Internal Addresses in the Mandelbrot Set and Irreducibility of Polynomials

Eike Lau and Dierk Schleicher

Abstract

For the polynomials $p_c(z) = z^d + c$, the periodic points of periods dividing n are the roots of the polynomials $P_n(z) = p_c^{\circ n}(z) - z$, where any degree $d \geq 2$ is fixed. We prove that all periodic points of any exact period k are roots of the same irreducible factor of P_n over $\mathbb{C}(c)$. Moreover, we calculate the Galois groups of these irreducible factors and show that they consist of all permutations of periodic points which commute with the dynamics. These results carry over to larger families of maps, including the spaces of general degree-d-polynomials and families of rational maps.

Main tool, and second main result, is a combinatorial description of the structure of the Mandelbrot set and its degree-d-counterparts in terms of *internal addresses* of hyperbolic components. Internal addresses interpret kneading sequences of angles in a geometric way and answer Devaney's question: *"How can you tell where in the Mandelbrot a given rational external ray lands, without having Adrien Douady at your side?"*

Contents

1 Introduction

We will investigate periodic points of exact period n for polynomials $p_c(z) = z^d + c$, where a degree $d \geq 2$ is fixed and suppressed in the notation. These periodic points are roots of the polynomial $P_n(z) = p_c^{\circ n}(z) - z$, which will be considered as a polynomial in z with coefficients in $\mathbb{C}[c]$. Dividing out the periodic points whose periods strictly divide n, we arrive at polynomials Q_n which are recursively defined by

$$P_n = \prod_{k|n} Q_k \ .$$

For fixed period n, the roots of $Q_n(c, z)$ are periodic points of exact period n of $p_c(z)$. Define algebraic curves

$$Z_n := \{(c, z) \,:\, Q_n(c, z) = 0\}$$

which we will consider as ramified covers over the complex c-plane. By the implicit function theorem, its ramification values are the roots of hyperbolic components of period n: parameters c for which $(\partial/\partial z)Q_n(c, z) = 0$ for $(c, z) \in Z_n$. Let $\pi_n : Z_n \to \mathbb{C}$ be the projection onto the c-coordinate. If the number of periodic orbits of exact period n is N_n, then the degree of π_n is nN_n.

One of our main results is the following.

Theorem 4.1 (Analytic Version)
The algebraic curve Z_n is irreducible for every $n \geq 1$.

We will in fact show more.

Theorem 4.1 (Algebraic Version)
For every $n \geq 1$, the polynomials Q_n are irreducible over $\mathbb{C}(c)$. Their Galois groups G_n consist of all the permutations of their roots which are compatible with the dynamics of p_c.

Remark. In other words, denoting the symmetric and cyclic groups on k elements by S_k and $\mathbb{Z}_k$, respectively, there exists a short exact sequence

$$0 \longrightarrow (\mathbb{Z}_n)^{N_n} \longrightarrow G_n \longrightarrow S_{N_n} \longrightarrow 0 \, .$$

The injection from $(\mathbb{Z}_n)^{N_n}$ into G_n corresponds to cyclic permutations of the N_n periodic orbits, while the surjection onto S_{N_n} corresponds to arbitrary permutations of the orbits. This can be made slightly more precise: The group S_{N_n} acts on the generators of $(\mathbb{Z}_n)^{N_n}$ by permutation, and G_n is the semi-direct product

$$(\mathbb{Z}_n)^{N_n} \rtimes S_{N_n}$$

for this action. This theorem has been proved algebraically by Thierry Bousch in his thesis [Bo] in the quadratic case $d = 2$. The analog for generic degree d polynomials $z^d + a_{d-1}z^{d-1} + \ldots +$

$a_1 z + a_0$ has been conjectured by Morton and Patel ([MP], Conjecture 2). This conjecture is an immediate consequence of the theorem above: since, for $d > 2$, polynomials $z^d + c$ do not represent every polynomial of degree d up to conformal conjugation, the possibility to achieve all the permutations within our restricted family is a stronger statement. Irreducibility of the polynomials Q_n for $d \geq 2$ has recently also been shown by Morton [Mo], using ideas from Bousch and simplifying some of his arguments.

A different way to look at the theorem is as follows: let E_n be the complex plane, punctured at roots of hyperbolic components of period n. Then the restriction $\pi_n : (\pi_n^{-1}(E_n)) \longrightarrow E_n$ is a covering map. Thus all the periodic points of exact period n can be continued analytically along paths in E_n. The result of this process depends only on the homotopy class of the path in E_n, so if we choose a base point $c_0 \in E_n$, then the fundamental group of (E_n, c_0) acts on the set of points of exact period n of p_{c_0} by permutation.

We will prove the theorem in the following algebraic-topological form.

Theorem 4.1 (Algebraic-Topological Version)
For any $c_0 \in E_n$, the fundamental group $\pi_1(E_n, c_0)$ acts transitively on the set of periodic points of exact period n. More precisely, it induces the full symmetric group on the set of periodic orbits of exact period n, and every orbit can independently be permuted cyclically, leaving all the other periodic points fixed.

Our proof will be constructive. We will give a base point and a set of generators of the fundamental group, in terms of which we can describe, for any dynamically possible permutation of periodic points of exact period n, a homotopy class of loops along which analytic continuation realizes this permutation. This problem had been posed repeatedly by Adrien Douady and John Hubbard.

A related result has been obtained by Blanchard, Devaney and Keen [BDK]: they describe the possible permutations of periodic points which can by obtained by analytic continuation in the *escape locus* of general degree-d-polynomials: the locus of all polynomials all the critical points of which escape to infinity. It turns out that one gets all the permutations which come from automorphisms of the shift space.

The second centerpiece of this paper is the development a new geometric description of the Mandelbrot set (and its higher degree counterparts) which we call *internal addresses*. It gives a geometric interpretation to kneading sequences of angles, including an answer to the question of which and how many rational external rays share the same kneading sequence. Internal addresses help describe the combinatorial structure of the Mandelbrot set, they display certain symmetries of sublimbs of the Mandelbrot set and correspondences to Julia sets and they give sufficient conditions on which symbolic sequences are realized as kneading sequences of angles. This will be an important ingredient in the explicit specification of homotopy classes of loops realizing desired permutations of periodic points.

Internal addresses also allow to answer Devaney's question where in the Mandelbrot set given rational external rays land. A different interpretation and answer can be given using the Spider algorithm as described in Hubbard and Schleicher [HS]: it takes the angle and produces a sequence of complex numbers which converges to the Misiurewicz point or the center of the hyperbolic component at which the given external ray lands. From this value, the Julia set can be drawn on a computer. Unlike the answer by the Spider algorithm, our answer is a finite procedure, giving a combinatorial result.

Informally, the idea of internal addresses is to describe the path from the origin to any given hyperbolic component, much like walking directions for a pedestrian walking in the Mandelbrot set: we specify the most important roadmark on every remaining part of the path. Roadmarks will be hyperbolic components, and they will be the more important the smaller their periods are. This idea is formalized in Definition 6.1. It turns out that this information is just enough to reconstruct easily as detailed walking directions as desired: one can find all the (usually infinitely many) hyperbolic components encountered along the way, including their order. All this information is encoded in the kneading sequence of any external argument of the hyperbolic component, and conversely allows to reconstruct the kneading sequence.

Internal addresses are based on hyperbolic components. We extend the discussion to Misiurewicz points as necessary, but stop short of extending our results sytematically to Misiurewicz points in parameter space and preperiodic points in the dynamical plane, or even more generally. This should be a project in its own right.

In several ways, interal addresses are related to addresses and veins ("nervures") as developed in exposés XX–XXII in Douady and Hubbard [DH1] and described in Douady [D2]. The construction itself is different, making this paper self-contained to a large extent. For background on complex dynamics, see Milnor [M2]. The used results about parameter space are described in Section 2 below, together with references. General references, besides [DH1], are the books by Beardon [Be], Carleson and Gamelin [CG] and Steinmetz [St].

The organization of this paper is as follows. After some background in Section 2, we develop the framework of the result on permutations in Section 3: we describe the action of analytic continuation along small loops and link these local results together by a global labelling scheme for periodic points using symbolic dynamics. The possible permutation groups are calculated in Section 4, using one result about the existence of kneading sequences which will be proved in Section 10. Similar results for rational maps are immediate corollaries.

In one sense, the remainder of the paper serves to establish exactly this existence result. This will be possible by investigating the geometry of kneading sequences. Their study is begun in Section 5 and translated into the geometric language of *internal addresses* in Section 6; we will also explain an algorithm how to tell where in the Mandelbrot set rational external rays land. No results about parameter space without a study of dynamical planes; this is done in Section 7. In Section 8, we relate dynamical and parameter planes. These results help, in Section 9, to prove that hyperbolic components are uniquely described by their internal addresses if certain angles are given additionally, and that the question whether or not a certain internal address is realized in the Mandelbrot set does not depend on these angles. In Section 10, we introduce a special kind of hyperbolic components which we call *purely narrow* and classify exactly their internal addresses and kneading sequences. It turns out that their kneading sequences are just enough to prove the permutation results of periodic points. A grand example is given in Section 11.

Our results are described in Sections 3–11 only in the quadratic case, although most results and proofs hold equally well for higher degrees. What changes is the language to describe them: every hyperbolic component splits up into $d-1$ sectors. In order to simplify the description and avoid cumbersome notation, we collect all the results about higher degrees in Section 12, stating and proving those which do not carry over in the obvious way, replacing every factor 2^k by d^k.

At several places, we supply new and simplified proofs of known results: of Lavaurs' Lemma that the arc between two hyperbolic components of equal period always contains a component of lower period (Lemma 3.8; in fact, we prove a stronger result in Theorem 9.2), of Levin's result that periodic points can always be continued analytically in the entire wake of the component in which they were on an attracting orbit (Lemma 3.7), and of the result of Douady and Hubbard that branch points in the Mandelbrot set are postcritically finite (Theorem 9.1). A consequence of the latter result is that local connectivity of the Mandelbrot set is equivalent to combinatorial rigidity (which means that any two different points in the Mandelbrot set, not belonging to the closure of the same hyperbolic component, are separated by a pair of rational rays) and in particular implies that every component is hyperbolic. A simple new proof about landing properties of rational external rays of the Mandelbrot set, which this paper relies on in an essential way, is given in Chapter 2 of [S].

This paper grew out of Chapters 3 and 4 of the second author's Ph.D. thesis [S] at Cornell University.

Acknowledgements. We would like to thank John Hubbard, thesis advisor of the second author, for many inspiring and interesting discussions. Discussions with many people have also been very helpful and are gratefully acknowledged; among them are Pau Atela, Christoph Bandt, Bodil Branner, Bob Devaney, Adrien Douady, Wolfgang Fuchs, Karsten Keller, Jiaqi Luo, Misha Lyubich, Jack Milnor, Patrick Morton, Shizuo Nakane, Chris Penrose, Carsten Petersen, Kevin Pilgrim, Pierrette Sentenac, John Smillie and Tan Lei. Both of us would like to express our gratitude to the Studienstiftung des Deutschen Volkes, and the second author would like to thank Cornell University in Ithaca, NY, the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, and the Mathematical Sciences Research Institute in Berkeley, CA (NSF grant no. DMS 90-22140), for their support.

2 Mandelbrot Set and Multibrot Sets

For quadratic polynomials $z \mapsto z^2 + c$, the space of all parameters c for which the Julia set is connected is called the *Mandelbrot set* or *quadratic connectedness locus*. We will call its generalizations for polynomials $z^d + c$ of degrees $d \geq 2$ the *Multibrot sets*, following a suggestion of Douady.

This section recalls some background about the analytic theory of these parameter spaces. The standard reference is Douady and Hubbard [DH0] and [DH1]; for an overwiev, see Branner [Br] and, for higher degrees, Devaney, Goldberg and Hubbard [DGH]. Combinatorial results are given in Douady [D1] and Milnor [M1]. Many results can be found, with simplified proofs, in Chapter 2 of Schleicher [S]; see also Appendix C of Goldberg and Milnor [GM].

In our context, *hyperbolic* polynomials can be simply described as those for which there exists an attracting or superattracting periodic orbit. If such an orbit exists, it is unique and has a well-defined *multiplier*: the derivative of the first return map of any periodic point on this orbit. Hyperbolic polynomials are contained in components of the Multibrot sets which presumably make up the entire interior. We will sometimes make this explicit by speaking of *hyperbolic components*, although we will never consider other components.

Definition 2.1 (PER) *Let $\mathcal{A}$ be a hyperbolic component. By its* period $\mathrm{PER}(\mathcal{A})$ *we mean the period of the attracting periodic orbit in the dynamical plane of any of the polynomials in $\mathcal{A}$.*

Any periodic point of period n is also periodic for all periods which are multiples of n. We will use the term *exact period* n for the smallest number of iterations it takes for the point to return. This is sometimes referred to as the *prime* or *primitive period* of a periodic point, although it need not be prime.

The dynamics on angles will always be multiplication by d modulo one. Whether a given rational angle is periodic or preperiodic depends of course on d.

Proposition 2.2 (Hyperbolic Components) *For every hyperbolic component W, the multiplier map $\mu : W \to \mathbb{D}$ is an analytic $d-1$-fold cover over the open unit disk, ramified only at a single inverse image c_0 of 0. It extends continuously as a $d-1$-fold cover of the boundary ∂W to $\mathbb{S}^1$.*

Definition 2.3 (Sectors and Roots of Hyperbolic Components)
The inverse images of the ray $[0,1]$ under μ separate every hyperbolic component into $d-1$ regions which we will call the sectors *of the component. The $d-1$ inverse images of 1 will be called the* root points *of the component. A root will be called* principal *if the number of external rays landing at it is two, and* non-principal *if this number is one.*

The next theorem says that the number of external rays at every root is either one or two, so the description of roots is complete. Every sector has two roots of the component on its boundary. The *wake* of the sector is the domain bounded by the sector and the two rays landing at its two roots, excluding the closure of the sector (if one of the roots is principal, so that two rays land there, we choose the ray which yields the smaller wake). The *limb* of the sector is the intersection of its wake with the connectedness locus. Limb and wake of an immediately bifurcating component at internal angle p/q will be called *p/q-sublimb* and *subwake*, respectively.

In case $d = 2$, there is no need to distinguish components and sectors, so one can always read "component" for "sector"; for higher degrees, however, the main characters in the play will be sectors rather than components.

A Misiurewicz point c is a parameter for which the critical point is strictly preperiodic. For given preperiod and period, there are only finitely many Misiurewicz point because they satisfy a certain polynomial equation.

Theorem 2.4 (The Multibrot Sets)
The Multibrot sets have the following properties:

1. *They are connected.*

2. *All external rays with n-periodic angles land at roots of hyperbolic components of period n, and no other external rays land at such roots.*

3. *Every hyperbolic component has $d-1$ roots, one of them principal and the others non-principal, so that exactly d external rays land at these roots.*

4. *If a root of a period-n-component is on the boundary of a different component, then that component has period k properly dividing n and in the dynamical plane two orbits of respective periods n and k coalesce. Such a root is always principal. At all other roots, exactly two orbits of exact periods n coalesce and the root may or may not be principal.*

5. *If the parameter c is a root of a hyperbolic component, then the angles of the external rays landing at c are exactly the same as the angles of the external rays in the dynamical plane of c which land at the parabolic periodic point on the closure of the critical value Fatou component and are adjacent to this component.*

6. *All preperiodic external rays of the Multibrot sets land at Misiurewicz points; in their dynamical planes, the same external rays land at the critical value.*

In order to make this precise for $n = 1$, we have to count the rays at angles 0 and 1 separately.

We will speak about the combinatorial structure of the Multibrot sets as formed by the hyperbolic components (or sectors) and Misiurewicz points. Each of them has a finite number of external rays landing at them, all of them having rational angles. For a component or sector, we consider only the rays landing at the roots unless mentioned otherwise. There is a natural partial order on these objects, describing the abstract tree which is formed by hyperbolic components and Misiurewicz points but not specifying the type of embedding into the plane.

Definition 2.5 (Order) *Let $\mathcal{A}$ and $\mathcal{B}$ be two different hyperbolic components, sectors, or Misiurewicz points. We say $\mathcal{A} \prec \mathcal{B}$ if two external rays landing at $\mathcal{A}$ separate $\mathcal{B}$ from the origin.*

Definition 2.6 (Combinatorial Arc) *For two hyperbolic components, sectors or Misiurewicz points $\mathcal{A} \prec \mathcal{B}$, we define the combinatorial arc $[\mathcal{A}, \mathcal{B}]$ to be the collection of all components, sectors and Misiurewicz points $\mathcal{C}$ satisfying $\mathcal{A} \prec \mathcal{B} \prec \mathcal{C}$, together with their natural ordering; the endpoints $\mathcal{A}$ and $\mathcal{B}$ should form part of the arc.*

Remark. There is in fact a topological arc from $\mathcal{A}$ to $\mathcal{B}$, at least in the quadratic case, as was pointed out by J. Kahn (see Douady [D2]). For our purposes, it suffices to use the combinatorial information collected in the definition above; it differs from a similar definition of Douady and Hubbard in that it looks at entire components, rather than at their centers and roots.

The following proposition is proved as Proposition 2.4.3 in [S] in the quadratic case; the general case is similar. The *width* of the wake of a sector will be the difference of the angles of the two external rays bounding this wake.

Proposition 2.7 (Width of Wakes) *Given a sector of period n and width δ, then the width of its p/q-subwake is*

$$\frac{(d^n-1)^2}{d^{qn}-1}\,\delta\ .$$

Corollary 2.8 *If a sector $\mathcal{A}$ has higher period than a sector $\mathcal{B}$ with $\mathcal{A} \prec \mathcal{B}$, then $\mathcal{B}$ sits in the $1/2$-sublimb of $\mathcal{A}$.*

Proof. Let n be the period of $\mathcal{A}$ and let δ be its width. Using the previous proposition, the width of a p/q-subwake with $q \geq 3$ is

$$\frac{(d^n-1)^2}{d^{qn}-1}\delta < \frac{d^{2n}}{d^{qn}}\delta < \frac{1}{d^n} < \frac{1}{d^n-1}\ .$$

If there was a sector of period less than n in the wake, the two external rays landing at its roots would have an angular distance of more than that. $\square$

3 Periodic Points and Symbolic Dynamics

From here on and until Section 11, we will restrict to the quadratic case, deferring the general case to Section 12. This section will provide the motor for the result on permutations of periodic points by relating local to global properties: we will introduce symbolic dynamics which permits to label periodic points globally, and we will describe the action of analytic continuation along certain local curves on the set of periodic points. We will need to introduce the concept of *kneading sequences*, which will play a prominent role in the entire paper. Finally, we will prove two results about the relative location of hyperbolic components in the Mandelbrot set. The results of this section are well known, with the possible exception of Lemma 3.5, but we give new proofs for some of them.

Let X be the exterior of the Mandelbrot set in $\mathbb{C}$ minus the positive real axis; this set is simply connected. For a parameter $c \in X$, let $\theta \neq 0$ be the external angle. The dynamical plane of c is separated by the two external rays at angles $\theta/2$ and $(\theta+1)/2$ which bounce into the critical point (compare Figure 1). The boundary meets neither the Julia set nor the critical value c. We label the part containing the critical value by `1`; the other part, containing the external ray at angle 0, will be labelled `0`. To every point of the Julia set, we associate an *itinerary* as the sequence of labels of the parts containing the chosen point and its successive forward images.

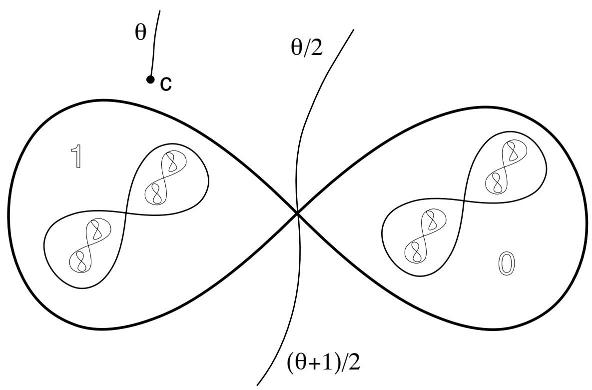

Figure 1: The partition which defines itineraries of points in completely disconnected Julia sets.

The following theorem is well known and will not be proved here.

Theorem 3.1 (Symbolic Dynamics) *For every parameter $c \in X$, the Julia set is a Cantor set which is homeomorphic to the set of infinite sequences on the set of the two symbols* `0` *and* `1`*. The itinerary of points in the Julia set gives an explicit homeomorphism. It conjugates the dynamics on the Julia set to the left shift on the shift space. Periodic points correspond bijectively to periodic sequences of the same period. When periodic points are continued analytically as the parameter varies within X, their itineraries remain invariant.* □

Let φ be a rational angle which is not on the inverse orbit of the external angle θ of a parameter $c \in X$. The corresponding external ray then lands on a (pre)periodic point z in the Julia set (compare Atela [At] and the appendix in Goldberg and Milnor [GM]). The itinerary of z equals the itinerary of the external ray φ with respect to the same partition. This combinatorial information can simply be read off from the external angles; it will be the θ-itinerary of φ.

