The Connected Isentropes Conjecture in a Space of Quartic Polynomials

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Abstract of the Dissertation

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This paper illustrates how dynamic complexity of a system evolves under deformations. The objects I considered are quartic polynomial maps of the interval that are compositions of two logistic maps. In the parameter space P^Q of such maps, I considered the algebraic curves corresponding to the parameters (λ, μ) for which critical orbits are periodic, and I called such curves left and right bones. Using quasiconformal surgery methods and rigidity I showed that the bones are simple smooth arcs that join two boundary points. I also analyzed in detail, using kneading theory,

how the combinatorics of the maps evolves along the bones. The behavior of the topological entropy function of the polynomials in my family is closely related to the structure of the bone-skeleton. The main conclusion of the paper is that the entropy level-sets in the parameter space that was studied are connected.

To my grandmother, Maria.

Mămăiţei, cu toata dragostea.

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Chapter 1

Introduction

1.1 General context

My dissertation illustrates how dynamic complexity of a system evolves under deformations. This evolution is in general only partly understood. Attempts to give a quantitative approach have considered simple examples of dynamical systems and have made use of the topological entropy $h(f)$ as a particularly useful measure of the complexity of f. However, the only results so far have been obtained in the case of interval polynomials of degree two and three.

The logistic family $Q = \{f_\mu(x) = \mu x(1-x)\,,\, \mu \in [0,4]\}$ illustrates many of the most important phenomena that occur in Dynamics. The theory in this case is the most complete: $\mu \to h(f_\mu)$ is continuous, monotonely increasing, and different values $h_0 = h(f_\mu)$ are realized for a single μ in some cases, or for infinitely many in other cases (see [D]).

The case of the cubic polynomials has been discussed in [DGMT] and [MT]. The family was parametrized by p in a compact set P using the pair of critical values provided by the 2-modal shape. The monotonicity property was generalized to connectedness of the level sets $\{p\ ,\ h(f_p) = h_0\}$.

In general, families of degree d polynomials depend on $d-1$ parameters, so the same concepts are harder to inspect for higher degrees. It is most natural to research next a family of quartic polynomials that depends only on two parameters. My paper is therefore focused on showing the *Connected Isentropes Conjecture* for the family of compositions of quadratic maps: $Q^2 = \{f_\mu \circ f_\lambda\ ,\text{where } f_\mu, f_\lambda \in Q\}$, which I called the Q-family.

I realized this by comparing the properties of the Q-family with those of a subset in the standard family of stunted sawtooth maps (see figure 1.3). These are continuous, piecewise monotone functions, very useful in kneading-theory because they are rich enough to encode in a canonical way all possible kneading-data of m-modal maps. The subfamily I used to mimic the behavior of these degree 4 polynomials in the ST-family $\{f^b \circ f^a$, where f^a, f^b are stunted tent maps with critical values a and $b\}$.

1.2 Brief description of proofs and results

I organized my paper as follows:

In chapter 1 I briefly study the more general combinatorics of $2n$-periodic orbits under a pair of $(+,-)$ unimodal maps (f_1, f_2) (i.e. orbits under alternate iterations of f_1 and f_2):

$$x_1 = x_{i_1} \xrightarrow{f_1} y_{j_1} \xrightarrow{f_2} \ldots \xrightarrow{f_1} y_{j_n} \xrightarrow{f_2} x_{i_1},$$

where $(x_i)_{1\le i\le n}$ and $(y_j)_{1\le j\le n}$ are increasing finite sequences in $I = [0,1]$.

I introduce a way to keep track of the succession of the orbit points in I by defining the *order-data* of the orbit to be the $(\sigma, \tau) \in S_n^2$ such that $f_1(x_i) = y_{\sigma_i}$ and $f_2(y_j) = x_{\tau_j}$. If the critical orbits are periodic under (f_1, f_2), their order-data turns out to be strongly connected to the kneading-data of the composition $f_2 \circ f_1$.

I consider the parameter spaces P^Q and P^{ST} corresponding to the Q-family and the ST-family. For a given order-data (σ, τ), I define the left bones in the parameter space P^Q to be the subset for which the critical point $\frac{1}{2} \in I$ has periodic orbit of order-data (σ, τ) under (f_λ, f_μ). We define the right bone as the subset of P^Q for which the critical point is periodic under (f_μ, f_λ) with order-data (σ, τ). Similarly I define the bones in P^{ST}.

In either family, the bones are algebraic curves , and clearly left bones $B_L(\sigma_1, \tau_1)$ can only intersect right bones $B_R(\sigma_2, \tau_2)$. I call a crossing:

• primary intersection, if $(\sigma_1, \tau_1) = (\sigma_2, \tau_2)$, and it corresponds to a pair of maps with common bicritical orbit.

• secondary intersection, if it corresponds to a pair of maps with disjoint critical orbits.

The properties of the bones in P^{ST} are easy to understand. I construct a diffeomorphic correspondence between P^{ST} and P^Q to get information on the behavior of the bones in P^Q. The combinatorial results made crucial use of Thurston's Uniqueness Theorem, and of an extension of it due to Poirier and interpreted by [MT].

Chapters 2 shows the following assertion:

Theorem: *The bones in P^Q are smooth 1-dimensional submanifolds that intersect transversally with other bones and the boundary of P^Q.*

Smoothness follows as in [M] at parameter points inside the hyperbolic components of P^Q. If the parameter point is outside these components, a quasiconformal surgery construction is necessary in order to smoothly perturb a map with a superattracting cycle to a map having an attracting cycle with small nonzero multiplier.

Chapter 3 completes the description of Q-bones with the following:

Theorem: *There are no bone-loops in P^Q.*

[MT] proved the similar assertion in the case of cubic polynomials, either assuming true the well-known Fatou Conjecture or using a weaker theorem due to Heckman. I used instead a quite new and interesting rigidity result of [KSvS], that delivers density of hyperbolicity in my parameter space.

The results complete the proof of:

Theorem: *For any $n \geq 1$, there is a homeomorphism η_n that takes P^{ST} to P^Q, carrying the boundary of P^{ST} to the boundary of P^Q, each left/right bone of order-data (σ, τ) in P^{ST} to a corresponding left/right bone in P^Q with the same order-data, and each primary/secondary intersection in P^{ST} to a similar one in P^Q.*

Moreover, we can define, in either parameter space, the n-skeleton to be the union of all bones of period at most $2n$, together with the boundary of the space. We call the skeletons $S_n^{ST} \subset P^{ST}$ and $S_n^Q \subset P^Q$. Put a dimension 2 topological cell structure on P^{ST} and P^Q as follows:

- the 0-cells are the intersections of bones in S_n and the boundary points;
- the 1-cells are the connected components of $S_n \backslash \{$ 0-cells $\}$;
- the 2-cells are the connected components of the complement of S_n.

The map η_n is then a homeomorphism of cell-complexes.

All these results converge in chapter 4 to finalize the proof of the central statement of the paper. The relation between entropy and the bones becomes apparent: two points in either parameter space correspond to distinct values of the entropy only if any path connecting them crosses infinitely many bones. The interval of entropy values is therefore the same within corresponding cells in the two parameter spaces. We are interested in the *isentropes*, i.e. the entropy level sets. Their properties in P^{ST} are just a particularization of the more general result explained extensively in [MT] for the stunted sawtooth family:

Theorem: *For each $h_0 \in [0, log(m+1)]$, the h_0-isentrope in the stunted sawtooth family is contractible.*

Transporting this topological property through the homeomorphism η_n is the last step towards proving my claim:

Main Theorem. *Isentropes in P^Q are connected.*

1.3 A discussion on the kneading-data

Let $h : I \to I$ be an *m-modal map* of the interval, i.e. there exist $0 < \mathbf{c_1} \leq \mathbf{c_2} \leq ... \leq \mathbf{c_m} < 1$ "folding" or "critical points" of h such that h is alternately increasing and decreasing on the intervals $H_0, ..., H_m$ between the folding points.

$$I = \bigcup_{j=0}^{m} H_j \cup \bigcup_{j=1}^{m} \{\mathbf{c_j}\}$$

We say that h is of shape $s = (+, -, +, ...)$ if h is increasing on H_0 and of shape $s = (-, +, -, ...)$ if h is decreasing on H_0. We say that h is strictly m-modal if there is no smaller m with the properties above.

We define the itinerary $\Im(x) = (A_0(x), A_1(x), ...)$ of a point $x \in I$ under h as a sequence of symbols in $\mathcal{A} = \{H_0, ..., H_m\} \cup \{\mathbf{c_1}, ..., \mathbf{c_m}\}$, where

$$\begin{cases} A_k(x) = H_j & \text{, if} \quad f^{\circ k}(x) \in H_j \\ A_k(x) = \mathbf{c_j} & \text{, if} \quad f^{\circ k}(x) = \mathbf{c_j} \end{cases}$$

The kneading sequences of the map h are defined as the itineraries of its folding values:

$$\mathcal{K}_j = \mathcal{K}(\mathbf{c_j}) = \Im(f(\mathbf{c_j})),\ j = \overline{1, m}$$

The kneading-data $\mathbf{K}$ of h is the m-tuple of kneading-sequences:

$$\mathbf{K} = (\mathcal{K}_1, ..., \mathcal{K}_m)$$

The simplest example of an m-modal map is a *sawtooth map* with m teeth (see figure 1.1(a)).

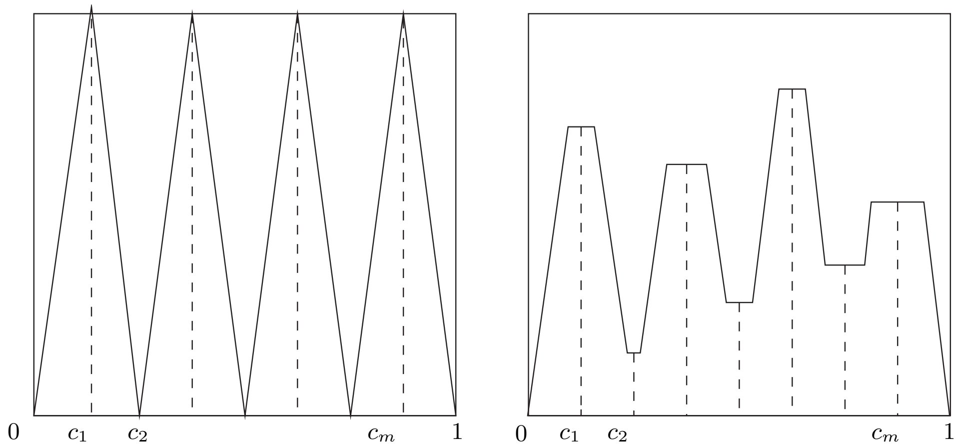


Figure 1.1: *(a)Sawtooth map of the interval. (b)Stunted sawtooth map*

We call a *stunted sawtooth map* a sawtooth map whose vertexes have been stunted by plateaus placed at chosen heights (see figure 1.1(b)).Its critical points are considered to be the centers of the plateaus. In the next sections we will focus our attention specifically on tent maps (1-modal sawtooth maps) and on their stunted version, which we will call stunted tent maps.

Another simple and rich example of m-modal maps is the collection of $(m+1)$-degree polynomials from I to itself. The "folding points" for each polynomial are in this case its regular critical points of odd order.

Definition 1.3.1. A polynomial map is called post-critically finite if the orbit of every critical point is periodic or eventually periodic.

Theorem 1.3.2. ***Thurston Uniqueness Theorem for Real Polynomial Maps:*** *A post-critically finite real polynomial map of degree m+1 with* *m distinct real critical points is uniquely determined, up to a positive affine conjugation, by its kneading data.*

We will also use a converse of this basic theorem of Thurston, due to Poirier (as interpreted by [MT]).

Definition 1.3.3. We say that a symbol sequence $\Im(x) = (A_0(x), A_1(x), ...)$ is *flabby* if some point of the associated orbit which is not a folding point has the same itinerary as an immediately adjacent folding point. A symbol sequence is called *tight* if it is not flabby. The kneading data of a map is tight if each of its kneading sequences is tight.

Lemma 1.3.4. *The kneading data of a stunted sawtooth map is tight if and only if the orbit of each folding point never hits a plateau except at its critical point.*

Theorem 1.3.5. *Suppose that the m-modal kneading data* $\mathbf{K}$ *is admissible for some shape s, with $K_i \neq K_j$ for all i. There exists a post-critically finite polynomial map of degree $m+1$ and shape s with kneading-data* $\mathbf{K}$ *if and only if each K_i is periodic or eventually periodic, and also tight. This polynomial is always unique when it exists, up to a positive affine change of coordinates, or as a boundary anchored map of the interval.*

1.4 Definitions and first goals

In the light of the general definition given in section 1.3, a (boundary anchored) *(+,-) unimodal map* of the unit interval is a $f : I = [0, 1] \to I$ such that $f(0) = f(1) = 0$ and such that there exists $\gamma \in (0,1)$, called *folding* or *critical* point, with f increasing on $(0,\gamma)$ and decreasing on $(\gamma,1)$. The orbit of a point $x \in I$ under a such f will be the sequence of iterates $(f^{\circ n}(x))_{n\geq 0}$. The itinerary of x under f is the sequence $(J_0, J_1, ...)$ of symbols L, R and Γ such that:

$$\begin{cases} J_j = L, & \text{if } f^{\circ j}(x) < \gamma \\ J_j = R, & \text{if } f^{\circ j}(x) > \gamma \\ J_j = \Gamma, & \text{if } f^{\circ j}(x) = \gamma \end{cases}$$

The first sections of this paper are dedicated to the study of combinatorics of the dynamical system I am considering: generated by alternate iterates of two unimodal interval maps. In this sense, it is convenient to consider two copies of the unit interval $I_1 = I_2 = I$ and think of our pair of maps (f_1, f_2) as a map from the disjoint union $I_1 \sqcup I_2 \to I_1 \sqcup I_2$ which carries I_1 to I_2 as f_1 and I_2 to I_1 as f_2, with critical points $\gamma_1 \in I_1$ and $\gamma_2 \in I_2$, respectively.

We call an *orbit* under the pair (f_1, f_2) a sequence:

$$x \to f_1(x) \to f_2(f_1(x)) \to f_1(f_2(f_1(x)))...$$

and we say a such orbit is *bicritical* if it contains both critical points γ_1 and γ_2. We call the *itinerary* of a point x under (f_1, f_2) the infinite sequence $\Im(x) = (J_k(x))_{k\geq 0}$ of alternating symbols in $\{L_1, \Gamma_1, R_1\}$ and $\{L_2, \Gamma_2, R_2\}$ that expresses the positions of the iterates of x in I_1 and I_2 with respect to γ_1 or γ_2. More explicitly, for $j \geq 0$:

$$J_{2j} = L_1, \quad \text{if } (f_2 \circ f_1)^{\circ j}(x) < \gamma_1 \qquad J_{2j+1} = L_2, \quad \text{if } (f_1 \circ f_2)^{\circ j}(f(x)) < \gamma_2$$
$$J_{2j} = R_1, \quad \text{if } (f_2 \circ f_1)^{\circ j}(x) > \gamma_1 \qquad J_{2j+1} = R_2, \quad \text{if } (f_1 \circ f_2)^{\circ j}(f(x)) > \gamma_2$$
$$J_{2j} = \Gamma_1, \quad \text{if } (f_2 \circ f_1)^{\circ j}(x) = \gamma_1 \qquad J_{2j+1} = \Gamma_2, \quad \text{if } (f_1 \circ f_2)^{\circ j}(f(x)) = \gamma_2$$

Clearly, not all arbitrary symbol sequences are in general admissible as itineraries of a point under a pair of given maps.

We would like to emphasize here the immediate connection between the combinatorics of the dynamical system considered (f_1, f_2) and of iterations of the composed map $f_2 \circ f_1$. The following statement shows the relation between the critical itineraries of (f_1, f_2) and the kneading-data of $f_2 \circ f_1$.

Theorem 1.4.1. *Consider (f_1, f_2) a pair of $(+,-)$ unimodal maps such that $f_2 \circ f_1$ has real critical points. The itineraries $\Im(\gamma_1)$ and $\Im(\gamma_2)$ of the two critical orbits under (f_1, f_2) determine completely the kneading-data of $f_2 \circ f_1$ and conversely.*

Proof. Let x be an iterate of γ_1 in I_1, i.e. $x = (f_2 \circ f_1)^{\circ k}(\gamma_1)$. $\Im(\gamma_1)$ gives us the L_1, R_1 or Γ_1 position of x in I_1. To tell its address from $\mathcal{A}$, we look at the position of $f_1(x)$ in I_2: if it is R_2, then $x \in (\mathbf{c_1}, \mathbf{c_3})$; if it is L_2, then $x \in [0, \mathbf{c_1}) \cup (\mathbf{c_3}, 1]$; if it is Γ_2, then $x \in \{\mathbf{c_1}, \mathbf{c_3}\}$. This gives us $\mathcal{K}_2$ of $\mathbf{c_2} = \gamma_1$. For $\mathcal{K}_1 = \mathcal{K}_3$ we have to look at the address from $\mathcal{A}$ of the iterates of the critical values $(f_2 \circ f_1)(\mathbf{c_1}) = (f_2 \circ f_1)(\mathbf{c_3}) = f_2(\gamma_2) \in I_1$ under $f_2 \circ f_1$. This is also clear, as $\Im_2$ gives us the L_1, R_1, Γ_1 position in I_1 and the L_2, R_2, Γ_2 position in I_2 of all iterates of γ_2 in I_1 and I_2, respectively.

Conversely, from $\mathcal{K}_1 = \mathcal{K}_3$ and $\mathcal{K}_2$ we know the address from $\mathcal{A}$ of any

$x \in I_1$ in either critical orbit of (f_1, f_2). Hence we will automatically have its L_1, R_1, Γ_1 position in I_1 and also the L_2, R_2, Γ_2 position of $f_1(x)$ in I_2. So the two itineraries will be known. $\square$

Corollary 1.4.2. *The theorem applies to pairs of stunted tent maps and to pairs of logistic maps.*

I will use the regular order on admissible itineraries:

$$\Im(x) < \Im(x')$$

if there exists $N \in \mathbb{N}$ such that $J_k(x) = J_k(x')$, for all $k = \overline{0, N-1}$ and either

- $J_N(x) < J_N(x')$ and there is an even number of R_1, R_2 among J_k, for $k \leq N-1$

or

- $J_N(x) > J_N(x')$ and there is an odd number of R_1, R_2 among J_k, for $k \leq N-1$.

This is a total order on admissible itineraries, consistent with the order on the real line, i.e.:

$$\Im(x) < \Im(x') \Rightarrow x < x'$$

$$x < x' \Rightarrow \Im(x) \leq \Im(x')$$

We say that the orbit of $x \in I_1$ is periodic of period $2n$ under (f_1, f_2) if n is the smallest positive integer such that $(f_2 \circ f_1)^{\circ n}(x) = x$ (i.e. x has period n under the composition $(f_2 \circ f_1)$) . I will use the following notation for a $2n$-periodic orbit under (f_1, f_2):

$$x_1 = x_{i_1} \xrightarrow{f_1} y_{j_1} \xrightarrow{f_2} x_{i_2} \xrightarrow{f_1} ... \xrightarrow{f_1} y_{j_n} \xrightarrow{f_2} x_{i_1} \tag{1.1}$$

where $(x_i)_{i=\overline{1,n}} \subset I_1$ and $(y_j)_{j=\overline{1,n}} \subset I_2$ are both increasing.

To study the combinatorics, it is essential to have a way of keeping track of the succession of points in a periodic orbit. We are particularly interested in the behavior of periodic orbits containing either critical point γ_1 or γ_2.

Definition 1.4.3. The *order-data* of the periodic orbit (1.1) is the pair (σ, τ) of permutations in S_n given by:

$$f_1(x_i) = y_{\sigma_i}$$

$$f_2(y_j) = x_{\tau_j}$$

so that $\sigma_{i_k} = j_k$ and $\tau_{j_k} = i_{k+1}$. (Here the subscripts must be understood as integers mod n, e.g. $i_{n+1} = i_1 = 1$.)

An admissible order-data is a $(\sigma, \tau) \in S_n^2$ which is achieved as order-data of a periodic orbit of some pair (f_1, f_2) of interval unimodal maps.

