(M_{i_r}).$$

Similarly, we have

$$\mathrm{CSN}_M(\Gamma) = \mathrm{CSN}_{M_{1,\ldots,k-1}}(\Gamma) \oplus \mathrm{CSN}_M(M_1) \oplus \cdots \oplus \mathrm{CSN}_M(M_{k-1})$$

$$\mathrm{CSN}_{M_{i_1,\ldots,i_r}}(\Gamma) = \mathrm{CSN}_{M_{1,\ldots,k-1}}(\Gamma) \oplus \mathrm{CSN}_M(M_{i_{r+1}}) \oplus \cdots \oplus \mathrm{CSN}_M(M_{i_{k-1}}),$$

where $\{i_{r+1}, \ldots, i_{k-1}\} = \{1, \ldots, k-1\} \setminus \{i_1, \ldots, i_r\}$ is the set of complementary indices. These decompositions give us trivialisations of the conformal symplectic normal bundles of Γ in M and $M_{i_1,\ldots,i_r}$, respectively, in such a way that each triple $\Gamma \subset M_{i_1,\ldots,i_r} \subset M$ satisfies the assumptions of Proposition 2.6. Therefore, the contact surgery on M along Γ can indeed be done in a way inducing the corresponding contact surgery on every submanifold $M_{i_1,\ldots,i_r}$. Furthermore, such a choice of framings guarantees that topologically our situation corresponds to the one from Example 2.6. □

Remark 2.21. In the original proof [20], the authors used the gluing from Example 2.13. Consequently, they had to show that the result of killing the fundamental group of the manifold

$$S^{2n-1}\#_{S^{2n-3}}(S^{2n-3} \times T^2)$$

is the sphere S^{2n-1} with its standard differentiable structure.

Remark 2.22. In the proof we perform two operations:

$$S^{2n-1} \xrightarrow{\text{Presas gluing}} S^{2n-2} \times S^1 \xrightarrow{\text{surgery on } S^1} S^{2n-1} .$$

As we have stressed before, topologically they are reverse surgeries. However, the second operation does not necessarily reverse the first one in the contact realm, meaning that the final result does not need to be contactomorphic to the sphere (S^{2n-1}, α_-) we begin with. In particular, we do not know whether the latter is PS-overtwisted.

As an easy corollary, we obtain the main result.

Theorem 2.23 (Niederkrüger – van Koert)**.** *Every contact manifold admits a PS-overtwisted, and hence non-fillable contact structure.*

Proof. Let (M, ξ) be a contact manifold of dimension $2n-1$ and (S^{2n-1}, ξ_{PS}) a contact sphere containing an embedded bLob P. Then the connected sum of (M, ξ) and (S^{2n-1}, ξ_{PS}) along a point in S^{2n-1} not belonging to P is a contact manifold which is diffeomorphic to M and contains a bLob. □

Remark 2.24. Entyre and Pancholi [9] have given another proof of the theorem. Recall that every contact three-manifold can be made overtwisted by applying a Lutz twist along a transverse knot. One can generalise the definition of a Lutz twist and obtain the same result in higher dimensions. This approach has an additional feature that the PS-overtwisted contact structure constructed in such a way is homotopic to the original contact structure through almost contact structures.

Appendix A. Almost complex geometry

In this appendix we collect basic definitions and facts about almost complex manifolds and pseudoholomorphic curves. We assume that the reader has already encountered most of them, hence our treatment will be rather terse. For a comprehensive study of almost complex geometry, see the book [5]. All the proofs can be also found in [19].

Let (W, J) be an almost complex manifold, i.e. J is an endomorphism of the tangent bundle TW satisfying $J^2 = -\mathrm{Id}$.

Definition A.1. A *J-holomorphic* or *pseudoholomorphic* curve in (W, J) is a smooth map $u\colon \Sigma \to W$ from a Riemann surface (Σ, i) (possibly with boundary) whose differential commutes with the almost complex structures, i.e. $du \circ i = J \circ du$. We will always assume the Riemann surface Σ to be compact and connected.

We will be particularly interested in the pseudoholomorphic curves with boundary satisfying the additional condition $u(\partial\Sigma) \subset A$, for a given subset $A \subset W$. In this case, we write

$$u\colon (\Sigma, \partial\Sigma) \to (W, A).$$

We also often require our curves to be somewhere injective.

