Lecture 31

Quadratic Inequalities

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Quadratic inequalities

We will solve inequalities of the following types:

\[ ax^2 + bx + c \geq 0, \quad ax^2 + bx + c > 0, \quad ax^2 + bx + c \leq 0, \quad ax^2 + bx + c < 0, \]

where \( a \neq 0 \), \( b \), \( c \) are given coefficients, and \( x \) is unknown.

For example, \( x^2 + 5x - 6 \leq 0 \) is a quadratic inequality.
Here \( a = 1 \), \( b = 5 \), \( c = -6 \).

The coefficient \( a \) is not zero, otherwise the inequality would be not quadratic, but rather linear.

What does it mean to solve inequality?
It means to find all the values of unknown \( x \) for which the inequality holds true.

Visualization

Let us draw a picture illustrating a quadratic inequality.

We know that the equation \( y = ax^2 + bx + c \) defines a parabola, and know how to draw this parabola.

If \( a > 0 \), then the parabola opens upward:

- two \( x \)-intercepts
- one \( x \)-intercept
- no \( x \)-intercepts

If \( a < 0 \), then the parabola opens downward:

- two \( x \)-intercepts
- one \( x \)-intercept
- no \( x \)-intercepts
Geometric solution

Let us solve the inequality \( ax^2 + bx + c > 0 \) in the case when \( a > 0 \).

Let \( y = ax^2 + bx + c \). Then \( ax^2 + bx + c > 0 \iff y > 0 \).

Thus, to solve the inequality \( ax^2 + bx + c > 0 \), we need to find where the parabola \( y = ax^2 + bx + c \) is above the \( x \)-axis.

For which \( x \) is the parabola above the \( x \)-axis?

For which \( x \) is the parabola below or on the \( x \)-axis?

Geometric solution

Now let us solve the inequality \( ax^2 + bx + c \leq 0 \) again in the case when \( a > 0 \).

Let \( y = ax^2 + bx + c \). Then \( ax^2 + bx + c \leq 0 \iff y \leq 0 \).

Thus, to solve the inequality \( ax^2 + bx + c \leq 0 \), we need to find where the parabola \( y = ax^2 + bx + c \) is below or on the \( x \)-axis.

For which \( x \) is the parabola below or on the \( x \)-axis?
What if \( a < 0 \)?

We have a choice:

- **either** to solve the inequality using a parabola, as we did in the case \( a > 0 \),

Don’t forget that the parabola \( y = ax^2 + bx + c \) with \( a < 0 \) opens down:

- or multiply both sides of the inequality by \(-1\), like

\[-3x^2 + x - 2 \geq 0 \iff 3x^2 - x + 2 \leq 0,\]

in order to make \( a \)-coefficient positive.

Don’t forget to reverse the sign of inequality!

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**Example 1**

Solve the inequality \( x^2 - 4x + 3 < 0 \).

**Solution.** The parabola \( y = x^2 - 4x + 3 \) opens **upward**, since \( a = 1 > 0 \).

Determine the \( x \)-intercepts. They are the **roots** of the equation \( x^2 - 4x + 3 = 0 \).

\[ x^2 - 4x + 3 = 0 \iff (x - 1)(x - 3) = 0 \iff x_1 = 1, x_2 = 3. \]

Therefore, the parabola looks as follows:

To solve the inequality \( x^2 - 4x + 3 < 0 \), we have to find **all** \( x \) for which the parabola is **below** the \( x \)-axis.

As we see, those \( x \) fill the interval \((1, 3)\).

The **answer** can be written in several ways:

\[ 1 < x < 3, \text{ or } x \in (1, 3), \text{ or simply } (1, 3). \]
Example 2
Solve the inequality \(9x^2 - 6x + 1 > 0\).

Solution. The parabola \(y = 9x^2 - 6x + 1\) opens upward, since \(a = 9 > 0\).

Determine the \(x\)-intercepts. They are the roots of the equation \(9x^2 - 6x + 1 = 0\).

\[
9x^2 - 6x + 1 = 0 \iff (3x - 1)^2 = 0 \iff x_1 = \frac{1}{3}.
\]

Therefore, the parabola looks as follows:

To solve the inequality \(9x^2 - 6x + 1 > 0\), we have to find all \(x\) for which the parabola is above the \(x\)-axis.

As we see, those \(x\) fill the whole line except the point \(\frac{1}{3}\).

The answer can be written as \((-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)\) or \(\mathbb{R} \setminus \{\frac{1}{3}\}\).

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Example 3
Solve the inequality \(-x^2 + 3x - 1 \leq 0\).

Solution. The parabola \(y = -x^2 + 3x - 1\) opens downward, since \(a = -1 < 0\).

Determine the \(x\)-intercepts. They are the roots of the equation \(-x^2 + 3x - 1 = 0\). Solve the equation:

\[
-x^2 + 3x - 1 = 0 \iff x^2 - 3x + 1 = 0 \iff x_{1,2} = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.
\]

Therefore, the parabola looks as follows:

To solve the inequality \(-x^2 + 3x - 1 \leq 0\), we have to find all \(x\) for which the parabola is below or on the \(x\)-axis.

Answer: \(x \in \left(-\infty, \frac{3 - \sqrt{5}}{2}\right] \cup \left[\frac{3 + \sqrt{5}}{2}, \infty\right)\).
Example 4

Solve the inequality $-x^2 - x - 1 > 0$.

**Solution.** Alternative 1. The parabola $y = -x^2 - x - 1$ opens downward, since $a = -1 < 0$.

Determine the $x$-intercepts. They are the roots of the equation $-x^2 - x - 1 = 0$.

$-x^2 - x - 1 = 0 \iff x^2 + x + 1 = 0 \iff x_{1,2} = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$. No real roots!

Therefore, the parabola looks as follows:

To solve the inequality $-x^2 - x - 1 > 0$, we have to find all $x$

for which the parabola is above the $x$-axis.

As we see, there are no such $x$. **Answer:** no solutions.

Example 4

Let us solve the inequality $-x^2 - x - 1 > 0$ in a different way.

Alternative 2. $-x^2 - x - 1 > 0 \iff x^2 + x + 1 < 0$.

Instead of solving $-x^2 - x - 1 > 0$, we will solve an equivalent inequality $x^2 + x + 1 < 0$.

The parabola $y = x^2 + x + 1$ opens upward since $a = 1 > 0$, and has no $x$-intercepts, since the discriminant $b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3$ is negative.

Therefore, the parabola is situated above the $x$-axis:

To solve the inequality $x^2 + x + 1 < 0$

means to find all values of $x$ for which the parabola is below the $x$-axis.

But there are no such $x$. **Answer:** the inequality has no solutions.
Summary

In this lecture, we have learned

✓ what a **quadratic inequality** is
✓ what it means to **solve** a quadratic inequality
✓ how to **visualize** a quadratic inequality by a **parabola**
✓ how to solve a quadratic inequality in terms of the **leading coefficient** and the **roots**
✓ how to write down the **answer**