Squares and square roots

A number and its opposite have the same square:
for example, \(3^2 = 9\) and \((-3)^2 = 9\).

Number 9 is called the square of 3 (or \(-3\)).

Numbers 3 and \(-3\) are called the square roots of 9.

Let \(a\) be a non-negative number. A square root of \(a\) is a number \(b\) such that \(b^2 = a\).

If \(a\) is positive, then there are two numbers, \(b\) and \(-b\), whose square is \(a\):

\[
\begin{array}{c}
\text{square roots} \\
\text{\(b\)} \\
\text{\(-b\)} \\
\text{square} \\
\end{array}
\]

If \(a = 0\), then there is only one number, 0, whose square is 0: \(0 = 0^2\).

Definition of radical

Let \(a\) be a non-negative number.

The principal square root of \(a\) is a non-negative number \(b\) such that \(b^2 = a\).

\[
\begin{array}{c}
\text{principal square root} \\
\text{\(b\)} \\
\text{\(-b\)} \\
\text{square} \\
\end{array}
\]

Notation for the principal square root: \(\sqrt{a} = b\)

The symbol \(\sqrt{a}\) is called a radical sign.

The formula \(\sqrt{a} = b\) reads “the square root of \(a\) is equal to \(b\).”

By definition, \(\sqrt{a} = b \iff b^2 = a\) for non-negative \(a\) and \(b\).
Radicals and perfect squares

Examples. \( \sqrt{0} = 0 \) since \( 0^2 = 0 \),
\( \sqrt{1} = 1 \) since \( 1^2 = 1 \),
\( \sqrt{4} = 2 \) since \( 2^2 = 4 \),
\( \sqrt{9} = 3 \) since \( 3^2 = 9 \),
\( \sqrt{16} = 4 \) since \( 4^2 = 16 \).

A number \( a \) is called a **perfect square** if \( \sqrt{a} \) is an integer.

Here are some perfect squares: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Precautions

- When we work with **real numbers**, the number under the radical sign should be **non-negative**: \( \sqrt{a} \) is defined only for \( a \geq 0 \).

  For example, \( \sqrt{-9} \) is not defined.

- A square root is always **non-negative**: \( \sqrt{a} \geq 0 \).

  For example, it is **incorrect** to write \( \sqrt{9} = -3 \), since \( \sqrt{9} \), by definition, should be non-negative.
Taking principal square root is opposite to squaring

\[
\begin{array}{c}
\text{squaring} \\
3 \\
\text{taking principal root} \\
9
\end{array}
\]

It means that \( \sqrt{3^2} = 3 \) and \((\sqrt{9})^2 = 9\).

For any non-negative \( a \), \( \sqrt{a^2} = a \) and \((\sqrt{a})^2 = a \).

**Example.** Find the value of the following expressions:

- \( \sqrt{5^2} \), \( \sqrt{(-5)^2} \), \( \sqrt{-5^2} \), \((\sqrt{5})^2\), \((\sqrt{-5})^2\).

**Solution.**

- \( \sqrt{5^2} = 5 \)
- \( \sqrt{(-5)^2} = \sqrt{5^2} = 5 \)
- \( \sqrt{-5^2} = \sqrt{-25} \) is undefined
- \((\sqrt{5})^2 = 5\)
- \((\sqrt{-5})^2 \) is undefined

**Properties of radicals**

Let \( a, b \) be non-negative numbers. Then \( \sqrt{a}\sqrt{b} = \sqrt{ab} \) and \( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \).

Indeed, \((\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab \). Therefore, \( \sqrt{a}\sqrt{b} = \sqrt{ab} \).

\[
\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b} \]

Therefore, \( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \).

**Example.** Simplify the following expressions: \( \sqrt{3}\sqrt{12} \), \( \sqrt{75} \), \( \frac{\sqrt{27}}{\sqrt{12}} \).

**Solution.**

- \( \sqrt{3}\sqrt{12} = \sqrt{3 \cdot 12} = \sqrt{3 \cdot 4} = \sqrt{3^2 \cdot 2} = 3 \cdot 2 = 6 \).
- Another way to calculate: \( \sqrt{3}\sqrt{12} = \sqrt{3 \cdot 12} = \sqrt{36} = \sqrt{6^2} = 6 \).

\( \sqrt{75} = \sqrt{3 \cdot 25} = \sqrt{3 \cdot 5^2} = \sqrt{3} \cdot 5 = 5\sqrt{3} \)

\( \frac{\sqrt{27}}{\sqrt{12}} = \frac{\sqrt{3 \cdot 9}}{\sqrt{3 \cdot 4}} = \frac{\sqrt{3} \cdot \sqrt{9}}{\sqrt{3} \cdot \sqrt{4}} = \frac{3}{\sqrt{2}} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3}{2} \).
What is $\sqrt{x^2}$?