Definition 3.2 (Itinerary and Kneading Sequence of Angles)
Fix an angle $\theta \in \mathbb{S}^1$. We define the θ-itinerary $\mathtt{I}_\theta(\varphi)$ of an angle $\varphi \in \mathbb{S}^1$ to be the following sequence:

$$n\text{-th entry of } \mathtt{I}_\theta(\varphi) := \begin{cases} \mathtt{0} & \text{if } 2^n\varphi \in]\frac{\theta+1}{2}, \frac{\theta}{2}[\\ \mathtt{1} & \text{if } 2^n\varphi \in]\frac{\theta}{2}, \frac{\theta+1}{2}[\\ {}^{\mathtt{1}}_{\mathtt{0}} & \text{if } 2^n\varphi = \frac{\theta}{2} \\ {}^{\mathtt{0}}_{\mathtt{1}} & \text{if } 2^n\varphi = \frac{\theta+1}{2}\ . \end{cases}$$

The kneading sequence *of θ is the itinerary of θ with respect to its own partition: $\mathtt{K}(\theta) := \mathtt{I}_\theta(\theta)$. The sequences $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$ are equal to $\mathtt{K}(\theta)$, except that every boundary symbol ${}^{\mathtt{t}}_{\mathtt{s}}$ is replaced by $\mathtt{t}$ or $\mathtt{s}$, respectively.*

The two intervals are taken with their orientation inherited from the positive orientation of $\mathbb{S}^1$ so that the first one contains the angle 0. The first symbol in a kneading sequence thus always is $\mathtt{1}$. Kneading sequences contain the boundary symbols ${}^{\mathtt{1}}_{\mathtt{0}}$ and ${}^{\mathtt{0}}_{\mathtt{1}}$ only for periodic angles, once within the period. We will sometimes lump them together and call them $\star$, as is done most often in the literature. The sequences $\mathtt{K}^\pm(\theta)$ can also be regarded as limits (with pointwise convergence): $\mathtt{K}^+(\theta) = \lim_{\theta' \searrow \theta} \mathtt{K}(\theta')$ and $\mathtt{K}^-(\theta) = \lim_{\theta' \nearrow \theta} \mathtt{K}(\theta')$.

Kneading sequences capture the symbolic dynamics of the critical orbit and have been introduced by Milnor and Thurston [MT] in the case of real quadratic polynomials; compare also Hubbard and Schleicher [HS] and the survey by Bullet and Sentenac [BS].

Observation 3.3 *The n-th entry of the kneading sequence is the following function on $\mathbb{S}^1$:*

$$n\text{-th entry of } \mathtt{K}(\theta) = \begin{cases} \mathtt{0} & \text{if } \theta \in]\frac{a}{2^n-1}, \frac{a+1}{2^n-1}[\text{ for an odd integer } a \\ \mathtt{1} & \text{if } \theta \in]\frac{a}{2^n-1}, \frac{a+1}{2^n-1}[\text{ for an even integer } a \\ {}^{\mathtt{1}}_{\mathtt{0}} & \text{if } \theta = \frac{a}{2^n-1} \text{ for an even integer } a \\ {}^{\mathtt{0}}_{\mathtt{1}} & \text{if } \theta = \frac{a}{2^n-1} \text{ for an odd integer } a \end{cases}$$

where we require $0 \le a < 2^n - 1$.

PROOF. If we vary $\theta \in \mathbb{S}^1$, the n-th entry of $\mathtt{K}(\theta)$ starts as $\mathtt{1}$ for θ close to 0 and changes between $\mathtt{1}$ and $\mathtt{0}$ whenever θ passes some periodic angle $a/(2^n-1)$. In other words, the n-th entry in the kneading sequence counts modulo 2 the number of nonnegative angles smaller than θ the period of which divides n. $\square$

Having defined a global labelling scheme for periodic points outside the Mandelbrot set, we will now describe analytic continuation as a local problem. For a fixed period n, all periodic points of exact period n can be continued analytically as long as they are all distinct. This fails exactly when the multiplier of the orbit equals 1, which happens at the roots of the finitely many hyperbolic components of exact period n. We need only describe the action of small loops around such roots along which periodic points are continued analytically. This will be done in the subsequent two lemmas.

Lemma 3.4 *Let c be the root of some hyperbolic component $\mathcal{A}$ of exact period n. There is a neighbourhood U around c such that analytic continuation along any closed curve which is completely inside $U - \{c\}$ and which winds around c once acts on the set of periodic points of period n as follows:*

If $\mathcal{A}$ is a primitive component, then two orbits are interchanged, and the actions of the loop and its inverse loop are the same.

If $\mathcal{A}$ is an immediate bifurcation from a hyperbolic component of period k at the end of the internal ray at angle p/q, then one orbit is permuted cyclically by $p'k$, where p' is the multiplicative inverse of p modulo q.

The affected periodic points are those which become parabolic when continued analytically as the parameter moves to c within U.

Proof. Let U be any simply connected neighbourhood of c which does not meet any other hyperbolic component of period n. If $\mathcal{A}$ is a primitive component, then two orbits of period n coalesce there. If the loop did not interchange them, both orbits could be continued analytically in a neighbourhood of c. Since the multiplier depends analytically on the parameter, both orbits must become attracting near c, so c would be a common root of two hyperbolic components. But hyperbolic components never share roots, see Theorem 2.4.

In case of a bifurcation, two orbits of periods k and $n = qk$ meet; the k-periodic points can be continued analytically because their multiplier is a root of unity different from 1. If the parameter c tends to the parabolic parameter c_0, q points each of the qk-cycle approach every of the k-periodic points. When c is near the parabolic parameter c_0, then these q points will, to first order, form a regular q-gon because the k-th iterate has the local form $z \mapsto \lambda z$ with $\lambda = e^{2\pi i p/q}$ a q-th root of unity.

When c turns once around c_0, analytic continuation induces a permutation $\mathcal{P}$ within every such set of q points of period qk. It must be compatible with the dynamics and is thus cyclic. Conversely, when any of the qk-periodic points turns around the corresponding k-periodic point $1/q$-th of a turn so as to reach the adjacent qk-periodic point, then the entire orbit and its multiplier will be restored to first order. The parameter c can be continued analytically; since there is only one hyperbolic component of period qk at c_0, c will turn a finite number s of times around c_0. Hence, s is not zero and the permutation $\mathcal{P}$ is not the identity. Because of holomorphy, s is positive. If it was greater than 1, then after only $1/s$ of the motion of the qk-periodic point, c would be restored to the initial position, and so would the qk-periodic points (as a set). This forces $s = 1$.

When c makes one turn around c_0, each of the qk-periodic points moves to the next one with respect to the position around the k-periodic points. Since the dynamics of the k-th iterate is a permutation by pk units, the statement follows. □

Lemma 3.5 *Let θ be a rational angle of exact period n and let c be the root of a hyperbolic component of period n where the external ray at angle θ lands. Consider a small loop around c as described in Lemma 3.4, starting and ending on the external ray at angle θ.*

If $\mathcal{A}$ is primitive, then the loop interchanges the two periodic points with itineraries $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$, together with their orbits.

If $\mathcal{A}$ is not primitive, then exactly one of the itineraries $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$ has exact period n, and the point with this itinerary is on the orbit which is permuted cyclically.

PROOF. Let c_1 and c_2 be two points in X on the loop such that their external arguments θ_1 and θ_2 satisfy $\theta_1 < \theta < \theta_2$ and such that these three angles are very close to each other. In the dynamical planes of c_i, the external rays at angle θ land at repelling periodic points z_i (for $i = 1, 2$). In the neighbourhood U of c given by Lemma 3.4, the analytic continuations of z_i cannot remain repelling because in the dynamical plane of the parameter c, the external ray at angle θ lands on the parabolic cycle.

By the discussion after Theorem 3.1, the itineraries of the points z_i in their Julia sets equal the itineraries of their external angles θ with respect to θ_i. They are n-periodic (although no necessarily of exact period n) because θ is, and they are in fact equal to $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$ provided the angles are close enough to each other. The sequences $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$ differ only at the n-th position within the period and cannot be cyclic permutations of each other because the numbers of symbols `0` and `1` in their itineraries differ by one. Since both in the primitive and the non-primitive cases, exactly two periodic orbits coalesce, exactly one of the points z_1 and z_2 is on each of these orbits.

If $\mathcal{A}$ is primitive, these points are interchanged, together with their orbits, and they both have period n. If $\mathcal{A}$ is not primitive, one of them is on the n-periodic orbit which is permuted cyclically. □

Together with the lemma, we have proved the following:

Corollary 3.6 *Let R_θ be an external ray at angle θ landing at the root of some hyperbolic component $\mathcal{A}$ of period n. The component is primitive iff both $\mathtt{K}^+(\theta)$ and $\mathtt{K}^-(\theta)$ have exact period n. Otherwise, one of the sequences has exact period n, and the period of the other sequence is the period of the component that $\mathcal{A}$ is immediately bifurcating from.* □

Using the results obtained so far, we can calculate the effect of analytic continuation of n-periodic points along any loop in $\mathbb{C}$ which avoids roots of hyperbolic components of period n because such a loop is homotopic (relative to the roots) to small loops as described in the previous two lemmas, connected by paths in X. A loop once around the Mandelbrot set can be interpreted as a collection of little loops around the roots of all the components of period

n; it turns out to interchange the symbols 0 and 1 in all the itineraries (as can be seen from Figure 1).

In principle, we have reduced the question which permutations among periodic points can be realized by analytic continuation to a purely combinatorial question: which symbolic sequences are realized as kneading sequences of periodic angles. This question is unsolved in its full generality. It might seem that the context of the Mandelbrot set is no longer necessary. It turns out, however, that it adds structure and geometry to the space of kneading sequences and permits to etablish the existence of sufficiently many kneading sequences. This was the initial motivation for the investigation of internal addresses. The interplay goes both ways: kneading sequencess also help describe the structure of the Mandelbrot set. This forms the content of the second part of this paper, starting with Section 5.

In the sequel, we will often need a lemma of Lavaurs' from [La]. There is a simple proof in our context. Along the way, we will give an easy proof of a result of Levin's ([Le], Theorem 7.3) which might have been known before to Hubbard and Lavaurs.

Lemma 3.7 (Levin) *Let $\mathcal{A}$ be a hyperbolic component. Every periodic point which is on the attracting cycle in $\mathcal{A}$ extends to an analytic function in the entire wake of $\mathcal{A}$.*

PROOF. We give a proof by contradiction. The only obstacle to the periodic point extending to the entire wake as an analytic function would be a root of a hyperbolic component $\mathcal{B}$ in which the continued periodic point lies on the attracting cycle. Let $\mathcal{A}'$ be the component bifurcating from $\mathcal{A}$ in the subwake of which $\mathcal{B}$ lies. It is enough to show that the periodic point under consideration can be continued analytically in the entire wake of $\mathcal{A}'$. When the parameter c is in $\mathcal{A}'$, the periodic point in question will be repelling. The angles of the external rays landing at its orbit will be a finite set Φ. Let φ_1 and φ_2 be the two external arguments of the root of $\mathcal{A}'$; they are the two supporting rays of the critical value Fatou component when $c \in \mathcal{A}'$. In particular, $\varphi_1, \varphi_2 \in \Phi$ and no other angle in Φ is between φ_1 and φ_2.

It follows that when c varies within the wake of $\mathcal{A}'$, its external argument in parameter space will never be in Φ. The same external rays will land at the considered periodic point as long as its orbit remains repelling, even when the Julia set is a Cantor set. If there was a component $\mathcal{B}$ in which this orbit became attracting, a loop around this component would interchange or permute the orbit and could not leave the external rays at the periodic orbit intact. $\square$

Lemma 3.8 (Lavaurs) *For two hyperbolic components $\mathcal{A} \prec \mathcal{B}$ of equal period, there is a hyperbolic component of lower period on the arc between $\mathcal{A}$ and $\mathcal{B}$.*

PROOF. Assume the lemma was false, so there were two hyperbolic components $\mathcal{A} \prec \mathcal{B}$ of equal period with no component of lower period on the arc between them. Since all the external rays of periods less then $\mathrm{PER}(\mathcal{A})$ come in pairs between the external rays landing at the roots of $\mathcal{A}$

and $\mathcal{B}$, the kneading sequences $\mathtt{K}(\mathcal{A})$ and $\mathtt{K}(\mathcal{B})$ would coincide except at the entries $n, 2n, \ldots$ (compare Observation 3.3). By Lemma 3.5, loops around their roots would affect the same periodic points, which would contradict Levin's Lemma 3.7. □

Lavaurs' Lemma is true even without the requirement $\mathcal{A} \prec \mathcal{B}$; this will be proved in Theorem 9.2 as "completeness of internal addresses".

4 Permutations and Galois Groups

In this section, we will show how irreducibility of the polynomials $Q_n(z)$ can be proved and their Galois groups can be calculated. The logical place of this section would be at the end of the paper. We bring it here because it motivates the subsequent sections and is independent of them, except for the following existence result which will be proved in Section 10. We order sequences on the two symbols 0 and 1 by the lexicographical order induced by $1 > 0$.

Corollary 10.8 *Let S be a periodic sequence of symbols 0 and 1 with exact period $n \geq 2$. Then its largest shift is realized as the kneading sequence $\mathtt{K}^+(\theta)$ or $\mathtt{K}^-(\theta)$ of some periodic angle θ of period n. If the sequence contains more than one symbol 0 within its period, then the external ray at angle θ lands at the root of a primitive hyperbolic component of the Mandelbrot set.*

In Section 10, we will describe exactly at which hyperbolic components such rays land.

Now we can prove the Main Theorem in the quadratic case.

Theorem 4.1 (T. Bousch)
For given n, let E_n be the complex plane punctured at the roots of hyperbolic components of period n. The fundamental group of E_n with respect to any base point acts on the set of periodic points of P_c of exact period n by analytic continuation. This action induces precisely those permutations π which commute with the dynamics, i. e. for which $P_c(\pi(z)) = \pi(P_c(z))$.

PROOF. Analytic continuation along a curve which avoids roots of period n will always induce a permutation which commutes with the dynamics. Thus no further permutations are possible and we have to show how all these permutations are realized.

There are two fixed points for every c; they are interchanged by loops around the root $c = 0.25$ of the main cardioid. The two periodic points of exact period 2 are on the same orbit and are interchanged by loops around the root $c = -0.75$ of the period-2-component. We can thus restrict attention to $n \geq 3$.

In the previous section, we have shown that we can represent periodic points by their itineraries in an invariant way throughout all of X, and we have described how loops around roots of n-periodic hyperbolic components act on these itineraries. We will now prove that, whenever the itinerary $\mathtt{I}$ of a periodic point z contains two or more symbols 0 within its period, then there is a loop which interchanges this point with another point in the itinerary of which one of the symbols `0` is replaced by a `1`. In fact, the largest shift of the itinerary `I` (with respect to lexicographical order) is realized as the kneading sequence $\mathtt{K}^{\pm}(\theta)$ of a periodic angle θ of period n which is one of the two external angles of a primitive component (by Corollary 10.8); the shifted itinerary corresponds to some point z' on the same orbit as z. The n-th entry in this sequence is `0` because otherwise its shift by $n-1$ digits would be larger. Since $\mathtt{K}^{+}(\theta)$ and $\mathtt{K}^{-}(\theta)$ differ exactly at the n-th position within each period, Lemma 3.5 shows that a loop around the root of this component turns the last `0` in the itinerary of z' into a `1`. Since entire orbits are interchanged, some `0` in the itinerary of z has turned into a `1`, too.

This implies that a finite set of such loops can turn any periodic orbit of exact period n onto the unique orbit the itinerary of which contains exactly one `0`. Thus the loops act transitively on the set of orbits. But a permutation group G on a finite set of objects which is generated by transpositions and which acts transitively is the full symmetric group: for a proof, it is enough to see that whenever the transpositions (a,b) and (b,c) are in G, then $(a,c)=(a,b)(b,c)(a,b)$ is also in G. We see that G contains all transpositions and is hence the symmetric group.

To see that all cyclic permutations of all the periodic orbits can be realized independenty, it is enough to have one loop which induces a cyclic permutation on one of the orbits by some number relatively prime to the period and which fixes all the other periodic points. By Lemma 3.4, any loop around the root of one of the period-n-components bifurcating immediately from the main cardioid does this job. $\square$

Remark. This proof is constructive: Whenever we want to turn a `0` into a `1` in the itinerary of a periodic point, we need a hyperbolic component with specified kneading sequence. In the subsequent sections we will describe how to turn a kneading sequence into an internal address, which is a description where in the Mandelbrot set an appropriate hyperbolic component can be found. Section 11 is devoted to a detailed example.

There are parameter spaces for which the analogs to the polynomials Q_n are in fact reducible. Indeed, it is easy to manufacture covering spaces over E_n such that the required loops around roots land on different sheets. The simplest example is the space of quadratic polynomials in the parametrization $\lambda z(1-z)$. The two fixed points are 0 and $1-1/\lambda$ and globally distinguishable, except at the root $\lambda=1$. A similar two-fold cover over the root -1.75 of the primitive period-3-component makes the two orbits of period 3 distinguishable and permits to solve for the periodic points; compare Giarrusso and Fisher [GF]. We know of no examples of analytic parameter spaces where the polynomials are reducible, or where the Galois groups are not maximal, but where no two maps are conformally conjugate within this parameter space. On the other hand, the space of polynomials z^d+c represents every polynomial with $c\neq 0$ exactly $d-1$ times up to a conformal change of variables, but we still have maximality of the Galois groups.

It is evident that all the results hold for parameter spaces which contain a subspace conformally equivalent to the parameter space of quadratic polynomials, and they can be adapted for covering spaces thereof. For example, consider the space of quadratic rational maps in the parametrization $Q_{\lambda,\mu}(z) = (\lambda z^2 + 2 - \lambda)/(\mu z^2 + 2 - \mu)$ with critical points at 0 and ∞ and an extra fixed point at 1. Since the critical points are always distinct and cannot be multiple fixed points, every quadratic rational map can be written in this form (Milnor [M3] is an excellent reference on quadratic rational maps). When $\mu = 0$, we obtain the space of quadratic polynomials $\lambda z^2/2 + 1 - \lambda/2$, which is conformally conjugate to $z^2 + c$ for $c = \lambda/2 - \lambda^2/4$. The $(\lambda, \mu = 0)$-plane is thus a double cover over the c-plane for quadratic polynomials, ramified over $c = 1/4$. Therefore, all dynamically permitted permutations of periodic points of any period $n > 1$ can be achieved by analytic continuation within the λ-plane. On the other hand, the fixed point $z = 1$ of $Q_{\lambda,\mu}$ has multiplier $\lambda - \mu$ and cannot be permuted with the other two fixed points by analytic continuation along curves in the entire (λ, μ)-space avoiding parameters with a parabolic fixed point.

The proof of Theorem 4.1 in the general case $d \geq 2$ can be given more conveniently when a few more notions are available. This will be done at the end of Section 12.

5 Kneading Sequences

In this section, we will lay the foundations of how kneading sequences determine the structure of the Mandelbrot set. In the next section, we will translate these results to a more convenient language which we call internal addresses.

Definition 5.1 (Visibility) *A hyperbolic component $\mathcal{A}$ is called* visible *from a hyperbolic component or Misiurewicz point $\mathcal{B}$ if $\mathcal{B} \prec \mathcal{A}$ or $\mathcal{A} \prec \mathcal{B}$ and if there is no hyperbolic component of lower period than $\mathcal{A}$ on the combinatorial arc between $\mathcal{B}$ and $\mathcal{A}$.*

This definition will be illustrated in Figure 2. Lavaurs' Lemma 3.8 can be interpreted as saying that no hyperbolic component is visible from a component of equal period.

Proposition 5.2 *If $\theta_1 < \theta_2$ are the two external rays landing at the root of a hyperbolic component $\mathcal{A}$, then $\lim_{\alpha \searrow \theta_1} \mathtt{K}(\alpha)$ and $\lim_{\alpha \nearrow \theta_2} \mathtt{K}(\alpha)$ exist and are equal, where the limit is taken separately at every place.*

All angles of external rays landing at the same Misiurewicz point have the same kneading sequence.

Proof. For two external rays landing at the same Misiurewicz point, the number of external rays of any period m between them will be even because they land in pairs at hyperbolic components, so the m-th entries in the kneading sequences will be equal, according to Observation 3.3.