The (+,-) unimodal shape of f_1 and f_2 imposes a set of necessary and sufficient conditions for a (σ, τ) to be "admissible":

$$(I) \begin{cases} \text{If } \sigma_{i+1} < \sigma_i \ , & \text{then } \sigma_{j+1} < \sigma_j, \forall j \geq i \\ \text{If } \tau_{i+1} < \tau_i \ , & \text{then } \tau_{j+1} < \tau_j, \forall j \geq i \end{cases}$$

(II) $\quad \tau \circ \sigma$ is a cyclic permutation (i.e. has no smaller cycles).

A first goal will be to research the relation between the itinerary and the order-data of a periodic orbit. To begin, we prove the following:

Theorem 1.4.4. *If the orbit of γ_1 is bicritical of period 2n under a pair of (+,-) unimodal maps (f_1, f_2), then the itinerary of γ_1 determines the order-data of the orbit and conversely.*

This result will turn out to be of a very practical use: It will give an "order" on the admissible order-data, inherited from the predefined order on itineraries, hence consistent with the order of numbers on the unit interval.

Before starting the proof, note that the order of points in a critical periodic orbit of a (+,-) unimodal map is *strictly* preserved in the order of their itineraries, as shown in the following lemma.

Lemma 1.4.5. *If x and x' are two distinct points of a critical periodic orbit under a pair of (+,-) unimodal maps (f_1, f_2), then $\Im(x) \neq \Im(x')$. Hence $x < x'$ implies $\Im(x) < \Im(x')$.*

Proof. For any arbitrary map h of the interval, two distinct points x and x' along a periodic orbit of period n of h can never map to the same value under iterations of h.

Suppose now that, in our case, $\Im(x) = \Im(x')$ under (f_1, f_2). The orbit being critical, this means that x and x' will both map to the same critical point after a finite number of iterates, hence $x = x'$ $\square$

Lemma 1.4.6. *Suppose γ_1 is periodic of period $2n$ under (f_1, f_2) and let $x_1 = x_{i_1} \rightarrow y_{j_1} \rightarrow ... \rightarrow y_{j_n} \rightarrow x_{i_1}$ be its orbit. Then its order-data*

$(\sigma, \tau) \in S_{2n}^2$ determines its itinerary via the position of the element in I_2 closest to γ_2. In other words, there are at most two critical itineraries corresponding to a given order-data.

Proof. We have $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$ the elements of the critical orbit in I_1 and I_2, respectively. We hence know that there exist a k such that $x_{i_k} = \gamma_1$ (take k for which $f_1(x_{i_k}) = y_n$), and an l such that y_{j_l} is closest to γ_2 (take l for which $f_2(y_{j_l}) = x_n$). Then, we will also know that $x_i < x_{i_k} = \gamma_1$, for all $i < i_k$ and that $x_i > x_{i_k} = \gamma_1$, for all $i > i_k$. Similarly, $y_j < \gamma_2$, for all $j < j_l$ and that $y_j > \gamma_2$, for all $j > j_l$. This only leaves ambiguous the position of y_{j_l}.

If, in particular, the orbit is bicritical, then $y_{j_l} = \gamma_2$ and the itinerary is completely defined. □

Proof of Theorem 1.4.4.

($\Rightarrow$) Suppose we have the itinerary $\Im$ of the bicritical orbit (subsequently, we have the itineraries of all points in the orbit, which differ from each other by shifts). For any two elements $x_i \neq x_j \in I_1$ in the orbit, we have $\Im(x_i) \neq \Im(x_j)$ (from lemma 1.4.5). Strict order of itineraries $\Im(x_i) < \Im(x_j)$ gives strict order of the points $x_i < x_j$. So we will know the order of occurrence of the orbit points in I_1, similarly in I_2, separately:

$$x_1 = x_{i_1} \xrightarrow{f_2 \circ f_1} x_{i_2} \to ... \to x_{i_n} = x_1$$

$$y_1 = y_{j_1} \xrightarrow{f_1 \circ f_2} y_{j_2} \to ... \to y_{j_n} = y_1$$

If $shift$ is the one-sided map that shifts sequences with one position to the right, then, for an arbitrary $l \in \overline{1,n}$ we clearly have:

$$shift(\Im(x_{i_l})) = \Im(f_1(x_{i_l}))$$

Again from lemma 1.4.5, there is a unique $k \in \overline{1,n}$ for which $\Im(f_1(x_{i_l})) = \Im(y_{j_k}))$, hence $f_1(x_{i_l}) = y_{j_k}$. Repeating the argument for all points, we get the order-data (σ, τ) of the orbit.

Note. The argument applies under the weaker assumption of critical orbit rather than bicritical.

($\Leftarrow$) Follows immediately from lemma 1.4.6. $\square$

I will end this introductory section with a focus on how these notions and properties apply to a particular family of maps that are subject of our interest, the stunted tent maps:

Recall that the tent map of the interval is:

$$f_{tent}(x) = \begin{cases} f_L(x) = 2x & \text{if } x \leq \frac{1}{2} \\ f_R(x) = 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

A stunted tent-map is obtained by cutting a plateau at an arbitrary height $0 \leq a \leq 1$:

$$f^a(x) = \begin{cases} 2x & \text{if } x \leq \frac{a}{2} \\ a & \text{if } \frac{a}{2} < x < 1 - \frac{a}{2} \\ 2 - 2x & \text{if } x \geq 1 - \frac{a}{2} \end{cases}$$

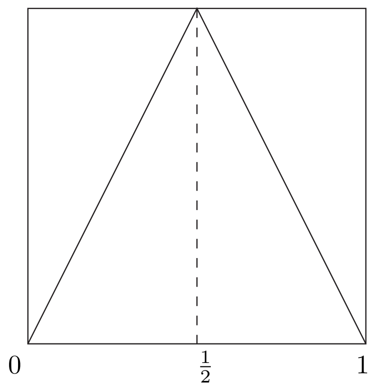


Figure 1.2: *Tent map of the interval. The critical point is $\frac{1}{2}$.*

Also recall that the "critical point" of such a stunted tent map was taken by convention to be the midpoint $\gamma = \frac{1}{2}$.

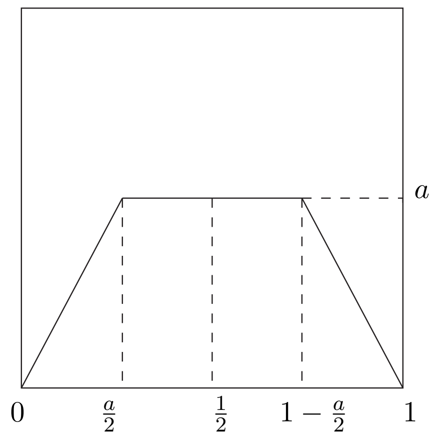


Figure 1.3: *The stunted tent map f^a, with critical point $\frac{1}{2}$ and critical value plateau at height a.*

The pair of (+,-) unimodal maps will be in this case a pair of two stunted tent maps:

$$f^a : I_1 \to I_2,\ \gamma_1 = \frac{1}{2}$$

$$f^b : I_2 \to I_1,\ \gamma_2 = \frac{1}{2}$$

We call the family of all such pairs (f^a, f^b) of stunted tent maps: the ST-family (where $(a, b) \in I_2 \times I_1$).

The behavior of the pairs in the ST-family is well understood and much easier to study than the behavior of pairs of quadratic polynomials. Although we would like to have the following result for pairs of quadratic maps, proving it first for the "approximations" in the stunted tent family is a good thing to do. This will be a strategy very frequently used for the proofs in the next few sections. We will eventually sustain with an argument why this approximation is "correct".

Theorem 1.4.7. *Given $(\sigma, \tau) \in S_n^2$ admissible order-data, there is a unique pair of stunted tent maps (f^a, f^b) with periodic bicritical orbit of order-data (σ, τ).*

We will first prove the following :

Proposition 1.4.8. *Let $\Im$ be a sequence of symbols in $\{L_1, R_1, \Gamma_1\}$ and $\{L_2, R_2, \Gamma_2\}$, admissible as a bicritical itinerary of period $2n$ under a pair of unimodal maps:*

$$\Im = (J_0 = \Gamma_1, J_1, J_2, ..., J_{2l}, J_{2l+1} = \Gamma_2, J_{2l+2}, ..., J_{2n-1}, J_{2n} = \Gamma_1, ...)$$

where $J_{2n+k} = J_k$ for all k and $J_k \neq \Gamma_1, \Gamma_2$, for all k nonequivalent to $1, ..., 2l$ mod $2n$. There exists a unique pair of stunted tent maps (f^a, f^b) that has a bicritical orbit of period $2n$:

$$x_1 = x_{i_1} \to y_{j_1} \to ...y_{j_n} \to x_{i_1} = x_1$$

having $\Im$ as its itinerary.

Proof. **Uniqueness**: Our required orbit is bicritical, hence there will be a k such that $x_{i_k} = \gamma_1 = \frac{1}{2}, \quad \Rightarrow f^b(y_{j_{k-1}}) = x_{i_k} = \gamma_1$. The itinerary gives us the position of $y_{j_{k-1}}$ with respect to γ_2, so we know which branch of f^b applies to $y_{j_{k-1}}$. This permits us to find $y_{j_{k-1}}$. If we continue to iterate backwards, we obtain b, the height of the second plateau. Similarly we get a, starting with γ_2 and iterating backwards. The uniqueness follows, as a stunted tent map is well-determined by the height of its plateau.

Existence: Consider the partial finite sequences

$$\overline{\Im}_1 = (J_1, J_2, ..., J_{2l+1} = \Gamma_2)$$

and

$$\overline{\Im}_2 = (J_{2l+2}, ..., J_{2n-1}, J_{2n} = \Gamma_1)$$

Clearly, $\Im$ is obtained by starting with Γ_1 and following it by $\overline{\Im}_1$ and $\overline{\Im}_2$, alternated infinitely.

We extend each of these two partial sequences with the itinerary of 1 (the critical value under a pair of tent maps):

$$\Im_1 = (J_1, J_2, ..., J_{2l+1} = \Gamma_2, R_1, L_2, L_1, L_2, ...)$$

and

$$\Im_2 = (J_{2l+2}, ..., J_{2n-1}, J_{2n} = \Gamma_1, R_2, L_1, L_2, L_1, ...)$$

There is an orbit of the tent map for each itinerary $\Im_1, \Im_2$, respectively. Indeed, in the same way we proceeded to prove uniqueness, we can start with $\gamma_1 = \frac{1}{2} \in I_1$ and $\gamma_2 = \frac{1}{2} \in I_2$ and work our way backwards to determine all elements of the orbit by means of solving linear equations. Doing so, we are guaranteed to have the points of the orbit on either L or R as required, because:

$$\begin{cases} f_L^{-1}(y) = \frac{y}{2} \in (0, \frac{1}{2}) & \text{, if} \quad y \in (0,1) \\ f_R^{-1}(y) = 1 - \frac{y}{2} \in (\frac{1}{2}, 1) & \text{, if} \quad y \in (0,1) \end{cases}$$

We obtain :

$$y_{j_k} \to x_{i_k+1} \to ... \to y_{j_l} = \gamma_2 \to 1 \to 0 \to 0 \to ...$$

$$x_{i_l+1} \to ... \to y_{j_{k-1}} \to x_{j_k} = \gamma_1 \to 1 \to 0 \to 0 \to ...$$

as the two orbits under the tent maps corresponding to $\Im_1$ and $\Im_2$.

y_{j_k} and x_{i_l+1} are the highest elements in each partial orbit, that is the largest elements to the left of $1 \to 0 \to 0...$ in the respective sequence. We cut the tent maps at heights y_{j_k} and x_{i_l+1}, respectively, to get the stunted maps f^a and f^b.

Clearly, $\Im$ will be the sequence for the bicritical orbit

$$x_{i_l+1} \to ... \to x_{i_k} \to y_{j_k}, ... \to y_{j_l}$$

of (f^a, f^b) of length $2n$, with $\gamma_1 = \frac{1}{2} \in I_1$ and $\gamma_2 = \frac{1}{2} \in I_2$ being the critical points. □

Proof of Theorem 1.4.7. Given an admissible order-data $(\sigma, \tau) \in S_n^2$ for a required bicritical orbit, we can determine the itinerary $\Im$ of the orbit. By Proposition 1.4.8, we can find a unique pair (f^a, f^b) of stunted tent maps with a bicritical orbit of length $2n$ and itinerary $\Im$. By Theorem 1.4.4, the order-data for the orbit we have found will be (σ, τ). □

1.5 Disjoint periodic critical orbits

To make the discussion a step more general, we look next at pairs of arbitrary unimodal maps for which both critical points γ_1 and γ_2 are periodic. There are two possible cases that can occur: a bicritical orbit (discussed in section 1.4) and two disjoint critical orbits. This section will extend the results in section 1.4 for the second case.

Definition 1.5.1. Let $(\sigma, \tau) \in S_{m+n}^2$ be a pair of permutations decomposable into two cycles: $(\sigma_1, \tau_1) \in S_m^2$ and $(\sigma_2, \tau_2) \in S_n^2$. We say that two disjoint periodic orbits o_1 and o_2 under a pair (f_1, f_2) of (+,-) unimodal maps have *joint order-data* (σ, τ) if:

1. o_1 has order-data (σ_1, τ_1) and o_2 has order-data (σ_2, τ_2);

2. the order of the points in I_1 and I_2 (see "order-type" [MT]) is given by $(\tau \circ \sigma)$ and $(\sigma \circ \tau)$ respectively.

We will say about a permutation $(\sigma, \tau) \in S_{m+n}$ that it is "admissible" as a joint order-data, if there exist two disjoint orbits under some pair of (+,-) unimodal maps which have joint order-data (σ, τ).

Theorem 1.5.2. *Let o_1 and o_2 be disjoint critical orbits under a pair (f_1, f_2) of (+,-) unimodal maps. Their itineraries determine their joint order-data and conversely.*

Proof. ($\Rightarrow$) For two points $x \neq x'$ of $o_1 \cup o_2$ in I_1, we have $\Im(x) \neq \Im(x')$. Indeed, if x and x' are both in the same critical orbit (either o_1 or o_2), then $\Im(x) \neq \Im(x')$ follows directly from lemma 1.4.5. Suppose $x \in o_1$ and $x' \in o_2$ such that $\Im(x) = \Im(x')$. Then there exists a $k \in \overline{1, m}$ such that $(f^b \circ f^a)^k(x) = \gamma_1$, i.e. $J_{2k}(x) = \Gamma_1$. Hence $J_{2k}(x') = J_{2k}(x) = \Gamma_1 \quad \Rightarrow (f^b \circ f^a)^{\circ k}(x') = \gamma_1$; so γ_1 is in o_2, which means $o_1 = o_2$ bicritical orbit, contradiction.

So the strict order of the itineraries of the points in I_1 of $o_1 \cup o_2$ is preserved in the strict order of the points themselves:

$$\Im(x) < \Im(x') \;\Rightarrow\; x < x'$$

In conclusion, all points of $o_1 \cup o_2$ in I_1 can be strictly ordered, and similarly in I_2. Also, by the note to the proof of theorem 1.4.4, the itineraries determine the order-data of each of the two disjoint orbits, separately. So the joint order-data is well defined.

($\Leftarrow$) Follows from lemma 1.4.6. $\square$

Theorem 1.5.3. *Given $(\sigma, \tau) = ((\sigma_1, \tau_1), (\sigma_2, \tau_2)) \in S^2_{m+n}$ admissible joint order-data, there exists a unique pair (f^a, f^b) of stunted tent maps with disjoint critical orbits $o_1 \ni \gamma_1$ and $o_2 \ni \gamma_2$ having joint order-data (σ, τ).*

Proof. The proof develops very similarly to the one for a common bicritical orbit in section 1.4. By theorem 1.5.2, from the joint order-data we can determine the itineraries for the two periodic orbits of lengths $2m$ and $2n$ under (f^a, f^b).

Uniqueness: Start with γ_1 and iterate backwards, using the branch of the stunted tent map imposed by the itinerary. This will determine uniquely the height b of the plateau of f^b. Similarly get the plateau a of f^a starting with γ_2 and iterating backwards.

Existence: To prove existence, shift the itineraries by one position, so that they start with the position of the critical values, then end them at the first Γ_1 , respective Γ_2 position. We complete them with the itineraries of the critical value 1 under a pair of tent maps : $\overline{\Im}_1 = (R_1, L_2, L_1, ...)$ and $\overline{\Im}_2 = (R_2, L_1, L_2, ...)$. We find two orbits under the tent maps having the given extended sequences, then we stunt the maps conveniently to make the orbits closed; we are guaranteed this way that they have the given itineraries and that their critical values are $a = f^a(\gamma_1)$ and $b = f^b(\gamma_2)$. By theorem 1.5.2, they will also have the required order-data. □

1.6 Description of bones in the ST-family

Fix an admissible order-data $(\sigma, \tau) \in S_n^2$.

By a *left bone* in the parameter space $I_2 \times I_1$ for the ST-family we mean the set of pairs $(a, b) \in I_2 \times I_1 = [0, 1]^2$ such that the critical point $\gamma_1 \in I_1$ has under (f^a, f^b) a periodic orbit of given period $2n$ and given order-data (σ, τ). We will use the notation $B_L^{ST}(\sigma, \tau)$ or B_L^{ST} if there is no ambiguity. We define a *right bone* symmetrically (i.e. we require γ_2 to be periodic of specified period and order-data) and we denote it by $B_R^{ST}(\sigma,\tau) = B_R^{ST}$. Later we will give a more comprehensive approach to the left and right bones and their properties.

Recall form theorem 1.4.7 that: *There is a unique pair $(a_0, b_0) \in B_L^{ST}$ such that the periodic orbit of γ_1 is bicritical (i.e. hits γ_2) under (f^{a_0}, f^{b_0}).*

Theorem 1.6.1. *For each admissible order-data (σ,τ), let (a_0, b_0) be the parameter pair for the associated bicritical orbit in the ST-family. Then there are unique numbers $a_1 < a_0 < a_2$ so that the left bone $B_L^{ST}(\sigma,\tau)$ is the union $\{a_1, a_2\} \times [b_0, 1] \cup (a_1, a_2) \times \{b_0\}$ of three line segments, as illustrated in figure 1.7. The description of the right bone $B_R^{ST}(\sigma,\tau)$ is completely analogous.*

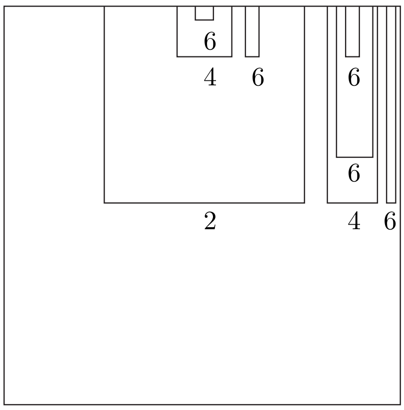


Figure 1.4: *Left bones in the ST-family of period at most 6.We marked by (2) the unique bone of period 2, corresponding to order-data in $(\sigma = (1), \tau = (1)) \in S_1^2$. (4) are the 2 bones of period 4 and having the two possible order-data $(\sigma = (12), \tau = (1)(2))$ or $(\sigma = (1)(2), \tau = (12)) \in S_2^2$. (6) are the bones of period 6 and one of the 5 admissible order-data: $(\sigma = (123), \tau = (231)$, $(\sigma = (132), \tau = (321)), (\sigma = (231), \tau = (231)), (\sigma = (321), \tau = (132))$ or $(\sigma = (231), \tau = (123))$.*

We will determine the shape of B_L^{ST}, hence prove 1.6.1, by constructive means, starting with the point (a_0, b_0).

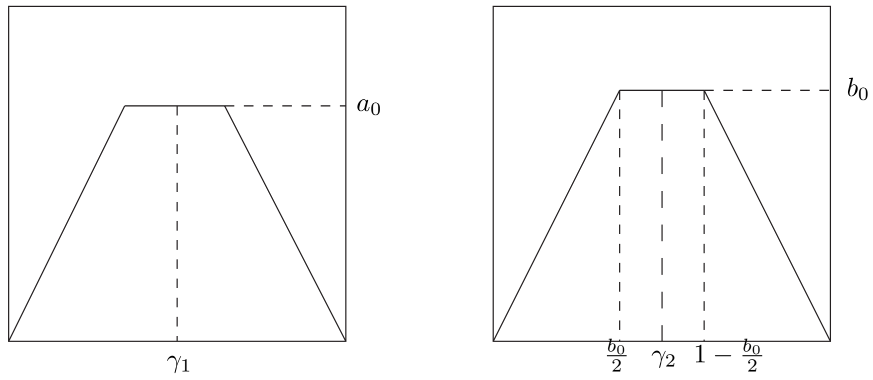


Figure 1.5: *The pair (f^a, f^b) has a bicritical orbit for $a = a_0$ and $b = b_0$.*

Under (f^{a_0}, f^{b_0}): $\gamma_1 \to a_0 \to ... \to \gamma_2 \to b_0 \to ... \to \gamma_1$

The bicritical orbit only hits each plateau once, at its center.