We know that $x^2$ is non-negative for any value of $x$. So $\sqrt{x^2}$ is defined.

Is it true that $\sqrt{x^2} = x$ for all $x$? No!

For non-negative $x$, $\sqrt{x^2} = x$ by definition of the radical.

For negative $x$, $\sqrt{x^2} = -x$, since $-x > 0$ and $(-x)^2 = x^2$.

Therefore, $\sqrt{x^2} = |x|$. Reminder: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Example 1. $\sqrt{(-5)^2} = | -5 | = 5$.

Example 2. Simplify the following expressions: $\sqrt{x^4}$, $\sqrt{x^6}$.

Solution. $\sqrt{x^4} = \sqrt{(x^2)^2} = |x^2| = x^2$  
$\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3| = x^2 \cdot |x| = x^2 \cdot |x|$

Why $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$?

It is not true that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ for arbitrary $x$, $y$.

Indeed, if $x = 9$ and $y = 16$, then $\sqrt{x+y} |_{x=9,y=16} = \sqrt{9+16} = \sqrt{25} = 5$, while $(\sqrt{x} + \sqrt{y}) |_{x=9,y=16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ and $5 \neq 7$.

Are there any $x$, $y$ for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$? Yes!

For example, $x = y = 0$: $\sqrt{0+0} = \sqrt{0} + \sqrt{0}$

or $x = 1$ and $y = 0$: $\sqrt{1+0} = \sqrt{1} + \sqrt{0}$.

Actually, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ only if at least one of $x$, $y$ is zero.
Simplest radical form

An expression involving radicals can be written in many different forms. For example,

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{3} \cdot 2}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{3}$$

It is a custom to write radical expressions in a special form, which is called **simplest radical form**.

In simplest radical form, the expression

- doesn’t contain perfect square factors: \(\sqrt{12}\) is not in the simplest form, but \(2\sqrt{3}\) is. (\(\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}\))

- doesn’t contain fractions under the radical: \(\sqrt{\frac{3}{4}}\) is not in the simplest form, but \(\frac{\sqrt{3}}{2}\) is. (\(\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}\))

- doesn’t contain radicals in denominators: \(\frac{1}{\sqrt{2}}\) is not in the simplest form, but \(\frac{\sqrt{2}}{2}\) is. (\(\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}\))

Example. Bring the following expressions in simplest radical form:

- \(\frac{1}{\sqrt{3}}\), \(\sqrt{\frac{2}{5}}\), \(\frac{1}{3 - \sqrt{2}}\)

Solution.

- \(\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}\)

- \(\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{\sqrt{10}}{(\sqrt{5})^2} = \frac{\sqrt{10}}{5}\)

- \(\frac{1}{3 - \sqrt{2}} = \frac{1 \cdot (3 + \sqrt{2})}{(3 - \sqrt{2})(3 + \sqrt{2})} = \frac{3 + \sqrt{2}}{3^2 - (\sqrt{2})^2} = \frac{3 + \sqrt{2}}{9 - 2} = \frac{3 + \sqrt{2}}{7}\)

Remember: \((a - b)(a + b) = a^2 - b^2\), so

\((3 - \sqrt{2})(3 + \sqrt{2}) = 3^2 - (\sqrt{2})^2\)
Operating with radical expressions

Example 1. Simplify the expression: $\sqrt{6}(\sqrt{18} - \sqrt{24})$

Solution. $\sqrt{6}(\sqrt{18} - \sqrt{24}) = \sqrt{6}\sqrt{18} - \sqrt{6}\sqrt{24} = \sqrt{6}\cdot 6\sqrt{3} - \sqrt{6}\cdot 6\sqrt{4} = 6\sqrt{3} - 6\sqrt{2} = 6\sqrt{3} - 12$.

Example 2. Bring the expression in simplest radical form: $\frac{\sqrt{6} - 3}{\sqrt{3} - \sqrt{2}}$.

Solution. $\frac{\sqrt{6} - 3}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3}\cdot 2 - (\sqrt{3})^2}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3}\sqrt{2} - (\sqrt{3})^2}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3}(\sqrt{2} - \sqrt{3})}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3}(-1)(\sqrt{3} - \sqrt{2})}{\sqrt{3} - \sqrt{2}} = -\sqrt{3}$.

Summary

In this lecture, we have learned

- what the square roots of a non-negative number are
- what the principal square root is
- what the perfect squares are
- the defining identities for radical: $\sqrt{a^2} = a$ and $(\sqrt{a})^2 = a$
- the properties of radicals: $\sqrt{a\sqrt{b}} = \sqrt{ab}$, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\sqrt{x^2} = |x|$ that for all $x$
- $\sqrt{x} + y \neq \sqrt{x} + \sqrt{y}$ for arbitrary $x, y$
- what the simplest radical form is
- how to operate with radical expressions