Definition A.2. Let $u\colon \Sigma \to W$ be a J-holomorphic curve. We call u *injective at a point* $x \in \Sigma$ if x is the only point mapped to $u(x)$ and the derivative the derivative du does not vanish at x. We call a J-holomorphic curve *somewhere injective* if it is injective at some point.

Definition A.3. A smooth function $f\colon W \to \mathbb{R}$ is called *J-convex* or *plurisubharmonic* if for every non-zero vector v tangent to W we have

$$-dd^J f(v, Jv) > 0.$$

Recall that $d^J f$ is a one-form on W defined by $d^J f = df \circ J$. Sometimes we also use notation $d^{\mathbb{C}}$ instead of d^J, particularly when the almost complex structure is integrable.

An easy computation shows that if $u\colon \Sigma \to W$ is a J-holomorphic curve, then in local holomorphic coordinates $z = x + iy$ on Σ, we have

$$u^*(-dd^J f) = \left(\frac{\partial^2 (f \circ u)}{\partial x^2} + \frac{\partial^2 (f \circ u)}{\partial y^2}\right) dx \wedge dy = \Delta\,(f \circ u)\ \ dx \wedge dy.$$

Hence, a J-convex function restricts to a weakly subharmonic function on any J-holomorphic curve. Furthermore, $f \circ u$ has strictly positive Laplacian at every point $x \in \Sigma$ such that the derivative du does not vanish at x.

Using the above observation and well-known facts about subharmonic functions, one proves the following important results. We use them extensively in section 1.

Lemma A.4 (Maximum principle)**.** *Let $u\colon \Sigma \to W$ be a J-holomorphic curve and $f\colon W \to \mathbb{R}$ a J-convex function. If the function $f \circ u\colon \Sigma \to \mathbb{R}$ attains a local maximum at an interior point of Σ, then u is a constant map.*

Lemma A.5 (Boundary point lemma)**.** *Let $u\colon \Sigma \to W$ be a J-holomorphic curve with boundary and $f\colon W \to \mathbb{R}$ a J-convex function. If the function $f \circ u\colon \Sigma \to \mathbb{R}$ attains its maximum at a point $z_0 \in \partial\Sigma$, then either u is constant or $d(f \circ u)(v) > 0$ for every vector $v \in T_{z_0}\Sigma$ positively transverse to $\partial\Sigma$.*

The above lemmas can be stated in a more geometric way using the notion of a pseudoconvex hypersurface.

Definition A.6. Let M be a co-oriented codimension one submanifold of W. The *distribution of complex tangencies* of M is the maximal J-invariant subbundle of TM. Explicitely, it is given by $\xi = TM \cap J(TM)$.

Let α be a one-form on M defining ξ, i.e. $\xi = \ker \alpha$. Fix the sign of α by assuming that M is co-oriented by a normal vector v and $\alpha(Jv) > 0$. Then the *Levi form* of M is a two-form on ξ given by $\omega_M = d\alpha\big|_\xi$. It is well-defined up to a multiplication by a positive function.

Definition A.7. A hypersurface M is called *J-convex* or *pseudoconvex* if its Levi form ω_M is positively defined on the distribution of complex tangencies ξ, i.e $\omega_M(v, Jv) > 0$ for every non-zero vector $v \in \xi$.

Example A.8. An odd-dimensional sphere S^{2n-1} is the pseudoconvex boundary of the even-dimensional ball B^{2n} viewed as a domain in $\mathbb{C}^n$. Pseudoconvex domains in $\mathbb{C}^n$ motivated the more general notion of J-convexity we consider here. They appear naturally in complex analysis as they are domains of holomorphy.

We have the following useful characterisation of pseudoconvexity.