By sliding the first plateau up and down, the orbit of γ_1 will change in a continuous way. For a fixed height a of the first plateau, call $y_l(a)$ the element in I_2 closest to γ_2 in the orbit of γ_1 under (f^a, f^{b_0}). Clearly, if $a = a_0$, then $y_l(a_0) = \gamma_2$.

We can move a continuously within an interval $[a_1, a_2] = [a_0 - \epsilon, a_0 + \epsilon]$, $\epsilon > 0$ such that $y_l(a)$ moves from $\frac{b_0}{2}$ to $1 - \frac{b_0}{2}$ (see figure 1.5). Along the process, the orbit stays periodic and the order of the occurrence of points remains consistent with (σ, τ).

Hence $[a_1, a_2] \times \{b_0\} \subset B_L^{ST}$.

It is not hard to see that also for any $b > b_0$: (a_1, b) and (a_2, b) are in B_L^{ST}. (Indeed, the orbit itself will not change in this case with the raising of the second plateau.)

If we call $\sqcup = \sqcup(\sigma, \tau) = \{a_1, a_2\} \times [b_0, 1] \cup (a_1, a_2) \times \{b_0\} \ni (a_0, b_0)$ we have

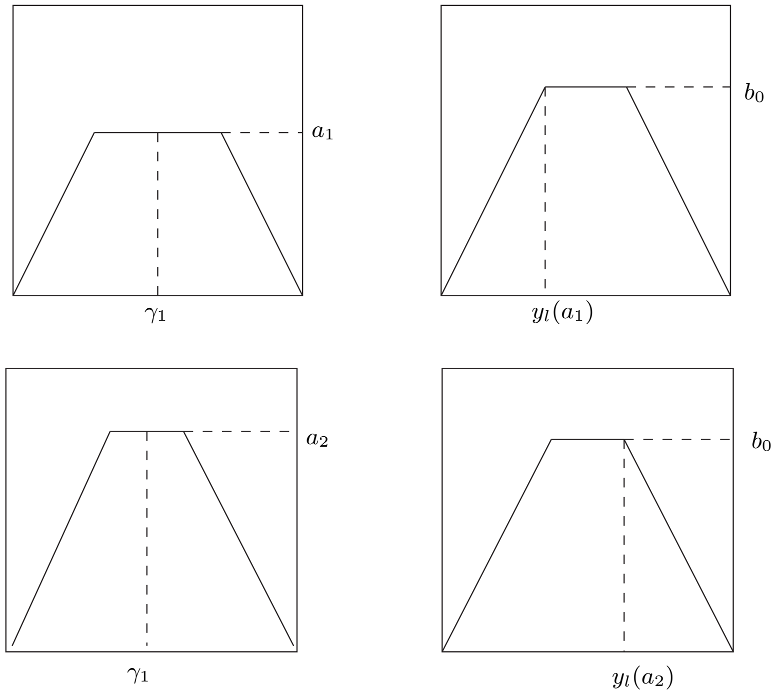


Figure 1.6: *By sliding the first plateau up and down between a_1 and a_2, the orbit point in I_2 closest to γ_2 slides continuously between the left and right endpoints of the second plateau*

just obtained :

Lemma 1.6.2. $\sqcup \subset B_L^{ST}$.

Lemma 1.6.3. *There are exactly two values $a = a_1$ and $a = a_2$ such that the orbit of γ_1 has given order-data (σ, τ) under (f^a, f^1) (i.e. there are exactly two points of B_L^{ST} on $[0,1] \times \{1\}$).*

Proof. The critical periodic orbit of γ_1 has order-data (σ, τ), so all symbol-positions of the orbit elements are precisely determined, via the position of y_l, the element in I_2 closest to γ_2. The ambiguity is between L_2 and R_2; Γ_2 is impossible, because it would mean bicritical orbit under (f^a, f^1).

We hence have two distinct itineraries for the orbit

$\gamma_1 \to a \to ... \to \gamma_1$

Each itinerary uniquely determines a value of a, iterating backwards from γ_1 through left or right branches of the tent map, according to the respective itinerary. So there are at most two values of a.

Since the orbits of γ_1 are not bicritical, there are exactly two values of a. □

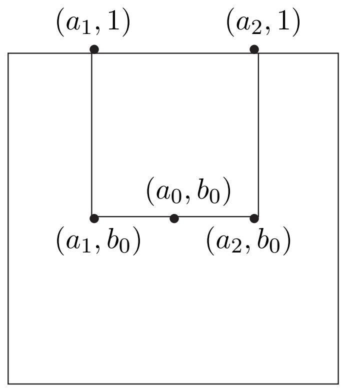


Figure 1.7: $B_L^{ST} = \sqcup = \{a_1, a_2\} \times [b_0, 1] \cup (a_1, a_2) \times \{b_0\} \ni (a_0, b_0)$

Theorem 1.6.4. $B_L^{ST} = \sqcup$

Proof. We have shown in lemma 1.6.2 that $\sqcup \subset B_L^{ST}$. We prove the inverse inclusion.

Take an arbitrary $(a, b) \in B_L^{ST}$. The orbit of γ_1 under (f^a, f^b) is periodic:

$$\gamma_1 \to a \to ... \to y_l \to ... \to \gamma_1$$

where again we call $y_l = y_l(a)$ the closest element to γ_2 in I_2 (such that

$f^b(y_l) \in I_1$ is the maximal element in I_1 of the orbit).

We distinguish two cases:

(I) $y_l < \frac{b}{2}$ or $y_l > 1 - \frac{b}{2}$, i.e. $f^b(y_l) < b$

(II) $y_l \in [\frac{b}{2}, 1 - \frac{b}{2}]$, i.e. $f^b(y_l) = b$

We discuss them separately.

(I) The critical orbit never hits the second plateau, so it is identical with the orbit of γ_1 under (f^a, f^1). By lemma 1.6.3, $a = a_1$ or $a = a_2$. It follows almost immediately that b can't be lower than b_0, for either $a = a_1$ or $a = a_2$.

Hence in this case $(a, b) \in \{a_1, a_2\} \times [b_0, 1] \subset \sqcup$.

(II) The orbit has the form:

$$\gamma_1 \to a \to ... \to y_l \to b \to ... \to \gamma_1$$

We slide the first plateau continuously. As $y_l(a)$ is the closest point in the orbit to γ_2 in I_2, it will reach γ_2 (for some $a = a'$) without changing the order of the points in the orbit. $(f^{a'}, f^b)$ has therefore a bicritical orbit of the same order-data (σ, τ) as the orbit of γ_1 under (f^a, f^b). By the uniqueness of such a pair, it follows that $a' = a_0$ and $b = b_0$.

Knowing $b = b_0$, we solve for a:

$$\frac{b_0}{2} < y_l < 1 - \frac{b_0}{2} \quad \Rightarrow \quad a_1 < a < a_2$$

(Recall that a_1 and a_2 were defined such that they corresponded to $y_l(a_1) = \frac{b_0}{2}$ and $y_l(a_2) = 1 - \frac{b_0}{2}$).

So in this case: $(a, b) \in (a_1, a_2) \times \{b_0\} \subset \sqcup$ $\square$

1.7 Important points on the bones

We aim to compare the parameter spaces for the two families ST={ pairs (f_1, f_2) of stunted tent maps } and Q={ pairs (f_1, f_2) of logistic maps } :

$$P^{ST} = \{(a, b) \;/\; a = f_1(1/2), b = f_2(1/2),\ (f_1, f_2) \in ST\} \ \in \ [0, 1]^2$$

$$P^{Q} = \{(\lambda, \mu) \;/\; \lambda = 4f_1(1/2), \mu = 4f_2(1/2),\ (f_1, f_2) \in Q\} \ \in \ [0, 4]^2$$

In either space, we defined the left and right $2n$-bones for a given admissible $(\sigma, \tau) \in S_n^2$ to be (see section 1.6):

$B_L(\sigma, \tau)$= the set of all parameters for which γ_1 has periodic orbit of order-data (σ, τ) under the corresponding pair of maps;

$B_R(\sigma, \tau)$= the set of all parameters for which γ_2 has periodic orbit of order-data (σ, τ) under the corresponding pair of maps.

For a fixed admissible $(\sigma, \tau) \in S_n^2$, I will call the bones in P^{ST}: B_L^{ST}, B_R^{ST} and the ones in P^Q: B_L^Q, B_R^Q.

Remarks: (1) By definition, any two left bones are disjoint and any two right bones are disjoint.

(2) It follows easily from theorem 1.6.1 that two bones in the ST-family can cross only at 0,2 or 4 points.

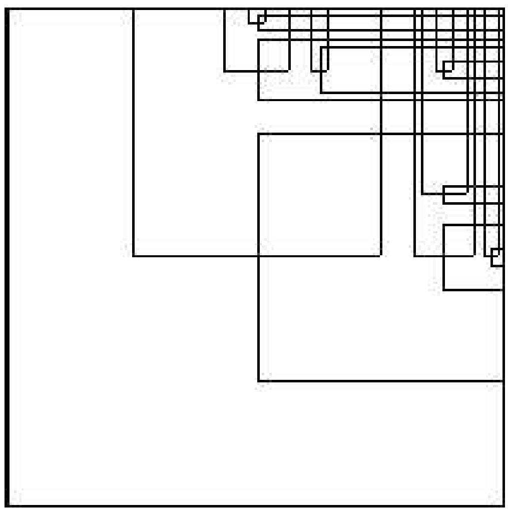


Figure 1.8: *Left and right bones in P^{ST} of all periods less or equal to 6 and all possible order-data. Left bones can intersect right bones only at 0, 2 or 4 points.*

Definition 1.7.1. In either parameter space, an intersection of $B_L(\sigma_1, \tau_1)$ and $B_R(\sigma_2, \tau_2)$ is called a *primary intersection* if $(\sigma_1, \tau_1) = (\sigma_2, \tau_2)$ and there is a bicritical orbit with this order-data under the pair of maps. It is called a *secondary intersection* if the two critical orbits are disjoint, of distinct order-data (σ_1, τ_1) and respectively (σ_2, τ_2), and joint order-data (σ, τ).

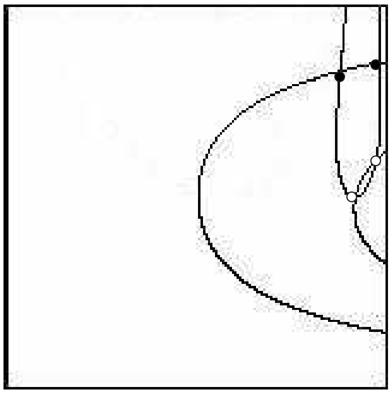


Figure 1.9: *The left* 4*-bone of order-data* $(\sigma, \tau) = ((12), (21))$ *crosses the right* 2*-bone at two secondary intersections with joint order-data* $((231), (321))$ *and* $((132), (231))$ *(filled dots) and crosses the corresponding right 4-bone at a primary intersection with order-data* (σ, τ) *and at a secondary intersection with joint order-data* $((1243), (3421))$ *(empty dots).*

A *capture point* on $B_L(\sigma_1, \tau_1)$ in either P^{ST} or P^Q is a pair of maps for which γ_2 eventually maps on γ_1 such that it has an eventually periodic, but not periodic, orbit (see picture). We define symmetrically a capture point on $B_R(\sigma_2, \tau_2)$.

Theorem 1.4.7 provided us with a bijection between admissible order-data and primary intersections in P^{ST}. In the following section, theorem 1.5.3 extended the result with a bijection between admissible joint order-data and secondary intersections. The next statement is a further extension for capture itineraries and can be proved similarly with the direct implication in theorem 1.5.3.

Theorem 1.7.2. *Suppose the two critical points of a pair of unimodal maps* *are such that one of them has a closed orbit and the other maps on this closed orbit after a finite number of iterates, but without being periodic itself. Let* $\Im_1$ *and* $\Im_2$ *be the itineraries of the two critical points. Then there exists at least a pair* (f^a, f^b) *with critical itineraries* $\Im_1$ *and* $\Im_2$*, respectively. (i.e.: There exists at least a capture point in* P^{ST} *with given "capture" critical itineraries.)*

All previous theorems concern bones crossings in the ST-family. One of our goals will be to establish similar results in the case of pairs of quadratic maps. This will bring us a better perspective on the shape of the bones in P^Q and their similarity with the bones in P^{ST}.

1.8 More on kneading-data

In this section we will construct a bijective correspondence of bones intersections between our two parameter spaces P^{ST} and P^Q. For the proof, it is necessary to view the composition $f_\mu \circ f_\lambda$ of the two logistic maps either as a 3-modal map with three critical points in $I = I_1$: $\mathbf{c_1} \leq \mathbf{c_2} \leq \mathbf{c_3}$, with $\mathbf{c_2} = \gamma_1$ and $f_\lambda(\mathbf{c_1}) = f_\lambda(\mathbf{c_3}) = \gamma_2$ or as a unimodal map with folding point γ_1, (in case $f(x) = \gamma_2$ has a double real root or two complex roots). I will use rigidity theorems that involve essentially some properties of the kneading-data.

Let us look in more detail at the possible kneading-data of the maps in P^{ST} and P^Q.

Maps in P^{ST}: For any $(a,b) \in P^{ST}$, the map $f^b \circ f^a$ could be considered 3-modal, with folding points $c_1 = \frac{1}{4}, c_2 = \gamma_1 = \frac{1}{2}$ and $c_3 = \frac{3}{4}$.

$$\mathcal{A}^{ST} = \{[0, \frac{1}{4}), \frac{1}{4}, (\frac{1}{4}, \frac{1}{2}), \frac{1}{2}, (\frac{1}{2}, \frac{3}{4}), \frac{3}{4}, (\frac{3}{4}, 1]\}$$

.

and

$$\mathbf{K} = (\mathcal{K}(c_1), \mathcal{K}(c_2), \mathcal{K}(c_3))$$

As appendix A shows, we can consider P^{ST} as made of three parts:

$$P^{ST} = P_1^{ST} \cup P_2^{ST} \cup P_3^{ST}$$

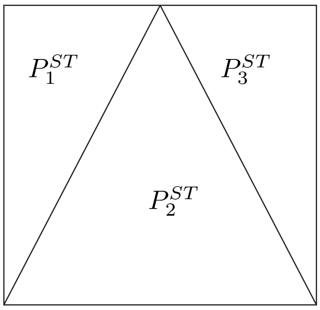


Figure 1.10: *The behavior of the composed map $f^b \circ f^a$ changes with the position of the parameter $(a, b) \in P^{ST}$, as shown in appendix A.*

I. Clearly there are no right bones in P_1^{ST}, hence no bones intersections.

Indeed: $2a < b \leq 1$, so $a < \frac{1}{2}$, hence γ_2 could not be hit by any orbit under (f^a, f^b).

II. Left bones in P_2^{ST} do not contain secondary intersections, because $\frac{b}{2} \leq a \leq 1 - \frac{b}{2}$, so $f^b(\gamma_2) = b = (f^b \circ f^a)(\gamma_1)$.

Moreover, if $(a,b) \in P_2^{ST}$ is a primary intersection, then the map $f^a \circ f^b$ is strictly 3-modal, with only one exception: $(a,b) = (\frac{1}{2}, \frac{1}{2})$.

Indeed, $2a \geq b$, hence $1 - \frac{a}{2} \leq 1 - \frac{b}{4}$. So either

$1 - \frac{b}{4} < b \;\Rightarrow\; (b,a) \in P_3^{ST}$, (see **III**), or:

$\frac{b}{4} < b < 1 - \frac{b}{4} \;\Rightarrow\; (f^a \circ f^b)^{\circ k}(\gamma_1) = b, \;\forall k$. Hence $b = \gamma_1 = \frac{1}{2}$, so $f^a(\gamma_1) = \gamma_2 \;\Rightarrow\; a = \gamma_2 = \frac{1}{2}$.

III. For $(a,b) \in P_3^{ST}$ we clearly have that $f^b \circ f^a$ is strictly 3-modal, that is $\mathcal{K}(c_1) = \mathcal{K}(c_3) \neq \mathcal{K}(c_2)$.

Maps in P^Q: The behavior of the degree 4 polynomials in the family P^Q is also different for distinct values of the parameters.

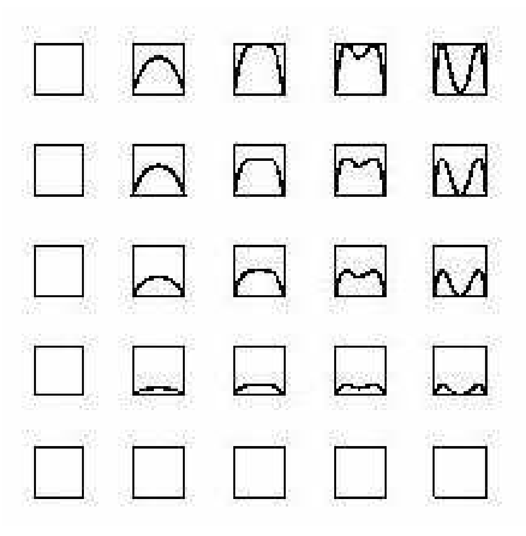


Figure 1.11: *A few examples of behavior of maps in P^Q. The critical points of the quartic map $f_\mu \circ f_\lambda$ are distinct and real for $\lambda > 2$, all coincide for $\lambda = 2$, while two of them are complex for $\lambda < 2$.*

I. If $\lambda < 2$, then $f_\mu \circ f_\lambda$ has only one real critical point $C_2 = \gamma_1 = \frac{1}{2}$ and two complex $C_1, C_3 \in \mathbb{C}\backslash\mathbb{R}$.

This parameter subset will be of somewhat less interest, as it does not cross any right bones, hence contains no bones intersections. Indeed, if $f_\lambda(x)$ is less than $\frac{1}{2}$, $\forall x \in I_1$, no orbit can go through γ_2.

II. If $\lambda = 2$, then $f_\mu \circ f_\lambda$ has a degenerated real critical point $C_1 = C_2 = C_3 = \gamma_1$. This line contains primary intersections with right bones. More precisely, if a left bone hits $\{\lambda = 2\}$, then the crossing point is its primary intersection. However, in this case $f_\lambda \circ f_\mu$ is strictly 3-modal, with the exception of $\lambda = \mu = 2$, which is the period 2 primary intersection.

III. If $\lambda > 2$, there are three distinct real critical points for $f_\mu \circ f_\lambda$:

$C_1 < C_2 = \gamma_1 < C_3$, with $f_\lambda(C_1) = f_\lambda(C_3) = \gamma_2$

The map is 3-modal:

$$\mathcal{A}^Q = \{[0, C_1), C_1, (C_1, C_2), C_2, (C_2, C_3), C_3, (C_3, 1]\}$$

.

and

$$\mathbf{K} = (\mathcal{K}(C_1), \mathcal{K}(C_2), \mathcal{K}(C_3))$$

Remark. We emphasize that $f_\mu \circ f_\lambda$ has complex critical points iff $\lambda < 2$.

If the point (λ, μ) is on a bone, it can't be in the region $\{\lambda < 2, \mu < 2\}$, so $\mu \geq 2$. Hence the map $f_\lambda \circ f_\mu$ corresponding to the symmetric point (μ, λ) has real critical points, non-degenerate if $\mu \neq 2$.

A correspondence is already apparent between the shape and position of the left bones in the two families P^{ST} and P^Q. For instance, the unique primary intersection of period two: $(a, b) = (\frac{1}{2}, \frac{1}{2}) \in P^{ST}$ clearly corresponds to its similar $(\lambda, \mu) = (2, 2) \in P^Q$. We will consider at least this case classified in our future analysis. The following theorems will therefore concern specifically the strictly 3-modal case (applicable for either $f^b \circ f^a$ and $f_\mu \circ f_\lambda$ or $f^a \circ f^b$ and $f_\lambda \circ f_\mu$).