The same is true for a hyperbolic component $\mathcal{A}$ of period n, say, if m is not a multiple of $\mathrm{Per}(\mathcal{A})$; the m-th entry will then be constant in neighbourhoods of the θ_1. For the same

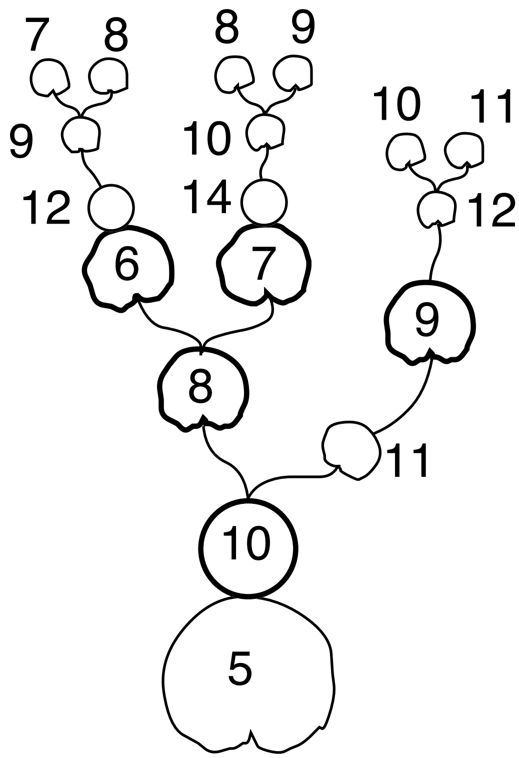

Figure 2: Visible and non-visible hyperbolic components: outlined components are visible from the drawn component of period 5. Only components in the wake of the period-5-component are shown.

reason, at positions which are multiples of n, both limits from inside will be equal (but different from the limits from outside, and different from the entries in the kneading sequence of θ_i). $\square$

This result suggests to consider kneading sequences as properties of hyperbolic components or Misiurewicz points.

Definition 5.3
(Kneading sequences of hyperbolic components and Misiurewicz points)
For a hyperbolic component $\mathcal{A}$ with external angles $\theta_1 < \theta_2$, we define the kneading sequence $\mathtt{K}(\mathcal{A})$ *to be one of the two equal limits from the previous proposition. The kneading sequence* $\mathtt{K}^-(\mathcal{A})$ just before $\mathcal{A}$ *will be the limit* $\lim_{\alpha \nearrow \theta_1} \mathtt{K}(\alpha) = \lim_{\alpha \searrow \theta_2} \mathtt{K}(\alpha)$. *For a Misiurewicz point $\mathcal{B}$, we define the kneading sequence $\mathtt{K}(\mathcal{B})$ to be the kneading sequence of any of its external arguments.*

Sometimes, notations will be simplified if we set $\mathtt{K}^-(\mathcal{B}) := \mathtt{K}(\mathcal{B})$ for a Misiurewicz point $\mathcal{B}$.

Remark. An equivalent way to define the kneading sequence of a hyperbolic component $\mathcal{A}$ is to take any external ray θ in the wake of $\mathcal{A}$ but not in any of its subwakes (such angles are irrational and have been shown to land on the boundary of $\mathcal{A}$ at irrational internal angles) and to define the kneading sequence $\mathtt{K}(\mathcal{A})$ to be the kneading sequence $\mathtt{K}(\theta)$. From the results of this section, it follows easily that this sequence is periodic of exact period n (despite the angle being irrational) and yields an equivalent definition.

Proposition 5.4 *Let two hyperbolic components $\mathcal{A} \prec \mathcal{B}$ be given. The following three conditions are equivalent:*

1. *$\mathcal{B}$ bifurcates immediately from $\mathcal{A}$.*
2. *The kneading sequence of $\mathcal{A}$ and the kneading sequence just before $\mathcal{B}$ are equal; in symbols: $\mathtt{K}(\mathcal{A}) = \mathtt{K}^-(\mathcal{B})$.*
3. *There is no hyperbolic component of period of lower period than $\text{Per}(\mathcal{B})$ on $[\mathcal{A}, \mathcal{B}]$, and $\text{Per}(\mathcal{A})$ divides $\text{Per}(\mathcal{B})$.*

If these conditions are not satisfied, or if $\mathcal{A}$ and/or B is a Misiurewicz point, let m be the position of the first entry where $\mathtt{K}(\mathcal{A})$ and $\mathtt{K}^-(\mathcal{B})$ differ. Then the hyperbolic component $\mathcal{C}$ of lowest period on the combinatorial arc between $\mathcal{A}$ and $\mathcal{B}$ has period m. Moreover, the first m entries of $\mathtt{K}(\mathcal{C})$ and $\mathtt{K}^-(\mathcal{B})$ coincide, and so do the first m entries of $\mathtt{K}^-(\mathcal{C})$ and $\mathtt{K}(A)$.

Proof. Consider two external arguments α of $\mathcal{A}$ and β of $\mathcal{B}$. All the hyperbolic components which are not on the combinatorial arc $[\mathcal{A}, \mathcal{B}]$ contribute two external angles of equal period between α and β, so they have no effect on the kneading sequences of these angles by Obervation 3.3. Components of any period k on this combinatorial arc do change the kneading sequence, however, at positions $k, 2k, 3k, \ldots$.

If $\mathcal{B}$ does not bifurcate immediately from $\mathcal{A}$, there exists a unique hyperbolic component $\mathcal{C}$ of least period on $[\mathcal{A}, \mathcal{B}]$ by Lavaurs' Lemma 3.8, so the first entry at which the kneading sequences $\mathtt{K}(A)$ and $\mathtt{K}^-(\mathcal{B})$ differ is at position $\text{Per}(\mathcal{C})$. This shows the second part of the proposition and the equivalence of the first two statements. If the period of $\mathcal{B}$ is a multiple of the period of $\mathcal{A}$ and the first $\text{Per}(\mathcal{B})$ positions of $\mathtt{K}(A)$ and $\mathtt{K}^-(\mathcal{B})$ coincide, then these two kneading sequences will in fact be equal, so there is no component at all between $\mathcal{A}$ and $\mathcal{B}$. This shows the third equivalence. □

Remark. Using this proposition, one can construct the lowest period of hyperbolic components on the combinatorial arc $[\mathcal{A}, \mathcal{B}]$; by repeated application, one gets the periods of all hyperbolic components on the combinatorial arc leading to a hyperbolic component or Misiurewicz point.

Corollary 5.5 (Exact Periods of Kneading Sequences)
Let $\mathcal{A}$ be a hyperbolic component of period n. Then its kneading sequence $\mathtt{K}(\mathcal{A})$ has exact period n; the kneading sequence $\mathtt{K}^-(\mathcal{A})$ has exact period n iff $\mathcal{A}$ is primitive; otherwise, its exact period is the period of the component that $\mathcal{A}$ is immediately bifurcating from.

If $\mathcal{B}$ is a Misiurewicz point the external angles of which become, after exactly l steps, periodic of period k, then its kneading sequence will also become periodic after exactly l steps, and the length of its period divides k (not necessarily properly).

PROOF. The statement about hyperbolic components is, in the primitive case, a reformulation of Corollary 3.6. If $\mathcal{A}$ is an immediate bifurcation from a component $\mathcal{B}$, then the kneading sequences $\mathtt{K}^-(\mathcal{A})$ and $\mathtt{K}(\mathcal{B})$ must be equal by the previous proposition, so $\mathtt{K}(\mathcal{A})$ has exact period n. For a Misiurewicz point with external angle θ, we have $2^l\theta = 2^{l+k}\theta$, of which $2^{l-1}\theta$ is the preperiodic inverse image and 2^{l-1+k} is the periodic one. They are different, so the lengths of preperiods of external angle and kneading sequence are the same. The statement about the periods is trivial. □

For a further discussion of kneading sequences, see Hubbard and Schleicher [HS].

6 The Geometry of Internal Addresses

In this section, we are going to define internal addresses which allow to describe hyperbolic components in a geometric way. Some fundamental existence and uniqueness theorems will be given in Section 9.

Definition 6.1 (Internal Address and Variants)
For a hyperbolic component or Misiurewicz point $\mathcal{A}$, consider all the hyperbolic components $\mathcal{B}$ on the combinatorial arc $[0,\mathcal{A}]$ which have the property that no hyperbolic component on the arc $[\mathcal{B},\mathcal{A}]$ has smaller period than $\mathcal{B}$. With respect to the order of these components on the combinatorial arc $[0,\mathcal{A}]$, their periods form a strictly increasing sequence of integers starting with 1. This sequence is called the internal address *of $\mathcal{A}$ and denoted $1 \to n_2 \to n_3 \to \ldots$.*

If we additionally give the internal angles by which the combinatorial arc $[0,\mathcal{A}]$ leaves every hyperbolic component appearing in the internal address, we obtain the angled internal address *of $\mathcal{A}$ and denote it by $1_{p_1/q_1} \to (n_2)_{p_2/q_2} \to (n_3)_{p_3/q_3} \to \ldots$.*

The periods of the entire collection of hyperbolic components on the combinatorial arc $[0,\mathcal{A}]$, together with their order along the arc, form the long internal address*, and the* angled long internal address *gives additionally all the internal angles by which the arc $[0,\mathcal{A}]$ leaves each component.*

Recall that combinatorial arcs contain their endpoints, so the internal address of a hyperbolic component of period n will be a finite sequence ending with n. The internal address of a Misiurewicz point is infinite, but the sequence of increments is preperiodic. Long internal addresses are in general infinite and not even well-ordered. Figure 3 illustrates internal addresses.

In Proposition 5.4 and the remarks thereafter, we have explained how to find successively the periods of smallest components on the combinatorial arc between two components with known kneading sequence. This leads to the following obvious algorithm to turn internal addresses into kneading sequences and back into (long) internal addresses.

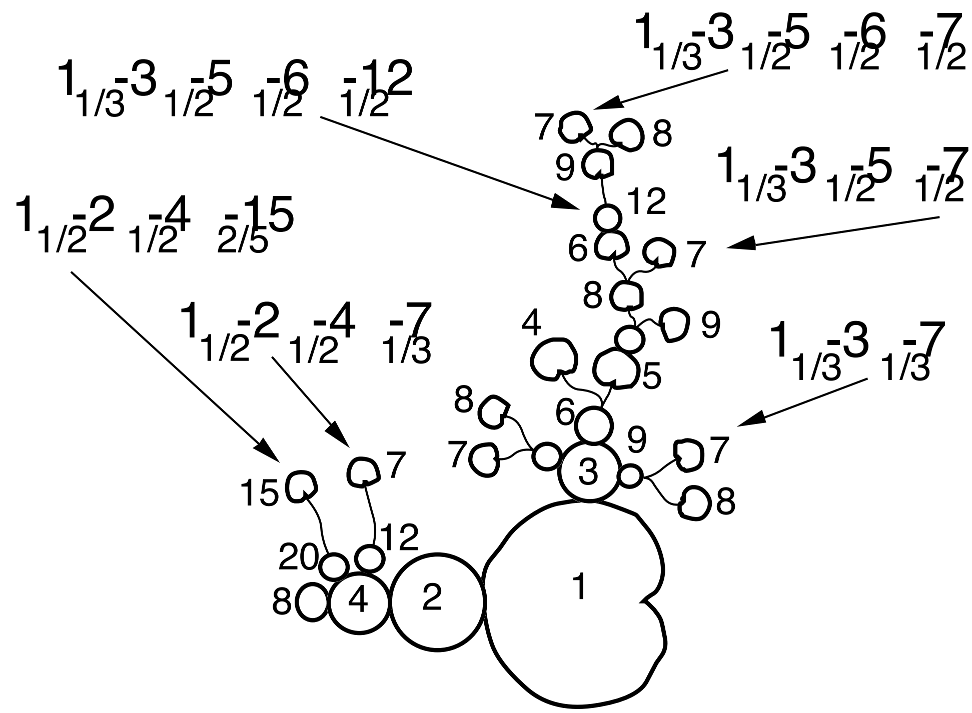

Figure 3: Hyperbolic components and their angled internal addresses.

Algorithm 6.2 (Turning Internal Addresses Into Kneading Sequences And Back) *Let $\mathcal{A}$ be a hyperbolic component of period n with the internal address $1 \to n_2 \to \ldots \to n_k$. Then the following recursive algorithm yields $\mathtt{K}(\mathcal{A})$:*

$\mathtt{K}(\mathcal{A})$ starts with $\mathtt{1}$. The first n_{i+1} entries of $\mathtt{K}(\mathcal{A})$ can be obtained by continuing the first n_i entries periodically and then changing the n_{i+1}-st entry. $\mathtt{K}(\mathcal{A})$ is periodic with period $n = n_k$.

Conversely, given the kneading sequence $\mathtt{K}(\mathcal{A})$ of a hyperbolic component, its internal address can be found by the same idea: start with $\mathtt{K} = \overline{\mathtt{1}}$ and compare it with $\mathtt{K}(\mathcal{A})$; the position n of first difference will be the next entry in the internal address. Now continue the comparison, taking $\mathtt{K}$ as the periodic continuation of the first n entries in $\mathtt{K}(\mathcal{A})$, and go on until the period of $\mathcal{A}$ is reached. □

We denote (pre)periodic symbolic sequences with an overbar over the symbols which should be repeated periodically.

Here is an example of how this algorithm works: There is a hyperbolic component of period 7 at external angles 23/127 and 24/127 which has the angled internal address $1_{1/3} \to 3_{1/2} \to 5_{1/2} \to 6_{1/2} \to 7$; the angles are, however, irrelevant for the kneading sequence. The component with internal address 1 has the kneading sequence $\overline{\mathtt{1}}$, so that the internal address $1 \to 3$ belongs to $\overline{\mathtt{110}}$, $1 \to 3 \to 5$ belongs to $\overline{\mathtt{11010}}$, and the kneading sequences of $1 \to 3 \to 5 \to 6$ and $1 \to 3 \to 5 \to 6 \to 7$ are respectively $\overline{\mathtt{110100}}$ and $\overline{\mathtt{1101000}}$. It is easy to verify that the external angles 23/127 and 24/127 indeed have this kneading sequence. The same procedure can be run backwards to turn the kneading sequence into the internal address.

There is a Misiurewicz point at external angle 13/60. The kneading sequence of this angle is $\mathtt{11}\overline{\mathtt{0010}}$, which translates into internal address $1 \to 3 \;\to\; 4 \to 6 \to 7 \;\to\; 8 \to 10 \to 11 \;\to \ldots$. This may be written more conveniently $1 \to 3 + 1 \to 2 \to 1$, giving the periodic sequence of increments after the "+".

REMARK. The algorithm can be extended to find the denominators of the angles in the angled long internal address: in order to find the angle at some component $\mathcal{B}$, find succesively the components of lowest period after $\mathcal{B}$ until the period of one of them it a multiple of $\mathrm{PER}(\mathcal{B})$; this component is then an immediate bifurcation and the denominator is the quotient of periods (since such a bifurcation always exists, this procedure will eventually stop). There is an a priori statement about the denominators, directly from the internal address, which leaves only two choices. Moreover, a denominator can be different from 2 only at components which appear in the internal address. This is the content of the following two statements.

Lemma 6.3 (Denominators) *If a hyperbolic component of period k is visible from a component $\mathcal{A}$ of period n, then it is in some p/q-sublimb of $\mathcal{A}$, where q is as follows:*

If k/n is an integer, then $q = k/n$.

If k/n is not an integer, then $k/n < q < k/n + 2$.

PROOF. The first statement is simply Proposition 5.4. For the second, the component of period k can only be visible if its period is smaller than the period qn of the immediately bifurcating component. On the other hand, an entire p/q-subwake contains less than $(2^n - 1)^2/(2^{qn} - 1)$ worth of angles by Proposition 2.7, while a component of period k needs at least $1/(2^k - 1)$. It remains a little calculation. □

Lemma 6.4 *If the combinatorial arc from the origin to some hyperbolic component or Misiurewicz point $\mathcal{A}$ leaves a hyperbolic component $\mathcal{B}$ at an internal angle different from $1/2$, then $\mathcal{B}$ occurs in the internal address of $\mathcal{A}$.*

PROOF. If $\mathcal{B}$ does not occur in the internal address of $\mathcal{A}$, there must be a hyperbolic component of period less than $\mathrm{PER}(\mathcal{B})$ on the combinatorial arc $[\mathcal{B}, \mathcal{A}]$ which can be only in the 1/2-sublimb of $\mathcal{B}$ by Corollary 2.8. □

Although Misiurewicz points have infinitely many hyperbolic components on their internal address, only finitely many of them can have an angle different from 1/2. This can be deduced from Proposition 8.4, and another proof appears at the end of Section 9.

Now we will give an answer to Devaney's question: *"How can you tell where a given rational external ray lands at the Mandelbrot set, without having Adrien Douady at your side?"* Let θ

be the angle and assume for now that it is periodic, such that the corresponding ray lands at the root of a hyperbolic component $\mathcal{A}$. The answer will then be in terms of the angled internal address of $\mathcal{A}$, plus the information which of the two rays landing there it is. In Theorem 9.2, we will prove that the angled internal address specifies a unique hyperbolic component. Recall from Algorithm 6.2 that this also determines the long angled internal address. The case of preperiodic angles will be discussed below.

The kneading sequence of θ can be found from the definition. The period of $\mathcal{A}$ is the period of θ, so Algorithm 6.2 gives the internal address $1 \to n_2 \to \ldots \to n_k$ of the hyperbolic component it lands at; by the remark after the algorithm, it also permits to find the denominators in the angled internal address. It remains to give the numerators p_i of the internal angles in $\mathcal{B}_i$ and to tell whether θ is the smaller or larger of the two angles landing at the root of $\mathcal{A}$. This will be done in the next two propositions.

Proposition 6.5 (Numerators) *Let $\mathcal{A}$ be a hyperbolic component or Misiurewicz point and let θ be the angle of one of the external rays which land at this Misiurewicz point or at the root of $\mathcal{A}$. Let k occur in the internal address of $\mathcal{A}$ and let p/q be the internal angle by which the combinatorial arc $[0, \mathcal{A}]$ leaves the corresponding hyperbolic component $\mathcal{B}$ of period k. Consider the $q-1$ angles $\theta, 2^k\theta, \ldots, 2^{(q-2)k}\theta$. Then p is the number of these angles which do not exceed θ (with respect to the order in the interval $(0,1)$).*

PROOF. Let $\mathcal{B}'$ be the hyperbolic component of period qk which bifurcates from $\mathcal{B}$ at internal angle p/q such that the combinatorial arc $[0, \mathcal{A}]$ runs from $\mathcal{B}$ into $\mathcal{B}'$ and let $\theta_1 < \theta_2$ be the angles of the external rays which land at the root of $\mathcal{B}'$. In the dynamical plane for some $c \in \mathcal{B}'$, the rays θ_1 and θ_2 land at the root of the critical value Fatou component adjacent to this component. The 0-th, k-th, $\ldots, (q-1)k$-th iterates of this component have a common root, which is periodic of period k and has combinatorial rotation number p/q. The 0-th, k-th, $\ldots, (q-2)k$-th iterates of $]\theta_1, \theta_2[$ are disjoint, and since $\theta \in]\theta_1, \theta_2[$, these iterates of θ also have this order. We do not know on which side of 0 the last image $2^{(q-1)k}\theta$ is, but the remaining rays suffice to determine p. $\square$

Proposition 6.6 (Right or Left Ray) *Let $\mathcal{A}$ be a hyperbolic component of period n and let $\theta_1 < \theta_2$ be the two external angles which land at the root of $\mathcal{A}$. The n-th digits in their binary expansions are different. The smaller angle has digit 1 iff the n-th entry of $\texttt{K}(\mathcal{A})$ is $\texttt{0}$.*

PROOF. The n-th digits in the binary expansions of θ_1 and θ_2 are different because the number of rays of periods dividing n in the wake of $\mathcal{A}$ is even. The kneading sequence $\texttt{K}(\mathcal{A})$ is defined as $\texttt{K}^+(\theta_1)$; its n-th entry is $\texttt{1}$ whenever the n-th entry in $\texttt{K}(\theta_1)$ is $\frac{\texttt{1}}{\texttt{0}}$; this happens when $2^{n-1}\theta_1$ is smaller than $1/2$ so that the n-th entry in the binary expansion is 0. The other case is similar. $\square$

As further example consider the period-15-component at external angle $\theta = 13492/(2^{15}-1)$. This angle has binary expansion $\overline{0110\,1001\,0110\,100}$ and kneading sequence $\mathtt{1011\,1011\,1011\,10}\genfrac{}{}{0pt}{}{\mathtt{1}}{\mathtt{0}}$. Internal address and kneading sequence of the component the ray lands at can be found as above; they are $1 \to 2 \to 4 \to 15$ and $\overline{\mathtt{1011\,1011\,1011\,100}}$. In order to find the angles in the internal address, we obtain from Lemma 6.3 denominators 2 for the first two places (which implies numerators 1) and either 4 or 5 at the period-4-component. By the remark after Algorithm 6.2, we find the components of consecutively lowest periods after the period-4-component: their periods are 17 (kneading sequence $\overline{\mathtt{1011\,1011\,1011\,1011\,0}}$), 18 (kneading sequence $\overline{\mathtt{1011\,1011\,1011\,1011\,11}}$) and 20, which is a multiple of four and thus the immediate bifurcation, so the denominator is 5. (In fact, the lowest period in between being 17 already ruled out the possibility of denominator 4 because that would have required a bifurcation into a period-16-component.) We find the remaining numerator at the period-4-component, according to Proposition 6.5, by looking at the four angles $\theta = 13492/(2^{15}-1), 2^4\theta \equiv 19270/(2^{15}-1), 2^8\theta \equiv 13417/(2^{15}-1), 2^{12}\theta \equiv 18070/(2^{15}-1)$ (all congruences modulo 1). Of these, the first and third do not exceed θ, so we get numerator 2 and the angled internal address is $1_{1/2} \to 2_{1/2} \to 4_{2/5} \to 15$. The 15-th entries in the kneading sequence of the component and the binary expansion of the angle are $\mathtt{0}$ and 0, respectively, so Proposition 6.6 says that our ray is the larger one of the two landing at the component.

