Lecture 20
Lines on a Plane. Part 2

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Linear equation \( y = mx + b \)

Consider a general linear equation \( Ax + By = C \) whose graph is a line.

If \( B \neq 0 \), then the equation can be rewritten as follows:

\[
Ax + By = C \iff By = -Ax + C \iff y = \frac{-A}{B}x + \frac{C}{B} \iff y = mx + b,
\]

where \( m = \frac{-A}{B} \) and \( b = \frac{C}{B} \).

If \( B = 0 \), then \( Ax + By = C \iff Ax = C \iff x = \frac{C}{A} \), and the graph is a vertical line.

Any non-vertical line can be described by the equation \( y = mx + b \).

The \( y \)-intercept

Consider a linear equation \( y = mx + b \). What do the coefficients \( m \) and \( b \) represent?

The coefficient \( b \) represents the \textbf{y-intercept} of the line \( y = mx+b \).

Indeed, if \( x = 0 \) then \( y = m \cdot 0 + b \iff y = b \) and \((0, b)\) is the \textbf{y-intercept}.

The coefficient \( b \) in the equation \( y = mx+b \) shows where the line meets the \textbf{y-axis}.

The coefficient \( b \) is called the \textbf{y-intercept}. 
**Slope-intercept equation of a line**

The coefficient $m$ in the equation $y = mx + b$ is called the slope of the line.

\[
y = mx + b
\]

The equation $y = mx + b$ is called the slope-intercept equation of a line.

What does the slope of the line represent?

**Slope measures the inclination of a line**

Let us study a line $y = mx$. The $y$-intercept is zero, therefore the line passes through the origin.

Here are several lines with positive slopes:

- $y = 2x$ \( \text{slope} = 2 \)
- $y = x$ \( \text{slope} = 1 \)
- $y = \frac{1}{2}x$ \( \text{slope} = \frac{1}{2} \)

The larger the slope, the steeper the line.

A line with positive slope rises as we move from left to right.

Lines $y = mx$ with positive $m$ are located in the first and third quadrants of the plane.
Negative slope. Zero slope

Here are several lines with negative slopes:

\[ y = -\frac{1}{2}x \]
\[ y = -x \]
\[ y = -2x \]

A line with negative slope falls as we move from left to right.

A line \( y = mx \) with negative \( m \) is located in the second and fourth quadrants.

If the slope \( m = 0 \), then
\[ y = mx + b \iff y = 0 \cdot x + b \iff y = b, \]
and the line is horizontal.

Slope of vertical line

The slope of a vertical line is undefined.
Parallel lines have the same slope

A line $y = mx + b$ is obtained from the line $y = mx$ by a **vertical shift** along the $y$-axis.

Two non-vertical lines are **parallel** if and only if they have the same slope.

Example of parallel lines

The equations of the parallel lines are:

- $y = \frac{1}{2}x + 3$
- $y = \frac{1}{2}x + 1$
- $y = \frac{1}{2}x$
- $y = \frac{1}{2}x - 2$
Parallel or not?

Example 1. Are the lines $3x - 2y = 1$ and $-6x + 4y = 5$ parallel?

Solution. To answer the question, we have to determine the slopes of the lines. For this, we rewrite the equations in the slope-intercept form $y = mx + b$.

$$3x - 2y = 1 \iff 2y = 3x - 1 \iff y = \frac{3}{2}x - \frac{1}{2}$$

$$-6x + 4y = 5 \iff 4y = 6x + 5 \iff y = \frac{3}{2}x + \frac{5}{4}.$$

Since the lines have the same slope of $\frac{3}{2}$, they are parallel.

Example 2. Are the lines $y = 2$ and $y = 2x$ parallel?

Solution. The slope of the line $y = 2$ is 0, since $y = 2 \iff y = 0 \cdot x + 2$.

The slope of line $y = 2x$ is 2. Since the lines have different slopes, they are not parallel.

Remark. $y = 2$ is a horizontal line, while $y = 2x$ is not. So the lines are not parallel.

Slope of a line through two given points

Theorem. A line passing through two points $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1 \neq x_2$ has the slope $rac{y_2 - y_1}{x_2 - x_1}$.

