Lecture 30

Parabolas

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Quadratic functions

A quadratic function is a function \( y = ax^2 + bx + c \), where \( a, b, c \) are given numbers and \( a \neq 0 \).

Examples of quadratic functions:
- \( y = x^2 \)
- \( y = x^2 + x \)
- \( y = -3x^2 + 2x - 5 \)
- \( y = \frac{1}{3}x^2 - \sqrt{2}x + 1 \)

Functions and, in particular, quadratic functions, are studied in the precalculus and calculus courses.

In this lecture, we will learn how to draw the graph of a quadratic function.

The graph of a function provides a visualization of various properties of the function, and helps to understand these properties.

What is the graph

The graph of a quadratic function \( y = ax^2 + bx + c \) is the set of all points on the plane whose coordinates \((x, y)\) satisfy the equation \( y = ax^2 + bx + c \).

The graph of a quadratic function is a plane curve, it is called a parabola.

Here are a few examples of parabolas:

In this lecture, we will learn how to draw a parabola by its equation.
Geometry of a parabola

Any parabola has certain geometric elements which are common for all parabolas. Let us have a look on a typical parabola:

Which geometric elements do we observe on this parabola?

Horns: upward or downward

A parabola has its “horns” turned upward or downward. (A parabola opens upward or downward.)

It is the coefficient \( a \) (called the leading coefficient) which is responsible for this.

- If \( a > 0 \), then the parabola opens upward
- If \( a < 0 \), then the parabola opens downward
Vertex and axis of symmetry

There is a characteristic point on a parabola, where the parabola makes a turn. This point is called the vertex.

The vertex is the lowest point on the parabola if \( a > 0 \), and the highest point if \( a < 0 \).

A vertical line passing through the vertex is called the axis of symmetry, because a parabola is symmetric about its axis of symmetry.

The \( x \)-intercepts

The points where the parabola intersects the \( x \)-axis, are called the \( x \)-intercepts.

A parabola may have two, one, or no \( x \)-intercepts.
The \textit{y}-intercept

A point where the parabola intersects the \textit{y}-axis is called the \textit{y}-intercept.

Each parabola has exactly one \textit{y}-intercept.

Wide or narrow?

Some parabolas are wider than others:

\[ y = x^2 \]
\[ y = 2x^2 \]
\[ y = \frac{1}{2}x^2 \]

\(|a|\) is responsible for the \textbf{width} of the parabola.

The smaller \(|a|\), the wider the parabola.
What do we need to sketch a parabola?

- the vertex
- the axis of symmetry
- the sign of \( a \) (upward or downward)
- the \( y \)-intercept
- the \( x \)-intercepts (if any)

**Example.** Sketch a parabola which opens downward, has the vertex at \((-1, 3)\), the \( y \)-intercept at \((0, 9/4)\), and the \( x \)-intercepts at \((-3, 0)\) and \((1, 0)\).

**Solution.**

![Graph of a parabola with vertex at (-1, 3), y-intercept at (0, 9/4), and x-intercepts at (-3, 0) and (1, 0).]

**How to find the vertex**

The vertex of the parabola \( y =ax^2 + bx + c \) is located at the point with coordinates \((-\frac{b}{2a}, -\frac{b^2}{4a} + c)\).

Why is this so? Rewrite the equation of the parabola using **completing the square**:

\[
y = ax^2 + bx + c \iff y = a\left(x + \frac{b}{2a}\right)^2 + \left(-\frac{b^2}{4a} + c\right)
\]

If \( a > 0 \), then the vertex is located at the **lowest** point on the parabola, that is at the point, where \( y \) takes the **minimal** value.

Since \( a\left(x + \frac{b}{2a}\right)^2 \geq 0 \) for all \( x \), the minimal value of \( y = a\left(x + \frac{b}{2a}\right)^2 + \left(-\frac{b^2}{4a} + c\right) \) occurs exactly when \( \left(x + \frac{b}{2a}\right)^2 = 0 \), that is when \( x = -\frac{b}{2a} \).

Therefore, the vertex is located at \((-\frac{b}{2a}, -\frac{b^2}{4a} + c)\).
How to find the vertex and the axis of symmetry

If $a < 0$, then the vertex is located at the highest point on the parabola, that is at the point, where $y$ takes the maximal value. Since $a \left( x + \frac{b}{2a} \right)^2 \leq 0$ for all $x$, the maximal value of $y = a \left( x + \frac{b}{2a} \right) + \left( -\frac{b^2}{4a} + c \right)$ occurs exactly when $\left( x + \frac{b}{2a} \right)^2 = 0$, that is when $x = -\frac{b}{2a}$.

Therefore, the vertex is located at $\left( -\frac{b}{2a}, -\frac{b^2}{4a} + c \right)$.

Remember that

The vertex of the parabola $y = ax^2 + bx + c$ is located at the point where $x = \frac{-b}{2a}$.

The axis of symmetry is the vertical line passing through the vertex.

Its equation is $x = -\frac{b}{2a}$.

Example. Find the vertex and the axis of symmetry of the parabola $y = x^2 - 4x + 1$.

Solution. The $x$-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-4)}{2 \cdot 1} = \frac{4}{2} = 2$.

To find the $y$-coordinate of the vertex, we plug in $x = 2$ into the equation of the parabola: $y = 2^2 - 4 \cdot 2 + 1 = 4 - 8 + 1 = -3$.

Therefore, the vertex of the parabola is at the point with coordinates $(2, -3)$.