It seems well possible to give an algorithm to turn an angled internal address into the Hubbard tree of the corresponding critically (pre)periodic polynomial. From there, it is easy to reconstruct the external angles, yielding a complete, even algorithmic, equivalence between the concepts Hubbard tree, external angle, kneading sequence (with some extra information about cyclic order) and angled internal address. Internal addresses and kneading sequences alone are too little information to reconstruct the polynomial, but our results show exactly how much extra information is necessary. The exact cyclic order of the forward images of the angle in the construction of the kneading sequence is already an overdetermination.

Another way to turn kneading sequences into external angles would be an iterative procedure, similar to the spider map (see Hubbard and Schleicher [HS]), but restricted to the set of angles. Depending on the starting point, it would converge to one of possibly several external angles.

It is well known that for every hyperbolic component $\mathcal{A}$ of the Mandelbrot set, there is a homeomorphism of the Mandelbrot set onto a subset of itself, sending the main cardioid onto $\mathcal{A}$ and every component of period k to a component of period $k\,\mathrm{PER}(\mathcal{A})$. The image is called the *tuned copy* of the Mandelbrot set at $\mathcal{A}$. The boundary of the tuned copy within the entire Mandelbrot set consists of images of dyadic Misiurewicz points under the tuning map (Misiurewicz points with external angles $a/2^s$ for some integers a and s). These results are due to Douady and Hubbard but, as far as we know, there is no complete reference for them in the literature. The foundations are laid in Douady and Hubbard [DH2], and the combinatorics are described in Douady [D1] and Milnor [M1]. Although we do not need the results about tuning anywhere in this paper, we describe here how to tell in which tuned copies a given hyperbolic component sits.

Proposition 6.7 (Tuned Copies) *A hyperbolic component $\mathcal{B}$ with internal address $1 \to n_2 \to \ldots \to n_k \to n_{k+1} \to \ldots \to n_{k+j}$ is within the tuned copy of a component $\mathcal{A}$ of internal address $1 \to n_2 \to \ldots \to n_k$ if and only if all the periods $n_{k+1} \ldots n_{k+j}$ are divisible by n_k.*

PROOF. The "only if" condition is obvious. For the converse, let A and A' denote the first n_k entries in the kneading sequences of $\mathcal{A}$ and just before $\mathcal{A}$, so that $\mathtt{K}(\mathcal{A}) = \overline{A}$ and $\mathtt{K}^-(\mathcal{A}) = \overline{A'}$. If all the numbers $n_{k+1} \ldots n_{k+j}$ are divisible by n_k, then it is easy to see that the corresponding kneading sequences are concatenations of blocks A and A'. Since both A and A' have length n and differ only at the n-position, it follows recursively that all the hyperbolic components on the combinatorial arc $[\mathcal{A}, \mathcal{B}]$ have kneading sequences consisting of the same two blocks and that their periods are divisible by n_k.

Assume that the component $\mathcal{B}$ was not in the tuned copy at $\mathcal{A}$ but that the component $\mathcal{B}'$ at internal address $1 \to n_2 \to \ldots \to n_k \to n_{k+1} \to \ldots \to n_{k+j-1}$ was. The combinatorial arc $[\mathcal{A}, \mathcal{B}]$ leaves the tuned copy at a Misiurewicz point $\mathcal{C}$. The inverse tuning map, which sends $\mathcal{A}$ to the main cardioid, maps $\mathcal{C}$ to a Misiurewicz point with dyadic external angle θ. Its kneading sequence will, after a finite initial sequence, consist only of symbols $\mathtt{0}$, so its internal address will eventually increase indefinitely by 1, using all sufficiently large positive integers. Tuning back, the internal address of $\mathcal{C}$ will contain all integers which are divisible by n_k and larger than some number N. Lavaurs' Lemma 3.8 excludes the existence of any component behind $\mathcal{C}$ which is visible from $\mathcal{C}$ such that its period is divisible by n_k and exceeds N. But such a component had to exist on the arc $[\mathcal{C}, \mathcal{B}]$ if $\mathcal{B}$ was not in the tuned copy at $\mathcal{A}$, a contradiction. $\square$

Consider a hyperbolic component of period n, so that its kneading sequence is periodic of period n. When using the Algorithm 6.2 to determine the internal address of $\mathcal{A}$, the n-th entry in the kneading sequence has to be such that an entry n is generated in the internal address of $\mathcal{A}$; this condition determines the entry uniquely. The other choice of the n-th entry in the kneading sequence would be the kneading sequence just before $\mathcal{A}$, which would either generate an infinite internal address without the entry n or a kneading sequence of lower period. For example, $\overline{\mathtt{10100}}$ corresponds to internal address $1 \to 2 \to 5$, while $\overline{\mathtt{10101}}$ would generate an infinite internal address $1 \to 2 \to 6 \to 7 \to 11 \to 12 \to \ldots$ and ist not an abstract kneading sequence.

Definition 6.8 (Abstract Internal Address and Abstract Kneading Sequence) *A finite strictly increasing sequence of integers starting with* 1 *will be called an* abstract internal address*. A sequence of symbols* $\{\mathtt{0}, \mathtt{1}\}$ *of exact period* n *will be called an* abstract kneading sequence *if it starts with* `1` *and generates a finite internal address ending with* n*, so that Algorithm 6.2 relates it bijectively to an abstract internal address.*

Not all abstract internal addresses are realized as internal address of some hyperbolic component; e. g. $1 \to 2 \to 4 \to 5 \to 6$ is not realized. The corresponding abstract kneading sequence is $\overline{\texttt{101 100}}$; if it was realized, the kneading sequence just before the component would be $\overline{\texttt{101 101}}$ with period 3, so the period-6-component would be a bifurcation, but the entry 3 does not appear in the internal address. Similar examples can be constructed for every primitive hyperbolic component and every bifurcation ratio $q \geq 2$.

The space of kneading sequences has been investigated by Penrose in his thesis [Pe1] and in a recent preprint [Pe2]. His "abstract abstract Mandelbrot set" consists of symbolic sequences of symbols `0` and `1`, whether or not they are abstract kneading sequences or even realized as kneading sequences (in his language: whether or not they are complex admissible). Internal addresses appear there implicitly as "principal nonperiodicity functions". He has a different way to look at the example above of a kneading sequence which is not realized ([Pe1], Example 2.2). Similar questions have been addressed in a series of papers by Bandt and Keller [BK1], [BK2], [Ke1], [Ke2], [Ke3].

7 The Dynamical Plane

In order to harvest more in parameter space, we need to plough in the dynamical planes. All Julia sets in this section will be postcritically finite and in particular connected, locally connected and thus pathwise connected (compare Milnor [M2]); the trivial case $c = 0$ will be excluded. The term "Julia set" will refer to the "filled-in Julia set" which contains all the bounded Fatou components in $\mathbb{C}$.

A theory of internal addresses for such Julia sets can be developed in much the same way as in parameter space; things are generally much easier. We give only those few results which will later be needed for the harvest. The role of hyperbolic components will be played by *precritical points:* points which map, after a finite nonnegative number of iteration steps, to the critical point.

Definition 7.1 (STEP) *Let P_c be postcritically finite. We define the* STEP *of a precritical point to be the number of iterations it takes to map it to the critical value.*

For example, $\textsc{Step}(0) = 1$ and, if c is periodic of period n, $\textsc{Step}(c) = n$.

If the critical point is periodic, there is a canonically defined coordinate system of internal rays in every Fatou component: they are images of radial lines under the Riemann map of the Fatou component taking the origin to the precritical point, rotated so that the dynamics on the critical Fatou component doubles angles and leaves angles invariant on all the other components. The first return map on periodic components acts then by angle doubling.

Definition 7.2 (Regular Arcs) *The* regular arc *$[z_1, z_2]$ between two points z_1 and z_2 in a Julia set of a postcritically finite parameter is an embedding of an interval into the Julia set connecting these two points, subject to the condition that it traverse Fatou components (if any) only along internal rays.*

Definition 7.3 (Hubbard Tree) *The Hubbard tree of a postcritically finite Julia set is the smallest collection of regular arcs which connect the entire critical orbit.*

It is well known and easy to see that Hubbard trees are forward invariant, that the critical point is one of its extremities, and that the critical point connects no more than two of its branches; in the periodic case, the Hubbard tree meets a Fatou component either not at all, on the internal ray at angle 0, or at the two internal rays at angles 0 and 1/2 (see the early sections in Douady and Hubbard [DH1] and Poirier [Po]). Often, this tree is considered an abstract topological tree with embedding class into the plane, rather than as subset of $\mathbb{C}$.

We will define visibility and internal addresses in Julia sets analogously to the definitions in parameter space, replacing PER by STEP. First we give a dynamical version of Lavaurs' Lemma 3.8. It is stronger because it does not require a particular order between z_1 and z_2.

Lemma 7.4 *Let P_c be postcritically finite. Let z_1 and z_2 be precritical points with the same* STEP*. Then there is an precritical point of lower* STEP *on the regular arc between z_1 and z_2.*

PROOF. Let $n = \text{STEP}(z_1) = \text{STEP}(z_2)$. If the lemma was false, then $P_c^{\circ(n-1)}$ would be a local homeomorphism on $[z_1, z_2]$ and $P_c^{\circ(n-1)}(z_1) = 0 = P_c^{\circ(n-1)}(z_2)$. This contradicts simple connectivity of the filled Julia set K_{P_c}. □

Definition 7.5 (Visibility and Internal Addresses in Julia Sets) *Let z be a precritical point in a postcritically finite Julia set.*

The point z will be called visible *from some point w in the Julia set if there is no precritical point of lower* STEP *than z on the interior of the arc $[w, z]$.*

Now consider all precritical points w on $[0, z]$ which have the property that z is visible from w and $\text{STEP}(w) < \text{STEP}(z)$. With respect to their order on the arc $[0, z]$, the STEP*s of these points form a finite strictly increasing sequence of integers, starting with $1 = \text{STEP}(0)$ and ending with $\text{STEP}(z)$. This sequence will be called the* internal address *of z.*

Definition 7.6 (Itinerary in the Julia set)
In the dynamical plane of any parameter c, the critical point and all the external rays landing there, or the closure of the critical value Fatou component together with all the rays landing at its boundary (according to whether the critical point is preperiodic or periodic), separate the complex plane into a collection of parts. We label the part containing the external ray at angle

0 by `0`*, the part containing the external ray at angle* $1/2$ *by* `1`*, and the critical point (respectively the closure of the critical value Fatou component) by* $\star$*. Any remaining parts do not get any label. For a point* z *in the Julia set which is contained in the three labelled parts, together with its entire forward orbit, we define its itinerary to be the sequence of labels corresponding to the sectors containing* $z, P_c(z), P_c(P_c(z)), \ldots$*.*

REMARK. The idea of the definition, in the case of a Misiurewicz point, is the following: we may choose an external argument θ of the critical value and define the itinerary of any point in the Julia set as the itinerary of any of its external angles with respect to the partition by $\theta/2$ and $(\theta+1)/2$ as in Definition 3.2. The given definition applies and yields the same result whenever this itinerary does not depend on the choice of θ or the external angle of the considered point. A similar statement holds when the critical point is periodic.

Lemma 7.7 *The set of points whose itinerary is defined is forward invariant, connected, and contains the critical orbit.*

PROOF. Forward invariance is built into the definition. That the itinerary is defined for the critical point follows from the fact that the Hubbard tree has at most two branches there which run into regions of defined itinerary. For connectedness, consider two points z_1 and z_2 for which the itinerary is defined. For two points in any of the three labelled parts, the regular arc connecting them is completely contained in these three parts. The forward image of the regular arc $[z_1, z_2]$ is either the regular arc connecting $P_c(z_1)$ and $P_c(z_2)$, or the union of the regular arcs $[P_c(z_1), c]$ and $[c, P_c(z_2)]$. The result now follows by induction. □

The following obvious observation permits to determine precritical points of lowest STEP between two points with given itinerary, in analogy to Algorithm 6.2 in parameter space. The symbol $\star$ should be considered a joker symbol which is not different from either `0` or `1`.

Proposition 7.8 *In the Julia set of a postcritically finite polynomial* P_c*, let two points* z, z' *be given such that their itineraries* `I` *and* `I'` *are defined and differ for the first time at the* k*-th entry. Then it takes exactly* $k-1$ *iterations for the arc* $[z, z']$ *to cover the origin as an interior point, so that* k *is the smallest* STEP *of precritical points on this arc. The itinerary of the unique point* z'' *of* STEP k *on this arc equals the first* k *entries in the itinerary of* z*, followed by the itinerary of* c*.* □

Whenever two or more external rays land at the same boundary point of a Fatou component with internal angle α, we call the region in the complex plane that they cut off from the component the α-*wake* of the component.

Proposition 7.9 *Let the critical point be periodic with period n. If two external rays land at the same point on the boundary of the critical value Fatou component, then the landing point has dyadic internal angle. Moreover, if k is the lowest* STEP *of precritical points in the $1/2$-wake, then the lowest* STEP *in any $a/2^q$-wake is $k+(q-1)n$ for odd a and $q \geq 1$.*

PROOF. The critical point is on the Hubbard tree and can thus only be in the wake at internal angle 0 or $1/2$ of any Fatou component. Any wake with internal angle α different from 0 and $1/2$ maps homeomorphically onto the wake at internal angle α or 2α of its image component, doubling the internal angle only when mapping the critical component. The distance between the angles of two external rays landing at the same boundary point of the Fatou component is doubled in every iteration step, so that the wake will eventually map over the critical point. This can only happen at internal angles 0 or $1/2$. $\square$

8 Dynamical and Parameter Planes

In this section, we will transfer several results about dynamical planes into parameter space. Such correspondences are a frequent phenomenon in complex dynamics; compare for example Tan Lei [TL] (from a different point of view).

Lemma 8.1 (Equal Insertion Algorithms) *Let $\mathcal{A}$ be a postcritically finite parameter and let $\mathcal{B}$ be a hyperbolic component of period n, say, which is visible from $\mathcal{A}$. In the dynamical plane of $\mathcal{A}$, let c be the critical value and let z be a precritical point of* STEP *n which is visible from c. If c is periodic, assume additionally that z is in the wake at internal angle 0 or $1/2$ of the critical value Fatou component. Then the lowest period of hyperbolic components on the interior of the arc $[\mathcal{A},\mathcal{B}]$ equals the lowest* STEP *of precritical points on the interior of $[c,z]$, unless $\mathcal{A}$ is a hyperbolic component, $\mathcal{A} \prec \mathcal{B}$, and $\mathcal{B}$ or the component of lowest period on the interior of $[\mathcal{A},\mathcal{B}]$ bifurcates immediately from $\mathcal{A}$.*

PROOF. If $\mathcal{B} \prec \mathcal{A}$, let $\mathtt{K}_{\mathcal{B}} := \mathtt{K}(\mathcal{B})$ and $\mathtt{K}_{\mathcal{A}} := \mathtt{K}^-(\mathcal{A})$; otherwise, let $\mathtt{K}_{\mathcal{B}} := \mathtt{K}^-(\mathcal{B})$ and $\mathtt{K}_{\mathcal{A}} := \mathtt{K}(\mathcal{A})$. The lowest period of components on the arc $[\mathcal{A},\mathcal{B}]$ is found by comparing the two sequences $\mathtt{K}_{\mathcal{A}}$ and $\mathtt{K}_{\mathcal{B}}$ and looking for the position s of the first difference (Proposition 5.4). We have $s > n$ because we assumed $\mathcal{B}$ visible from $\mathcal{A}$. The sequence $\mathtt{K}_{\mathcal{B}}$ has exact period $n = \mathrm{PER}(\mathcal{B})$ (by Corollary 5.5, it could fail to have this period only if $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B}$ was not primitive; but then either $\mathcal{B}$ is not visible or bifurcates immediately from $\mathcal{A}$). Write $\mathtt{K}_{\mathcal{B}} = \overline{\mathtt{k}_{\mathcal{B}}}$ where $\mathtt{k}_{\mathcal{B}}$ are the first n symbols of $\mathtt{K}_{\mathcal{B}}$ and the overbar denotes periodic repetition.

The itinerary of z is defined: since the point z is visible from c, the first $n-1$ forward images of the interior of the regular arc $[c,z]$ avoid the critical point; if some forward image of c lands on the critical point, then it is periodic and the extra condition says that the forward image of the arc leaves the critical Fatou component by internal angle 0 or 1/2 into a region of defined itinerary.

Proposition 7.8 says that the lowest STEP of precritical points on the interior of the arc $[c, z]$ is the position of the first difference between the itineraries of c and z (disregarding differences due to symbols $\star$). The itinerary of c is $\mathtt{K}_{\mathcal{A}}$ (except that, if $\mathcal{A}$ is a hyperbolic component of period k, it contains a $\star$ at positions $k, 2k, 3k, \ldots$), while the itinerary of z is $\mathtt{k}_{\mathcal{B}}\mathtt{K}_{\mathcal{A}}$ (again ignoring symbols $\star$ at position n and possibly at $n+k, n+2k, \ldots$): indeed, the first n entries in the itinerary of z agree with those of c because of visibility of z, and the itinerary of c is $\mathtt{K}_{\mathcal{A}}$ and coincides with $\overline{\mathtt{k}_{\mathcal{B}}}$ for at least n entries. After n iterations, z maps onto c and inherits its itinerary.

The sequences $\mathtt{K}_{\mathcal{A}}$ and $\overline{\mathtt{k}_{\mathcal{B}}}$ differ for the first time at the s-th position (with $s > n$), so the sequences $\mathtt{k}_{\mathcal{B}}\mathtt{K}_{\mathcal{A}}$ and $\mathtt{k}_{\mathcal{B}}\overline{\mathtt{k}_{\mathcal{B}}}$ have their first difference at position $n+s$. It follows that the first difference between the sequences $\mathtt{k}_{\mathcal{B}}\mathtt{K}_{\mathcal{A}}$ and $\mathtt{K}_{\mathcal{A}}$ occurs at position s. It follows that the lowest STEP of precritical points on $[c, z]$ is s, provided the itineraries of c and z do not have a $\star$ at this position. This is what remains to show.

If $\mathcal{A}$ is a Misiurewicz point, the only $\star$ occurs at position $n < s$, so this case presents no problem.

If $\mathcal{A}$ is a hyperbolic component with period k, there are symbols $\star$ at positions $n, n+k, n+2k, \ldots$ or $k, 2k, 3k, \ldots$, but only position $s > n$ is of concern. If there is no precritical point of STEP less than $n+k$ on the interior of $[c, z]$, there is none at all, because the itineraries are periodic of period n immediately or after k steps; but then we would have $z = c$. It remains the case that s is a multiple of k. Let $\mathcal{C}$ be the component of period s found on the arc $[\mathcal{A}, \mathcal{B}]$. Then the kneading sequences $\mathtt{K}_{\mathcal{A}}$ and $\mathtt{K}(\mathcal{C})$ (or $\mathtt{K}^{-}(\mathcal{C})$, if $\mathcal{A} \prec \mathcal{B}$) agree forever, so there can be no component in between. This means that $\mathcal{C}$ bifurcates directly from $\mathcal{A}$ if $\mathcal{A} \prec \mathcal{B}$ and is a contradiction otherwise. □

Proposition 8.2 (Equal Internal Addresses) *Let $\mathcal{A}$ be some hyperbolic component or Misiurewicz point. Its internal address equals the internal address of the critical value in the dynamical plane of (the center of) $\mathcal{A}$.*

PROOF. This proposition follows recursively from Lemma 8.1: we start using the main cardioid with PER $= 1$ for $\mathcal{B}$ and $z = 0$ with STEP $= 1$ and obtain a component $\mathcal{B}'$ and a precritical point z' the period and STEP of which agree. Using the lemma again with $\mathcal{B}'$ and z', we continue to reconstruct the internal addresses of $\mathcal{A}$ in parameter space and of c in the dynamical plane. □

REMARK. Although proved only in the postcritically finite case, this proposition holds for every quadratic polynomial, whether or not the internal address is finite. The given proof goes through when the Julia set is pathwise connected; otherwise, the regular arc has to be replaced by a combinatorial counterpart, much like the combinatorial arc in parameter space.

