Lecture 17

Linear Inequalities

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What a linear inequality is

There are four inequality signs: \(<\), \(\leq\), \(>\), \(\geq\).

- \(a < b\) \(a\) is less than \(b\)
- \(a \leq b\) \(a\) is less than or equal to \(b\)
- \(a > b\) \(a\) is greater than \(b\)
- \(a \geq b\) \(a\) is greater than or equal to \(b\)

A linear inequality consists of two linear expressions connected by one of the inequality signs.

For example, \(3(x - 1) \leq 4 + 5x\) is a linear inequality in one variable.

Evaluation of both sides of an inequality at a number gives rise to a numerical inequality, which may be either true or false.

For example, at \(x = 0\) the inequality above holds true:

\[
3(0 - 1) \leq 4 + 5 \cdot 0 \iff -3 \leq 4 \checkmark
\]

Solution

To solve an inequality means to find all values of the variable, for which the inequality holds true.

These values form a solution set.

A linear inequality is very similar to a linear equation.

As we remember, the solution set of a linear equation
- either consists of a single number (when the equation has one solution),
- or is empty (when the equation has no solutions),
- or is the entire number line (when the equation has infinitely many solutions).

The solution set of a linear inequality is quite different.

Consider a simple inequality \(x \leq 2\). Its solution set consists of all numbers \(\leq 2\) and is denoted by \(\{x \mid x \leq 2\}\). One can graph the solutions on the number line:

The solution set is an interval. It is denoted by \((-\infty, 2]\).
Intervals

Let us review intervals that we may encounter solving linear inequalities.

<table>
<thead>
<tr>
<th>inequality</th>
<th>solution</th>
<th>graph</th>
<th>interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; a$</td>
<td>${ x \mid x &lt; a }$</td>
<td>$\bullet$</td>
<td>$(-\infty, a)$</td>
</tr>
<tr>
<td>$x \leq a$</td>
<td>${ x \mid x \leq a }$</td>
<td>$\bullet$</td>
<td>$(-\infty, a]$</td>
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<tr>
<td>$x &gt; a$</td>
<td>${ x \mid x &gt; a }$</td>
<td>$\circ$</td>
<td>$(a, \infty)$</td>
</tr>
<tr>
<td>$x \geq a$</td>
<td>${ x \mid x \geq a }$</td>
<td>$\circ$</td>
<td>$[a, \infty)$</td>
</tr>
</tbody>
</table>

Equivalent inequalities

Two inequalities are called equivalent if they have the same solution sets.

It means that each solution of the first inequality is a solution of the second one, and vice versa: each solution of the second inequality is a solution of the first one.

If two inequalities are equivalent, we write the equivalence sign “ $\iff$ ” between them, like this

$$x + 1 > 3 \iff x > 2.$$  

How to transform an inequality into an equivalent inequality?

To this end, we will use three elementary transformations.
Add the same to both sides

Any inequality is equivalent to the inequality obtained from it by adding the same expression to both sides.

**Example 1.** Consider the inequality \( x - 1 > 2 \). If we add 1 to both sides, then we get an equivalent inequality:

\[
\begin{align*}
    x - 1 > 2 & \iff x - 1 + 1 > 2 + 1 \\
    & \iff x > 3
\end{align*}
\]

**Example 2.**

\[
5 - x \leq 0 \iff 5 - x + x \leq 0 + x \iff 5 \leq x \iff x \geq 5
\]

**Example 3.**

\[
5 - x < 2 \iff 5 - x + (x - 2) < 2 + (x - 2) \iff 5 - x + x - 2 < 2 + x - 2 \iff 3 < x \iff x > 3
\]

Similarly, subtracting the same expression from both sides of an inequality gives rise to an equivalent inequality:

\[
x + 2 \geq 6 \iff x + 2 - 2 \geq 6 - 2 \iff x \geq 4
\]

Fast track

There is a trick that may help you to operate more efficiently with inequalities. The subtraction of \( x \) from both sides of the inequality \( 2x - 1 \leq 5 + x \), namely

\[
2x - 1 \leq 5 + x \iff 2x - 1 - x \leq 5 + x - x \iff x - 1 \leq 5
\]

is equivalent to relocation \( x \) from the right hand side (RHS) of the inequality to the left hand side (LHS) with the opposite sign:

Look how fast we can solve the inequality:

\[
2x - 1 \leq 5 + x \iff x - 1 \leq 5 \iff x \leq 6.
\]
Multiply both sides by the same positive number

Any inequality is equivalent to the inequality obtained from it by multiplying both sides by the same positive number.