1.9 The correspondence of the bones intersections

I will use Thurston's Theorem and its extension for boundary anchored polynomials of degree four and shape (+,-,+,-). In this section, I plan to construct a bijection between bones crossings in the two parameter spaces. More precisely, we **claim:**

1. For each primary intersection in P^{ST}, there is a unique one in P^Q with the same order-data and conversely (hence each bone in either family has a unique primary intersection with the dual bone).

2. For each secondary intersection in P^{ST}, there is a unique one in P^{Q} with the same joint order-data and conversely (hence bones in either family cross at 0, 2 or 4 points).

Remark. By theorem 1.4.7, $(\sigma, \tau) \in S_n^2$ is admissible order-data is equivalent to existence of a pair (f^a, f^b) with bicritical orbit of order-data (σ, τ), hence equivalent to the existence of a primary intersection in P^{ST} with order-data (σ, τ). By theorem 1.5.3, $(\sigma, \tau) \in S_{m+n}^2$ is admissible joint order-data is equivalent to existence of a pair (f^a, f^b) with disjoint critical orbits of joint order-data (σ, τ), hence equivalent to the existence of a secondary intersection in P^{ST} with joint order-data (σ, τ). So we can restate our claim in the form of the following two theorems.

Theorem 1.9.1. *Let $(\sigma, \tau) \in S_n^2$ be admissible order-data. There is a unique primary intersection (λ, μ) in P^{Q} with this data and conversely.*

Proof. **Uniqueness**: Suppose we have a pair $(\lambda, \mu) \in P^{Q}$ with a bicritical orbit of order-data (σ, τ). We implicitly know the itinerary of the bicritical orbit, hence the kneading sequences of the three real distinct critical points $C_1 < C_2 = \frac{1}{2} < C_3$ of $f_\mu \circ f_\lambda$ (if $\lambda > 2$) or $f_\lambda \circ f_\mu$ (if $\mu > 2$). By Thurston's Theorem, the boundary anchored polynomial of degree 4 with the expected kneading data is unique, implying the uniqueness of the pair (f_λ, f_μ) with the given order-data.

Existence: Let (f^a, f^b) be the pair of stunted tent maps with bicritical orbit of order-data (σ, τ). We know by theorem 1.4.4 that we can determine the itinerary of this bicritical orbit. If we exclude the case $a = b = \frac{1}{2}$, which is already classified, then either $f^b \circ f^a$ or $f^a \circ f^b$ is strictly 3-modal (say $f^b \circ f^a$, to fix our ideas). Theorem 1.4.1 says we can obtain the kneading-data **K** for $f^b \circ f^a$, which should also be the kneading-data for the polynomial $f_\mu \circ f_\lambda$ that we want to find. We hence need to prove existence of a polynomial of degree 4 with the required kneading-data **K** and then show that it can be written as a composition of two logistic maps f_λ and f_μ. We will finally show that the pair (f_λ, f_μ) we found has indeed the given order-data.

I will use the extended version of Thurston's theorem, so I aim to show:

1. Each two consecutive kneading-sequences of **K** are distinct.

2. All kneading sequences of **K** are tight.

1. The critical points of $f^b \circ f^a$ are $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$ and $c_3 = \frac{3}{4}$. Clearly, $\mathcal{K}(c_1) = \mathcal{K}(c_3)$. The pair (f^a, f^b) has a bicritical orbit, so the orbits of both c_1 and c_3 go through c_2 and either one of them closes through c_2. But, by lemma 1.4.5, we can't have two distinct elements of a periodic critical orbit follow identical itineraries. So $\mathcal{K}(c_1) \neq \mathcal{K}(c_2)$.

2. To use lemma 1.3.4 we only need to show that each $\mathcal{K}(c_i)$ hits each plateau at most once, above its corresponding critical point. This also follows almost directly from lemma 1.4.5, because $f^b(f^a(c_1)) = f^b(f^a(c_3))$.

By Thurston's Theorem, (1) and (2) imply existence and uniqueness of a polynomial P with kneading-data **K**, of shape (+,-,+,-) and conditions at the boundary $P(0) = 0$ and $P(1) = 0$. A boundary anchored polynomial P of degree 4, shape (+,-,+,-) and real distinct critical points $0 < C_1 < C_2 < C_3 < 1$ is a composition of logistic maps if and only if $P(C_1) = P(C_3)$ (see appendix).

We know that the kneading sequences $\mathcal{K}(C_1) = \mathcal{K}(c_1)$ and $\mathcal{K}(C_3) = \mathcal{K}(c_3)$ are identical. Suppose $v_1 = P(C_1) < P(C_3) = v_3$. Then the whole interval $[v_1, v_3]$ will have the same (bicritical) itinerary, as $\mathcal{K}(C_1) = \mathcal{K}(C_3)$, so, after a finite number of iterations under P, it will all map to C_2, contradiction. So $P(C_1) = P(C_3)$, hence there exists a pair of quadratic maps such that $P = f_\mu \circ f_\lambda$.

The kneading data $\mathbf{K}$ determines the itinerary of the bicritical orbit which, according to theorem 1.4.4, determines the order-data. So the polynomial map we found can only have the given order-data (σ, τ). $\square$

Theorem 1.9.2. *Let $(\sigma, \tau) \in S^2_{m+n}$ admissible joint order-data. There is a unique secondary intersection in P^Q with this data and conversely.*

Proof. **Existence**: Consider the secondary intersection $(a, b) \in P^{ST}$ with joint order-data (σ, τ). It determines the kneading data of $f^b \circ f^a$. Clearly $\mathcal{K}(c_1) = \mathcal{K}(c_3) \neq \mathcal{K}(c_2)$, otherwise the orbits would coincide into a bicritical one, which doesn't happen at a secondary intersection. The tightness condition also follows. Hence there exists a polynomial of degree four with the respective kneading-data, which can be easily shown to be of the form $f_\mu \circ f_\lambda$, with $(\lambda, \mu) \in P^Q$. Its joint order-data is (σ, τ).

Uniqueness: Suppose we have a point $(\lambda, \mu) \in P^Q$ with disjoint periodic critical orbits of joint order-data $(\sigma, \tau) \in S^2_{m+n}$, hence distinct real critical points. This will determine the critical itineraries under (f_λ, f_μ), hence the kneading-data of the polynomial $f_\mu \circ f_\lambda$.

By directly applying Thurston's Theorem, there is at most one boundary anchored polynomial of degree four with this set of kneading data, which in particular gives us the required uniqueness of the pair (f_λ, f_μ) with the given joint data. $\square$

1.10 The correspondence of the boundary points

Fix $(\sigma_1, \tau_1) \in S_n^2$. The left bone $B^{ST} = B_L^{ST}(\sigma_1, \tau_1)$ in P^{ST} with order-data (σ_1, τ_1) is as an algebraic curve in $P^{ST} = I_2 \times I_1 = I^2$. Its boundary consists of two points:

$$\delta B^{ST} = B^{ST} \cap \delta P^{ST} = B^{ST} \cap (I_2 \times \{1\}) = \{(a_1, 1), (a_2, 1)\}$$

with $a_1 < a_2$.

For any (a, b), I will call $\Im(x)(a, b)$ the itinerary of x under (f^a, f^b) and $\mathbf{K}(a, b)$ the kneading-data of $f^b \circ f^a$.

The itineraries of the critical points γ_1 and γ_2 under (f^{a_1}, f^1) and (f^{a_2}, f^1) are respectively:

$\Im(\gamma_1)(a_1, 1) \neq \Im(\gamma_1)(a_2, 1)$ (as will become explicit later)

$\Im(\gamma_2)(a_1, 1) = \Im(\gamma_2)(a_2, 1) = (\Gamma_2, R_1, L_2, L_1, L_2, L_1, ...) = (\Gamma_2, R_1, \overline{L_2, L_1})$

At any $(a, b) \in B^{ST}$, γ_1 has a periodic orbit o_1 of period $2n$ and order-data (σ_1, τ_1). At the points $(a_1, 1)$ and $(a_2, 1)$ in $\delta B^{ST} \subset P^{ST}$, the orbit o_2 of γ_2 is not periodic, although it is finite.

Statements in previous sections referred to primary or secondary intersections of bones. I will need some extensions of these statements to apply to boundary points of left bones in either parameter space. As we have noted, these boundary points are not bones crossings.

Theorem 1.10.1. (extension of 1.5.3) *Suppose we have two sequences admissible as critical itineraries under a pair of unimodal maps: a $2n$-periodic $\Im_1$ and $\Im_2 = (\Gamma_2, R_1, \overline{L_2, L_1})$. Then there exists a unique pair of stunted tent maps (f^a, f^b) with $\Im(\gamma_1)(a,b) = \Im_1$ and $\Im(\gamma_2)(a,b) = \Im_2$. Moreover, $b = 1$.*

Proof. **Uniqueness**: If the iterates of a point in I_2 under (f^a, f^b) stay indefinitely on L_1 and L_2 in both I_1 and I_2, then the point has to be zero (otherwise it will get pushed away after a certain number of iterates). Hence $\Im_2 = \Im(\gamma_2) = (\Gamma_2, R_1, \overline{L_2, L_1})$ implies that $(f^a \circ f^b)(\gamma_2) = 0$ and $f^b(\gamma_2)$ is R, so clearly $f^b(\gamma_2) = 1$, i.e. $b = 1$. We obtain a by starting with γ_1 and using $\Im_1 = \Im(\gamma_1)$ to iterate backwards.

Existence: There exists a point $x \in I_1$ with orbit of length $2n$ and itinerary $shift(\Im_1)$ under a pair of tent maps (f_1, f_2). Take a to be the highest value in I_1 in the orbit of x under (f_1, f_2) and take $b = 1$. The required itineraries follow for the pair of stunted maps (f^a, f^b). $\square$

Remark. Theorem 1.10.1 shows in particular why $\Im(\gamma_1)(a_1, 1) \neq \Im(\gamma_1)(a_2, 1)$.

We expect the boundary of the corresponding quadratic left bone $B^Q = B^Q(\sigma_1, \tau_1)$ to look similarly. The next theorem gives us the first information we have so far on the general shape of a quadratic bone.

Theorem 1.10.2. $\delta B^Q = \{(\lambda_1, 4), (\lambda_2, 4)\}$, *with* $\lambda_1 < \lambda_2$.

Remark. In fact, we will show that an arbitrary quadratic bone B^Q is a simple arc joining the two boundary points. The proof will be based on the following assertion, detailed in chapter 3:

Theorem A: There are no bone-loops in $\mathbf{P^Q}$.

Any bone in P^Q is an algebraic variety, hence it is a disjoint union of connected components which can be simple arcs with end-points on δP^Q or can contain loops (simple closed curves). **Theorem A** states that a bone in P^Q can't contain any loops.

Proof of theorem 1.10.2. We will show that the boundary of B^Q consists of exactly two distinct points in $[0, 4] \times \{4\} \subset \delta P^Q$.

Consider the maps $f^{a_i} \circ f^1$ with their kneading-data $\mathbf{K}(1, a_i)$. For each i, the adjacent kneading-sequences are distinct.

For each $i \in \{1, 2\}$, the pair of critical itineraries at $(1, a_i)$ determines the respective kneading-data $K(1, a_i)$, by theorem 1.4.1. Note that $\Im(\gamma_1)(a_1, 1) \neq \Im(\gamma_1)(a_2, 1)$, so $\mathbf{K}(1, a_1) \neq \mathbf{K}(1, a_2)$. The kneading-data also satisfies for each i the conditions in the extended version of Thurston's theorem:

1. the kneading sequences are finite;

2. $K(1, a_i)(c_1) = K(1, a_i)(c_3) \neq K(1, a_i)(c_2)$;

3. they are tight.

Hence for each i there exists a point $(\mu_i, \lambda_i) \in P^Q$ such that $f_{\lambda_i} \circ f_{\mu_i}$ has kneading-data $K(1, a_i)$, and subsequently (theorem 1.4.1) the same critical itineraries as (f^{a_i}, f^1). In consequence:

$$\Im(\gamma_1)(\lambda_i, \mu_i) = \Im(\gamma_1)(a_i, b_i)$$

$$\Im(\gamma_2)(\lambda_i, \mu_i) = \Im(\gamma_2)(a_i, 1) = (\Gamma_2, R_1, \overline{L_2, L_1})$$

As we stated in the proof of theorem 1.4.4, the order-data of a critical orbit under a pair of unimodal maps is well-determined by its itinerary (not conversely). The order-data at $(1, a_i)$ of γ_1 is (σ_1, τ_1); hence the order-data of the orbit of γ_1 under $(f_{\lambda_i}, f_{\mu_i})$ is also (σ_1, τ_1). (λ_i, μ_i) must therefore be in the left bone $B^Q_L = B^Q$ in P^Q corresponding to $B^{ST}_L = B^{ST}$ in P^{ST}.

We also know that the itinerary of γ_2 under $(f_{\lambda_i}, f_{\mu_i})$ is $(\Gamma_2, R_1, \overline{L_2, L_1})$. Zero is a repeller for the composition $f_\mu \circ f_\lambda$, for any $(\lambda, \mu) \in [0, 4]^2$ such that $\mu\lambda > 1$, and this will be the case if we are situated on a left quadratic bone (see chapter 3 for proof). So the only way for the itinerary of a point to stay indefinitely on L_1 and L_2 is for the point to map to zero after a number of iterates. To be consistent with the required itinerary, we need to have $(f_{\lambda_i} \circ f_{\mu_i})(\gamma_2) = 0$ and $f_{\mu_i}(\gamma_2)$ is R, so $f_{\mu_i}(\gamma_2) = 1$, hence $\mu_i = 4$, for both values of i.

In conclusion: for the two points $(a_1, 1), (a_2, 1) \in \delta B^{ST}$ we found two points $(\lambda_1, 4), (\lambda_2, 4) \in \delta B^Q$ with the same kneading-data for $f_{a_i} \circ f_1$ and $f_{\lambda_i} \circ f_4$. We prove that the two points $(\lambda_1, 4)$ and $(\lambda_2, 4)$ we found in δB^Q are the only two boundary points of B^Q.

Indeed, suppose that δB^Q is made of more than two points (λ, μ), all clearly in $[0, 4] \times \{4\} \subset \delta P^Q$. By Thurston's theorem, the 3-modal maps $f_\lambda \circ f_\mu$ have distinct kneading-data. For each of the kneading-data, we have $\Im(\gamma_2) = (\Gamma_2, R_1, \overline{L_2, L_1})$ and necessarily distinct periodic itineraries $\Im(\gamma_1)$. From theorem 1.10.1, there will be a unique pair $(a, b) \in \delta P^{ST}$ for each $\Im(\gamma_1)$; all these pairs (a, b) have to have the same order-data (σ_1, τ_1) for the orbit of γ_1, yet different itineraries for γ_1. This contradicts lemma 1.4.6, which permits at most two such points. $\square$

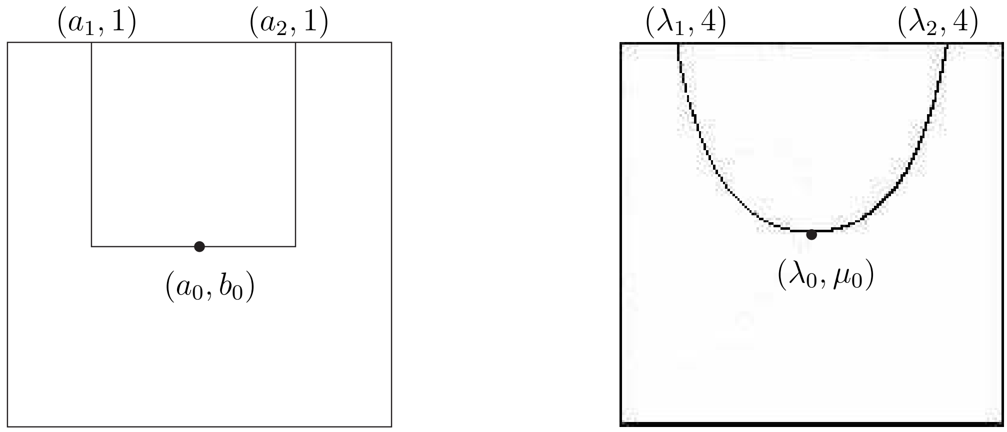


Figure 1.12: *For given (σ_1, τ_1), both $B_L^{ST}(\sigma_1, \tau_1)$ and $B_L^Q(\sigma_1, \tau_1)$ have exactly two boundary points. The itinerary of γ_1 changes only at the primary intersection along either bone. Moreover, $\Im(\gamma_1)(a_1, 1) = \Im(\gamma_1)(\lambda_1, 4)$ and $\Im(\gamma_1)(a_2, 1) = \Im(\gamma_1)(\lambda_2, 4)$.*

Remark. We will put aside the proof of theorem A until later. As an easier, more immediate goal, we plan to show that the succession of crossings along $B^Q = B^Q(\sigma_1, \tau_1) \subset P^Q$ is same as along the corresponding bone $B^{ST} =$

$B^{ST}(\sigma_1, \tau_1) \subset P^{ST}$. This requires a closer look at the properties of a bone in the ST family.

1.11 A more complete description of bones in $\mathbf{P^{ST}}$

In this section we show a combinatorial result concerning the order of occurrence of the primary and secondary intersections along a bone in P^{ST} with fixed order-data (σ_1, τ_1). To fix our ideas, all proofs and results are developed for left bones $B^{ST} = B^{ST}_L$, hence we will omit writing the index L unless it causes ambiguity.

The following idea has been mentioned several times before. I would like to state it and prove it as a simple but important result in its own right.

Lemma 1.11.1. *Following (a, b) along the bone B^{ST}, the itinerary $\Im(\gamma_1)(a, b)$ changes at the primary intersection point with the corresponding dual bone and only there.*

Proof. Recall that the order-data of γ_1 at a point $(a, b) \in B^{ST}$ determines the itinerary $\Im(\gamma_1)(a, b)$ except from the position L_2, R_2 or Γ_2 of the point $y_l \in I_2$ closest to $\gamma_2 = \frac{1}{2} \in I_2$.

Sliding along the horizontal part of the bone from (a_1, b_0) to (a_2, b_0), y_l continuously changes from one side of γ_2 to the other, passing through γ_2 when $a = a_0$. Hence $\Im(\gamma_1)$ only changes at the primary intersection (a_0, b_0). $\square$

Therefore, $B_*^{ST} = B^{ST} \setminus \{(a_0, b_0)\}$ can be divided into two halves, each corresponding to a different itinerary of γ_1 under (f^a, f^b); call B_-^{ST} the left half, containing $(a_1, 1) \in \delta B^{ST}$ and B_+^{ST} the one containing $(a_2, 1) \in \delta B^{ST}$. Clearly:

$$B^{ST} = B_*^{ST} \cup \{(a_0, b_0)\} = B_-^{ST} \cup \{(a_0, b_0)\} \cup B_+^{ST}$$

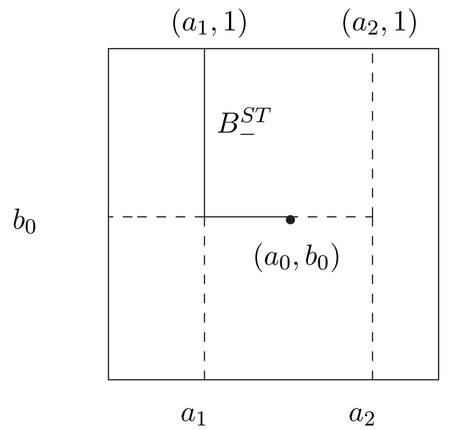


Figure 1.13: *We divide a left bone in P^{ST} in two halves, according to the two possible itineraries of γ_1: $B^{ST} = B_-^{ST} \cup \{(a_0, b_0)\} \cup B_+^{ST}$*

To fix our ideas, we look at B_-^{ST}; the results and the proofs should work symmetrically for B_+^{ST}. B_-^{ST} is composed of a vertical segment and a horizontal one:

$$B_-^{ST} = \{a_1\} \times [b_0, 1] \cup [a_1, a_0] \times \{b_0\} = B_{-,v}^{ST} \cup B_{-,h}^{ST}$$

We can now state our claim for this section in more precise terms:

Theorem 1.11.2. *The secondary intersections occur along $B_{-,v}^{ST}$ in the strictly decreasing order of their itinerary $\Im(\gamma_2)$, as b decreases from 1 to b_0.*

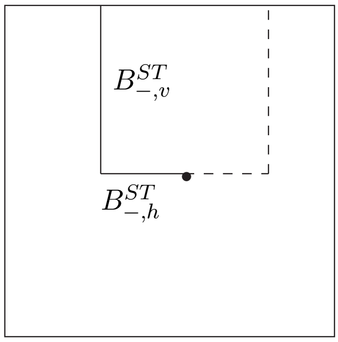


Figure 1.14: *We divide the left half B^{ST}_{-} of a left bone in P^{ST} into a vertical segment $B^{ST}_{-,v}$ and a horizontal segment $B^{ST}_{-,h}$. All secondary intersections occur along $B^{ST}_{-,v}$. The horizontal part is composed only of capture points.*

Let's take a brief look at our case: the itinerary of γ_1 under (f^{a_1}, f^b) remains constant all along $B^{ST}_{-,v}$:

$$\Im(\gamma_1)(a_1, b) = \Im(\gamma_1)(a_1, 1),\ \forall b_0 \leq b \leq 1$$

and the itinerary of γ_2 should intuitively change at the secondary crossings, but not only. So we propose to study a slightly wider problem: the succession of all points along $B^{ST}_{-,v}$ where the itinerary $\Im(\gamma_2)(a_1, b)$ changes. To follow is the description of such points.