A different approach is as follows. Consider periodic points which have the property that two external rays landing at this periodic point separate the critical point from the critical value, and no other point on the same orbit separates this point from the critical value. Now we can define the internal address of the critical value in the Julia set similarly as before, replacing precritical points and their STEPs with such periodic points and their period. It turns out that we get the same internal address. This definition even works for connected Julia sets which are not pathwise connected. In fact, much more than equality of internal addresses is true: all the hyperbolic components on the combinatorial arc $[0, \mathcal{A}]$ to some hyperbolic component or Misiurewicz point $\mathcal{A}$ have their counterparts in periodic points which satisfy the property given above, and conversely. This has been found by Lavaurs and follows easily from Levin's Lemma 3.7. It can be rephrased in Thurston's language as "monotonicity of the periodic lamination". The reason we did not use this definition of internal addresses in Julia sets in the first place is that it applies only to the critical value, and there is no analog to the Correspondence Principle (see below) or to the subsequent lemma.

A result formally similar to Proposition 8.2 has been obtained by Lavaurs for his theory of addresses of veins as developed in exposés XX–XXII of [DH1].

Lemma 8.3 *Let θ_1 and θ_2 be the two external arguments of the root of a hyperbolic component with center c, or two external arguments of a Misiurewicz point c such that there is no other external argument of this point on the oriented interval $I = [\theta_1, \theta_2] \subset \mathbb{S}^1$. The lowest period of hyperbolic components with external arguments in I then equals the lowest* STEP *of precritical points in the dynamical plane of c in the subset of the Julia set cut out by the external rays θ_1 and θ_2 and corresponding to the interval I of external arguments.*

PROOF. If $\theta_1 > \theta_2$, then the main cardioid with period 1 and the critical point with STEP 1 trivially satisfy the statement, so we may assume $\theta_1 < \theta_2$. Set $\delta := \theta_2 - \theta_1$ and let n be the smallest positive integer such that $1/(2^n - 1) \leq \delta$.

Then there certainly is an external angle of period n in the interval I and hence there is a hyperbolic component of period n with external arguments in I; however, a hyperbolic component of lower period could not have both external arguments in I.

In the dynamical plane, the forward images of the considered interval have widths 2δ, 4δ, ... The smallest STEP of precritical points in this wake having value n means that, after $n-1$ iterates, the image of the region between these two rays contains the origin. But all the four inverse images of the two rays at angles θ_1 and θ_2 land at the critical point (or critical Fatou component) and their angles are not contained in any interval of length less than $1/2 + \delta/2$; this yields the necessary condition $2^{n-1}\delta > 1/2 + \delta/2$ or $1/(2^n - 1) < \delta$. On the other hand, whenever n satisfies this inequality, the corresponding forward image of the wake surrounds the origin. Hence the smallest PER and STEP in corresponding wakes are indeed equal. □

Proposition 8.4 (Weak Correspondence Principle)
Let $\mathcal{A}$ be a hyperbolic component of period $n > 1$. The lowest period of hyperbolic components in a p/q-subwake and the lowest Step *of precritical points in any $a/2^{q-1}$-wake of the critical value Fatou component in the dynamical plane of $\mathcal{A}$ are then equal.*

Proof. As usual, we assume that p and q are coprime and that a is odd. According to Proposition 2.7, the width of the p/q-subwake of $\mathcal{A}$, measured in terms of external angles, does not depend on p, and the lowest period of hyperbolic components in the subwake does not, either. In the dynamical plane, nothing depends on a, so that we can restrict to the cases $p = a = 1$ in order to simplify notation.

Let $\mathcal{A}'$ be the hyperbolic component bifurcating from $\mathcal{A}$ at internal angle $1/q$. According to Lemma 8.3, the lowest period of hyperbolic components in the wake of $\mathcal{A}'$ (which might be $\mathcal{A}'$ itself) equals the lowest Step of precritical points in the entire wake of the critical value Fatou component in the dynamical plane of $\mathcal{A}'$. We want to translate these statements about $\mathcal{A}'$ into statements about $\mathcal{A}$.

The component of lowest period in the $1/q$-subwake of $\mathcal{A}$ is the component of lowest period in the wake of $\mathcal{A}'$, so there is nothing to translate in parameter space. The angles bounding the wake of $\mathcal{A}'$ are the angles of the two supporting rays of the critical value Fatou component in the dynamical plane of $\mathcal{A}'$. In the dynamical plane of $\mathcal{A}$, the rays at these angles land at the critical value Fatou component at internal angles $1/(2^q-1)$ and $2/(2^q-1)$; as $\mathcal{A}$ bifurcates into $\mathcal{A}'$, their landing points coalesce. The lowest Steps of precritical points between these two external rays are the same before or after bifurcation, so we are home if we show that the precritical point of lowest Step between the internal rays at angles $1/(2^q-1)$ and $2/(2^q-1)$ actually occurs in the $1/2^{q-1}$-wake.

Between these two internal rays, there is no ray at angle $s/2^t$ for $t < q-1$; the angle $1/2^{q-1}$ is there, but no ray at angle $s/2^q$, either, as can be verified easily. Of course, there are lots of rays at angles $s/2^t$ for $t \geq q+1$. After $(q-1)n$ iterations, the $1/2^{q-1}$-wake maps to the 0-wake and certainly covers the origin, so it contains a precritical point of Step at most $(q-1)n+1$. According to Proposition 7.9, any wake at angle $s/2^t$ for $t \geq q+1$ maps after $(q-1)n$ iterations homeomorphically to the wake at angle $s/2^{t-(q-1)}$, which certainly cannot contain the origin. So the first precritical point indeed occurs in the $1/2^{q-1}$-wake. $\square$

Corollary 8.5 (Weak Translation Principle)
Let $\mathcal{A}$ be a hyperbolic component of period n and let s be the minimal period of hyperbolic components in its $1/2$-wake. The minimal period of hyperbolic components in any p/q-wake is then $s + (q-2)n$.

Proof. This is trivial for the main cardioid. For the others, it is an immediate consequence of Proposition 7.9 and the Weak Correspondence Principle 8.4. $\square$

Here we give conjecturally general "Correspondence" and "Translation Principles".

Conjecture 8.6 (Correspondence Principle)
For every hyperbolic component of period n, the tree of visible components in any p/q-sublimb in parameter space and the tree of visible precritical points of STEP *less than nq in any $a/2^{q-1}$-sublimb of the critical value Fatou component in its dynamical plane coincide, including the embedding into the plane.*

Conjecture 8.7 (Translation Principle)
The trees of visible hyperbolic components in sublimbs of denominators q_1 and q_2 of a hyperbolic component $\mathcal{A}$ coincide, including the embedding into the plane, when all periods of visible components in the q_1-sublimbs are increased by $(q_2 - q_1)\text{PER}(\mathcal{A})$.

One can formulate similar statements about the trees of visible components and precritical points for Misiurewicz points.

The significance of these principles is that they display a lot of symmetry, and they reduce the determination of trees of visible components, and thus the possible continuations of internal addresses, to the tree at internal angle 1/2 in the dynamical plane, where things generally are easiest.

It is easy to show that the Correspondence Principle implies the Translation Principle. In fact, it would be enough to know that the Correspondence principle holds in the 1/2-subwake for components of periods less than n, and for the corresponding precritical points. It is not hard to show that the tree of precritical points in the dynamical plane is a subtree of the tree of visible components in parameter space.

In Section 10, we will deal with "narrow components": hyperbolic components which contain no components of lower periods in their 1/2-wake. Their analysis is much simpler than the general case. We will prove the Translation Principle for those components; there is also a simple proof for the Correspondence Principle.

9 Internal Addresses

In this section, we are going to prove two fundamental results about internal addresses: angled internal addresses completely specify a unique hyperbolic component (if the address exists), and the existence of an internal address is independent of the angles.

We need the fact that combinatorial arcs in the Mandelbrot set branch off only at centers of hyperbolic components or at Misiurewicz points. This has been shown by Douady and Hubbard ([DH1], exposé XXII.3) after a long argument. A different proof follows from looking more generally at the space of abstract kneading sequences (see Definition 6.8). One can topologize it appropriately and show that its branch points are sequences which become eventually periodic.

This has been done by Penrose [Pe2], Theorem 4.2, and is also the content of an unpublished manuscript by Bandt. Using this, the result about branch points in the Mandelbrot set can be shown using Thurston's No Wandering Triangles theorem (see [T], Theorem II.5.2).

Here, we give a new, more direct proof which also uses the No Wandering Triangles Theorem.

Theorem 9.1 *For two postcritically finite parameters c_1 and c_2 in the Mandelbrot set, the intersection of the combinatorial arcs $[0, c_1]$ and $[0, c_2]$ is the combinatorial arc $[0, c_0]$ for some postcritically finite parameter c_0. More precisely, we either have $c_1 \prec c_2$ or $c_2 \prec c_1$, or c_0 is a Misiurewicz point the external angles of which separate 0, c_1 and c_2, or c_0 is the center of a hyperbolic component such that c_1 and c_2 are in two different of its subwakes.*

PROOF. If $c_1 \prec c_2$ or conversely, nothing is to prove, so we assume these two parameters are not comparable by "$\prec$". Let Γ be the combinatorial arc $[0, c_1]$ in the Mandelbrot set and let H_0 be a sequence of hyperbolic components on the combinatorial arc Γ which starts with 0 and is strictly monotonically increasing with respect to the order "$\prec$" and such that every component in this sequence is smaller than both c_1 and c_2. Moreover, assume that this sequence is maximal in the following sense: there is no component on Γ which is smaller than both c_1 and c_2 and which is a strict upper bound for the sequence H_0. Let H_1 be a similar strictly decreasing sequence of components on Γ which are smaller than c_1 but not smaller than c_2; let it start with c_1. Assume again maximality: every component on Γ which is a strict lower bound of H_1 should be smaller than c_2. The union of all the combinatorial arcs within H_0 and within H_1 then contains all hyperbolic components on Γ.

The external angles of the components in H_0 form one strictly increasing and one strictly decreasing sequence of rational numbers; denote their limits by α_0 and β_0, respectively; they satisfy $\alpha_0 < \beta_0$. Similarly, let $\alpha_1 < \beta_1$ be the two limits of the external angles of components in H_1. The objective is to show that the angles are rational and belong to a branch point c_0 in the Mandelbrot set. It is not hard to show that, whenever one of these angles is periodic, then all of them are, and the rays at angles α_0 and β_0 land at the root of a hyperbolic component $\mathcal{A}_0$ which is the maximum in H_0, while the rays at angles α_1 and β_1 land at the root of a component $\mathcal{A}_1$ which is the minimum in H_1 and bifurcates directly from $\mathcal{A}_0$. The sought point c_0 is then the center of $\mathcal{A}_0$. We will assume in the sequel that all the four angles are strictly preperiodic or irrational.

In the dynamical plane of c_1, let γ be the regular arc $[0, c_1]$. For every hyperbolic component in H_0, its two external arguments determine two rays in the dynamical plane of c_1 which land at a repelling periodic point on γ. Let S_0 be the sequence of these points, which is mononic along the arc γ, and let z_0 be their limit point. Since the Julia set of c_1 is locally connected, the external rays at angles α_0 and β_0 land at z_0. Similarly, construct a sequence S_1 of periodic points on γ from the sequence H_1 of components and let z_1 be their limit. It will be the landing point of the rays at angles α_1 and β_1. The first step is to show that $z_0 = z_1$.

Since there is no hyperbolic component on Γ strictly between the sequences H_0 and H_1, the limits of the kneading sequences of the components in H_0 and "just before" the components in H_1 will coincide. This implies $\mathtt{K}^-(\beta_0) = \mathtt{K}^+(\alpha_0) = \mathtt{K}^-(\alpha_1) = \mathtt{K}^+(\beta_1)$.

Let $\alpha < \beta$ be the two external arguments of a component in H_1. In the corresponding dynamical plane, no forward image of α or β ever maps into the interval (α, β): this is the statement that the critical value is an extremity of the Hubbard tree, translated into external angles. If z is the corresponding periodic point in S_1, the external angles α and β land at z and separate the Julia set into two parts. The entire forward orbit of z will stay on the closure of the side containing the origin. Since the critical value c_1 is on the other side, this orbit will avoid the interior of the arc $[z, c_1]$ and also its inverse image $[z', z'']$ bounded by the two inverse images of z. The closer z is to z_1, the larger will the avoided arc be, and in the limit it follows that the orbit of z_1, which need not be finite, avoids the arc $[z_1', z_1'']$ between its two inverse images. It will not even map onto the boundary of this arc because z_1 is not periodic. It is even easier to see that no point in S_0, nor the limit point z_0, ever maps onto the closure of the arc $[z_0', z_0'']$ between the two inverse images of z_0, which includes the arc $[z_1', z_1'']$.

Since both z_0 and z_1 are on the Hubbard tree, the itinerary of every point on $[z_0, z_1]$ is defined by Lemma 7.7. For any point z on this arc close to z_0, the itinerary will coincide with $\mathtt{K}^+(\alpha_0)$ for the more iterations the closer z is to z_0; similarly, for points on the arc arbitrarily close to z_1, the itinerary will coincide with $\mathtt{K}^-(\alpha_1)$ for arbitrarily many iterations. Since $\mathtt{K}^+(\alpha_0) = \mathtt{K}^-(\alpha_1)$, the itinerary of all the points on the arc is the same for all times and the arc will never cover the critical point. In particular, there is no Fatou component on the arc. Moreover, all the forward images of the arc $[z_0, z_1]$ avoid the arc $[z_1', z_1'']$ containing the critical point. If z_0 was different from z_1, this would contradict expansivity on the Julia set (compare for example Milnor [M2], Section 14; recall that the dynamics is postcritically finite).

We conclude that $z_0 = z_1$. All the rays $\alpha_0, \beta_0, \alpha_1, \beta_1$ land at this point. It may happen that $\alpha_0 = \alpha_1$ or $\beta_0 = \beta_1$, but not both at the same time because in parameter space we assumed that the parameter c_2 and its external angles were either between α_0 and α_1 or between β_0 and β_1. Hence at least three different rays land at z_0, so all their angles are rational by Thurston's No Wandering Triangles Theorem (see [T], Theorem II.5.2). Since we assumed the angles not to be periodic, we can now assume that they are strictly preperiodic.

We have shown that, in the dynamical plane of c_1, there exists a (pre)periodic point z at which at least three external rays land. Their angles include $\alpha_0, \alpha_1, \beta_0, \beta_1$, which separate the external angles of c_1 and c_2. Had we chosen a different postcritically finite parameter c_1' on $[0, c_1]$ closer to 0 but still not smaller than c_2, the same result would follow with the same angles α_i and β_i. In particular, we can suppose c_1 to have external angles arbitrarily close to α_1 and β_1.

Among all the rays landing at z in the dynamical plane of c_1, let α and β be the ones closest to the external arguments of c_1 on both sides. All these external rays landing at z will continue to land at a common landing point for every parameter c which is not separated from c_1 by the collection of rays at any of the external angles of z or their forward orbits. By continuity, the same is true for the parameter c_0 at the endpoints of the rays at the preperiodic angles α and β. This point is a Misiurewicz point at which the external rays at angles $\alpha_0, \alpha_1, \beta_0$ and β_1 land. □

REMARK. Douady and Hubbard had proved this result in order to show that local connectivity of the Mandelbrot set implies that hyperbolicity is dense among quadratic polynomials. The easy argument is in exposé XXII.4 in [DH1].

Theorem 9.2 (Completeness of Internal Addresses) *Angled internal addresses describe hyperbolic components and Misiurewicz points completely, that is, no two of them share the same angled internal address.*

PROOF. We consider the case of hyperbolic components first. Assume two of them shared the same angled internal address and call them $\mathcal{A}$ and $\mathcal{B}$. Consider the combinatorial arcs $[0, \mathcal{A}]$ and $[0, \mathcal{B}]$ to their respective centers. According to the previous theorem, their intersection is the combinatorial arc $[0, \mathcal{C}]$ for some postcritically finite parameter $\mathcal{C}$ which is either the center of $\mathcal{A}$, of $\mathcal{B}$, of a different hyperbolic component, or it is a Misiurewicz point. We will exclude these cases in order.

In the first two cases, one of the two arcs is a subset of the other, so Lavaurs' Lemma 3.8 would give a unique hyperbolic component of lowest period on the combinatorial arc $[\mathcal{A}, \mathcal{B}]$ which would change the kneading sequences and hence the internal addresses of the components.

If the two combinatorial arcs separated at the center of some hyperbolic component, they would leave by different internal angles which would appear in the angled internal address by Lemma 6.4.

The only serious case is that of a Misiurewicz point $\mathcal{C}$. If two hyperbolic components had the same internal address, they would also have the same long internal address. This will now be excluded.

The external rays landing at $\mathcal{C}$ separate the complex plane into some finite number of regions which we will call the *wakes* of $\mathcal{C}$; one of them will be the *main wake* containing the origin. The combinatorial arc $[0, \mathcal{A}]$ can be written as the union of $[0, \mathcal{C}]$ in the main wake of $\mathcal{C}$ and $[\mathcal{C}, \mathcal{A}]$ outside the main wake, and similarly for $\mathcal{B}$. By assumption, the intersection of $[\mathcal{C}, \mathcal{A}]$ and $[\mathcal{C}, \mathcal{B}]$ is just the point $\mathcal{C}$.

Let $\mathcal{C}^0_{\mathcal{A}}$ be the hyperbolic component of lowest period in the wake containing $\mathcal{A}$. Consider the sequence $\mathcal{C}^1_{\mathcal{A}}$, $\mathcal{C}^2_{\mathcal{A}}$, ...of hyperbolic components such that every $\mathcal{C}^{n+1}_{\mathcal{A}}$ is the component of lowest period on the arc $[\mathcal{C}, \mathcal{C}^n_{\mathcal{A}}]$. From some index n_0 on, all these components will be on the combinatorial arc $[\mathcal{C}, \mathcal{A}]$.

Now we do the analogous construction in the dynamical plane: the rays at the same angles will land at the critical value and separate the dynamical plane into the same number of wakes.

Consider the wake containing the external arguments of $\mathcal{A}$ and construct first the precritical point $z^0_{\mathcal{A}}$ of lowest STEP in this wake and then a sequence of precritical points such that every $z^{n+1}_{\mathcal{A}}$ is the precritical point of lowest STEP on $[c, z^n_{\mathcal{A}}]$. By Lemma 8.3, $\mathrm{PER}(\mathcal{C}^0_{\mathcal{A}}) = \mathrm{STEP}(z^0_{\mathcal{A}})$, and now Lemma 8.1 shows recursively that for all n, $\mathrm{PER}(\mathcal{C}^n_{\mathcal{A}}) = \mathrm{STEP}(z^n_{\mathcal{A}})$.

We do the same construction in the wakes in parameter space and dynamical plane corresponding to $\mathcal{B}$. Since the long internal addresses of $\mathcal{A}$ and $\mathcal{B}$ coincide, the constructed sequences above will be equal after a finite number of initial steps (which might be different for $\mathcal{A}$ and $\mathcal{B}$) and we obtain a sequence of visible precritical points of equal STEPs in two different wakes of the critical value in the dynamical plane of $\mathcal{C}$. This contradicts Lemma 7.4 and proves the theorem for hyperbolic components.

If $\mathcal{A}$ and $\mathcal{B}$ were Misiurewicz points with identical angled internal address, their angled long internal address would also be the same, so one could pick two different hyperbolic components with equal angled internal addresses on the combinatorial arcs leading to the Misiurewicz points. □

REMARK. A *combinatorial class* of the Mandelbrot set is a subset which is not separated by rational external rays. As pointed out in the introduction, Theorem 9.1 implies that local connectivity of the Mandelbrot is equivalent to the statement that combinatorial classes are hyperbolic components (with part of their boundary) or single points. Theorem 9.2 can be extended to say that internal addresses describe combinatorial classes uniquely, using local connectivity of the Mandelbrot set at Misiurewicz points or [TL].

The following result says that the internal address says everything about the denominators but nothing about the numerators of its possible angles.

Theorem 9.3 (Independence of Angles) *From an internal address, the denominators q_i of the angled internal address can be determined uniquely. If the internal address is at all realized, then for every choice of numerators p_i coprime to q_i, the corresponding angled internal address is realized.*

REMARK. In other words, the tree structure formed by all the hyperbolic components, visible or not, in two limbs at angles p/q and p'/q of a hyperbolic component $\mathcal{A}$ is the same. These trees have, however, a different embedding into the complex plane.

In order to prove the theorem, one can start with a hyperbolic component and explicitly construct a component with different numerators in the angles of its angled internal address. There are several ways to do this: one can use Thurston's theorem [DH3] or its offspring, Poirier's Realization Theorem [Po] and the Spider Theorem [HS] (compare also Bielefeld, Fisher and Hubbard [BFH] for preperiodic polynomials). Another way is to construct the polynomial with "cut and paste" techniques around "local α-fixed points" and use quasiconformal surgery similarly as in [BD]. Here is a combinatorial proof.

PROOF OF THEOREM 9.3. The denominators q_i in the angles can be reconstructed from the internal address itself, so they do not carry any information. To prove that the existence is independent of the numerators p_i, we will give an algorithm which constructs from the internal address of some hyperbolic component $\mathcal{A}$ and from the width δ of its wake the tree of all visible hyperbolic components and the widths of their wakes. This algorithm will not make use of the numerators.

We need the following observation: If two external rays in parameter space land at the same point, the difference between their angles alone permits to determine the number of external rays of any given period between them. This is because such rays have to come in pairs in order to land at the same root of a hyperbolic component.

