**Factoring polynomials**

To **factor** a polynomial means to present this polynomial as a product of polynomials of degree less than the original polynomial.

For example, $x^2 - 1 = (x - 1)(x + 1)$ is a factoring, but

$3x^2 + 3 = 3(x^2 + 1)$ is not a polynomial factoring, since the degree of $x^2 + 1$ is not less than the degree of $3x^2 + 3$.

Factoring is an important algebraic tool that helps to solve various problems. The same polynomial can be factored in different ways.

For example, $x^3 - x$ can be factored as $x(x^2 - 1)$ or as $x(x - 1)(x + 1)$ or as $2x(x - 1)\left(\frac{1}{2}x + \frac{1}{2}\right)$.

**Monomials**, that is, polynomials of type $ax^n$, are easy to factor.

For example, $4x^3 = 4x^2 \cdot x$ or $4x^3 = 4x \cdot x \cdot x$.

In this lecture we will learn how to factor quadratic **binomials** and **trinomials**.

---

**Irreducible polynomials**

If a polynomial can’t be factored, it is called **irreducible**.

Polynomials of degree one are irreducible, they can’t be factored:

we can’t present a polynomial of degree one as a product of polynomials of degrees less than one.

Some polynomials are easy to factor: $x^2 - 4 = x^2 - 2^2 = (x - 2)(x + 2)$.

The factors, $x - 2$ and $x + 2$, contain only integer coefficients. Such factoring is called factoring over the integers.

Consider another factoring: $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$.

Here the factors, $x - \sqrt{3}$ and $x + \sqrt{3}$, have real coefficients, such factoring is called factoring over the reals.

The polynomial $x^2 - 3$ can’t be factored over the integers. It is irreducible over the integers.

The polynomial $x^2 + 1$ is irreducible over the reals, but can be factored over the complex numbers: $x^2 + 1 = (x - i)(x + i)$.
Factoring quadratic binomials

Quadratic binomials are expressions of type $ax^2 + bx$ or $ax^2 + c$, they are special types of quadratic polynomials.

It’s easy to factor the binomial $ax^2 + bx$: $ax^2 + bx = x(ax + b)$

The binomial $ax^2 + c$ can be factored over the reals only if the coefficients $a$ and $c$ have opposite signs.

If $a$ and $c$ are of the same sign (both positive or both negative) then $ax^2 + c$ is irreducible.

Example. Factor the following polynomials: $9x^2 - 4$, $9x^2 + 4$.

Solution. $9x^2 - 4 = (3x)^2 - 2^2 = (3x - 2)(3x + 2)$.

The polynomial $9x^2 + 4$ is irreducible.

For the rest of the course, we will say that a polynomial is irreducible, if it is irreducible over the reals.

Factorization theorem for quadratic trinomials

Theorem. Let $ax^2 + bx + c$ be a quadratic polynomial with non-negative discriminant, that is, $b^2 - 4ac \geq 0$.

Then $ax^2 + bx + c = a(x - x_1)(x - x_2)$

where $x_1$, $x_2$ are the roots of the polynomial, that is, the solutions of the equation $ax^2 + bx + c = 0$.

Remarks.

1. If the discriminant is 0, then $x_1 = x_2$ is the only root of the equation, and the factoring is $ax^2 + bx + c = a(x - x_1)(x - x_1) = a(x - x_1)^2$.

2. Factoring is simple when $a = 1$:

$$x^2 + bx + c = (x - x_1)(x - x_2).$$

3. If the discriminant is negative, then the polynomial is irreducible.
Proving factorization formula

By completing the square,

\[ ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a^2} = \]

\[ a \left( x + \frac{b}{2a} \right)^2 - \left( \frac{\sqrt{b^2 - 4ac}}{2a} \right)^2 = \]

\[ a \left( x + \frac{b}{2a} - \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left( x + \frac{b}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) = \]

\[ a \left( x - \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left( x - \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) = a(x - x_1)(x - x_2), \]

as required.

Factoring by finding roots

Example 1. Factor \( 2x^2 - x - 1 \).

Solution. By the factoring theorem,

\( 2x^2 - x - 1 = 2(x - x_1)(x - x_2) \), where

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{1 \pm \sqrt{1 + 8}}{4} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4} \]

So \( x_1 = \frac{1 + 3}{4} = 1 \) and \( x_2 = \frac{1 - 3}{4} = -\frac{1}{2} \).

The factoring is

\( 2x^2 - x - 1 = 2(x - 1) \left( x - \left( \frac{-1}{2} \right) \right) = 2(x - 1) \left( x + \frac{1}{2} \right) = (x - 1)(2x + 1) \).
Factoring by finding roots

Example 2. Factor \( x^2 - x - 1 \).