Proposition A.9. *Let $M \subset W$ be a co-oriented hypersurface. Then the following conditions are equivalent:*

(1) *M is J-convex, i.e. its Levi form is positively defined.*
(2) *There exists a smooth function $f\colon W \to \mathbb{R}$ such that f is J-convex on a neighbourhood of M and f is a regular equation for M, i.e. $M = f^{-1}(0)$ and df does not vanish along M.*
(3) *The distribution of complex tangencies ξ is a contact structure on M whose natural orientation is compatible with the given co-orientation of M and whose conformal symplectic structure tames $J\big|_\xi$.*

The third point of the above characterisation is the cornerstone of an intimate connection between almost complex manifolds with pseudoconvex boundaries and symplectic fillings. We explore this connection at the beginning of section 1.

The maximum principle and boundary point lemmas impose some geometric constraints on the behaviour of pseudoholomorphic curves near a pseudoconvex boundary.

Proposition A.10. *Let (W, J) be an almost complex manifold with J-convex boundary $M = \partial W$. Then no non-constant J-holomorphic curve in W can touch M at an interior point. If such a curve touches M at a boundary point, then it is transverse to M at this point.*

Proof. Let $f\colon W \to (-\infty, 0]$ be a smooth function such that $M = f^{-1}(0)$, df does not vanish on M and f is J-convex on a neighbourhood of M. Such a function exists by the previous characterisation of J-convex hypersurfaces.

Let $u\colon \Sigma \to W$ be a non-constant J-holomorphic curve. If there was an interior point $x \in \Sigma$ such that $u(x) \in M$, then the function $f \circ u$ would have a global maximum at x. This is impossible by the maximum principle. On the other hand, if $x \in \partial\Sigma$ and $u(x) \in M$, then by the boundary point lemma we have

$$df(u_* v) = d(f \circ u)(v) > 0$$

for a vector $v \in T_x\Sigma$ pointing out of $\partial\Sigma$. This shows that the vector $u_*(v)$ points out of M and hence the map u is transverse to M at x. □

Apart from pseudoconvex hypersurfaces, another important class of submanifolds of almost complex manifolds are totally real ones. In the context of symplectic fillings they appear naturally as submanifolds equipped with a Legendrian foliation. We discuss such submanifolds in the part of section 1 devoted to bordered Legendrian open books (see particularly Lemma 1.10).

Definition A.11. A submanifold $L \subset W$ is called *totally real* if
$$TW\big|_L = TL \oplus J(TL).$$
Equivalently, $\dim L = \frac{1}{2}\dim W$ and $TL \cap J(TL) = 0$.

Example A.12. A basic example is the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$.

Example A.13. If W is equipped with a symplectic structure ω compatible with J, then every Lagrangian submanifold $L \subset X$ is totally real, since $J(TL)$ is the orthogonal complement of TL with respect to the Riemannian metric $\omega(J\cdot, \cdot)$.

From our point of view, the most important property of totally real submanifolds is that they impose certain constraints on pseudoholomorphic curves. To prove that, we need an auxiliary statement that a totally real submanifold can be described as a global minimum of a J-convex function.

Proposition A.14. *If L is a closed totally real submanifold of an almost complex manifold (W, J), then there exists a smooth function $f\colon W \to [0, \infty)$ such that $L = f^{-1}(0)$ and f is J-convex on a neighbourhood of L. In particular, every point of L is a global minimum of L and $df = 0$ along L.*

As a corollary, we obtain the following important result.

Proposition A.15. *Let L be a closed totally real submanifold of an almost complex manifold (W, J). There exists an open neighbourhood $U \subset W$ of L such that every J-holmorphic curve $u\colon (\Sigma, \partial\Sigma) \to (W, L)$ with $u(\Sigma) \subset U$ is constant.*

Proof. Let $f\colon W \to [0, \infty)$ be a function such that $L = f^{-1}(0)$ and f is J-convex on a neighbourhood U of L. Since we have assumed that $u(\Sigma) \subset U$, the composition $f \circ u$ is a well-defined subharmonic function on Σ and hence it attains its global maximum at the boundary $\partial\Sigma$. At the same time, the boundary image $u(\partial\Sigma)$ lies in L, where f vanishes. Therefore, $f \circ u$ identically vanishes on Σ and $u(\Sigma) \subset L$. If $x \in \Sigma$ is a point such that $du_x \neq 0$, then $du_x(T_x\Sigma) \subset T_{u(x)}L$ is a two-dimensional subspace closed under the action of J. As L is totally real, this cannot happen, hence $du = 0$ everywhere and u is constant. □

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