Proof. Let $y = mx + b$ be an equation of the line. We have to prove that the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Since the points $(x_1, y_1)$ and $(x_2, y_2)$ are on the line $y = mx + b$,

their coordinates satisfy the equation $y = mx + b$:

$$y_1 = mx_1 + b \quad \text{and} \quad y_2 = mx_2 + b.$$

Subtracting the first equality from the second one, we get

$$y_2 - y_1 = (mx_2 + b) - (mx_1 + b) \iff y_2 - y_1 = m(x_2 - x_1) \iff m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Notice that $x_2 - x_1 \neq 0$ since $x_1 \neq x_2$. 
**Slope as a ratio**

Let us give a **geometric** interpretation of this result:

A line \( y = mx + b \) passing through the points \((x_1, y_1)\) and \((x_2, y_2)\) with \(x_1 \neq x_2\) has the slope \( m = \frac{y_2 - y_1}{x_2 - x_1} \).

![Graph showing slope as a ratio](image)

When we move along the line from a point \((x_1, y_1)\) to another point \((x_2, y_2)\), the difference \(x_2 - x_1\) shows the change in \(x\)-coordinate, and the difference \(y_2 - y_1\) shows the change in \(y\)-coordinate.

The slope is the ratio of the change: \( \text{slope} = \frac{\text{change in } y}{\text{change in } x} \)

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**Examples**

**Example 1.** Find the equation of the line passing through the points \((1, -1)\) and \((-3, 7)\).

**Solution.** Let \( y = mx + b \) be the equation of the line. We have to determine the coefficients \( m \) and \( b \).

The slope \( m \) of the line passing through the points \((x_1, y_1) = (1, -1)\) and \((x_2, y_2) = (-3, 7)\) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-1)}{-3 - 1} = \frac{8}{-4} = -2.
\]

Our line has the equation \( y = -2x + b \).

To determine \( b \), we plug in any of two given points into this equation.

Plugging in \((x_1, y_1) = (1, -1)\), we get

\[
y_1 = -2 \cdot \frac{1}{x_1} + b \iff -1 = -2 + b \iff b = 1.
\]

Therefore, the line has equation \( y = -2x + 1 \).
Examples

Example 2. Find the equation of the line passing through the points \((2, -1)\) and \((2, 3)\).

Solution. The given points have the same \(x\)-coordinate. Therefore, they belong to a vertical line. The equation of the line is \(x = 2\).

Example 3. Find the equation of the line passing through the points \((-1, 3)\) and \((4, 3)\).

Solution. The given points have the same \(y\)-coordinate. Therefore, they belong to a horizontal line. The equation of the line is \(y = 3\).

Point-slope equation

Theorem. A line that has a slope of \(m\) and passes through the point \((x_1, y_1)\) has the equation \(y - y_1 = m(x - x_1)\).

Proof. Let us show that the equation above describes a line that has a slope of \(m\) and passes through \((x_1, y_1)\).

Rewrite the equation in a slope-intercept form:

\[
y - y_1 = m(x - x_1) \iff y = \frac{m}{\text{slope}} x + (-mx_1 + y_1).
\]

The coefficient in front of \(x\) is the slope \(m\). Moreover, the point \((x_1, y_1)\) satisfies the equation \(y - y_1 = m(x - x_1)\): \(y_1 - y_1 = m(x_1 - x_1) \iff 0 = 0\), so it belongs to the line.

Example. Find a slope-intercept equation of a line that has a slope of \(3\) and passes through the point \((-1, 2)\).

Solution. Using the point-slope equation \(y - y_1 = m(x - x_1)\), we get \(y - 2 = 3(x - (-1)) \iff y - 2 = 3(x + 1) \iff y = 3x + 5\).
**Perpendicular lines**

**Theorem.** Two non-vertical lines are perpendicular if the product of their slopes is $-1$.

**Proof.**

The triangles are congruent.

The lines are perpendicular.

**Example.** Prove that the lines $x - 2y = 1$ and $6x + 3y = 2$ are perpendicular.

**Solution.**

$x - 2y = 1 \iff 2y = x - 1 \iff y = \frac{1}{2}x - \frac{1}{2}$

$6x + 3y = 2 \iff 3y = -6x + 2 \iff y = -2x + \frac{2}{3}$

The slopes $\frac{1}{2}$ and $-2$ are negative reciprocals of each other.

Therefore, the lines are perpendicular.

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**Summary**

In this lecture, we have learned

- ✔ the **slope-intercept equation** of a line $y = mx + b$
- ✔ what the **slope** of a line represents
- ✔ that **parallel lines** have the same slope
- ✔ how to find equation of a line passing through two points
- ✔ what the **point-slope equation** of a line is $y - y_1 = m(x - x_1)$
- ✔ that **perpendicular lines** have negative reciprocals slopes