Solution. By the factoring theorem,

\[
x^2 - x - 1 = (x - x_1)(x - x_2),
\]

where

\[
x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}
\]

So \( x_1 = \frac{1 + \sqrt{5}}{2} \) and \( x_2 = \frac{1 - \sqrt{5}}{2} \).

The factoring is

\[
x^2 - x - 1 = \left(x - \frac{1 + \sqrt{5}}{2}\right) \left(x - \frac{1 - \sqrt{5}}{2}\right).
\]

Factoring by finding roots

Example 3. Factor \( x^2 - 4x + 4 \).

Solution. By the factoring theorem,

\[
x^2 - 4x + 4 = (x - x_1)(x - x_2),
\]

where

\[
x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4 \pm 0}{2} = 2
\]

So \( x_1 = x_2 = 2 \).

The factoring is \( x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2 \).

Remark. If you recognize a perfect square trinomial formula in the expression \( x^2 - 4x + 4 \), then the factoring can be achieved faster:

\[
x^2 - 4x + 4 = x^2 - 2 \cdot x \cdot 2 + (2)^2 = (x - 2)^2.
\]
Factoring by finding roots

Example 4. Factor $3x^2 - x + 1$.

Solution. The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11 < 0,$$

therefore, the polynomial has no roots and is irreducible.

Vieta’s theorem

Theorem. If $x_1, x_2$ are the roots of the equation $ax^2 + bx + c = 0$,

then $x_1 + x_2 = \frac{-b}{a}$ and $x_1 \cdot x_2 = \frac{c}{a}$.

Proof. By the Factorization theorem,

$ax^2 + bx + c = a(x - x_1)(x - x_2)$.

Let us expand RHS of this identity:

$a(x - x_1)(x - x_2) = a(x^2 - x_1x - x_2x + x_1x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2$.

Therefore, $ax^2 + bx + c = ax^2 - a(x_1 + x_2)x + ax_1x_2$.

By comparison of the coefficients of these two polynomials, we get

$b = -a(x_1 + x_2)$ and $c = ax_1x_2$. From this,

$x_1 + x_2 = \frac{-b}{a}$ and $x_1x_2 = \frac{c}{a}$, as required.

Vieta’s theorem relates the roots and the coefficients of a quadratic equation.
Vieta’s theorem for finding roots

Vieta’s theorem is especially simple if \( a = 1 \). In this case,

- the roots \( x_1, x_2 \) of the equation \( x^2 + bx + c = 0 \) satisfy
  \[ x_1 + x_2 = -b \quad \text{and} \quad x_1x_2 = c. \]

Vieta’s theorem may be used for finding the roots of a quadratic equation, provided that the coefficients of the equation and the roots are integers.

Example. Solve the equation \( x^2 + x - 6 = 0 \).

Solution. If \( x_1 \) and \( x_2 \) are the roots of \( x^2 + x - 6 = 0 \), then

\[ x_1 + x_2 = -b = -1 \quad \text{and} \quad x_1x_2 = c = -6. \]

Let us guess two numbers, whose sum equals \(-1\) and the product equals \(-6\). The numbers are 2 and \(-3\).

\[ \text{Answer. } x = 2 \text{ or } x = -3. \]

⚠️ Warning. Guessing out the roots may be not a good idea.

It may happen that the equation has irrational roots or no roots at all. Although Vieta’s theorem is valid, it can’t be used to find the roots in these cases.

Don’t waste your time guessing!

---

Solving quadratic equations by factoring

Example. Solve the equation \( x^2 - 2x - 15 = 0 \).

Solution. For this equation, \( a = 1, b = -2, c = -15 \).

The discriminant of the equation is \( b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-15) = 64 \), which a perfect square.

It means that the roots are rational numbers, and we may guess them out.

The factoring is

\[ x^2 - 2x - 15 = (x - ?)(x - ?) \]

By guessing, we get

\[ x^2 - 2x - 15 = (x - 5)(x + 3). \]

So \( x^2 - 2x - 15 = 0 \iff (x - 5)(x + 3) = 0 \iff x - 5 = 0 \text{ or } x + 3 = 0 \iff x = 5 \text{ or } x = -3. \)

Answer. \( x = 5 \) or \( x = -3 \)
Summary

In this lecture, we have learned
✓ what it means to factor a polynomial
✓ what an irreducible polynomial is
✓ how to factor quadratic binomials
✓ how to factor quadratic trinomials \( ax^2 + bx + c = a(x - x_1)(x - x_2) \)
✓ how to prove the factorization formula
✓ Vieta’s theorem
✓ how to use Vieta’s theorem for solving quadratic equations
✓ how to solve quadratic equations by factoring