Let some q be given; we will now construct the tree of the visible components in some p/q-wake of $\mathcal{A}$. Let $\mathcal{B}$ be the hyperbolic component bifurcating from $\mathcal{A}$ at internal angle p/q. By Proposition 2.7, the width of the wake of $\mathcal{B}$ depends only on the width δ of the wake of $\mathcal{A}$ and on q, so we know the number of periodic rays of every period in that wake. We start constructing the tree by the component of minimal period and proceed recursively. Suppose we know all components of period up to k, including the widths of their wakes. Then we can tell how many rays of period $k+1$ are in each of these wakes, and we know their total number in the wake of $\mathcal{B}$. Those rays in the wake of $\mathcal{B}$ but not in any of the wakes of components of period up to k will land at visible components of period $k+1$; since internal addresses describe components uniquely (Theorem 9.2), there is at most one such component. If there is one, we can find which of the components of periods up to k are enclosed by the new component of order $k+1$ by determining their long internal addresses, using Algorithm 6.2. This way, we know where to insert the new component into the tree of the previously constructed components (but we do not know the embedding of the tree into the complex plane). In order to keep the induction going, we need to know the width of the new component. This is easy to tell because we know the number of rays of period $k+1$ it encloses. $\square$

The correspondence between trees can in fact be extended to a homeomorphism between the entire limbs, which again does not preserve the embedding into the plane and thus does not extend to a neighbourhood. Once the combinatorial Theorem 9.3 is proved, the homeomorphisms follow from the fact they obviously should map entire tuned copies of the Mandelbrot to other tuned copies, preserving the embedding, so the homeomorphism between them can be defined using the tuning map or quasiconformal surgery (as in Branner and Douady [BD]). The remaining points are non renormalizable, so Yoccoz' "points are points" result applies (see Hubbard [HY], Theorem III) and allows to piece the homeomorphism together, similarly as in the construction of J. Kahn as described in Douady [D2]. It is interesting to compare these homeomorphisms to the ones obtained by quasiconformal surgery, for example as in the recent work of Branner and Fagella: the latter ones preserve the embedding into the plane but change periods of hyperbolic components.

This result permits to tell exactly how often some (pre)periodic kneading sequence is realized by rational angles provided it is realized at all: convert it into an internal address and find the denominators in the angled internal address. The kneading sequence is realized for exactly as many hyperbolic components or Misiurewicz point as there are possible selections of numerators coprime to the denominators. This provides another proof that Misiurewicz points can have only finitely many angles different from 1/2. However, this does not allow to tell whether or not the kneading sequence is realized.

10 Narrow Hyperbolic Components

We call a hyperbolic component *narrow* if it contains no component of equal or lesser period in its wake. The angles of the two external rays landing at its root then differ by $1/(2^n - 1)$, where n is the period of the component, so the wake of the component is as narrow as possible for components of that period.

For the proof that analytic continuation acts transitively on the set of periodic points of exact period n, we will need the existence of hyperbolic components with sufficiently many kneading sequences. It is far from true that most periodic symbolc sequences are realized as kneading sequences of hyperbolic components. There are 2^{n-2} abstract kneading sequences of period n and equally many abstract internal addresses, while there are 2^{n-1} hyperbolic components of that period (to be precise, one needs to subtract sequences and components of periods strictly dividing n). This does not imply that every kneading sequence is realized exactly by two hyperbolic components. In the preceding section, we have shown that existing kneading sequences tend to be realized quite often, which decreases the ratio of existing sequences accordingly. We do not know whether there exists a positive lower bound on this ratio independent of n.

Many of the questions which are of interest to us are easier to show for narrow hyperbolic components; in particular, we will be able to establish the existence of sufficiently many kneading sequences using only narrow components. This is what this section is about.

It is not too hard to show that more than half of the hyperbolic components are narrow for every period n. The following lemma says that narrow components are "usually" primitive.

Lemma 10.1 *A narrow component is either primitive or it bifurcates directly from the main cardioid. In the latter case, and only then, is its kneading sequence* $\overline{\texttt{11}\ldots\texttt{10}}$.

Proof. Consider a component $\mathcal{A}$ of period qn which bifurcates immediately from a component $\mathcal{B}$ of period n. If the width of the wake of $\mathcal{B}$ is $b/(2^n - 1)$, then the width of $\mathcal{A}$ is, by Proposition 2.7, equal to

$$\frac{(2^n-1)^2}{2^{qn}-1} \cdot \frac{b}{2^n-1} = \frac{b(2^n-1)}{2^{qn}-1} \;.$$

The component $\mathcal{A}$ is narrow whenever the width of its wake is $1/(2^{qn}-1)$, which forces $b = n = 1$. On the other hand, this condition is always satisfied for direct bifurcations from the main cardioid. The corresponding internal addresses are $1 \to n$ and yield the given kneading sequence. □

Theorem 10.2 *For every narrow hyperbolic component $\mathcal{A}$ of period n, there exist visible hyperbolic components of every period greater than n. More precisely, every sublimb of internal angle p/q contains exactly n visible hyperbolic components: one of every period $(q-1)n+1 \dots qn$.*

REMARK. This translates into the following statement about internal addresses: *If $1 \to n_2 \to \dots \to n_k$ is the internal address of a narrow hyperbolic component, then the abstract internal address $1 \to n_2 \to \dots \to n_k \to n_{k+1}$ is realized for any $n_{k+1} > n_k$. If q is the smallest integer not less than n_{k+1}/n_k, then every sublimb of internal angle p/q contains such a component.*

We will prove a much stronger result about kneading sequences: *If $\mathtt{K}$ is a sequence of n symbols $\mathtt{0}$ or $\mathtt{1}$ such that the periodic sequence $\overline{\mathtt{K}}$ has exact period n and belongs to a narrow hyperbolic component of period n, then every abstract kneading sequence $\mathtt{K}'$ is realized which consists of a finite number $q-1$ of blocks $\mathtt{K}$, followed by an arbitrary block of at most n symbols and then repeated periodically. A corresponding component exists in every p/q-sublimb.* (Note that the requirement that $\mathtt{K}'$ be an abstract kneading sequence gives a condition on the last entry in the arbitrary block at the end.)

PROOF. First we show that every subwake of $\mathcal{A}$ of denominator q contains exactly 2^k external rays with angles $a/(2^{(q-1)n+k}-1)$ for $1 \le k \le n$, including the two rays bounding the wake. In fact, Proposition 2.7 says that the width of the wake is $(2^n-1)/(2^{qn}-1)$, so the number of rays one expects by comparing widths of wakes is

$$\frac{(2^n-1)(2^{(q-1)n+k}-1)}{2^{qn}-1} =: 2^k + \alpha,$$

where an easy calculation shows that $-1 \le \alpha < 1$ and that $\alpha = -1$ can occur only for $k = n$. The actual number of rays can differ from this expected value by no more than one and is even, hence equal to 2^k. Moreover, no such ray of angle $a/(2^{(q-1)n+k}-1)$ can have period smaller than $(q-1)n+k$ because it would land at a hyperbolic component of some period dividing $(q-1)n+k$ — but in the considered wake there would not be room enough to contain a second ray of equal period.

This shows that, for any $k \le n$, the number of hyperbolic components of period $m = (q-1)n+k$ in any subwake of $\mathcal{A}$ of denominator q equals 2^{k-1}. This is exactly the number of abstract kneading sequences of period m which consist of $q-1$ blocks $\mathtt{K}$, followed by k arbitrary symbols and then repeated periodically, subject to the condition that it be an abstract kneading sequence. To show that all of them actually occur, it suffices to exclude that any two such hyperbolic components in any one subwake have the same kneading sequence. If they did, the combinatorial arcs leading to them would have to separate within this subwake at some other hyperbolic component $\mathcal{B}$ of period at least $(q-1)n+1$ and leave it with an internal angle of some denominator $q' \geq 3$. The hyperbolic component of least period in the $1/2$-subwake of $\mathcal{B}$ has period at least $(q-1)n+1$, so the Weak Translation Principle 8.5 entails that all the components in the q'-subwake have periods at least $2(q-1)n+2 > m$, a contradiction. □

We will now investigate the tree of hyperbolic components which are visible from a given hyperbolic component $\mathcal{A}$ in the sublimb of internal angle p/q. We will only be interested in the combinatorics of this tree, disregarding the embedding into the plane. In other words, this tree represents the finite collection of visible hyperbolic components in this sublimb, including the partial ordering with respect to "$\prec$" but ignoring the imbedding in the plane and the information which components are immediate bifurcations from others. If $\mathcal{A}$ is a narrow hyperbolic component of period n, then this tree contains exactly n components and is independent of p, according to Theorems 10.2 and 9.3. In the next proposition, we will show that it does not even depend on q when an appropriate multiple of n is added to all the periods on this tree, as illustrated in Figure 4. We believe that this result is in fact valid for every hyperbolic component, whether or not it is narrow (Conjecture 8.7).

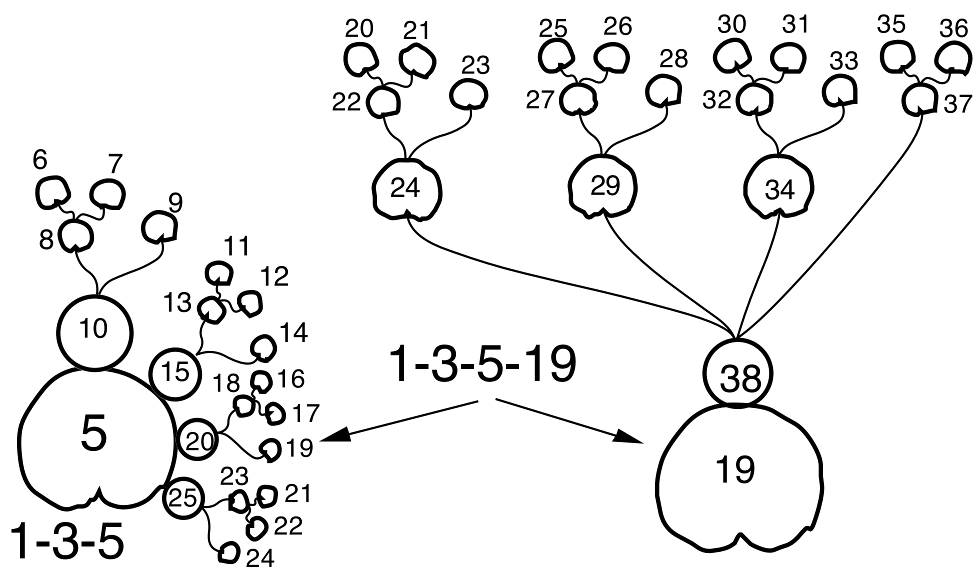

Figure 4: The Translation Principle 10.3, and how trees grow in Proposition 10.4.

Proposition 10.3 (Translation Principle)
The trees of visible hyperbolic components in sublimbs of denominators q_1 and q_2 of a narrow hyperbolic component $\mathcal{A}$ coincide when all periods of visible components in the q_1-sublimb are increased by $(q_2-q_1)\mathrm{P{\small ER}}(\mathcal{A})$.

Proof. Theorem 10.2 establishes the existence of visible hyperbolic components of periods $(q-1)n+1\ldots qn$ in every p/q-sublimb. For every visible hyperbolic component $\mathcal{B}$ of period

$(q-1)n+k$ with $0 \le k \le n-1$, let $(q-1)n+m$ be the lowest period of hyperbolic components on the combinatorial arc $[\mathcal{A},\mathcal{B}]$, excluding its ends. To prove the Translation Principle, it suffices to show that m does not depend on q. We do not have to consider the case $k = n$ because it occurs exactly for the components bifurcating directly from $\mathcal{A}$.

The lowest period on the arc can be found, using Proposition 5.4, by comparing the kneading sequence $\mathtt{K}(\mathcal{A})$ with the kneading sequence just before $\mathcal{B}$ which consists of the first $(q-1)n+k$ entries of $\mathtt{K}(\mathcal{A})$, continued periodically. Since the bifurcation component of period qn certainly exists, we know that the first difference occurs between $(q-1)n$ and qn. The result is now immediate. $\square$

Remark. In fact, the Translation Principle also preserves the embedding of the tree into the plane, but we do not need this result here.

Now we want to describe how the trees of visible components "grow" along the internal address.

Proposition 10.4 (Growing of Trees) *Let $\mathcal{A} \prec \mathcal{B}$ be narrow hyperbolic components such that $\mathcal{B}$ is visible from $\mathcal{A}$. The tree $T_{\mathcal{B}}$ of visible hyperbolic components in the $1/2$-sublimb of $\mathcal{B}$ consists of one component each of periods $\mathrm{Per}(\mathcal{B})+k$ for $k=1,\dots,\mathrm{Per}(\mathcal{B})$. The structure of the tree made up of these components is inherited from the components which are visible from $\mathcal{A}$ as follows: take the trees of visible components from $\mathcal{A}$ in all the $1/q$-sublimbs which have periods less than $\mathrm{Per}(\mathcal{A})+\mathrm{Per}(\mathcal{B})$, and add $\mathrm{Per}(\mathcal{B})-\mathrm{Per}(\mathcal{A})$ to all the periods. This will be a finite collection of trees, to which a common root of period $2\,\mathrm{Per}(\mathcal{B})$ is added to form a single new tree. This gives the tree of visible components in the $1/2$-sublimb of $\mathcal{B}$.*

Remark. This proposition is illustrated in Figure 4. The Translation Principle 10.3 also gives the trees of visible components in all the other sublimbs of $\mathcal{B}$. The reason we took only the $1/q$-sublimbs of $\mathcal{A}$ is convenience: all sublimbs with equal q have the same tree, as shown in Theorem 9.3, and we need to use exactly one of them.

Proof. The tree structure is completely specified by the partial ordering "$\prec$" on the components. First, the component of period $2\,\mathrm{Per}(\mathcal{B})$ is an immediate bifurcation from $\mathcal{B}$ so that it is smaller with respect to "$\prec$" than all the other components and is thus the root of the tree. In the proof of the Translation Principle we have seen that, for a narrow hyperbolic component of period n, the tree of visible components of periods $n+1,\dots,n+s$ depends only on the first s entries in the kneading sequence of the component, for any $s \ge 1$ (not necessarily less than n). Since the kneading sequence of $\mathcal{B}$ coindices with that of $\mathcal{A}$ for $\mathrm{Per}(\mathcal{B})-1$ entries, the proposition follows. $\square$

Definition 10.5 (Purely Narrow Hyperbolic Components) *A narrow hyperbolic component $\mathcal{A}$ is called* purely narrow *if all hyperbolic components in its internal address are narrow.*

That means, if the internal address of $\mathcal{A}$ is $1 \to n_2 \to \ldots \to n_k$, then all hyperbolic components with internal addresses $1 \to n_2$, $1 \to n_2 \to n_3$, $\ldots$, $1 \to n_2 \to \ldots \to n_{k-1}$ should be narrow.

As mentioned before, it is easier to control the behaviour of narrow components. For a purely narrow hyperbolic component, we have this control at every step. This permits us to classify all purely narrow hyperbolic components by their kneading sequences.

Theorem 10.6 (Classification of Purely Narrow Components) *Suppose an abstract kneading sequence has the property that its k-th entry is* `0` *for every $k > 1$ occurring in the corresponding internal address. Then this abstract kneading sequence is realized as the kneading sequence of a purely narrow hyperbolic component, and every purely narrow hyperbolic component has a kneading sequence satisfying this property.*

Before proving this statement, we give examples of what it means.

Example 10.7 *The abstract internal address $1 \to 3 \to 8 \to 21$ corresponds to the abstract kneading sequence $\overline{\texttt{110 110 10 110 110 10 110 10}}$. All the entries at positions 3, 8 and 21 in this abstract kneading sequence are* `0`*, so this abstract internal address is realized and belongs to a purely narrow hyperbolic component.*

The abstract internal address $1 \to 3 \to 8 \to 22$ corresponds to the abstract kneading sequence $\overline{\texttt{110 110 10 110 110 10 110 111}}$; its entries at positions 3, 8, and 22 are `0`*,* `0`*, and* `1`*, respectively, so it does not belong to a purely narrow hyperbolic component. Since the internal address $1 \to 3 \to 8$ belongs to a purely narrow hyperbolic component, all continuations of this internal address are realized by Theorem 10.2, so the internal address $1 \to 3 \to 8 \to 22$ is realized for a hyperbolic component which is not narrow.*

Proof of Theorem 10.6. We will argue recursively and show the following statement: *If $\mathcal{B}$ is a purely narrow hyperbolic component, then a hyperbolic component $\mathcal{C}$ which is visible from $\mathcal{B}$ is narrow if and only if the entry at position $\mathrm{Per}(\mathcal{C})$ in the kneading sequence of $\mathcal{B}$ is* `1` (so that the corresponding entry in the kneading sequence of $\mathcal{C}$ will be `0`).

The main component is purely narrow, its visible components are all its immediate bifurcations and are narrow. Their kneading sequences are of the form $\overline{\texttt{11}\ldots\texttt{10}}$ and have the mentioned property. Now let $\mathcal{A}$ be a purely narrow hyperbolic component for which the statement holds and let $\mathcal{B}$ be a narrow hyperbolic component which is visible from $\mathcal{A}$; we will prove the statement for $\mathcal{B}$. By the Translation Principle 10.3, it suffices to consider the 1/2-sublimb of $\mathcal{B}$.

Proposition 10.4 implies that, for $1 \le k \le \mathrm{Per}(\mathcal{B}) - 1$, a component of period $\mathrm{Per}(\mathcal{B}) + k$ which is visible from $\mathcal{B}$ is narrow if and only if the components of period $\mathrm{Per}(\mathcal{A}) + k$ which are visible from $\mathcal{A}$ are narrow. For these components, the result follows by induction because the first $\mathrm{Per}(\mathcal{B}) - 1$ entries in the kneading sequences of $\mathcal{A}$ and $\mathcal{B}$ coincide.

Finally, the component of period $2\,\textsc{Per}(\mathcal{B})$ is an immediate bifurcation and not narrow. Since $\mathcal{B}$ is purely narrow, the entry at position $\textsc{Per}(\mathcal{B})$ in its kneading sequence is $\mathtt{0}$. $\square$

Now we are at last in a position to prove the result which was needed in Section 4 to establish transitive action of analytic continuation on the set of periodic points. Recall that we compare symbolic sequences by lexicographic order induced from $\mathtt{1} > \mathtt{0}$.

Corollary 10.8 *Let S be a periodic sequence of symbols $\mathtt{0}$ and $\mathtt{1}$ with exact period $n \geq 2$. Then its largest shift is realized as the kneading sequence $\mathtt{K}^+(\theta)$ or $\mathtt{K}^-(\theta)$ of some periodic angle θ of period n. If the sequence contains more than one symbol $\mathtt{0}$ within its period, then the external ray at angle θ lands at a primitive hyperbolic component.*

Proof. Let $\mathtt{K}_0$ be the largest shift of S; it starts with $\mathtt{1}$. First we need to check that $\mathtt{K}_0$ is an abstract kneading sequence. Suppose it was not. When we try to find its internal address, we get $1 \to n_2 \to n_3 \to \ldots \to n_k \to n_{k+1} \ldots$ with $n_k < n < n_{k+1}$ because of the assumption that it was not an abstract kneading sequence. Let B be the sequence consisting of the first n_k symbols and A the sequence consisting of less than n_k symbols such that $\mathtt{K}_0 = \overline{BB\ldots BA}$; then A is an initial subsequence of B so that $B = AC$ (where AC means concatenation of finite sequences). We have $\mathtt{K}_0 = \overline{AC\ldots ACA}$; consider $\mathtt{K}_1 = \overline{AAC\ldots AC}$ and $\mathtt{K}_2 = \overline{CA\ldots CAA}$ which are both shifts of $\mathtt{K}_0$ and different from $\mathtt{K}_0$ (if they were equal to $\mathtt{K}_0$, then the exact period of S would strictly divide n). We have either $AC < CA$ or $AC > CA$ and accordingly either $\mathtt{K}_1 < \mathtt{K}_0 < \mathtt{K}_2$ or $\mathtt{K}_2 < \mathtt{K}_0 < \mathtt{K}_1$, contradicting the choice of $\mathtt{K}_0$.

So $\mathtt{K}_0$ is an abstract kneading sequence; let $1 \to n_2 \to \ldots \to n_k$ be the corresponding abstract internal address. It satisfies the conditions of Theorem 10.6: if it did not, some n_i-th entry of $\mathtt{K}_0$ would be $\mathtt{1}$, but then the shift of $\mathtt{K}_0$ by n_{i-1} symbols would be greater than $\mathtt{K}_0$. Hence there exists a hyperbolic component with this kneading sequence; if it contains at least two symbols $\mathtt{0}$ within its period, in addition to the $\mathtt{1}$ it starts with, then Lemma 10.1 says that the component is primitive. $\square$

Remark. The converse is also true: The kneading sequence of a purely narrow hyperbolic component is the largest of all of its shifts (other shifts may be realized as well, but not for purely narrow components).