Recall that *capture points* in $B^{ST}_{-,v}$ are the pairs $(a_1, b) \in B^{ST}_{-,v}$ for which γ_2 maps to γ_1 after a finite number of iterates under (f^{a_1}, f^b).

So, at such a point, the orbit of γ_2 is eventually periodic, but not periodic, hence not bicritical. The itinerary of γ_2 is therefore finite at all capture points, but these points are not bones intersections.

We introduce some notations. For all $m \geq 1$, we call:

$\mathcal{C}^m = \{(a_1, b) \in B^{ST}_{-,v} \; / \; \gamma_2$ maps to γ_1 after $2m-1$ iterates under $(f^{a_1}, f^b)\}$

$\mathcal{S}^m = \{(a_1, b) \in B^{ST}_{-,v} \; / \; \gamma_2$ maps to itself after $2m$ iterates under (f^{a_1}, f^b), i.e. γ_2 has a critical orbit of length $2m\}$

$\mathcal{D}^m = \mathcal{C}^m \cup \mathcal{S}^m$

Also:

$\mathcal{C} = \bigcup_{m \geq 1} \mathcal{C}^m = \{(a_1, b) \in B^{ST}_{-,v}$ capture points $\}$

$\mathcal{S} = \bigcup_{m \geq 1} \mathcal{S}^m = \{(a_1, b) \in B^{ST}_{-,v}$ secondary intersections $\}$

$\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}^m = \mathcal{C} \cup \mathcal{S}$

I will call the points in D the *distinguished points* of $B^{ST}_{-,v}$. I will use, for an arbitrary $b \in [b_0, 1]$, the following notation for the itinerary of γ_2 under (f^{a_1}, f^b):

$$\Im(\gamma_2)(a_1, b) = \Im(\gamma_2)(b) = (J^b_0, J^b_1, J^b_2, ..., J^b_{2m}, J^b_{2m+1}, ...)$$

The even indexes correspond to iterates in I_2, hence to elements of $\{L_2, R_2, \Gamma_2\}$; the odd indexes stand for iterates in I_1, hence for elements of $\{L_1, R_1, \Gamma_1\}$.

Remark. If $(a_1, b) \in \mathcal{D}^m$, then $J^b_{2k-1} \neq \Gamma_1$ and $J^b_{2k} \neq \Gamma_2$, for all $1 \leq k \leq m-1$.

Moreover, either $J^b_{2m-1} = \Gamma_1$, if $(a_1, b) \in \mathcal{C}^m$, or $J^b_{2m} = \Gamma_2$ and $J^b_{2m-1} \neq \Gamma_1$, if $(a_1, b) \in \mathcal{S}^m$.

Definition 1.11.3. If $\Im = (J_0, J_1, ...)$, call $\Im^m = (J_0, J_1, ..., J_{2m})$ the finite symbol sequence obtained by truncating $\Im$ to the first $2m$ positions. If $\Im = \Im(\gamma_2)(b)$ for some $b \in [b_0, 1]$, then we use the notation:

$$\Im^m(\gamma_2)(b) = (J^b_0, J^b_1, ..., J^b_{2m-1}, J^b_{2m})$$

On the set of truncated itineraries of length k, we put the order naturally inherited from the space of infinite itineraries.

Properties:

(a) If $\Im_1$ and $\Im_2$ are infinite itineraries such that for some m we have $\Im^m_1 < \Im^m_2$, then $\Im_1 < \Im_2$.

(b) If $\Im_1 \leq \Im_2$, then $\Im^m_1 \leq \Im^m_2, \quad \forall m \geq 1$.

(c) If $(J^*_0, ..., J^*_{2m-1}, J_{2m}) < (J^*_0, ..., J^*_{2m-1}, \overline{J_{2m}})$ are such that $J_{2m}, \overline{J_{2m}} \in \{L_2, R_2\}$, then:

$(J^*_0, ..., J^*_{2m-1}, J_{2m}) < (J^*_0, ..., J^*_{2m-1}, \Gamma_2) < (J^*_0, ..., J^*_{2m-1}, \overline{J_{2m}})$.

(c') If $(J^*_0, ..., J^*_{2m}, J_{2m+1}) < (J^*_0, ..., J^*_{2m}, \overline{J_{2m+1}})$ are such that $J_{2m+1}, \overline{J_{2m+1}} \in \{L_1, R_1\}$, then:

$$(J_0^*, ..., J_{2m}^*, J_{2m+1}) < (J_0^*, ..., J_{2m}^*, \Gamma_1) < (J_0^*, ..., J_{2m}^*, \overline{J_{2m+1}}).$$

In the following, fix $m \in \mathbb{N}$.

Lemma 1.11.4. *$\Im^m(\gamma_2)(a_1, b)$ changes at all points in $\bigcup_{k \le m} \mathcal{D}^k$ (which are finitely many) and only at these points, as b goes from 1 to b_0.*

Proof. Clearly, $\Im^m(\gamma_2)(a_1, b)$ changes at any $(a_1, b) \in \bigcup_{k \le m} \mathcal{D}^k$. The second part follows: suppose $\Im^m(\gamma_2)(a_1, b) \ne \Im^m(\gamma_2)(a_1, \bar{b})$ for some $b < \bar{b}$. Then there exists a $1 \le k \le 2m$ such that $J_k^b \ne J_k^{\bar{b}}$. By continuity, there is a $b^* \in [b, \bar{b}]$ such that $J_k^{b^*} = \Gamma_1$ or Γ_2 (depending on the parity of k), i.e. $(a_1, b^*) \in \bigcup_{k \le m} \mathcal{D}^k$. □

Consider now the function that assigns to each point $(a_1, b) \in B_{-,v}^{ST}$ the truncated itinerary of length $2m$ of γ_2 under (f^{a_1}, f^b):

$$\psi^m : B_{-,v}^{ST} \rightarrow \{\Im^m(\gamma_2)(b) \; / \; b \in [b_0, 1]\}$$

$$\psi^m(a_1, b) = \Im^m(\gamma_2)(b)$$

ψ^m is locally constant on $B_{-,v}^{ST} \setminus \bigcup_{k \le m} \mathcal{D}^k$, from lemma 1.11.4. More precisely, ψ^m is constant on the intervals between consecutive points of $\bigcup_{k \le m} \mathcal{D}^k$ and it does change at all these points (its constant value has to change from an interval to the next through an intermediate value at the point in $\bigcup_{k \le m} \mathcal{D}^k$) .

We will say that a function on $B_{-,v}^{ST}$ with values in an ordered set is *increasing* if it is increasing as a function of its second variable $b \in [b_0, 1]$. In this context:

Proposition 1.11.5. *For any fixed $m \geq 1$, ψ^m is increasing* (with the previously defined order on finite symbol sequences) *and locally constant on $B^{ST}_{-,v} \setminus \bigcup_{k\leq m} \mathcal{D}^k$, with actual changes at all points in $\bigcup_{k\leq m} \mathcal{D}^k$.*

Proof. We only need to prove ψ^m increasing.Take $b < \bar{b}$ such that there exists $b^* \in [b, \bar{b}]$, with $(a_1, b^*) \in \bigcup_{k\leq m} \mathcal{D}^k$. WLOG, we can take b such that the first $2m$ iterates of $f^b(\gamma_2) = b$ in I_1 under (f^{a_1}, f^b) do not hit again the second plateau value b. Same take $\bar{b}$ such that $f^{\bar{b}}(\gamma_2)$ doesn't return to the plateau in less than $2m$ iterates under $(f^{a_1}, f^{\bar{b}})$. We then have that the orbit of $b = f^b(\gamma_2)$ under (f^{a_1}, f^b) is identical on the first $2m$ positions with the orbit of b under (f^{a_1}, f^1) and the orbit of $\bar{b} = f^{\bar{b}}(\gamma_2)$ under $(f^{a_1}, f^{\bar{b}})$ coincides on the first $2m$ positions with the orbit of $\bar{b}$ under (f^{a_1}, f^1):

$$\Im^m(\gamma_2)(a_1, b) = \Im^{m-1}(f^{a_1}(b))(a_1, b) = \Im^{m-1}(f^{a_1}(b))(a_1, 1)$$

$$\Im^m(\gamma_2)(a_1, \bar{b}) = \Im^{m-1}(f^{a_1}(\bar{b}))(a_1, \bar{b}) = \Im^{m-1}(f^{a_1}(\bar{b}))(a_1, 1)$$

But: $b < \bar{b}$, hence $f^{a_1}(b) < f^{a_1}(\bar{b}) \Rightarrow \Im(f^{a_1}(b))(a_1, 1) < \Im(f^{a_1}(\bar{b}))(a_1, 1)$. So, from property (b):

$$\Im^{m-1}(f^{a_1}(b))(a_1, 1) \leq \Im^{m-1}(f^{a_1}(\bar{b}))(a_1, 1)$$

Hence:

$$\Im^{m-1}(f^{a_1}(b))(a_1,b) \leq \Im^{m-1}(f^{a_1}(\bar{b}))(a_1,\bar{b})$$

In other words:

$$\Im^{m}(\gamma_2)(a_1,b) \leq \Im^{m}(\gamma_2)(a_1,\bar{b})$$

By lemma 1.11.4, this automatically means:

$$\Im^{m}(\gamma_2)(a_1,b) < \Im^{m}(\gamma_2)(a_1,\bar{b})$$

If $b^* \in \bigcup_{k\leq m} \mathcal{D}^k$, then for arbitrary $b, \bar{b} \notin \bigcup_{k\leq m} \mathcal{D}^k$ such that $b < b^* < \bar{b}$, it follows from the above and from property (c) that:

$\Im(\gamma_2)(b) < \Im(\gamma_2)(b^*) < \Im(\gamma_2)(\bar{b})$. □

Theorem 1.11.6. *ψ^m is increasing on $B^{ST}_{-,v}$.*

Proof. Follows from proposition 1.11.5 □

Define:

$\psi : B^{ST}_{-,v} \longrightarrow \{\Im(\gamma_2)(b) \ / \ b \in [b_0, 1]\}$

$\psi(a_1, b) = \Im(\gamma_2)(b)$

Theorem 1.11.7. *ψ is increasing on $B^{ST}_{-,v}$.*

Proof. Follows from theorem 1.11.6 and property (a). □

Corollary 1.11.8. *ψ is strictly increasing on D.*

Proof to theorem 1.11.2. The secondary intersections S on $B^{ST}_{-,v}$ are a subset of D, so in particular ψ is strictly increasing on S. $\square$

Remark. Theorem 1.11.2 makes it possible to identify the order of occurrence of the distinguished points (in particular of the secondary intersections) along $B^{ST}_{-,v}$ by looking at the itinerary of γ_2. From the construction of the stunted bones in section 1.6 it is also easy to see that there are no secondary intersections on the horizontal segment of $B^{ST}_{-,h}$. In fact, all points of $B^{ST}_{-,h}$ are capture points and $\Im(\gamma_2)(a, b_0)$ is constant for $a \in [a_1, a_0]$.

1.12 A more careful look at the bone-arcs in $\mathbf{P^Q}$

We want to shift our attention now to the quadratic bone-arc corresponding to $B^{ST}_L(\sigma_1, \tau_1) \in P^{ST}$ for our arbitrarily fixed order-data. More precisely, we look at the left bone in P^Q having the same order-data $(\sigma_1, \tau_1) \in S_n^2$ for the $2n$-periodic orbit of γ_1. It is composed of a simple arc connecting its boundary points and possible bone-loops. We will continue to call B^Q the arc-component of this left bone. There will be no need to take into account the bone-loops within this section. To avoid ambiguity, we will use more precise notations for the distinguished points along B^{ST} and B^Q: $\mathcal{D}^m_{ST} \subset \mathcal{D}_{ST} \subset B^{ST} \subset P^{ST}$ and respectively $\mathcal{D}^m_Q \subset \mathcal{D}_Q \subset B^Q \subset P^Q$, for all $m \geq 1$.

B^Q is a connected arc joining two boundary points $(\lambda_1, 4)$ and $(\lambda_2, 4)$ and having a unique primary intersection (λ_0, μ_0). As before, the itinerary of γ_1

under (f_λ, f_μ) changes only at (λ_0, μ_0) as we move (λ, μ) along B^Q. Hence we can divide B^Q into two halves: left of (λ_0, μ_0), containing $(\lambda_1, 4)$ and right of (λ_0, μ_0), containing $(\lambda_2, 4)$.

$$B^Q = B^Q_- \cup \{(\lambda_0, \mu_0)\} \cup B^Q_+$$

I will study the left half, comparatively with the vertical left half $B^{ST}_{-,v}$.

The itinerary $\Im(\gamma_1)$ of γ_1 has the same form along both partial bones $B^{ST}_{-,v}$ and B^Q_-.Go along the half bones from the boundary point towards the primary intersection. For any fixed m, the itinerary $\Im^m(\gamma_2)$ changes at each distinguished point in $\bigcup_{k\leq m} \mathcal{D}^k$ and stays constant on the intervals between such points. There is an actual change in $\Im^m(\gamma_2)$ from one such interval to the adjacent one (through the critical state that occurs at the distinguished point). We proved that in P^{ST} this change is an actual decrease. In P^Q such a change is required in order for Thurston's uniqueness condition to hold. Indeed, as we move along the bone, an iterate of γ_2 may reach either critical point γ_1 or γ_2 at a value of the parameter (σ, τ) in some $\mathcal{D}^k$. This iterate can't bounce back to the left or right interval it came from.

We also know that there is a bijective correspondence between secondary intersections along $B^{ST}_{-,v}$ and B^Q_- that associates to each intersection in $B^{ST}_{-,v}$ one with identical $\Im(\gamma_2)$ in B^Q_- .We would like to prove that these secondary intersections occur on both $B^{ST}_{-,v}$ and B^Q_- in the same decreasing order of $\Im(\gamma_2)$, going from the boundary towards the primary intersection. In other words, we prove that the bijection is order preserving.

Fix $m \geq 1$. Call (l_1, m_1) the first distinguished point in $\bigcup_{k\leq m} \mathcal{D}^k_Q$ on B^Q_-

(from $(\lambda_1, 4)$ along the connected curve, with the regular order inherited by the order on $(0,1) \subset \mathbb{R}$).

Using theorems 1.5.3 and 1.7.2, there is a corresponding distinguished point $(\alpha, \beta) \in \bigcup_{k \leq m} \mathcal{D}^k_{ST} \subset B^{ST}_{-,v}$ with the same critical itineraries :

1. $\Im(\gamma_1)(\alpha, \beta) = \Im(\gamma_1)(l_1, m_1)$ and
2. $\Im(\gamma_2)(\alpha, \beta) = \Im(\gamma_2)(l_1, m_1)$

Claim. (α, β) is the first point to occur in $\bigcup_{k \leq m} \mathcal{D}^k_{ST}$ along $B^{ST}_{-,v}$.

Suppose not. Then there exists a point $(a^*, b^*) \in \bigcup_{k \leq m} \mathcal{D}^k_{ST}$ between $(a_1, 1)$ and (α, β). We then have (see picture):

$$\Im^m(\gamma_2)(a_1, 1) > \Im^m(\gamma_2)(\alpha^*, \beta^*) > \Im^m(\gamma_2)(\alpha, \beta)$$

$$\Im^m(\gamma_2)(a_1, 1) = \Im^m(\gamma_2)(\lambda_1, 4)$$

$$\Im^m(\gamma_2)(l_1, m_1) = \Im^m(\gamma_2)(\alpha, \beta)$$

The contradiction follows easily. (Note, for instance, that the conditions imply that the pair of critical itineraries at $(a_1, 1)$ has to be the same as the pair at a point right before (α, β)).

So the distinguished point in B^{ST}_{-} with itinerary $\Im(\gamma_2)(l_1, m_1)$ is the first to occur in $\bigcup_{k \leq m} \mathcal{D}^k_{ST}$. Continuing the procedure shows that the order of occurrence of all points in $\bigcup_{k \leq m} \mathcal{D}^k_{ST}$ along $B^{ST}_{-,v}$ is the same as the order of points in $\bigcup_{k \leq m} \mathcal{D}^k_{Q}$ along B^{Q}_{-} (i.e. the decreasing order of the itinerary $\Im^m(\gamma_2)$). We can state this as follows.

Theorem 1.12.1. *For a fixed $m \geq 1$, going along $B^{ST}_{-,v}$ from (a_0, b_0) to $(a_1, 1)$ and along B^Q_- from (λ_0, μ_0) to $(\lambda_1, 4)$, the itinerary $\Im^m(\gamma_2)$ is monotonely increasing, with actual changes occurring at each distinguished point in $\bigcup_{k\leq m} D^k_{ST}$ and $\bigcup_{k\leq m} D^k_Q$, respectively. Hence the infinite itinerary $\Im(\gamma_2)$ is monotonely increasing along $B^{ST}_{-,v}$.*

Proof. Follows from the previous considerations and from properties (a) and (b). □

Theorem 1.12.2. *For a fixed $m \geq 1$, going along $[0, 1] \times \{1\} \subset \partial P^{ST}$ and $[0, 4] \times \{4\} \subset \partial P^Q$, the itinerary $\Im(\gamma_2) = (\Gamma_2, R_1, \overline{L_2, L_1})$ stays constant, but the itinerary $\Im(\gamma_1)$ increases monotonically, with an actual change at each end-point of a bone of period $2k \leq 2m$.*

Proof. Similar to previous theorem. □

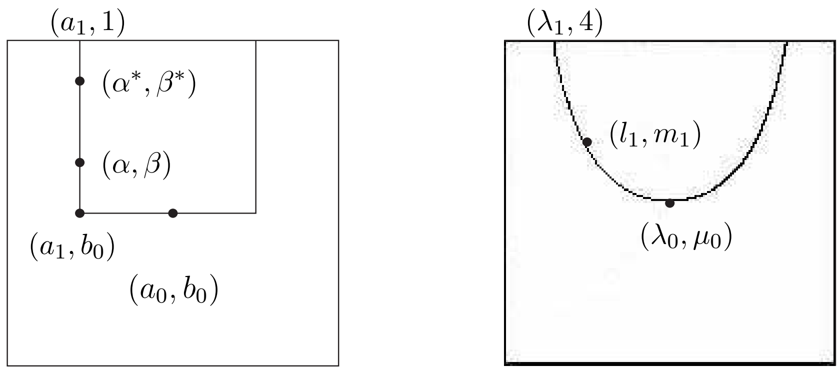


Figure 1.15: *If $\Im(\gamma_2)$ changes along $B^{ST}_{-,v}$ at (α^*, β^*) between $(a_1, 1)$ and (α, β), then $\Im^m(\gamma_2)$ would have to change along B^Q_- between $(\lambda_1, 4)$ and (l_1, m_1). This contradicts our assumption.*

1.13 The big picture

<u>**Overview of results**</u>:

Fix $m \geq 1$. Going along B_-^{ST} from (a_0, b_0) to $(a_1, 1)$ and along B_-^Q from (λ_0, μ_0) to $(\lambda_1, 4)$, the truncated itinerary $\Im^m(\gamma_2)$ increases monotonically, with an actual increase at each crossing with a right bone. Hence there is a one-to-one correspondence between the crossing points of bones of period at most $2m$ in the two families, correspondence that preserves the order of critical itineraries (i.e. of the joint order-data).