Example 1. \( \frac{x}{2} > 3 \iff \frac{x}{2} \cdot 2 > 3 \cdot 2 \iff x > 6 \)

Example 2. \( 3x \leq 5 \iff 3x \cdot \frac{1}{3} \leq 5 \cdot \frac{1}{3} \iff x \leq \frac{5}{3} \)

Similarly, dividing both sides of an inequality by the same positive number gives rise to an equivalent inequality:

\[
2x \geq 8 \iff \frac{2x}{2} \geq \frac{8}{2} \iff x \geq 4
\]

Multiply by negative number and reverse the sign

What happens if we multiply an inequality by a negative number?
Consider the inequality \( x > 2 \). Move \( x \) to RHS, and move 2 to LHS (don’t forget to change the signs):

\[
x > 2 \iff -2 > -x.
\]

This inequality says that \(-2\) is greater than \(-x\). This is the same as \(-x\) is less than \(-2\):

\[
-2 > -x \iff -x < -2.
\]

Therefore, \( x > 2 \iff -x < -2 \).

In general, if we multiply both sides of an inequality by a negative number, we have to reverse the sign of the inequality.

Example 1. \( \frac{-x}{3} < 2 \iff (-3) \cdot \frac{-x}{3} > (-3) \cdot 2 \iff x > -6 \).

The same rule is valid if we divide an inequality by a negative number.

Example 2. \( -2x \leq 6 \iff \frac{-2x}{-2} \geq \frac{6}{-2} \iff x \geq -3 \).
Elementary transformations

Elementary transformations of an inequality are

- **adding** the same expression to both sides of an inequality,
- **multiplying** both sides by the the same **positive** number, and
- **multiplying** both sides by the the same **negative** number and **reversing** the sign of the inequality.

See how a sequence of elementary transformations brings an inequality to a simple equivalent inequality.

Examples

Example 1. Solve the inequality $7x - 5 \leq 2x + 1$. Give the answer in interval notation. Show the solution on the number line.

Solution. Move $2x$ to the LHS: $7x - 2x - 5 \leq 1$

Simplify: $5x - 5 \leq 1$

Move $-5$ to the RHS: $5x \leq 1 + 5$

Simplify: $5x \leq 6$

Divide by 5: $x \leq \frac{6}{5}$

Answer. $\left(-\infty, \frac{6}{5}\right]$
**Examples**

**Example 2.** Solve the inequality \(-\frac{x}{2} + 3 < x + 4\). Give the answer in interval notation. Show the solution on the number line.

**Solution.**

Move 3 to the RHS: \(-\frac{x}{2} < x + 4 - 3\)

Simplify: \(-\frac{x}{2} < x + 1\)

Multiply by \((-2)\): \((-2) \left(-\frac{x}{2}\right) > (-2)(x + 1)\)

Simplify: \(x > -2x - 2\)

Move \(-2x\) to the LHS: \(x + 2x > -2\)

Simplify: \(3x > -2\)

Divide by 3: \(x > -\frac{2}{3}\)

Writing down the answer

The answer can be written as an inequality \(x > -\frac{2}{3}\),

or as a set \(\left\{ x \mid x > -\frac{2}{3} \right\}\),

or as an interval \((-\frac{2}{3}, \infty)\) on a number line:
Systems of linear inequalities

Two inequalities with the same single variable may form a system. To solve a system means to find all the values of the variable that satisfy both inequalities.

Example. Solve the system \[
\begin{align*}
3x - 2 &\leq 2x - 1 \\
-2x + 3 &< 4.
\end{align*}
\]

Write the answer in interval notation. Show the solution on the number line.

Solution.

\[
\begin{align*}
3x - 2 &\leq 2x - 1 \\
-2x + 3 &< 4
\end{align*} \iff \begin{align*}
3x - 2x &\leq -1 + 2 \\
-2x &< 1
\end{align*} \iff \begin{align*}
x &\leq 1 \\
x &> -\frac{1}{2}
\end{align*} \iff -\frac{1}{2} < x \leq 1
\]

Answer: \([-\frac{1}{2}, 1]\)

Solution of a system

Geometrically, the solution of a system of two linear inequalities in one variable is the intersection of two intervals.

The intersection consists of all points belonging to both intervals.

As the intersection, we may get a finite interval, for example, \(a \leq x < b\):

\[
\begin{array}{c}
\hline
a & b \\
\hline
\end{array}
\]

\([a, b]\)

an infinite interval, for example \(x \leq a\):

\[
\begin{array}{c}
\hline
a & b \\
\hline
\end{array}
\]

\((−\infty, a]\)

or the empty set (when the system has no solutions):

\[
\begin{array}{c}
\hline
a & b \\
\hline
\end{array}
\]

\(\emptyset\)
Summary

In this lecture, we have learned

- what a **linear inequality** is
- what the **solution** of an inequality is
- which **intervals** on a real line may appear as solutions of inequalities
- which inequalities are called **equivalent**
- what **elementary transformations** of inequalities are
  - adding the same expression to both sides
  - multiplying both sides by the same **positive** number
  - multiplying both sides by the same **negative** number and **reversing the sign** of the inequality
- how to solve inequalities **efficiently**
- how to **write down** the solution of an inequality
- how to show the solution on a **number line**
- how to solve a **system** of inequalities