Equalities

An (algebraic) equality consists of two algebraic expressions connected by the equality sign “=”.

For example, \( x^2 - 3x + 1 = x + 2 \),
\[
\begin{align*}
1 + 1 &= 2, \\
0 &= 1, \\
a + b &= b + a, \\
(x - y)^2 &= x^2 - 2xy + y^2.
\end{align*}
\]

An algebraic equality with a variable becomes a numerical one if we evaluate the expressions on both sides of the equality at some number.

For example, if we substitute \( x = 1 \) into both sides of the equality \( x^2 = x \), it turns into a numerical equality \( 1^2 = 1 \), which is true.

If we substitute \( x = -1 \), then we get \((-1)^2 = -1\), which is false.

**True or false**

<table>
<thead>
<tr>
<th>equalities</th>
<th>numerical</th>
<th>with variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 1 = 2</td>
<td>( a(b + c) = ab + ac )</td>
<td></td>
</tr>
</tbody>
</table>

**numerical equalities**

<table>
<thead>
<tr>
<th>true</th>
<th>false</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 4 = 2 + 3</td>
<td>1 + 4 = 2 + 5</td>
</tr>
</tbody>
</table>

**equalities with variables**

<table>
<thead>
<tr>
<th>always true</th>
<th>always false</th>
<th>true or false depending on values of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - y^2 = (x - y)(x + y) )</td>
<td>( x = x + 1 )</td>
<td>( x + 2 = 3 )</td>
</tr>
</tbody>
</table>
Identities

Here are some important identities that we have learned:

\[ a + b = b + a \]  (commutativity of addition)
\[ a(b + c) = ab + bc \]  (distributive law)
\[ x^n \cdot x^m = x^{n+m} \]  (multiplication rule for powers)
\[ x^2 - y^2 = (x - y)(x + y) \]  (difference of squares)
\[ (x + y)^2 = x^2 + 2xy + y^2 \]  (short multiplication)

Proving identities

A typical problem about an identity is to prove it.

That is, to prove that the equality is true for all values of the variables.

Example. Prove that \((x + 1)^3 = x^3 + 3x^2 + 3x + 1\) for all values of \(x\).

Solution. Work on the left hand side:
\[
(x + 1)^3 = (x + 1)(x + 1)^2 = (x + 1)(x^2 + 2x + 1)
= x^3 + 2x^2 + x + x^2 + 2x + 1
= x^3 + 3x^2 + 3x + 1,
\]
which is the right hand side of the identity.

Therefore, \((x + 1)^3 = x^3 + 3x^2 + 3x + 1\) for all values of \(x\), and the identity is proven.
Equation and its solution

Often we use the word “equation” instead of “equality”. This happens when we are interested to find the values of variables at which the equality turns to a true numerical equality.

A solution of an equation with a single variable is the value of the variable which turns the equation into a true numerical equality.

**Example.** Consider the equation $x + 2 = 3x$. At $x = 1$, the equation turns into a true numerical equality:

$$1 + 2 = 3 \cdot 1.$$ 

If we substitute $x = 0$, then the equation turns into a false numerical equality:

$$0 + 2 = 3 \cdot 0.$$ 

Therefore, $x = 1$ is a solution of the equation $x + 2 = 3x$, while $x = 0$ is not a solution.

All the solutions

It may happen that an equation has no solutions.

For example, the equation $0 \cdot x = 1$ has no solution, since $0 \cdot x \neq 1$ no matter what $x$ is.

Some equations have infinitely many solutions.

For example, the equation $0 \cdot x = 0$ has infinitely many solutions. Any number is a solution.

To solve an equation means to find all its solutions, that is to find all values of the variable which turn the equation into a true numerical equality.

The variable in the equation is called unknown.

To solve an equation means to make this unknown known.
Several unknowns

An equation may have several unknowns.

For example, \( x + 2y = 7 \) is an equation with two unknowns \( x \) and \( y \).

The equation turns into a true numerical equality if we plug in \( x = 1 \) and \( y = 3 \):
\[
1 + 2 \cdot 3 = 7.
\]

Plugging in \( x = 1 \) and \( y = 2 \) results into a false equality:
\[
1 + 2 \cdot 2 = 7.
\]

A solution of such equation is a pair of numbers which turns the equation into a true numerical equality. For example, the pair \( x = 1 \) and \( y = 3 \) is a solution.

Another solution is \( x = -1, \ y = 4 \). Indeed:
\[
(-1) + 2 \cdot 4 = 7.
\]

As we will learn later, equations like this have infinitely many solutions.

Summary

In this lecture, we have learned

- what an equality is
- that there are numerical equalities and equalities with variables
- what an identity is
- what a contradiction is
- what an equation is
- what a solution of an equation is
- what it means to solve an equation