The end-points of left bones have the same critical itineraries along the upper boundary of the two parameter spaces ($[0,1] \times \{1\} \subset \partial P^{ST}$ and $[0,4] \times \{4\} \subset \partial P^Q$). There is a one-to-one correspondence between all boundary points of bones of period smaller than $2m$ in the two families, correspondence that preserves the order of the critical itineraries.

We want to restate the results in terms of kneading-data.

Take two points (λ_1, μ_1) and (λ_2, μ_2) on the same B_-^Q. They will have the same itinerary $\Im(\gamma_1)$ and different itineraries for γ_2. "$<$" is a total order on the space of itineraries of γ_2 along B_-^Q. Suppose $\Im^1(\gamma_2) << \Im^2(\gamma_2)$. If the left bone-arc B_-^Q is in the region $\{\lambda \leq 2\}$, we can think of the maps $f_{\mu_1} \circ f_{\lambda_1}$ and $f_{\mu_2} \circ f_{\lambda_2}$ as unimodal, with folding point γ_1 and identical kneading-data $\mathbf{K_1} = \mathbf{K_2}$. If not, the maps are 3-modal, and for either $i = 1$ or $i = 2$ their critical points are $C_1^i < C_2 = \gamma_1 < C_3^i$. The collection of interval-symbols is $\mathcal{A}^Q = \{I_0, C_1, I_1, C_2, I_2, C_3, I_3\}$ and kneading-data:

$$\mathbf{K}^i = (\mathcal{K}^i(C_1^i), \mathcal{K}^i(\gamma_1), \mathcal{K}^i(C_3^i))$$

Lemma 1.13.1. *Consider* $(\lambda, \mu) \in B_-^Q$ *such that* $f_\mu \circ f_\lambda$ *has critical points* $C_1 < \gamma_1 < C_3$*, and call*

$\Im(\gamma_2) = (J_0, J_1, ..., J_{2k}, J_{2k+1}, ...)$

$\mathcal{K}(C_1) = (A_1, A_2, ..., A_k, ...)$

Then, for any $k \geq 1$*:*

$A_k = I_0$ *iff* $(J_{2k-1}, J_{2k}) = (L_2, L_1)$

$A_k = I_1$ *iff* $(J_{2k-1}, J_{2k}) = (L_2, R_1)$

$A_k = I_2$ *iff* $(J_{2k-1}, J_{2k}) = (R_2, R_1)$

$A_k = I_3$ *iff* $(J_{2k-1}, J_{2k}) = (R_2, L_1)$

$A_k = C_1$ *iff* $(J_{2k-1}, J_{2k}) = (L_2, \Gamma_1)$

$A_k = C_3$ *iff* $(J_{2k-1}, J_{2k}) = (R_2, \Gamma_1)$

$A_k = C_2$ *iff* $J_{2k-1} = \Gamma_2$

Proof. The proof is an easy exercise. □

Lemma 1.13.2. *For* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in B_-^Q$ *we have* $\Im^1(\gamma_2) < \Im^2(\gamma_2)$ *implies* $\mathbf{K^1} << \mathbf{K^2}$*.*

Proof. $\Im^1(\gamma_2)$ and $\Im^2(\gamma_2)$ coincide up to a position k. If the common part contains a Γ_1 or Γ_2, then $\Im^1(\gamma_2) = \Im^2(\gamma_2)$, hence $\mathbf{K^1} = \mathbf{K^2}$. If not, the parity of the number of Rs in this common part is going to be reflected in the parity of the number of groups (R_2L_1) and (L_2R_1) in the corresponding common part of the kneading sequences. The result follows easily from the lemma. □

Remark. (1) A similar statement works for the boundary $[0, 4] \times \{4\}$, on which $\Im(\gamma_2)$ is constant. If $(\lambda_1, 1)$ and $(\lambda_2, 1)$ are such that $\Im^1(\gamma_1) < \Im^2(\gamma_1)$, then $\mathbf{K^1} << \mathbf{K^2}$.

(2) The results were proved for left half of bone-arcs, but are similar for right halves.

We restate two important conclusions.

Theorem 1.13.3. *For parameters $(\lambda, \mu) \in P^Q$, the kneading-data of the maps $f_\mu \circ f_\lambda$ increases along a left bone-arc from its primary intersection towards either boundary point and increases along the upper boundary interval $[0, 4] \times \{4\} \in \partial P^Q$ from left to right (see picture). A symmetric statement holds for right bones and the right boundary interval.*

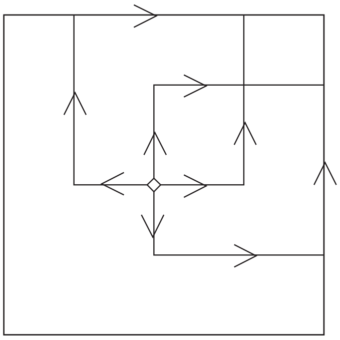
We know (see appendix D) that the order of the kneading-data of two maps is preserved into the order of their topological entropies. Hence:

Theorem 1.13.4. *The topological entropy increases in P^Q along each bone-arc from its primary intersection towards the boundary ∂P^Q and along the boundary segments $[0, 4] \times \{4\}$ and $\{4\} \times [0, 4]$ towards the upper right corner (see picture).*

Remark. The equivalent result for P^{ST} is discussed in detail in appendix B.

1.14 Important conclusions

To end this chapter and the combinatoric discussion, we want to point out two major results, most important for our later goals.



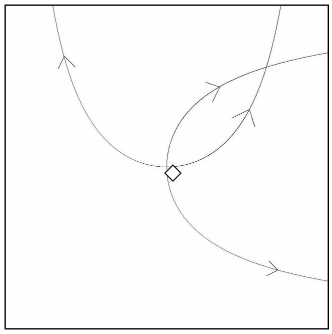


Figure 1.16: *The arrows show the direction of increasing entropy along the bones and the boundary in P^{ST} and P^Q.*

We showed that every bone in P^Q is composed of a bone-arc (that we called B^Q in a previous section) and possible loop components. These loops will eventually turn out not to be valid objects. For the time being, a step towards this conclusion follows as a consequence of Thurston's uniqueness: for any arbitrary left bone in P^Q, the bone-arc B^Q contains all possible post-critically finite kneading data (itineraries) admissible for the given bone. In consequence, any loop component that the bone may have can not contain any post-critically finite points.

Definition 1.14.1. Fix $n \in \mathbb{N}$. We define the n-skeleton in either parameter space to be :

S_n^{ST} = *the union of all (left and right) bones* $B_{2k}^{ST} \subset P^{ST}$ *of period* $2k \leq 2n$, *together with the boundary* ∂P^{ST};

S_n^Q = *the union of all (left and right) bones* $B_{2k}^Q \subset P^Q$ *of period* $2k \leq 2n$, *together with the boundary* ∂P^Q.

By a vertex of either skeleton we mean either an end-point of its bones or a (primary or secondary) intersection point.

Theorem 1.14.2. *For any fixed $n \in \mathbb{N}$, there is a homeo*

$$\eta_n : P^{ST} \longrightarrow P^Q$$

which maps S_n^{ST} onto S_n^Q, carrying ∂P^{ST} to ∂P^Q, carrying bones to corresponding bones and and vertexes to vertexes with the same data.

Proof. We use the result that will be proved independently in the next two chapters: the bones in P^Q are smooth $\mathcal{C}^1$ curves, intersecting transversally with each other and with the boundary. There are no bone loops in P^Q, so each bone is a smooth arc connecting two boundary points. Moreover, each such bone-arc contains all post-critically finite kneading-data existing on the corresponding bone in P^{ST}, in the same order of occurrence.

The construction of the homeomorphism is topologically straightforward. Define η_n on the set of vertexes by corresponding to each vertex in S_n^{ST} the unique one in S_n^Q with the same data. Along each bone, η_n preserves the order of the vertexes. Hence we can extend it continuously to the intervals on the bones or boundary between each two vertexes, then to each skeleton-enclosed region. This can easily be done in such a way that the resulting continuous map $\eta_n : P^{ST} \longrightarrow P^Q$ is a homeomorphism. $\square$

We can associate to the n-skeleton in either parameter space a topological cell-structure as follows:

- the 0-cells are points, more precisely the vertexes of the n-skeleton;
- the 1-cells are the connected components of the bones obtained by deleting the vertexes, hence they are homeo to open intervals;
- the 2-cells are the connected components of the complement of the n-

skeleton in the respective parameter space, hence they are homeo to open discs.

We will also use the closures of such cells, which are homeo to points, closed intervals and closed discs respectively.

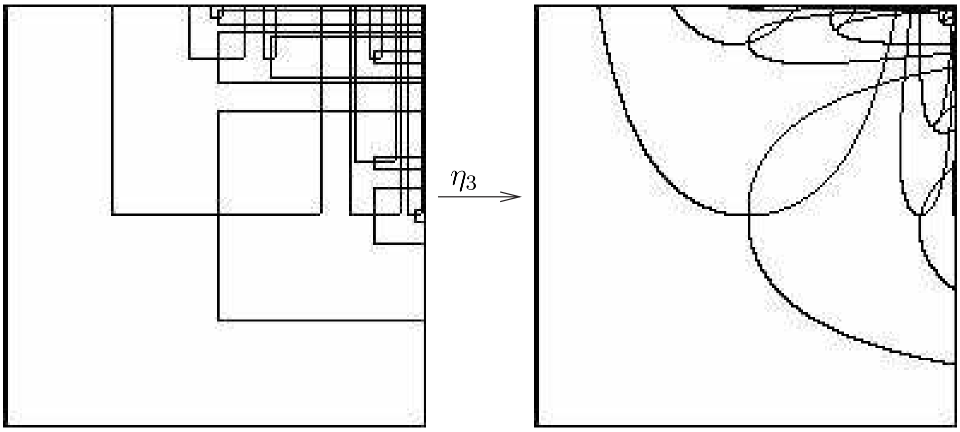


Figure 1.17: *The n-skeletons define topological cell-complexes in both parameter spaces. The map η_n is a homeomorphism between these complexes. The picture illustrates $n = 3$.*

We call the resulting complexes: P_n^{ST} in P^{ST} and P_n^Q in P^Q. The map $\eta_n : P_n^{ST} \longrightarrow P_n^Q$ is a homeomorphism of cell complexes, taking each cell in P_n^{ST} to a corresponding cell in P_n^Q by carrying vertexes to vertexes with the same entropy and edges to edges with the same interval of entropies.

Chapter 2

Smoothness of the Q-bones

2.1 Preliminary remarks

As we have stated before, a bone in P^Q is an algebraic variety with two boundary points in ∂P^Q. As far as we presently know, the bone curves may not even be connected. We will rule this out in chapter 3, where we show independently that a bone can't contain any loops. For now, we dedicate this paragraph to proving that:

Theorem 2.1.1. *The bones are smooth $\mathcal{C}^1$ curves that intersect transversally.*

Recall that we use the notations $B^Q_{L,2n}$ and $B^Q_{R,2n}$ for a left/right bone in P^Q of period $2n$ and given order-data. Fix an arbitrary point $p_0 = (\lambda_0, \mu_0)$ on a left bone $B^Q_{L,2n}$. We want to show that $B^Q_{L,2n}$ is smooth at $p_0 = (\lambda_0, \mu_0)$. For any $p = (\lambda, \mu)$ close to p_0, the corresponding map $f_\mu \circ f_\lambda$ has a unique periodic point close to $\gamma_1 = \frac{1}{2}$. The multiplier $m(\lambda, \mu)$ of this periodic point depends holomorphically on (λ, μ). We must prove that the partial derivatives $\frac{\partial m}{\partial \lambda}$ and $\frac{\partial m}{\partial \mu}$ are not simultaneously zero.

For the map $h = f_{\mu_0} \circ f_{\lambda_0}$, γ_1 has a superattracting periodic orbit of period $2n$. Let $U_h = U_h(\gamma_1)$ be the immediate attracting basin of γ_1. Hence, if $K(h)$ is the filled Julia set of h, then $U_h \subset K(h)$ is a simply connected bounded open neighbourhood of γ_1 that is carried to itself by $h^{\circ n}$. We point out the two cases that could appear, depending on the behavior of the other two (complex) critical points of h, called C_1 and C_3.

(1) The map h is not hyperbolic (i.e. C_1 and C_3 are not attracted to attracting cycles).

(2) The map h is hyperbolic (i.e. C_1 and C_3 are attracted). But each hyperbolic component in P^Q is biholomorpfic to a Blaschke model. Within each of these components, the locus of the maps with a specific superattracting orbit is a smooth complex manifold (see section 3.1.3 or [M1]).

2.2 Plan of proof for case 1

We will use quasiconformal surgery in the neighbourhood of our fixed map $h \in P^Q$. No iterates of the other two critical points of h belong to U_h, the immediate attracting basin of γ_1, hence U_h is isomorphic to the open unit disc, parametrized by its Bottcher coordinate. I.e., there exists a biholomorphic isomorphism that conjugates $h^{\circ n}$ to the squaring map $z \longrightarrow z^2$:

$\beta : U_h \longrightarrow \mathbb{D}$

$\beta(h^{\circ n}(z)) = (\beta(z))^2$

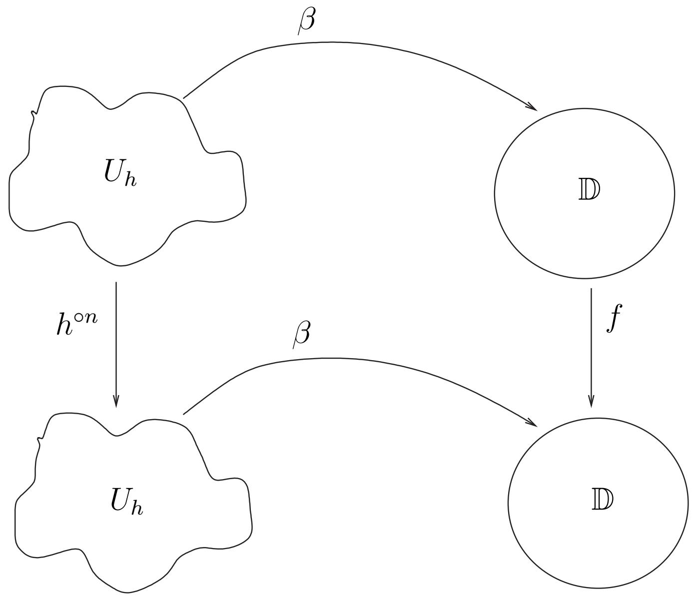


Figure 2.1: *The Bottcher isomorphism β conjugates $h^{\circ n}$ with the squaring map $f(z) = z^2$*

We want to replace the superattracting basin U_h by a basin with small positive multiplier Λ. For each Λ in a small disc centered at zero, we will construct a new map h_Λ corresponding to a $(\lambda_\Lambda, \mu_\Lambda) \in P^Q$ in such a way that $\Lambda \longrightarrow h_\Lambda \sim (\lambda_\Lambda, \mu_\Lambda)$ is analytic and that $h_0 = h$.

The composition of smooth (analytic) maps

$$\Lambda \longrightarrow h_\Lambda \sim (\lambda_\Lambda, \mu_\Lambda) \in P^Q \longrightarrow m(h_\Lambda)$$

is the identity. (Here m denotes again the function that assigns to each map in P^Q its multiplier at the specified attracting point). It follows that the partial derivatives $\frac{\partial m}{\partial \lambda}, \frac{\partial m}{\partial \mu}$ can't be simultaneously zero on a small neighbourhood of $h \in P^Q$. By the Implicit Function Theorem, the bone curve is smooth $\mathcal{C}^1$ on a small neighbourhood of h.

Consider the map $f(z) = z^2$ on the open unit disk $\mathbb{D}$ (which is the Bottcher parametrization of $h^{\circ n}$). Its unique critical point in $\mathbb{D}$ is the origin. Fix a small $\epsilon > 0$ (along the proof we will make specific requirements of how small we want ϵ to be) and let Λ be an arbitrary complex number such that $0 \leq \mid \Lambda \mid \leq \epsilon$.

We want to perturb the map f to a new degree 2 map g_Λ such that:

- g_Λ has the same dynamics as $f_\Lambda(z) = z^2 + \Lambda z$ inside a small disc around zero; in particular, the origin will be fixed, with multiplier Λ;

- g_Λ has the same dynamics as f outside a larger disc around zero.

We carry out the construction as follows:

Fix an $r < 1$, small. We will use concentric discs of radii $r^2 < r/2 < r$, so we want $r < \frac{1}{2}$ from the start.

For the fixed Λ, we consider the map:

$$f_\Lambda : \Delta_{r^2} \longrightarrow \mathbb{C}\ ,\ f_\Lambda(z) = z^2 + \Lambda z$$

We want to chose r such that we are sure that, for any Λ with $0 \leq \mid \Lambda \mid \leq \epsilon$, the following conditions, (1) and (2), are satisfied:

(1) f_Λ maps Δ_{r^2} into Δ_{r^2}

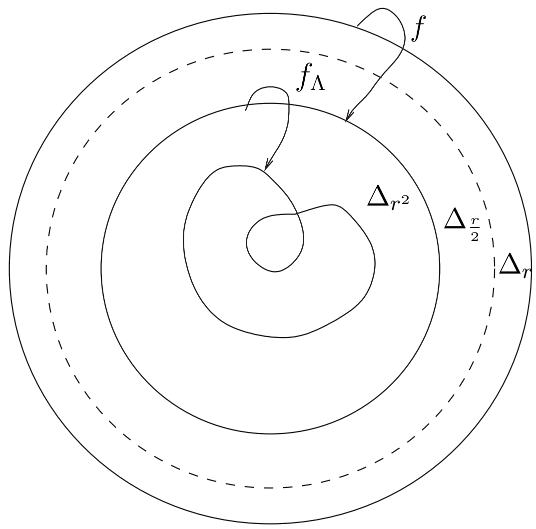


Figure 2.2: *We construct a map that coincides with f_Λ on the disc Δ_{r^2} and with the squaring map f on the exterior of the disc $\Delta_{\frac{r}{2}}$ (where $r < 1$ is chosen conveniently).*

But $\mid f_\Lambda(z) \mid = \mid z^2 + \Lambda z \mid = \mid z \mid\mid z + \Lambda \mid \leq r^2(r^2 + \epsilon)$

So it suffices to ask that $r^2 < 1 - \epsilon$, to have: $f_\Lambda(z) \in \Delta_{r^2},\ \forall z \in \Delta_{r^2}$

(2) the critical point of f_Λ is in Δ_{r^2}

We need $\mid -\frac{\Lambda}{2} \mid \leq r^2$. It is sufficient to ask for $\frac{\epsilon}{2} \leq r^2$.

Overall, it is sufficient to make our choice of the radius r such that:

$$\frac{\epsilon}{2} \leq r \leq min(\frac{1}{2}, 1 - \epsilon)$$

Using a partition of unity, we construct a smooth map that is identical with $f_\Lambda(z)$ inside Δ_{r^2} and with $f(z)$ outside $\Delta_{r/2}$. To make an explicit construction,start with:

$\xi : \mathbb{R} \longrightarrow \mathbb{R}$, $\xi(x) = e^{-\frac{1}{x}}, \quad \forall\, x > 0$

Then define:

$$\rho(z) = \begin{cases} \frac{\xi(\frac{r}{2}-|z|)}{\xi(\frac{r}{2}-|z|)+\xi(|z|-r^2)} & \text{, if } r^2 <\mid z \mid< r/2 \\ 0 & \text{, if } \mid z \mid\geq r/2 \\ 1 & \text{, if } \mid z \mid\leq r^2 \end{cases}$$

It is an easy exercise to show that $\rho : \mathbb{C} \longrightarrow \mathbb{R}$ is a smooth $\mathcal{C}^1$ map. We define $g_\Lambda : \mathbb{C} \longrightarrow \mathbb{C}$ as:

$$g_\Lambda(z) = z^2 + \Lambda\rho(z)z$$

Clearly g_Λ is $\mathcal{C}^1$ smooth and

$g_\Lambda(z) = f_\Lambda(z)$ in Δ_{r^2}

$g_\Lambda(z) = f(z)$ in ${}^C\Delta_{r/2}$

We would like to also have that: (1) g_Λ is a 2-to-1 map on $\mathbb{D}$ and that (2) it carries $\Delta_{r/2} \backslash \Delta_{r^2}$ into Δ_{r^2}.