The axis of symmetry is the vertical line $x = 2$. 

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How to find the \( x \)-intercepts

The \( x \)-intercepts are the points where the parabola meets the \( x \)-axis.

A parabola may have two, one or no \( x \)-intercepts. At \( x \)-intercept, the \( y \)-value is equal to 0. Therefore,

To find the \( x \)-intercepts of the parabola \( y = ax^2 + bx + c \), solve the equation \( ax^2 + bx + c = 0 \).

If the quadratic equation \( ax^2 + bx + c = 0 \) has two roots, then the parabola intersects the \( x \)-axis at two points.

If the equation has one root, then the parabola touches the \( x \)-axis at one point.

If the equation has no roots, then the parabola does not intersect the \( x \)-axis.

How to find the \( y \)-intercept

The \( y \)-intercept is easy to find.

When we plug \( x = 0 \) into the equation of the parabola \( ax^2 + bx + c \), we get \( y = a \cdot 0^2 + b \cdot 0 + c = c \).

Therefore,

The \( y \)-intercept of the parabola \( y = ax^2 + bx + c \) is located at the point \((0, c)\).
Step-by-step instruction for drawing a parabola

To draw the parabola \( y = ax^2 + bx + c \),

- Determine the vertex. It’s located at the point where \( x = \frac{-b}{2a} \).

- Draw the axis of symmetry. It’s the vertical line \( x = \frac{-b}{2a} \).

- Determine if the parabola opens upward (\( a > 0 \)) or downward (\( a < 0 \)).

- Determine the \( y \)-intercept. It’s located at the point \((0, c)\).

- Determine the \( x \)-intercepts (if any). They are located at the points \((x_{1,2}, 0)\), where \( x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

- Draw the parabola, using the information above.
  Make sure that your parabola is smooth and symmetric.

Example 1

Example 1. For the parabola \( y = x^2 - x - 2 \), determine the vertex, the axis of symmetry, the intercepts, and draw the graph.

Solution.
- The vertex is at \( x = \frac{-b}{2a} = \frac{-(-1)}{2} = \frac{1}{2} \). The \( y \)-coordinate of the vertex is \( y = \left( \frac{1}{2} \right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4} \). So the vertex is located at \((1/2, -9/4)\).

Draw the vertex.
- The axis of symmetry is the vertical line \( x = 1/2 \).

Draw the axis of symmetry.
Example 1

- \( a = 1 > 0 \), therefore, the parabola opens **upward**. 

Draw a small **sprout** of a parabola at the vertex.

- The **y**-intercept is at \((0, c) = (0, -2)\).
- The **x**-intercepts are the roots of \( x^2 - x - 2 = 0 \).

\[
x^2 - x - 2 = 0 \iff (x + 1)(x - 2) = 0 \iff x = -1, \ x = 2.
\]

So the **x**-intercepts are \((-1, 0)\) and \((2, 0)\). 

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Example 1

Now we are ready to draw the parabola:

- Be neat: the parabola should be smooth and symmetric.
Example 2.

Example 2. For the parabola \( y = -x^2 - 2x - 2 \), determine the vertex, the axis of symmetry, the intercepts, and draw the graph.

Solution. The vertex is at \( x = \frac{-b}{2a} = \frac{-(2)}{2 \cdot (-1)} = -1 \).

The \( y \)-coordinate of the vertex is \( y = -(-1)^2 - 2 \cdot (-1) - 2 = -1 + 2 - 2 = -1 \). By this, the vertex is \((-1, -1)\).

The axis of symmetry is \( x = -1 \).

\( a = -1 < 0 \), so the parabola opens downward.

The \( y \)-intercept is \((0, c) = (0, -2)\).

For the \( x \)-intercepts, solve the equation \(-x^2 - 2x - 2 = 0\):

\[-x^2 - 2x - 2 = 0 \iff x^2 + 2x + 2 = 0.\]

The discriminant is \( b^2 - 4ac = 4 \cdot 1 \cdot 2 = -4 < 0 \).

Therefore, there are no solutions, and the parabola doesn’t meet the \( x \)-axis.

Example 2.

Now put all the information on the graph.
The graph of a quadratic monomial

What do we know about the graph of the parabola $y = ax^2$?

- The vertex at the origin $(0,0)$, since $\frac{-b}{2a} = 0$.
- The axis of symmetry is the line $x = 0$, that is, the $y$-axis.
- The parabola opens upward if $a > 0$, and downward if $a < 0$.
- The $y$-intercept is $(0,0)$.
- The only $x$-intercept is $(0,0)$.

This information is not sufficient for a drawing.

We may need to plot a support point, say, $(x, y) = (1, a)$ belonging to the parabola.

By symmetry, we get another point $(x, y) = (-1, a)$ on the parabola.

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The graph of a quadratic monomial

Let us draw several parabolas $y = ax^2$ with different coefficients $a$.

![Graph of quadratic monomials](image)
Summary

In this lecture, we have learned

✓ what the **graph** of a quadratic function is
✓ what a **parabola** looks like
✓ what the essential **geometric elements** of the parabola are (vertex, axis of symmetry, intercepts)
✓ when a parabola opens **upward** \( a > 0 \) or **downward** \( a < 0 \)
✓ how to find the **vertex** and the **axis of symmetry** of a parabola
✓ how to find the **x-intercepts** (if any) and the **y-intercept** of a parabola
✓ how to **draw** the parabola from its equation
✓ how to draw the graph of a **quadratic monomial**