11 An Example

Here we give an example of permutations at work for period 5. Fix a basepoint $c \in X$. Suppose we want to find a loop to turn the periodic point with itinerary $\overline{\mathtt{00001}}$ into the periodic point with itinerary $\overline{\mathtt{00101}}$.

The strategy is to find loops along which analytic continuation turns symbols $\mathtt{0}$ into $\mathtt{1}$, so that the periodic point with itinerary $\overline{\mathtt{00001}}$ lands on the orbit with itinerary $\overline{\mathtt{11110}}$. We can then cyclically permute this orbit arbitrarily, and then connect it back to the orbit $\overline{\mathtt{00101}}$.

The largest shift of $\overline{\texttt{00001}}$ is $\overline{\texttt{10000}}$ which corresponds, according to Corollary 10.8, to a purely narrow hyperbolic component; its internal address is $1 \to 2 \to 3 \to 4 \to 5$. A loop around the root of this component (of which there exists only one, as the angles of bifurcation are all 1/2) turns the considered periodic point into the one with itinerary $\overline{\texttt{00011}}$. This time, the largest shift of the sequence is $\overline{\texttt{11000}}$ which corresponds to the internal address $1 \to 3 \to 4 \to 5$ (such components exist in the two wakes of angles 1/3 and 2/3 of the main cardioid). A loop around the root of such a component turns our point into the one with kneading sequence $\overline{\texttt{00111}}$. The next time around, we use a component with kneading sequence $\overline{\texttt{11100}}$, corresponding to internal address $1 \to 4 \to 5$, to turn the point into $\overline{\texttt{01111}}$, which is the unique standard orbit. A finite number of turns around any component with internal address $1 \to 5$, bifurcating from the main cardioid, will induce any desired cyclic permutation of the standard orbit. We will determine later which one we need.

Analogous considerations, starting with the periodic point of itinerary $\overline{\texttt{00101}}$, show that this point is permuted onto the standard orbit by loops around roots of components with kneading sequence $\overline{\texttt{10100}}$ (internal address $1 \to 2 \to 5$, yielding $\overline{\texttt{01101}}$) and with kneading sequence $\overline{\texttt{11010}}$ (internal address $1 \to 3 \to 5$, yielding $\overline{\texttt{11101}}$).

We thus see that, after we have turned our initial periodic orbit with itinerary $\overline{\texttt{00001}}$ into $\overline{\texttt{01111}}$, we need to permute it cyclically by two steps to obtain $\overline{\texttt{11101}}$ (for example by a double loop counterclockwise around the root which bifurcates from the main cardioid at internal angle 1/5, or by a simple loop around the component at internal angle 2/5), and can then it turn into $\overline{\texttt{00101}}$ by using loops around roots of components of internal address $1 \to 3 \to 5$ and $1 \to 2 \to 5$, in that order. This loop is illustrated in Figure 5.

Our collection of loops has not left the other orbits invariant. To interchange two specified periodic points, together with their orbits, while leaving all the other periodic points fixed, requires many more loops but is easy to achieve. The necessary sequence of transpositions of orbits can be figured out easily; whenever some orbit is turned along the way into the standard orbit $\overline{\texttt{11110}}$, this is a good opportunity to adjust its cyclic permutation to whatever is needed eventually.

12 Internal Addresses for Higher Degrees

All the results described in Sections 3–11 have natural generalizations for polynomials $z \mapsto z^d + c$, but it is not always a priori clear which generalization the natural one to take is. There are many remarkable relations between the Mandelbrot set and higher degree Multibrot sets, most of them we will not touch on; compare [DGH]. In our context, the results and proofs for the quadratic case go through in many instants by merely replacing factors 2^k with d^k. We will explain in this section where significant differences occur.

The definition of kneading sequences, as given in Section 3, is now done with respect to

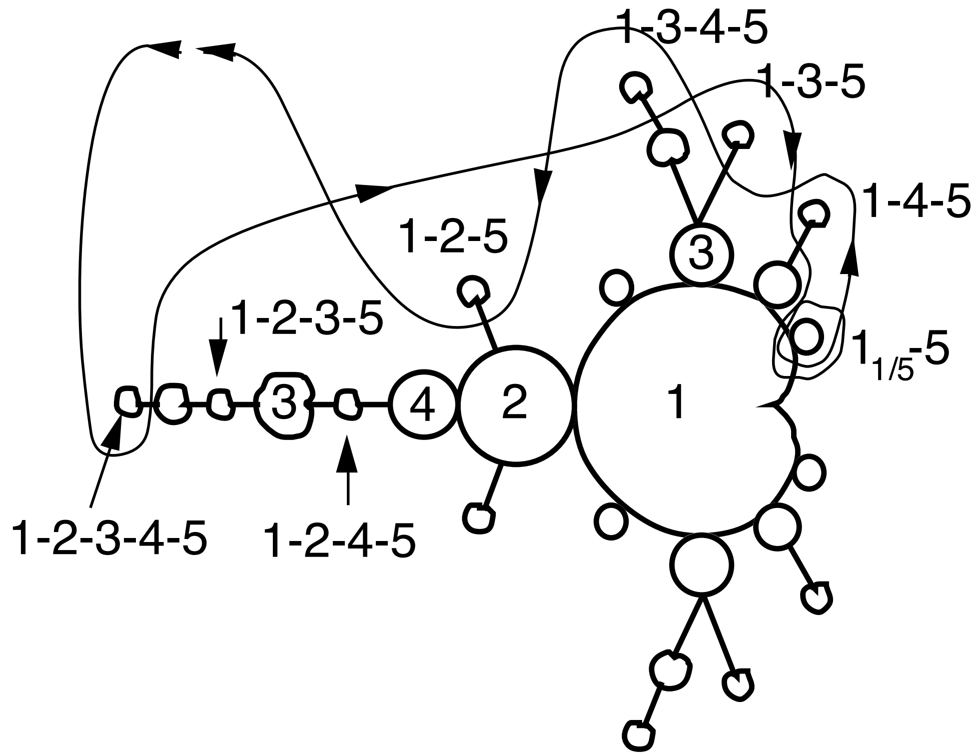

Figure 5: A loop turning the periodic point with itinerary $\overline{\texttt{00001}}$ into the periodic point with itinerary $\overline{\texttt{00101}}$.

multiplication of angles by d modulo one, rather than angle doubling. Kneading sequences consist of symbols $\texttt{0}, \texttt{1}, \ldots, (\texttt{d}-\texttt{1})$ and can start with any of these symbols except $\texttt{0}$. The local analysis of small loops around roots of hyperbolic components in Lemmas 3.4 and 3.5 remains the same, except that the cases to distinguish are whether or not the root is at an immediate bifurcation between components: if it is, one orbit is permuted cyclically, while two orbits are interchanged otherwise. The specification of the affected orbits in Lemma 3.5 remains the same.

The fundamental objects in parameter space are no longer hyperbolic components but their sectors, $d-1$ of which make up a component, because all the sectors within a component are quite independent. In particular, the decorations from sectors of the same component are in general fundamentally different. For a proof, consider any component $\mathcal{A}$ which is not connected to the period-1-component by a chain of immediate bifurcations, so that there are infinitely many hyperbolic components on the combinatorial arc leading to $\mathcal{A}$. All of them have further sectors, and they cannot all lead to a component of the same period as $\mathcal{A}$.

We define visibility between sectors in the same way as in Section 5. If $\theta_1 < \theta_2$ are the angles of the two rays bounding a sector of period n, their kneading sequences $\texttt{K}(\theta_i)$ and $\texttt{K}^{\pm}(\theta_i)$ will also be periodic with period n, the latter possibly with lower exact period. Since external rays of every period land in groups of d at components, the kneading sequences $\texttt{K}(\theta_1)$ and $\texttt{K}(\theta_2)$ will coincide on the $n-1$ initial symbols, while the n-th symbols will be consecutive boundary symbols $\genfrac{}{}{0pt}{}{\mathtt{s}}{\mathtt{r}}$ and $\genfrac{}{}{0pt}{}{\mathtt{t}}{\mathtt{s}}$ (where $\mathtt{r}$, $\mathtt{s}$ and $\mathtt{t}$ are consecutive symbols with respect to cyclic order). This gives rise to a natural way to define the kneading sequence of the sector, which is periodic of period n and contains the symbol $\mathtt{s}$ at the n-th entry. The kneading sequence of a Misiurewicz point is defined similarly.

The two rays landing at the principal root of a component cut the plane into two parts, one of which contains the component; it will be considered the *wake* of the component; the other part will be considered the *forbidden wake* (or *forbidden sector*) of the component.

We will distinguish the $d-1$ sectors of every hyperbolic component of period n in two ways: we will number them consecutively in a counterclockwise fashion from 1 through $d-1$, starting and ending at sectors near the principal root of the component; this defines the *number* of the sector. Its *label* will be the n-th entry in its kneading sequence. The forbidden sector gets number 0 or d and the label which is omitted by the other sectors; it will fit properly with respect to cyclic order. This sector gets its own kneading sequence in the obvious way; it will also be called the kneading sequence "just before" the component in analogy to Section 5. All these concepts are illustrated in Figure 6. The kneading sequence just before a Misiurewicz point will again be the same as the sequence of the point itself.

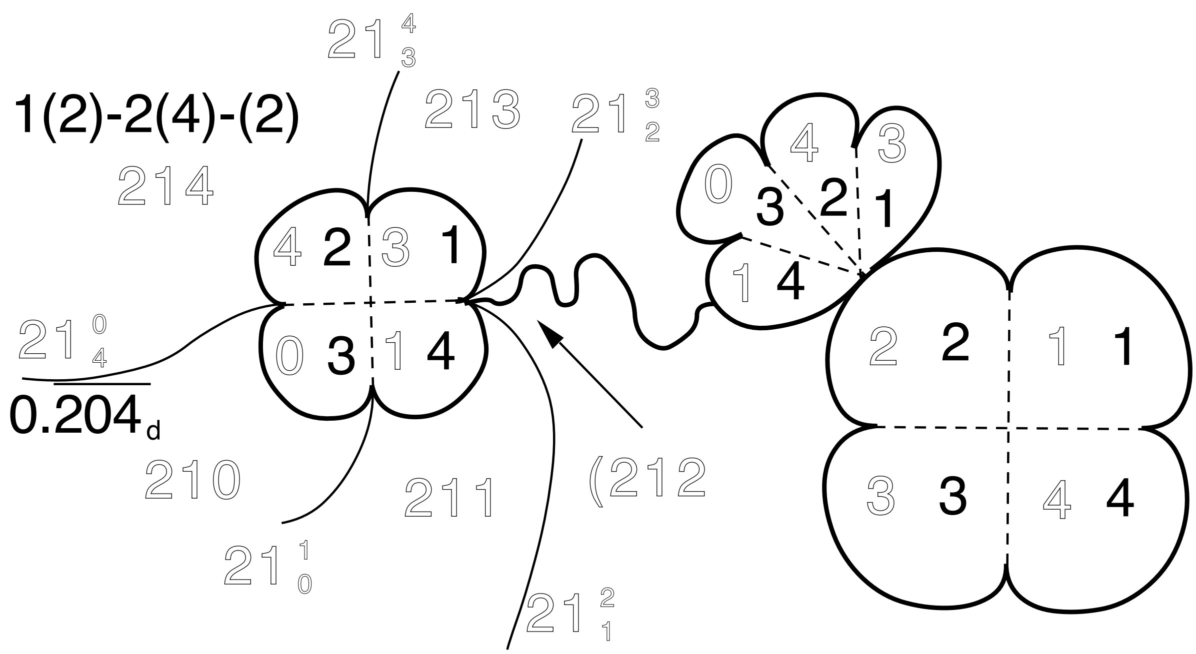

Figure 6: Sectors, their external rays, kneading sequences, and internal addresses, illustrated for $d = 5$. Three hyperbolic components, of periods 1, 2, and 3, are displayed. In every sector, its number (usual font) and its label (outline font) are given. For the component of period 3, kneading sequences of its sectors and of the rays landing at the roots are indicated; they are all periodic of period 3. The external ray at angle $54/124 = 0.\overline{204}$ (in base 5) is also labelled.

As before, the kneading sequence of every sector, except the forbidden one, will have the same exact period as the dynamics for parameters within the component; for Misiurewicz points, the analogous statement holds for the preperiods just as in Corollary 5.5. The insertion procedure in Proposition 5.4 then works literally in the same way, replacing components by sectors.

We may define internal addresses of sectors in a similar way as in Definition 6.1 in the quadratic case, but we have to keep track of the used sectors along the way: their numbers will be given in parentheses after the periods. Here is a precise recursive definition.

Definition 12.1 (Internal Address of a Sector)
The internal address of the sector numbered k of the period 1 component is $1(k)$. If A is the internal address of some sector $\mathcal{A}$ and $\mathcal{B}$ is a sector of period n which is visible from $\mathcal{A}$ and has higher period than $\mathcal{A}$ such that $\mathcal{A} \prec \mathcal{B}$ and the sector $\mathcal{B}$ has number k within its component, then the internal address of $\mathcal{B}$ is $A \to n(k)$.

As in the quadratic case, we define the angled internal address *of a sector by specifying additionally the internal angles of the bifurcations at all the sectors which occur in the internal address.*

A few examples are given in Figure 6. The forbidden sector of a component could be given its internal address by specifying a 0 in the last parenthesis. We may also define (angled) long internal addresses analogously to the quadratic case, specifying all the sectors along the way, together with their order.

The analog to Algorithm 6.2 is as follows.

Algorithm 12.2 (Kneading Sequences and Internal Addresses)
The following algorithm turns the kneading sequence of a sector $\mathcal{A}$ into its internal address: if the first entry in the kneading sequence is $\mathtt{s}$, then the internal address starts with $1(s)$. For the recursive step, assume the last entry generated in the internal address was $n_k(s_k)$. Then compare the kneading sequence of $\mathcal{A}$ with the periodic repetition of its first n_k symbols. If the first difference occurs at position n_{k+1} such that $\mathtt{K}(\mathcal{A})$ has symbol $\mathtt{s}$ and the periodic repetition has symbol $\mathtt{t}$ there, then the internal address is extended by $n_{k+1}(s-t)$ (where the sector number is to be interpreted modulo d). Repeat this step until the period of the sector is reached; by that time, there will be no more difference at all.

This algorithm can be inverted in the obvious way to turn the internal address of a sector into its kneading sequence, and it can be extended to find the long internal address of a sector.
□

The remainder of Section 6 can be generalized in the obvious way, together with Sections 7, 8 and 9. In particular, we obtain the statements that angled internal addresses describe sectors of hyperbolic components completely and that the numerators in existing angled internal addresses may be changed arbitrarily, and we get a similar algorithm to tell where given rational external rays land. At one key step, we had used Thurston's No Wandering Triangles Theorem. As far as we know, this theorem has not been generalized to the full space of degree d polynomials, as interesting and helpful as this would be. For polynomials $z^d + c$ (in Thurston's language, for laminations with only one critical gap) the original proof goes through with only the obvious modifications.

We want to prove existence of sufficiently many kneading sequences of sectors in order to obtain the permutation results for periodic points. In the quadratic case, this was done by looking at narrow components. Now we have to investigate narrow sectors: sectors which contain no component of lower period in their wakes. The lowest periods of components in the wakes of the $d-1$ sectors of a component may all be different.

Just like in the quadratic case, in can be shown that the visible components in every p/q-sublimb of a narrow sector of period n consist of exactly one component of periods $(q-1)n+1 \dots qn$ each; in the proof, one simply has to replace every number 2^k by d^k. This is also true for the Translation Principle for narrow sectors, but it requires keeping track of the labels of sectors.

Proposition 12.3 (Translation Principle)
For narrow sectors, the trees of visible components in two sublimbs of denominators q_1 and q_2 of a narrow sector $\mathcal{A}$ coincide when all periods of visible components in the q_1-sublimb are increased by $(q_2 - q_1)\mathrm{PER}(\mathcal{A})$. Moreover, the labels of corresponding components coincide in the sense that their forbidden sectors have the same symbol. □

We are only interested in the abstract tree structure made up by these components, not in their imbedding class in the plane.

Now we want to construct the trees of visible components from sectors, always restricting to the narrow case. In addition to determining the tree structure, we want to tell which of the sectors of visible components are again narrow. This is the main difference to the quadratic case.

Proposition 12.4 (Growing of Trees)
Let $\mathcal{A} \prec \mathcal{B}$ be narrow sectors and let $\mathcal{B}$ be visible from $\mathcal{A}$. Let $m := \mathrm{PER}(\mathcal{A})$ and $n := \mathrm{PER}(\mathcal{B})$. For $k = 1 \dots n$, let $\mathcal{A}_k$ be the component of period $m+k$ which is visible from $\mathcal{A}$ and which is its $1/q$-sublimb for some $q \geq 2$. Let $\mathcal{A}' = \mathcal{A}_n$. We define components $\mathcal{B}_k$ analogously; all of them are in the $1/2$-sublimb of $\mathcal{B}$. The immediate bifurcation is the component $\mathcal{B}' := \mathcal{B}_n$.

The tree of visible components and sectors in the $1/2$-sublimb of $\mathcal{B}$, i.e., the partial ordering "$\prec$" on the components $\mathcal{B}_k$ for $1 \leq k \leq n$ and the labels of their sectors, can be obtained as follows from the components $\mathcal{A}_k$ and their sectors:

- *The components $\mathcal{B}_1 \dots \mathcal{B}_{n-1}$ and their sectors have the same tree structure as the components $\mathcal{A}_1 \dots \mathcal{A}_{n-1}$ and their sectors (they may not form one connected tree). This includes the labelling of sectors.*

- *The component $\mathcal{B}'$ encloses all the components $\mathcal{B}_1 \dots \mathcal{B}_{n-1}$.*

In order to tell which of the sectors of $\mathcal{B}'$ are narrow, we will specify which of (the subtrees made up by) the components $\mathcal{B}_1 \dots \mathcal{B}_{n-1}$ sit in which of the sectors of $\mathcal{B}'$. We will compare it with the components $\mathcal{A}'$ and $\mathcal{A}_1, \dots, \mathcal{A}_{n-1}$: for $k \in \{1, \dots, n-1\}$, the sector of $\mathcal{B}'$ in whose wake $\mathcal{B}_k$ sits and the sector of $\mathcal{A}'$ in whose wake $\mathcal{A}_k$ sits bear the same label. Such a sector may be the forbidden sector in case of $\mathcal{A}'$ (indicating that $\mathcal{A}_k$ does not sit in the wake of the component $\mathcal{A}'$).

The label of the forbidden sector of $\mathcal{B}'$ is the n-th entry in the kneading sequence $\mathtt{K}(\mathcal{B})$ and the label of the forbidden sector of $\mathcal{A}'$ is the n-th entry in $\mathtt{K}(\mathcal{A})$; they are different.

REMARK. Having described the tree of visible components in the $1/2$-subwake of $\mathcal{B}$, we know this tree in every subwake of $\mathcal{B}$, using the Translation Principle. Note that $\mathcal{A}$ and $\mathcal{B}$ are sectors, but $\mathcal{A}'$, $\mathcal{B}'$ and all the $\mathcal{A}_k$, $\mathcal{B}_k$ are components.

The statement can be interpreted as follows: when comparing the d sectors of $\mathcal{A}'$ and $\mathcal{B}'$ (including the forbidden one), their labels "rotate" cyclically with respect to their numbers in the component, taking the components in their wakes with them. The rotation is such that the forbidden sector in $\mathcal{B}'$ gets the label of the sector $\mathcal{B}$ within its component, while the forbidden sectors of $\mathcal{A}'$ and $\mathcal{B}$ have the same labels. No component $\mathcal{A}_k$ is in the sector of $\mathcal{A}'$ which "rotates into the forbidden sector".

PROOF OF PROPOSITION 12.4. First we show that the components $\mathcal{B}_k$ have the same tree structure as the components $\mathcal{A}_k$. For a narrow sector of period n, the tree of visible components and sectors of periods $n+1$, $n+2$, ..., $n+s$ depends only on the first s entries of the kneading sequence; this includes the labels of sectors and the information in the wake of which sector the components sit. This holds for any positive integer s which need not be smaller than n and has been explained in the proof of the Translation Principle 10.3 in the quadratic case. Since the first $n-1$ entries in the kneading sequences of $\mathcal{A}$ and $\mathcal{B}$ coincide, the statement about the components $\mathcal{B}_k$, for $k \le n-1$, follows. All these components sit in the $1/2$-limb of the sector $\mathcal{B}$, so they are in the wake of its bifurcation $\mathcal{B}'$ at internal angle $1/2$.