$\mid g_\Lambda(z) \mid = \mid z^2 + \Lambda\rho(z)z \mid \leq \mid z \mid (\mid z \mid + \epsilon)$

To get condition (2) it suffices to ask for:

$$\frac{r}{2}(\frac{r}{2} + \epsilon) \leq r^2, \text{ i.e. } \frac{2\epsilon}{3} \leq r$$

For our fixed ϵ, fix an r such that $\frac{2\epsilon}{3} \leq r \leq 1 - \epsilon$. Then take ϵ smaller (if necessary), such that g_Λ has no critical point outside Δ_{r^2}, for any $0 \leq \mid \Lambda \mid \leq \epsilon$. (Recall that the critical point of $g_0(z) = f(z) = z^2$ is $0 \in \Delta_{r^2}$ and the dependence $\Lambda \longrightarrow g_\Lambda$ is smooth for $\mid \Lambda \mid \leq \epsilon$). This will insure that the map g_Λ is 2-1 on $\mathbb{D}$, for all Λ less in absolute value than the chosen small ϵ.

To summarize: For any fixed $\mid \Lambda \mid \leq \epsilon$, the map $g_\Lambda : \mathbb{D} \rightarrow \mathbb{D}$ constructed above is a 2-to-1 $\mathcal{C}^1$ smooth map that carries $\Delta_r \backslash \Delta_{r^2}$ into Δ_{r^2} and carries Δ_{r^2} into itself. g_Λ coincides with f_Λ inside Δ_{r^2} and with f outside of $\Delta_{\frac{r}{2}}$ (in particular it is conformal outside Δ_r) and has no critical points in $\Delta_r \backslash \Delta_{r^2}$. We would like to emphasize that, as $\Delta_r \backslash \Delta_{r^2}$ is mapped by g_Λ directly into Δ_{r^2}, the annulus $\Delta_r \backslash \Delta_{r^2}$ is intersected at most once by any orbit under g_Λ.

We pull g_Λ back to U_h through the Bottcher biholomorpic diffeomorphism β:

$$G_\Lambda = \beta^{-1} \circ g_\Lambda \circ \beta \, : U_h \rightarrow U_h$$

The new map G_Λ is 2-to-1 and $\mathcal{C}^1$ smooth, and has similar properties as the ones stated above for g_Λ (see figure):

It carries X_h to W_h and W_h to itself. G_Λ coincides with $\beta^{-1} \circ f \circ \beta = h^{\circ n}$

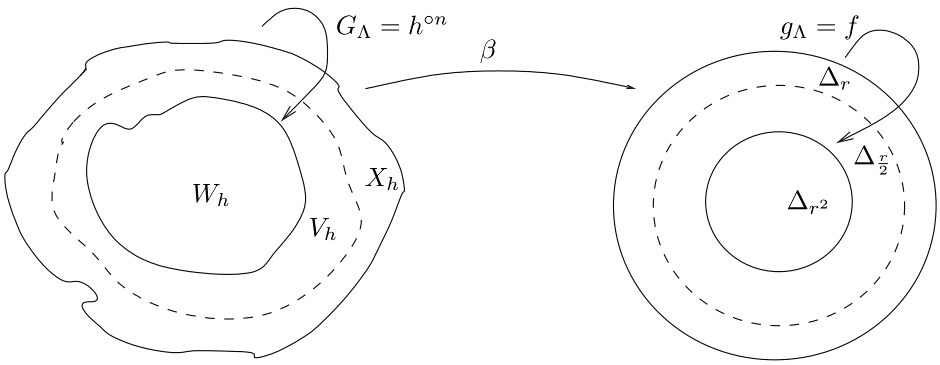


Figure 2.3: *X_h, V_h and W_h are the preimages under the Bottcher map β of Δ_r, $\Delta_{\frac{r}{2}}$ and Δ_{r^2}, respectively. The map $g_\Lambda : \mathbb{D} \to \mathbb{D}$ pulls back as the $\mathcal{C}^1$-map G_Λ, that acts as $h^{\circ n}$ outside V_h and carries V_h to W_h.*

outside V_h. It has a critical point inside W_h and none in the annular region $X_h \backslash W_h$. Also, any orbit under G_Λ crosses $X_h \backslash W_h$ at most once.

Recall that what we ultimately need is a small distortion of $h \in P^Q$ to a $h_\Lambda \in P^Q$ with similar dynamics, that replaces the superattracting cycle of h by a cycle with multiplier Λ.

We notice that $h : \mathbb{C} \to \mathbb{C}$ carries

$$U_h \to h(U_h) \xrightarrow{\sim} ... \xrightarrow{\sim} h^{\circ(n-1)}(U_h) \xrightarrow{\sim} h^{\circ n}(U_h) = U_h$$

(acting as a diffeo except on U_h).

We define H_Λ as

$H_\Lambda = h$ outside V_h

and such that

$h^{\circ(n-1)} \circ H_\Lambda = G_\Lambda$ inside X_h (i.e. $H_\Lambda = h^{\circ(1-n)} \circ G_\Lambda$)

The new H_Λ is $\mathcal{C}^1$ (notice that the two definitions coincide on $X_h \backslash V_h$) and

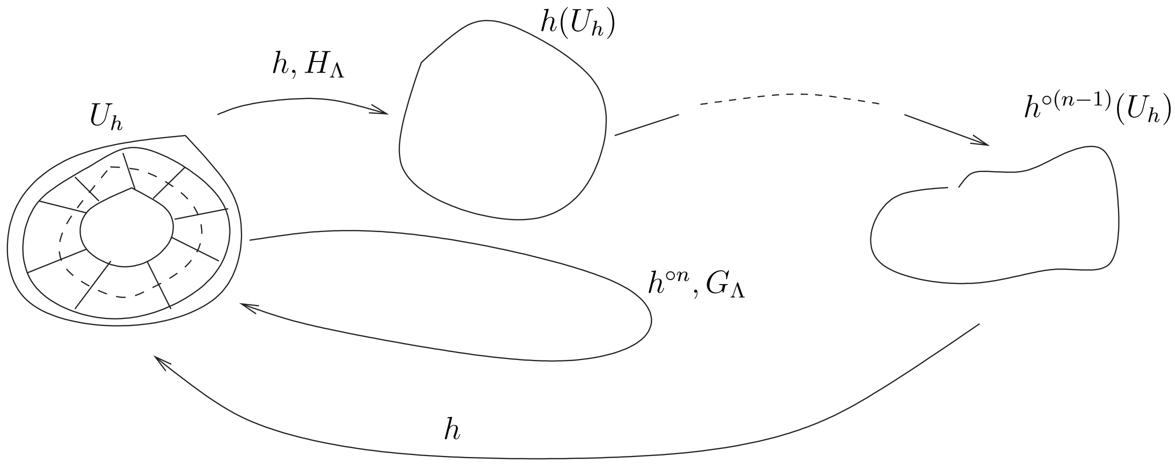


Figure 2.4: *Using $G_\Lambda : U_h \to U_h$ to replace $h^{\circ n}$, we can construct $H_\Lambda : U_\Lambda \to h(U_\Lambda)$ to replace h. Extend it as $H_\Lambda = h$ on $\mathbb{C} \backslash U_h$.*

has the desired dynamical behavior. However, it may fail to be analytic, hence it may not be a map in P^Q. The rest of the construction aims to transform H_Λ into a polynomial $h_\Lambda \in P^Q$, preserving the dynamics.

The Beltrami dilatation of H_Λ is:

$$\mu_{H_\Lambda}(z) = \frac{(H_\Lambda)_{\overline{z}}}{(H_\Lambda)_z}$$

$H_\Lambda = h$ outside V_h, hence $\mu_{H_\Lambda}(z) = 0$ outside V_h.

$h^{\circ(n-1)} \circ H_\Lambda = G_\Lambda$ inside X_h, and $h^{\circ(n-1)}$ is conformal, hence $\mu_{H_\Lambda} = \mu_{G_\Lambda}$ on X_h.

But $\beta \circ G_\Lambda = g_\Lambda \circ \beta$ and β is conformal, so:

$\mu_{\beta \circ G_\Lambda}(z) = \mu_{G_\Lambda}(z)$ and

$\mu_{g_\Lambda \circ \beta}(z) = \mu_{g_\Lambda}(\beta(z))\frac{\overline{\beta'(z)}}{\beta'(z)}$

Hence on X_h:

$$\mu_{H_\Lambda}(z) = \mu_{G_\Lambda}(z) = \mu_{g_\Lambda}(\beta(z))\frac{\overline{\beta'(z)}}{\beta(z)}$$

Recall that g_Λ has no critical point in $\Delta_r \backslash \Delta_{r^2}$, so $(g_\Lambda)_z \neq 0$ on $\Delta_r \backslash \Delta_{r^2}$. Hence the denominator of:

$$\mu_{g_\Lambda}(z) = \frac{(g_\Lambda)_{\overline{z}}}{(g_\Lambda)_z}$$

never vanishes. Moreover, for fixed z, both top and bottom above are linear in Λ, so it follows easily that:

$$\Lambda \rightarrow \mu_{g_\Lambda}$$

is an analytic dependence.

Under iteration of g_Λ, points hit the annulus $\Delta_r \backslash \Delta_{r^2}$ at most once, hence μ_{g_Λ} is bounded less than 1 in modulus.

$\mu_{H_\Lambda}(z)$ therefore depends itself analytically on Λ and :

$\mid \mu_{H_\Lambda}(z) \mid = \mid \mu_{g_\Lambda}(\beta(z)) \mid \mid \frac{\overline{\beta'(z)}}{\beta'(z)} \mid = \mid \mu_{g_\Lambda}(\beta(z)) \mid \leq 1$ on $X_h \backslash W_h$ and

$\mu_{H_\Lambda}(z) = 0$ outside $X_h \backslash W_h$.

We define an ellipse field starting with circles inside W_h and outside all preimages of X_h under H_Λ and pulling it back invariantly under H_Λ. All orbits hit $X_h \backslash W_h$ (the annular region where H_Λ is not analytic) at most once, so the ellipse field is distorted at most once along any orbit. Let μ_Λ be the coefficient of this field. The dependence of μ_Λ on Λ is holomorphic on $\mid \Lambda \mid \leq \epsilon$.

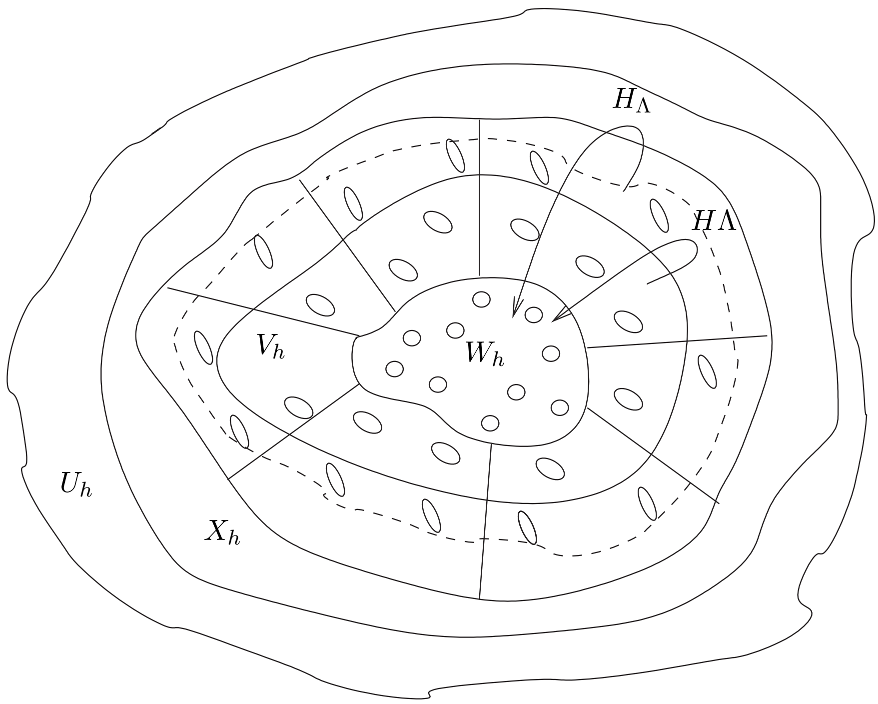


Figure 2.5: *The infinitesimal circles inside W_h are pulled back under H_Λ^{-1} to ellipses in V_h (the region with radial marks).*

Let ϕ_Λ solve the Beltrami equation:

$$\frac{\phi_{\overline{z}}}{\phi_z} = \mu_\Lambda$$

determined uniquely by the normalization $\phi_\Lambda(0) = 0$, $\phi_\Lambda(1) = 1$, $\phi_\Lambda(\infty) = \infty$,

With this choice for ϕ_Λ, $h_\Lambda = \phi_\Lambda \circ H_\Lambda \circ \phi_\Lambda^{-1}$ is a quartic complex polynomial. To validate the construction, we have to show that, for $\Lambda \in \mathbb{R}$, $\mid \Lambda \mid < \epsilon$, h_Λ is a polynomial in our family P^Q. Start by showing that h_Λ has real coefficients, in other words:

$$\overline{h_\Lambda(\overline{z})} = h_\Lambda(z),\ \forall z \in \mathbb{C}$$

To begin with, recall that $g_\Lambda(z) = z^2 + \Lambda\rho(z)z$, where $\rho(z) = \rho(\overline{z}) \in \mathbb{R}$. Hence:

$g_\Lambda(\overline{z}) = \overline{z}^2 + \Lambda\rho(z)\overline{z} = \overline{g_\Lambda(z)}$

Pulling back through the Bottcher coordinate, we remark that $\beta(\overline{z}) = \overline{\beta(z)}$, and so all three domains W_h, V_h and X_h are symmetric (i.e. $z \in W_h \Leftrightarrow \overline{z} \in W_h$ etc). On X_h:

$$G_\Lambda(\overline{z}) = \beta^{-1}(g_\Lambda(\beta(\overline{z}))) = \beta^{-1}(g_\Lambda(\overline{\beta(z)})) = \beta^{-1}(\overline{g_\Lambda(\beta(z))}) =$$

$$= \overline{\beta^{-1}(g_\Lambda(\beta(z)))} = \overline{G_\Lambda(z)}$$

Clearly, the property transfers to H_Λ:

$H_\Lambda(\overline{z}) = h(\overline{z}) = \overline{h(z)} = \overline{H_\Lambda(z)}$ outside V_h and

$H_\Lambda(\overline{z}) = h^{\circ(1-n)}(G_\Lambda(\overline{z})) = \overline{H_\Lambda(z)}$ inside $X_h \backslash V_h$

So $H_\Lambda : \mathbb{C} \to \mathbb{C}$ commutes with complex conjugation. We want to prove that ϕ_Λ has the same property.

We use the chain rule:

$\overline{\phi_z} = \overline{\phi}_{\overline{z}}$ and $\overline{\phi_{\overline{z}}} = \overline{\phi}_z$

$(\phi(\overline{z}))_z = \phi_{\overline{z}}(\overline{z})$ and $(\phi(\overline{z}))_{\overline{z}} = \phi_z(\overline{z})$

As G_Λ is symmetric, we have:

$$\mu_{G_\Lambda}(\overline{z}) = \frac{(G_\Lambda)_{\overline{z}}}{(G_\Lambda)_z}(\overline{z}) = \frac{(G_\Lambda(\overline{z}))_z}{(G_\Lambda(\overline{z}))_{\overline{z}}} = \frac{(\overline{G_\Lambda(z)})_z}{(\overline{G_\Lambda(z)})_{\overline{z}}} = \frac{\overline{(G_\Lambda)_{\overline{z}}(z)}}{\overline{(G_\Lambda)_z(z)}} =$$

$$= \overline{(\frac{(G_\Lambda)_{\overline{z}}}{(G_\Lambda)_z})} = \overline{\mu_{G_\Lambda}(z)}$$

Hence μ_{H_Λ} is symmetric. μ_Λ is also clearly symmetric. Recall that ϕ_Λ is the solution of the Beltrami equation

$$\frac{\phi_{\overline{z}}}{\phi z} = \mu_\Lambda$$

normalized by fixing three points. We claim that ϕ_Λ is symmetric. Indeed:

$$\frac{\overline{\phi_\Lambda(\overline{z})}_{\overline{z}}}{\overline{\phi_\Lambda(\overline{z})}_z} = \frac{\overline{(\phi_\Lambda(\overline{z}))_z}}{\overline{(\phi_\Lambda(\overline{z}))_{\overline{z}}}} = \frac{(\overline{\phi_\Lambda})_{\overline{z}}(\overline{z})}{(\overline{\phi_\Lambda})_z(\overline{z})} = \overline{(\frac{(\phi_\Lambda)_{\overline{z}}}{(\phi_\Lambda)_z})}(\overline{z}) =$$

$$= \overline{\mu_\Lambda}(\overline{z}) = \mu_\Lambda(z) = \frac{(\phi_\Lambda)_{\overline{z}}}{(\phi_\Lambda)_z}$$

By the uniqueness of solution of the Beltrami equation:

$$\overline{\phi_\Lambda(\overline{z})} = \phi_\Lambda(z)$$

Hence ϕ_Λ is symmetric, and so $h_\Lambda = \phi_\Lambda \circ H_\Lambda \circ \phi_\Lambda^{-1}$ is a symmetric quartic polynomial, i.e. a quartic polynomial with real coefficients.

Furthermore: $h_\Lambda(0) = (\phi_\Lambda \circ H_\Lambda \circ \phi_\Lambda^{-1})(0) = 0$ and $h_\Lambda(1) = (\phi_\Lambda \circ h \circ \phi_\Lambda^{-1})(1) = 0$.

Also, h_Λ has a critical point C_2 inside $\phi_\Lambda(X_h)$. The other two, C_1 and C_3, are outside $\phi_\Lambda(X_h)$ and given by the images under ϕ_Λ of the critical points in $\mathbb{C}\backslash X_h$ of $H_\Lambda = h$. Hence they are such that $h_\Lambda(C_1) = h_\Lambda(C_3)$. By appendix A, $h_\Lambda \in P^Q$.

Since H_Λ has an attracting cycle of length n , so does h_Λ, and the multiplier of the cycle is the same: $m(h_\Lambda) = \Lambda$.

Finally, from a well known theorem of Ahlfors [AB], ϕ_Λ depends holomorphically on the parameter Λ, as the coefficient μ_Λ depends holomorphically on Λ. Hence the solution ϕ_Λ depends holomorphically on Λ, so h_Λ depends smoothly on the parameter Λ, which completes our construction and the proof of case 1 of the theorem.

2.3 The mapping schema of a hyperbolic map

In section 2.2, we considered an arbitrary parameter point $p = (\lambda, \mu)$ situated on a left bone $B = B^Q_{L,2n}$ (i.e. such that γ_1 is periodic superattracting under $f_\mu \circ f_\lambda$). We were able to carry out a construction that proved smoothness of the bone curve B at p, provided that p was chosen such that there were no iterates of the other critical points of $f_\mu \circ f_\lambda$ in the immediate attracting basin of γ_1. Although such parameters p also cover some hyperbolic cases, we prefer to discuss the hyperbolic maps separately in the present section, to obtain consistent results that will also be of further use in chapter 3.

We would like to detail the description of hyperbolic maps in P^Q by defining the notion of a hyperbolic component and by classifying the types of hyperbolic components that can appear along a left bone in our parameter space P^Q.

Definition 2.3.1. *Let M be a finite disjoint union of copies of $\mathbb{C}$ and let $f : M \longrightarrow M$ be a proper holomorphic map of degree ≥ 2 on each component of M. We say that f is hyperbolic if every critical orbit converges to an attracting cycle.*

Let f be a hyperbolic map as above. Let $W(f)$ be the union of the basins of attraction of all attracting cycles of f. f carries each component $W_\alpha \subset W(f)$ onto a component W_β by a map of degree $d_\alpha \geq 1$. Also let $W^c(f)$ be the union of all critical components $W_\alpha \subset W(f)$, that is of all W_α that contain critical points of f.

We define the reduced mapping schema $\overline{S}(f) = (\mid S \mid, F, w)$ associated to f as the triplet made of:

• a set of vertexes $\mid S \mid$, obtained by associating a vertex α to each critical component $W_\alpha \subset W^c(f)$;

• a weight function $w :\mid S \mid \longrightarrow \mid S \mid$, defined as $w(\alpha) =$ the number of critical points of f in W_α;

• a set of edges $F :\mid S \mid \longrightarrow \mid S \mid$, $F(\alpha) = \beta$, where W_β is the image of W_α under the first return map to $W^c(f)$.

