Subtraction is the opposite of addition ........................................... 2
No commutativity for subtraction .................................................. 3
No associativity for subtraction .................................................... 4
Division is the opposite of multiplication ....................................... 5
Negative one ................................................................. 6
Why division by zero does not make sense .................................. 7
No commutativity for division ..................................................... 8
No associativity for division ...................................................... 9
Summary ................................................................. 10
Subtraction is the opposite of addition

Subtraction is the operation which is opposite to addition:

\[
\begin{array}{c}
3 \\
\hspace{1cm} +2 \\
\hline
5 \\
\hspace{1cm} -2
\end{array}
\]

This means that \((3 + 2) - 2 = 3\) and \((5 - 2) + 2 = 5\).

Recall that numbers \(a\) and \(-a\) are called opposite to each other.
For example, \(-2\) is opposite to \(2\), and \(2\) is opposite to \(-2\).

Subtraction of a number is addition of its opposite:
\[5 - 2 = 5 + (-2) = 3\quad \text{and} \quad 5 - (-2) = 5 + 2 = 7\,.
\]
Therefore, we can express any subtraction as addition of the opposite quantity:
\[a - b = a + (-b)\quad \text{for any} \quad a, b.
\]

No commutativity for subtraction

We know that addition is commutative: \(a + b = b + a\) for any \(a, b\).

Subtraction is not commutative: it is not true that \(a - b = b - a\) unless \(a = b\).

Indeed, take \(a = 1\) and \(b = 2\). Then \(a - b = 1 - 2 = -1\),
but \(b - a = 2 - 1 = 1\).

In general, \(a - b\) and \(b - a\) are opposite to each other: \(b - a = -(a - b)\).

So subtraction is not commutative.

But expressing subtraction \(a - b\) in terms of addition \(a + (-b)\),
we may apply the commutativity of addition to get:
\[a - b = a + (-b) = -b + a\quad \text{for any} \quad a, b.
\]
No associativity for subtraction

We know that addition is associative:

\[(a + b) + c = a + (b + c)\] for any \(a, b, c\).

Subtraction is not associative:

\[(a - b) - c \neq a - (b - c)\].

For example, if \(a = 3\), \(b = 1\) and \(c = 1\), then

\[(a - b) - c = (3 - 1) - 1 = 2 - 1 = 1,\]
but \(a - (b - 1) = 3 - (1 - 1) = 3 - 0 = 3\).

So subtraction is not associative.

But expressing subtraction \((a - b) - c\) in terms of addition \((a + (-b)) + (-c)\),
we may apply the associativity of addition to get:

\[(a - b) - c = (a + (-b)) + (-c) = a + ((-b) + (-c)) = a + (-b - c).\]

Recall that \(a - b - c\) has to be understood as \((a - b) - c\).

Division is the opposite of multiplication

Division is the operation which is opposite to multiplication:

\[\frac{3}{2} \times 2 = 6 \quad \text{and} \quad \frac{6}{2} \div 2 = 3.\]

This means that \((3 \cdot 2) \div 2 = 3\) and \((6 \div 2) \cdot 2 = 6\).

Recall that numbers \(a\) and \(1/a\) are called reciprocals.

For example, \(2\) and \(1/2\) are reciprocals.

Division by a non-zero number is multiplication by its reciprocal:

\[6 \div 2 = 6 \cdot \frac{1}{2} = 3 \quad \text{and} \quad 6 \div \frac{1}{2} = 6 \cdot 2 = 12.\]

(Keep in mind that the reciprocal of \(1/2\) is \(2\).)

In general: \(a \div b = a \cdot \frac{1}{b}\) for any \(a\) and non-zero \(b\).
Negative one

The reciprocal of $-1$ is $-1$, that is $\frac{1}{-1} = -1$. Indeed, $(-1)(-1) = 1$.

Sometimes negative one is slightly hidden: $-a = (-1)a$.

It is helpful to keep this in mind.

For example, $\frac{-a}{-b} = \frac{a}{b}$, because $\frac{-a}{-b} = \frac{(-1)a}{(-1)b} = \frac{a}{b}$.

Another example: $\frac{a}{-b} = \frac{a}{(-1)b} = \frac{1}{-1} \frac{a}{b} = (-1) \frac{a}{b} = -\frac{a}{b}$.

Why division by zero does not make sense

Let us try to divide some number, say $1$, by $0$.

We do not know what result will be. Let us call it $x$: $1 \div 0 = x$.

If $1 \div 0 = x$, then $x$ is a number such that $x \cdot 0 = 1$.

Which is impossible since $x \cdot 0 = 0$ for any $x$.

Never divide by zero! It doesn’t make sense.
No commutativity for division

We know that multiplication is commutative: \( ab = ba \) for any \( a, b \).

Division is not commutative:

in general, it is not true that \( a \div b = b \div a \).

For example, if \( a = 2 \) and \( b = 1 \), then \( a \div b = 2 \div 1 = 2 \),
but \( b \div a = 1 \div 2 = \frac{1}{2} \).

The expressions \( a \div b \) and \( b \div a \) are reciprocal to each other.

Indeed, \( a \div b = a \cdot \frac{1}{b} \) and \( b \div a = b \cdot \frac{1}{a} \). Therefore

\[
(a \div b)(b \div a) = \left( a \cdot \frac{1}{b} \right) \cdot \left( b \cdot \frac{1}{a} \right) = a \left( \frac{1}{b} \cdot b \right) \frac{1}{a} = a \cdot 1 \cdot \frac{1}{a} = a \cdot \frac{1}{a} = 1
\]

In fractional notation, this may be written as \( \frac{b}{a} = \frac{1}{a/b} \).

---

No associativity for division

We know that multiplication is associative:

\( (ab)c = a(bc) \) for any \( a, b, c \).

Division is not associative: \( (a \div b) \div c \neq a \div (b \div c) \).

Or, in fractional notation, \( \frac{a/b}{c} \neq \frac{a}{b/c} \).

For example, if \( a = 8 \), \( b = 4 \) and \( c = 2 \), then

\[
(a \div b) \div c = (8 \div 4) \div 2 = 2 \div 2 = 1 ,
\]
but \( a \div (b \div c) = 8 \div (4 \div 2) = 8 \div 2 = 4 \).

So division is not associative.

But expressing division \( a \div b \div c \) in terms of multiplication \( a \cdot \frac{1}{b} \cdot \frac{1}{c} \),
we may apply the associativity of multiplication to get:

\[
(a \div b) \div c = \left( a \cdot \frac{1}{b} \right) \cdot \frac{1}{c} = a \cdot \left( \frac{1}{b} \cdot \frac{1}{c} \right) = a \cdot \frac{1}{b \cdot c} = a \div (b \cdot c).
\]
Summary

In this lecture, we have learned that

✓ subtraction is the **opposite** of addition
✓ subtraction can be **expressed** as addition of the opposite: \( a - b = a + (-b) \)
✓ subtraction is **neither** commutative **nor** associative
✓ division is the **opposite** of multiplication
✓ division can be **expressed** as multiplication by the reciprocal: \( a \div b = a \cdot \frac{1}{b} \)
✓ division by zero **does not make sense**
✓ division is **neither** commutative **nor** associative