Now we consider the label of the sector of $\mathcal{A}'$ in the wake of which some $\mathcal{A}_k$ sits. If it has a predecessor with respect to "$\prec$" among the $\mathcal{A}_i$ for $i \le n-1$, then it sits in the same sector, so it suffices to consider the case that it does not have a predecessor. There is then no component of period up to $m+n-1$ on $[\mathcal{A}, \mathcal{A}_k]$ and the first $m+n-1$ entries in the kneading sequences of $\mathcal{A}$ and just before $\mathcal{A}_k$ coincide ($\mathcal{A}_k$ may even be an immediate bifurcation from $\mathcal{A}$). On the combinatorial arc $[\mathcal{A}', \mathcal{A}_k]$, there is no sector of period up to $m+n$ (except perhaps the sector $\mathcal{A}$ of period n). It follows that the label of the sector in $\mathcal{A}'$ in which $\mathcal{A}_k$ sits, which is the $m+n$-th entry in the kneading sequence of that sector, is the same as the $m+n$-th entry in the kneading sequence just before $\mathcal{A}_k$. The latter is periodic with period $m+k$, so we can as well take the $n-k$-th entry. Since the kneading sequence just before $\mathcal{A}_k$ and the kneading sequence of $\mathcal{A}$ coincide for at least $n+m-1$ entries, the sector of $\mathcal{A}'$ in which $\mathcal{A}_k$ sits bears the label which occurs at the $n-k$-th position in $\mathtt{K}(\mathcal{A})$.

The same reasoning holds for the component $\mathcal{B}'$. By equality of the trees for $k<n$ as established above, none of the $\mathcal{B}_i$ is between $\mathcal{B}_k$ and $\mathcal{B}$ for $i<n$, so that the first $2n-1$ entries in the kneading sequences of $\mathcal{B}$ and just before $\mathcal{B}_k$ coincide. On the combinatorial arc $[\mathcal{B}',\mathcal{B}_k]$, there is no sector of period up to $2n$. So we can find the label of the sector of $\mathcal{B}'$ in which $\mathcal{B}_k$ sits as the $2n$-th entry in the kneading sequence just before $\mathcal{B}_k$ or, by periodicity, as the $2n-(n+k)=n-k$-th entry in that kneading sequence. But this is also the $n-k$-th entry in the kneading sequence of $\mathcal{B}$, because these kneading sequences agree for $2n-1$ entries. Since the kneading sequence of $\mathcal{B}$ differs from that of $\mathcal{A}$ for the first time at the n-th position, the label is the same as the $n-k$-th entry in $\mathtt{K}(\mathcal{A})$, which is the label found above for the sector of $\mathcal{A}'$.

Finally, we want to show that the forbidden labels of $\mathcal{A}'$ and $\mathcal{B}'$ differ. They are respectively the $m+n$-th entry in $\mathtt{K}(\mathcal{A})$ and the $2n$-th entry in $\mathtt{K}(\mathcal{B})$. By their respective periodicities, we find the same labels at the n-th entries of $\mathtt{K}(\mathcal{A})$ and of $\mathtt{K}(\mathcal{B})$. But since the sector $\mathcal{B}$ is visible from $\mathcal{A}$, the algorithm to turn the internal address into the kneading sequence changes exactly this n-th position. $\square$

Since we have good control at narrow sectors, it is natural to single out those sectors whose internal address runs only through narrow sectors: we call a sector *purely narrow* if all the sectors specified by initial sequences of its internal address are narrow.

Let $\mathcal{S}_d$ be the set of symbols $\mathtt{0},\mathtt{1},\mathtt{2},\ldots,(\mathtt{d}-\mathtt{1})$. In order to classify kneading sequences of purely narrow sectors, we will decorate sequences on such symbols by attaching a set of symbols to every position. These sets can be generated from the sequence and thus do not add any new information. The first symbol in our sequences should be one of $\mathtt{1},\ldots,(\mathtt{d}-\mathtt{1})$ (recall that kneading sequences never start with $\mathtt{0}$), and it will be decorated with the empty set. We turn the sequence into an internal address using Algorithm 12.2 (this is a slight abuse, as we do not know whether or not or sequence is in fact the kneading sequence of a sector; consequently, we do not know whether the internal address obtained will be a finite sequence). Next we run the reverse procedure to obtain the same sequence back (which works even if the internal address was not finite) together with its decorations: whenever in the algorithm the symbols are continued periodically, we also continue the corresponding sets of symbols. At the position of the first difference, the symbol is increased by the number of the sector in the internal address (as before), and the set at this position is enlarged by the old symbol at this position. We illustrate this with an example.

Example 12.5 *Consider the kneading sequence* $\overline{\texttt{113 113 14 113 113 14 113 110}}$. *Its corresponding internal address is* $1(1) \to 3(2) \to 8(3) \to 22(d-3)$. *When decorated with sets, this kneading sequence becomes*

$$\overline{\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{3}_{\{\texttt{1}\}}\,\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{3}_{\{\texttt{1}\}}\,\texttt{1}_{\{\}}\texttt{4}_{\{\texttt{1}\}}\;\;\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{3}_{\{\texttt{1}\}}\,\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{3}_{\{\texttt{1}\}}\,\texttt{1}_{\{\}}\texttt{4}_{\{\texttt{1}\}}\;\;\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{3}_{\{\texttt{1}\}}\,\texttt{1}_{\{\}}\texttt{1}_{\{\}}\texttt{0}_{\{\texttt{1 3}\}}}\,.$$

The kneading sequence $\overline{\texttt{113 113 14 113 113 14 113 111}}$ *corresponds to the internal address* $1(1) \to 3(2) \to 8(3) \to 22(d-2)$. *Its decoration with sets is the same as above, except that the last entry in the period is* $\texttt{1}_{\{\texttt{1 3}\}}$, *rather than* $\texttt{0}_{\{\texttt{1 3}\}}$.

Finally, the sequence $\overline{\texttt{113 113 14 113 113 14 113 113}}$ *would generate an infinite internal address and belongs to the forbidden sector. (One could attempt to write this internal address as* $1(1) \to 3(2) \to 8(3) \to 22(0)$*).*

The following theorem will show that the first kneading sequence corresponds to a purely narrow sector, while the second one does not (but it is realized, because the internal address $1(1) \to 3(2) \to 8(3)$ is realized by a purely narrow sector from which all continuations of internal addresses are possible).

Theorem 12.6 (Classification of Purely Narrow Sectors) *Suppose that a sequence on the symbols* $\mathcal{S}_d$ *has exact period* n *and does not start with* `0`. *Suppose furthermore that this sequence has the property that, when decorated with sets as described above, no symbol is contained in the set attached to it. This sequence is then realized as the kneading sequence of a purely narrow sector, and every purely narrow sector has a kneading sequence satisfying this property.*

Remark. It is sufficient to verify this condition at the positions given by the internal address because the others are repetitions.

In the quadratic case, all the symbols `0` will contain a symbol `1` in their attached set. If a number k appears in an internal address for $d = 2$, then the set attached to the k-th symbol in the kneading sequence will contain at least the symbol `1`, so this statement reduces to Theorem 10.6 for $d = 2$.

Proof. Although the generalization from the corresponding statement in the quadratic case is less than obvious, the proof proceeds similarly. The statement is clearly true for the main component. Now let $\mathcal{A} \prec \mathcal{B}$ be purely narrow sectors such that $\mathcal{B}$ is visible from $\mathcal{A}$, and denote the respective periods of $\mathcal{A}$ and $\mathcal{B}$ by $m < n$. Assume that all the sectors which are visible from $\mathcal{A}$ have the mentioned property; we will show it for the components which are visible from $\mathcal{B}$, using Proposition 12.4 on the growing of the trees of visible components behind $\mathcal{A}$ and $\mathcal{B}$. By the Translation Principle 12.3, it suffices to consider the 1/2-sublimb of $\mathcal{B}$. We use the notations $\mathcal{A}_k$ and $\mathcal{B}_k$ for components of periods $m + k$ and $n + k$ which are visible from $\mathcal{A}$ and $\mathcal{B}$, respectively, as introduced in the proof of Proposition 12.4 for $1 \le k \le n$, and set again

$\mathcal{A}' := \mathcal{A}_n$ and $\mathcal{B}' := \mathcal{B}_n$. For $k \le n-1$, a sector in $\mathcal{B}_k$ is narrow if and only if the corresponding sector in $\mathcal{A}_k$ is narrow. For such a component $\mathcal{B}_k$, let S_k be the set of labels corresponding to position $n+k$ in its kneading sequence. It equals the set at position k of $\mathcal{B}$, except that the k-th entry of the kneading sequence of $\mathcal{B}$ has been added, which is the forbidden sector of the component containing $\mathcal{B}$. It was not in there before because $\mathcal{B}$ was a purely narrow sector, using the inductive hypothesis.

The labelling of sectors in any $\mathcal{B}_k$ is the same as in the corresponding $\mathcal{A}_k$ (which means that the forbidden sectors carry the same label). Since the first $n-1$ entries in the kneading sequences of $\mathcal{A}$ and $\mathcal{B}$ coincide, including their associated sets, the result for $k < n$ follows by induction.

For the component $\mathcal{B}'$ of period $2n$ which bifurcates from $\mathcal{B}$, we get the result by comparing with the component $\mathcal{A}'$ of period $m+n$ which is visible from $\mathcal{A}$. A sector of $\mathcal{B}'$ is not narrow if and only if there is a component $\mathcal{B}_k$ with $k < n$ in its wake. Such a sector carries the same label as the sector of $\mathcal{A}'$ which contains the component $\mathcal{A}_k$ in its wake. So the set of symbols labelling non-narrow sectors in $\mathcal{B}'$ is the same as in $\mathcal{A}'$, except that the label of the forbidden sector in $\mathcal{B}'$ has been added: there is none of the $\mathcal{B}_k$ in the forbidden sector of $\mathcal{B}'$ because it bifurcates immediately from $\mathcal{B}$, so the sector with this label is narrow for $\mathcal{A}'$. The forbidden sector in $\mathcal{B}'$ has the label of the sector $\mathcal{B}$ within its component, that is the n-th entry in the kneading sequence of $\mathcal{B}$.

On the other hand, the set of labels for non-narrow sectors in $\mathcal{A}'$ is the same as the set for $\mathcal{B}$, using the periodicity of the the kneading sequence of $\mathcal{A}$ and its sets. The set of labels of $\mathcal{B}'$ is enlarged by the sector of $\mathcal{B}$ within its component, so we see that the construction carries over in the inductive step. □

We will describe kneading sequences of purely narrow sectors in terms of largest shifts of sequences, as we did in Corollary 10.8. However, we are free to choose any order on the set $\mathcal{S}_d$ subject to the condition that $\mathtt{0}$ not be greatest element; this order need not be compatible with the circular order. The set of sequences on these d symbols then becomes totally ordered by the corresponding lexicographical order.

Corollary 12.7 *Suppose some symbolic sequence* $\mathtt{K}$ *on the set* $\mathcal{S}_d$ *is periodic with exact period* $n > 1$ *and is the largest of all its shifts. Then this sequence generates a finite internal address ending with* n *which is realized as the kneading sequence of a purely narrow sector of period* n.

Proof. We generate the internal address $1(s_1) \to n_2(s_2) \to \ldots$ corresponding to $\mathtt{K}$; the proof that it is finite is the same as in the case $d = 2$ in Corollary 10.8. We then decorate the kneading sequence with sets of symbols and claim that every symbol is smaller than all the elements in its attached set. It suffices again to consider symbols and sets at all the positions n_k which occur in the internal address. We turn the internal address back into a kneading sequence and suppose the condition was violated for the first time at period n_k, say. The periodic continuation of the first n_{k-1} symbols of $\mathtt{K}$ is being compared with $\mathtt{K}$, finding the first difference at position n_k such that $\mathtt{K}$ has some symbol $\mathtt{s}$ whereas the periodic continuation had a symbol $\mathtt{t} \neq \mathtt{s}$. If $\mathtt{s}$ was smaller than $\mathtt{t}$, the shift of $\mathtt{K}$ by n_{k-1} symbols would exceed $\mathtt{K}$, contradicting the assumption. Since we may recursively assume that every element in the set has been smaller than $\mathtt{t}$, we see indeed that every symbol is smaller than all the elements in its set. Theorem 12.6 then applies and finishes the proof. $\square$

At last, we can prove Theorem 4.1 in the general case.

Proof of Theorem 4.1 in the general case. The idea of the proof will be the same as in the quadratic case. With respect to the chosen order, we want to turn any periodic point into the unique point with maximal itinerary. It is again enough to prove transitive action on the set of periodic orbits; the rest follows just like in Section 4.

Pick any periodic point of exact period n and let $\mathtt{I}$ be the maximal shift of its itinerary, corresponding to some point on the orbit of the given point. By Corollary 12.7, it is realized as the kneading sequence of a purely narrow sector. The angles of all the d rays landing at the $d-1$ roots of the component containing this sector have kneading sequences with exact period n which coincide with $\mathtt{I}$ for $n-1$ entries; at their n-th entries, just before the period repeats, all the d boundary symbols appear exactly once. By a collection of loops approaching the Multibrot set along some of these rays and turning around the corresponding roots at the endpoints, connected by paths outside the Multibrot set, the n-th entry in $\mathtt{I}$ can be turned into the kneading sequence of any of the other sectors within the same component (not all of these sectors may be narrow); if the component is primitive, then also the kneading sequence of the forbidden sector may be achieved. The n-th symbol in the sequence $\mathtt{I}$ may thus be replaced by every symbol except the label of the forbidden sector in case of a bifurcation (exactly in that case, the period of the sequence would become smaller). We are home if we can increase this n-th symbol because we can then keep increasing itineraries, always replacing some symbol by a greater one, until we land on the orbit of the point with maximal itinerary.

The n-th symbol of $\mathtt{I}$ is not the maximal symbol because of maximality of $\mathtt{I}$. We can increase the sequence $\mathtt{I}$ except if the n-th symbol in $\mathtt{I}$ is the second largest symbol and the component is not primitive with the forbidden sector having the largest symbol as its label. In that case, we decorate the sequence $\mathtt{I}$ with sets and see that the set at position n contains the largest symbol. Let $\mathcal{A}$ be the sector specified by the itinerary $\mathtt{I}$ and let $\mathcal{A}'$ be the sector obtained by truncating the internal address by the last entry; then $\mathcal{A}'$ is also purely narrow. If its period is n', then its kneading sequence has the largest symbol at position n'. But the corresponding set must then be empty by Corollary 12.7. This entails that $\mathcal{A}'$ is the component of period one and the itinerary $\mathtt{I}$ was already maximal, finishing the proof. $\square$

References

[At] P. Atela: *Bifurcations of dynamic rays in complex polynomials of degree two.* Ergod. Th. Dynam. Sys. 12 (1992), 401–423.

[BD] B. Branner, A. Douady: *Surgery on complex polynomials.* In: *Holomorphic dynamics* (Lecture Notes in Mathematics 1345), R. Gomez-Mont (editor). Springer Verlag (1986).

[BDK] P. Blanchard, R. Devaney, L. Keen: *The dynamics of complex polynomials and automorphisms of the shift.* Invent. Math. 104 (1991), 545–580.

[Be] A. Beardon: *Iteration of rational functions.* Springer Verlag (1991).

[BFH] B. Bielefeld, Y. Fisher, J. Hubbard: *The classification of critically preperiodic polynomials as dynamical systems.* Jour. AMS, Vol. 5, No. 4 (1992).

[BK1] C. Bandt, K. Keller: *Symbolic dynamics for angle-doubling on the circle I. The topology of locally connected Julia sets.* In: *Ergodic theory and related topics* (Lecture Notes in Mathematics 1514), U. Krengel, K. Richter, V. Warstat (editors). Springer Verlag (1992), 1–23.

[BK2] C. Bandt, K. Keller: *Symbolic dynamics for angle-doubling on the circle II. Symblic description of the abstract Mandelbrot set.* Nonlinearity 6 (1993), 377–392.

[Bo] T. Bousch: *Sur quelques problèmes de la dynamique holomorphique.* Thèse, Université de Paris-Sud (1992).

[Br] B. Branner: *The Mandelbrot set.* In: *Chaos and fractals: the mathematics behind the computer graphics.* R. L. Devaney and L. Keen (editors), Proceedings of symposia in applied mathematics, Vol. 39, Amer. Math. Soc. (1989), 75–105.

[BS] S. Bullett, P. Sentenac: *Ordered orbits of the shift, square roots, and the devil's staircase.* Math. Proc. Camb. Phil. Soc. **115** (1994), 451–481.

[CG] L. Carleson, T. Gamelin: *Complex dynamics.* Springer Verlag (1993).

[D1] A. Douady: *Algorithms for computing angles in the Mandelbrot set.* In: *Chaotic dynamics and fractals,* Atlanta 1984, 155–168. M. Barnsley, S. Demko (editors). Academic Press (1986).

[D2] A. Douady: *Compacts in* $\mathbb{C}$. In: *Topological methods in modern mathematics.* Publish or perish, Houston (1992).

[DGH] R. Devaney, L. Goldberg, J. Hubbard: *A dynamical approximation to the exponential map by polynomials.* Preprint, Mathematical Sciences Research Institute, Berkeley (1986).

[DH0] A. Douady, J. Hubbard: *Itération des polynômes quadratiques complexes.* Comptes Rendus Acad. Sci. Paris 294 (1982), 123–126.

[DH1] A. Douady, J. Hubbard: *Etude dynamique des polynômes complexes.* Publications mathématiques d'Orsay 84-02 (1984) (première partie) and 85-02 (1985) (deuxième partie).

[DH2] A. Douady, J. Hubbard: *On the dynamics of polynomial-like mappings.* Ann. Aci. Ec. Norm. Sup. 18 (1985), 287–343.

[DH3] A. Douady, J. Hubbard: *A proof of Thurston's topological characterization of rational functions.* Acta Math., 171 (1993), 263–297.

[GF] D. Giarrusso, Y. Fisher: *A parametrization of the period* 3 *hyperbolic components of the Mandelbrot set.* Submitted to the Journal of the AMS.

[GM] L. Goldberg, J. Milnor: *Fixed points of polynomial maps I/II.* Ann. Scient. Ec. Norm. Sup., 4^{e} série, t. 25/26 (1992/93).

[HS] J. Hubbard, D. Schleicher: *The Spider Algorithm.* In: *Complex dynamics: the mathematics behind the Mandelbrot and Julia sets.* AMS Lecture Notes (1994).

[HY] J. Hubbard: *Three theorems of J.-C. Yoccoz.* In: *Topological methods in modern mathematics.* Publish or perish, Houston (1992).

[Ke1] K. Keller: *Symbolic dynamics for angle-doubling on the circle III. Sturmian sequences and the quadratic map.* Ergod. Th. Dyn. Sys., to appear.

[Ke2] K. Keller: *Symbolic dynamics for angle-doubling on the circle IV. Equivalence of abstract Julia sets.* Atti del Seminario dell'Universita de Modena XLII (1994), 301–321.

[Ke3] K. Keller: *Invariante Faktoren, Juliaäquivalenzen und die abstrakte Mandelbrotmenge.* Habilitationsschrift, Universität Greifswald, in preparation.

[La] P. Lavaurs: *Une description combinatoire de l'involution definie par M sur les rationnels à dénominateur impair.* Comptes Rendus Acad. Sci. Paris (4) 303 (1986), 143–146.

[Le] G. Levin: *Disconnected Julia set and rotation sets.* Preprint, Hebrew University (May 1993).

[M1] J. Milnor: *Self-similarity and hairiness in the Mandelbrot set.* In: Computers in geometry and topology, M. C. Tangora (editor). Lecture Notes in Pure an Applied Mathematics, Vol. 114, Dekker (1989), 211–257.

[M2] J. Milnor: *Dynamics in one complex variable: introductory lectures.* Preprint #5, Institute for Mathematical Sciences at Stony Brook (1990).

[M3] J. Milnor: *Remarks on quadratic rational maps.* Experimental Mathematics 2, No. 1 (1993), 37–83.

[Mo] P. Morton: *On certain algebraic curves related to polynomial maps.* Manuscript, Wellesley College (1994).

[MP] P. Morton, P. Patel: *The Galois theory of periodic points of polynomial maps.* Proc. London Math. Soc. (3) 68 (1994), 225–263.

[MT] J. Milnor, W. Thurston: *On iterated maps on the interval*. Lecture Notes in Mathematics 1342, Springer Verlag (1988), 465–563.

[Pe1] C. Penrose: *On quotients of the shift associated with dendrite Julia sets of quadratic polynomials.* Thesis, University of Coventry (1990).

[Pe2] C. Penrose: *Quotients of the shift associated with dendrite Julia sets.* To be published (1994).

[Po] A. Poirier: *On postcritically finite polynomials*, parts I and II. Preprints #5 and #7, Institute for Mathematical Sciences at Stony Brook (1993).

[S] D. Schleicher: *Internal addresses in the Mandelbrot set and irreducibility of polynomials.* Thesis, Cornell University (1994).

[St] N. Steinmetz: *Rational iteration.* De Gruyter Studies in Mathematics, 16 (1993).

[T] W. Thurston: *On the Geometry and Dynamics of Iterated Rational Maps.* Preprint, Princeton University.

[TL] Tan Lei: *Similarity between the Mandelbrot set and Julia sets.* Comm. Math. Phys., 134 (1990), 587–617.

Eike Lau, Holsteiner Chaussee 313b, D-22457 Hamburg, Germany. email: *eike@lau.sh.sub.org*.
Dierk Schleicher, Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA. email: *dierk@msri.org*.