The critical weight of $\overline{S}(f)$ is defined as $w(f) = \sum_{\alpha} w(\alpha)$

2.4 Hyperbolic maps in $\mathbf{P^Q}$

Let us return to our space, containing real quartic polynomials that are compositions $f_\mu \circ f_\lambda$ of logistic maps $f_\lambda : I_1 \longrightarrow I_2$ and $f_\mu : I_2 \longrightarrow I_1$. $f_\mu \circ f_\lambda$ is hyperbolic if all three critical points are attracted to attracting cycles under iterations of the map. Alternatively, we can express this as follows:

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two copies of the complex plane and consider $f_\lambda : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ and $f_\mu : \mathbb{C}_2 \longrightarrow \mathbb{C}_1$ the complex extensions of two fixed logistic maps of the interval. We define a new map: $f_\lambda^\mu : \mathbb{C}_1 \sqcup \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \cup \mathbb{C}_2$, acting as f_λ on $\mathbb{C}_1$ and as f_μ on $\mathbb{C}_2$.

Let $W(f_\lambda^\mu) \subset \mathbb{C}_1 \sqcup \mathbb{C}_2$ be the open set consisting of all complex numbers in $\mathbb{C}_1$ and $\mathbb{C}_2$ whose forward orbit under f_λ^μ converges to an attracting periodic orbit of f_λ^μ.

Under iteration of f_λ^μ, each component of $W(f_\lambda^\mu)$ is mapped onto a component of $W(f_\lambda^\mu)$. As before, we will say that f_λ^μ is hyperbolic if both $\gamma_1 \in I_1 \subset \mathbb{C}_1$ and $\gamma_2 \in I_2 \subset \mathbb{C}_2$ are contained in $W(f_\lambda^\mu)$. Such hyperbolic maps can be roughly classified into the three following types (see [M3]):

(1) **Bitransitive case**: γ_1 and γ_2 belong to $U_1 \subset \mathbb{C}_1$ and $U_2 \subset \mathbb{C}_2$ such that: U_1 is mapped to U_2 under q_1 iterates of f_λ^μ and U_2 is mapped to U_1 under q_2 iterates. All primary intersections of bones in P^Q provide us with particular examples of such maps.

(2) **Capture case**: $\gamma_1 \in U_1 \subset \mathbb{C}_1$ and $\gamma_2 \in U_2 \subset \mathbb{C}_2$ such that U_1 is

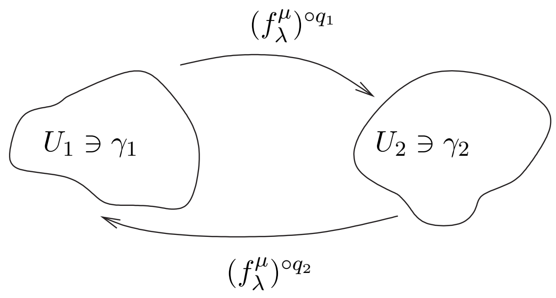


Figure 2.6: *The behavior of a bitransitive hyperbolic map.*

periodic and U_2 is not, but some forward image of U_2 coincides with U_1. Also its symmetric case. Example: all capture points along a bone.

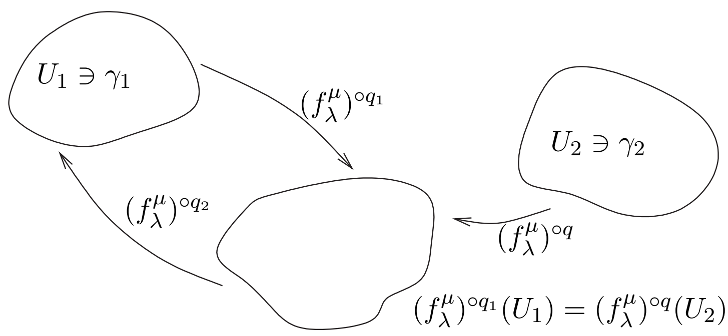


Figure 2.7: *The behavior of a map in the capture case.*

(3) **Disjoint periodic sinks**: $\gamma_1 \in U_1$ and $\gamma_2 \in U_2$, where U_1 and U_2 are periodic of periods q_1 and q_2, but no forward image of U_1 coincides with U_2 and vice-versa. Example: secondary intersections along a bone.

Remark. If our parameter $p = (\lambda, \mu)$ is on a left bone in P^Q and falls under **case 3** (disjoint sinks), then there is no iterate of another critical point of f_λ^μ in the immediate superattracting basin of γ_1. Hence this case could also be

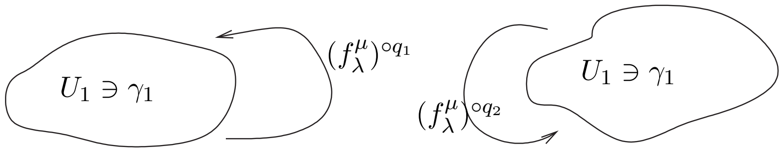


Figure 2.8: *The behavior of a map in the disjoint sinks case.*

classified by the quasiconformal surgery argument in section 2.2, as we claimed at the start of this section.

[M1] provides us with enough information to understand the structure of the hyperbolic components in P^Q. However, all results are stated and proved for families of monic centered polynomials of arbitrary fixed degree d (we call these normal polynomials of fixed degree d). So it is convenient to show the correspondence between our family of pairs of real quadratic maps, parametrized by $(\lambda, \mu) \in P^Q$ and the family of degree 2 normal polynomials. More precisely, we will show that each map $f_\mu \circ f_\lambda : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ is conjugated by a complex affine map L to a composition of maps $z \longrightarrow z^2 + a_1$ and $z \longrightarrow z^2 + a_2$. Moreover, the correspondence $(\lambda, \mu) \to (a_1, a_2)$ is "nice" enough to permit us to carry over to P^Q properties we have in the space of normal forms.

2.5 A re-parametrization of $\mathbf{P^Q}$

Theorem 2.5.1. *Let U be the subset of P^Q consisting of pairs (λ, μ) with $\lambda\mu > 1$. For each such pair $(\lambda, \mu) \in U$ there is a unique pair $(A, B) \in \mathbb{R}^2$ such that $f_\mu \circ f_\lambda$ is linearly conjugate to $z \longrightarrow z^4 + Az^2 + B$; there also exists a* *unique pair $(a_1, a_2) \in \mathbb{R}^2$ so that $f_\mu \circ f_\lambda$ is linearly conjugate to the composition of $z \longrightarrow z^2 + a_1$ and $z \longrightarrow z^2 + a_2$.*

Furthermore, let us define the connectedness locus $\mathcal{C}_\mathbb{R} \subset \mathbb{R}^2$ to be the subset of parametes $(a_1, a_2) \in \mathbb{R}^2$ for which the complex critical points of $(z^2+a_1)^2+a_2$ have bounded orbits. The correspondence described above:

$$\Xi : U \longrightarrow \mathcal{C}_\mathbb{R}$$

$$\Xi(\lambda, \mu) = (a_1, a_2)$$

is a bijective diffeomorphism.

Proof. Consider $(\lambda, \mu) \in U$. The map $f_\mu \circ f_\lambda : \mathbb{C} \to \mathbb{C}$ is conjugate to a quartic complex polynomial in the normal form $P(z) = z^4 + Az^2 + B$. More precisely: there exists a unique linear $L(z) = mz + b$ such that

$$L^{-1} \circ (f_\mu \circ f_\lambda) \circ L = P$$

The map L can be computed explicitely:

$$m = -\frac{1}{\sqrt[3]{\lambda^2\mu}} \text{ and } b = \frac{1}{2}$$

and so the pair (A, B) could be expresses as

$$A = \frac{\mu\lambda(2-\lambda)}{2\sqrt[3]{\lambda^2\mu}} \text{ and } B = \frac{[8-\mu\lambda(4-\lambda)]\sqrt[3]{\lambda^2\mu}}{16}$$

Then we can write:

$$z^4 + Az^2 + B = (z^2 + \frac{A}{2})^2 + (B - \frac{A^2}{4})$$

In conclusion: each $f_\mu \circ f_\lambda$ with $(\lambda, \mu) \in P^Q$ is conjugated by the affine map $L(z) = -\frac{1}{\sqrt[3]{\lambda^2\mu}}z + \frac{1}{2}$ to a composition of the two monic centered quadratic complex maps: $z \longrightarrow w = z^2 + a_1$ and $w \longrightarrow z = w^2 + a_2$, where $a_1(\lambda, \mu) = \frac{A(\lambda,\mu)}{2}$ and $a_2(\lambda, \mu) = B - \frac{A^2(\lambda,\mu)}{4}$.

Appendix C shows that the correspondence

$$\Phi : U \longrightarrow \mathbb{R}^2, \; \Phi(\lambda, \mu) = (A, B)$$

is a diffeomorphism onto its image; hence $\Xi(\lambda, \mu) = (a_1, a_2)$ is also a diffeo onto its image. We want to show that the image $\Xi(U)$ is exactly the connectedness locus $\mathcal{C}_\mathbb{R}$.

To do this, we consider an arbitrary map in $\mathcal{C}_\mathbb{R}$:

$$P(z) = (z^2 + a_1)^2 + a_2 = z^4 + Az^2 + B$$

We want to find a pair $(\lambda, \mu) \in P^Q$ such that the map $f_\mu \circ f_\lambda$ is linearly conjugate to P. We look at the map $Q(z) = -P(-z) = -P(z) = -z^4 - Az^2 - B$. Q is symmetric, linearly conjugate to P and may have three real critical points $C_1 \le C_2 = 0 \le C_3 = -C_1$ or one real critical point $C_2 = 0$ and the other two complex. However, as we started with $P \in \mathcal{C}_\mathbb{R}$, the orbits of the critical points of Q are also bounded.

Suppose $C_1 \le C_2 \le C_3$ real. Then there exists a $\xi \in (-\infty, C_1)$ fixed point of Q such that $Q'(\xi) > 1$. Otherwise $Q(x) \leq x$, $\forall x \in \mathbb{R}$, hence the orbits of the critical points are unbounded, contradiction.

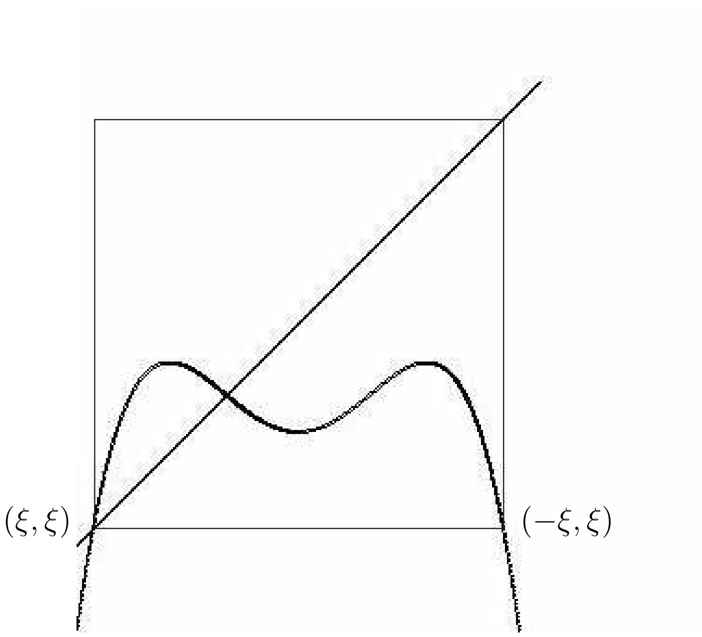


Figure 2.9: *Q takes the interval $[\xi, -\xi]$ into itself, and the restriction of Q to $[\xi, -\xi]$ is boundary anchored.*

So we have that $Q(\xi) = Q(-\xi) = \xi$. If we look at the restriction of Q to the interval $[\xi, -\xi]$, clearly Q is boundary anchored and $Q([\xi, -\xi]) \subset [\xi, -\xi]$, because poins outside $[\xi, -\xi]$ are repelled to ∞ under iterations of Q.

We conjugate Q to a polynomial S by a linear map $L(z) = mz + b$ such that $S(0) = S(1) = 0$. Explicitely, we look for L such that $L(0) = \xi$ and $L(1) = -\xi$, which gives us $L(z) = \xi(1 - 2z)$. The map S can be written as a composition $f_\mu \circ f_\lambda$ (see appendix A). Moreover, the boundary of the unit interval is repelling for S, hence $(\lambda, \mu) \in U$.

We follow the same procedure if C_1 and C_3 are complex. We find $\xi \in (-\infty, C_2)$ such that $Q'(\xi) > 1$. We conjugate Q to an S, boundary anchored on the unit interval, by $L(z) = \xi(1-2z)$. The proof of lemma A2 in appendix A shows how $S = f_\mu \circ f_\lambda$, with $(\lambda, \mu) \in [0,4]^2$. Automatically, $(\lambda, \mu) \in U$.

So we proved that, for each $(a_1, a_2) \in \mathcal{C}_{\mathbb{R}}$, there exists a pair $(\lambda, \mu) \in U$ such that $\Xi(\lambda, \mu) = (a_1, a_2)$. $\square$

Remarks. (1) Some maps Q in the proof above have two fixed points ξ_1, ξ_2 in the interval $(-\infty, C_1)$, with $Q'(\xi_1) > 1$ and $Q'(\xi_2) < 1$. This allows us two constructions: of a map S_1 that is a composition $f_{\mu_1} \circ f_{\lambda_1}$, with $(\lambda_1, \mu_1) \in U$, and of a second map S_2 such that $S_2 = f_{\mu_2} \circ f_{\lambda_2}$, with $(\lambda_2, \mu_2) \in^C \overline{U}$ (see figure 3.2).

(2) *All bones in P^Q are contained in U.* Indeed, suppose there is a (λ, μ) on a bone such that $(\lambda, \mu) \notin U$. The fixed origin is not repelling for the map $f_\mu \circ f_\lambda$ with negative Schwarzian derivative, so it attracts all critical points, hence (λ, μ) can't be on a bone, contradiction.

To continue, we will prove smoothness of bones in the family of normal maps, using [M1]. The result will automatically follow in P^Q.

2.6 The hyperbolic components model

Recall that we defined in 2.3 the mapping schema of a hyperbolic map (also see [M1]). All hyperbolic maps that interest us have reduced mapping schemata of critical weight 2, so we will only analyze the cases that appear for $w = 2$.

To a fixed mapping schema with $w = 2$, we associate the universal polynomial model space $\mathcal{P}$. This will be the space of all maps f from $\mathbb{C}_1 \sqcup \mathbb{C}_2$ to itself such that the restriction of f to each copy of $\mathbb{C}$ is a monic centered polynomial of degree 2. More precisely:

$$f(z) = z^2 + a_1, \text{ for all } z \in \mathbb{C}_1$$
$$f(z) = z^2 + a_2, \text{ for all } z \in \mathbb{C}_2$$

where $a_1, a_2 \in \mathbb{C}$.

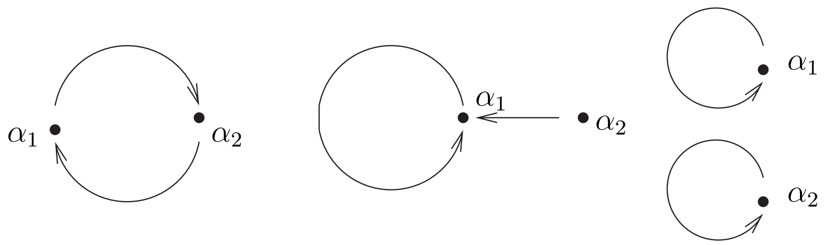
Remark Our P^Q family, re-parametrized as a family of normal polynomials, is the subset of real polynomials in $\mathcal{P}$.

We say that a map $f \in \mathcal{P}$ belongs to the connectedness locus $\mathcal{C}$ if its filled Julia set $K(f)$ intersects both $\mathbb{C}_1$ and $\mathbb{C}_2$ in a connected set. The hyperbolic connectedness locus $\mathcal{H} \subset \mathcal{C}$ is the open set of all $f \in \mathcal{P}$ for which the orbits of both critical points $0 \in \mathbb{C}_1$ and $0 \in \mathbb{C}_2$ converge to attracting periodic orbits.

For maps $f \in \mathcal{H}$, we may consider their reduced mapping schemata $\overline{S}(f)$. These schemata will all have critical weight 2, but not all are isomorphic. However, all maps in each connected component of $\mathcal{H}$ clearly have isomorphic schemata. Furthermore, by theorem 4.1 in [M1]:

Theorem 2.6.1. *If $H_\alpha \subset \mathcal{C}$ is a hyperbolic component of $\mathcal{H}$ with maps having reduced schemata isomorphic to S, then H_α is diffeomorphic to a model space $B(S)$. In particular, any two hyperbolic components H_α and H_β with schemata isomorphic to S are diffeomorphic. Moreover, each H_α contains a unique post-critically finite map f_α, called its center.*

Looking at the classification of hyperbolic components shown in section 2.4, we have a different reduced schema for each case as shown in figure 2.10.



(1) (2) (3)

Figure 2.10: *(1) Bitransitive case:* $\mid S \mid = \{\alpha_1, \alpha_2\}, \quad F(\alpha_1) = \alpha_2,\ F(\alpha_2) = \alpha_1, \quad \omega(\alpha_1) = \omega(\alpha_2) = 1.$ *(2) Capture case:* $\mid S \mid = \{\alpha_1, \alpha_2\}, \quad F(\alpha_1) = \alpha_1,\ F(\alpha_2) = \alpha_1, \quad \omega(\alpha_1) = \omega(\alpha_2) = 1.$ *(3) Disjoint sinks case:* $\mid S \mid = \{\alpha_1, \alpha_2\}, \quad F(\alpha_1) = \alpha_1,\ F(\alpha_2) = \alpha_2, \quad \omega(\alpha_1) = \omega(\alpha_2) = 1$

In other words, the hyperbolic components in each class have identical mapping schemata, hence within each class (bitransitive, capture and disjoint sinks) the hyperbolic components are diffeomorphic to each other. The center points in each case will be respectively primary intersections, capture points and secondary intersections. The characterization of the model space for each schema, presented in [M1],gives us a complete description of all hyperbolic components in $\mathcal{H}$.

However, we are not quite done yet. We still have to translate everything for polynomials with real coefficients, in order to have the results apply to our original family.

Definition 2.6.2. *A real form of the mapping schema S is an antiholomorphic involution $\rho : \mathbb{C}_1 \sqcup \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \sqcup \mathbb{C}_2$ which commutes with the special map $f_0^S : \mathbb{C}_1 \sqcup \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \sqcup \mathbb{C}_2,\ f_0^S(z) = z^2$. The collection of maps $f \in \mathcal{P}$ that commute with ρ is an affine space $\mathcal{P}_{\mathbb{R}}(\rho)$, which we call the real form of $\mathcal{P}$ associated with ρ. We also define the corresponding real connectedness locus* *and the real hyperbolic locus as:*

$$\mathcal{C}_{\mathbb{R}}(\rho) = \mathcal{C} \cap \mathcal{P}_{\mathbb{R}}(\rho)$$
$$\mathcal{H}_{\mathbb{R}}(\rho) = \mathcal{H} \cap \mathcal{P}_{\mathbb{R}}(\rho)$$

For each mapping schema of weight 2, [M1] shows that there are exactly two real forms. The form $\rho_0(z) = \overline{z}$ corresponds to the space $\mathcal{P}_{\mathbb{R}}(\rho_0)$ of real polynomials in $\mathcal{P}$. If we restate theorem 6.4 of [M1] in our particular case, we obtain:

Theorem 2.6.3. *Any hyperbolic component in $\mathcal{C}_{\mathbb{R}} = \mathcal{C}_{\mathbb{R}}(\rho_0) \subset \mathcal{P}_{\mathbb{R}}(\rho_0)$ is a topological 2-cell with a unique "center point" and is real analytically homeomorphic to a space of Blaschke products $\beta_{\mathbb{R}}(S, \rho_0)$.*

In other words, any hyperbolic component in $\mathcal{C}_{\mathbb{R}}$ is diffeomorphic to a "principal" hyperbolic component $H^S_{0\mathbb{R}}(\rho_0)$. For example, all bitransitive components are diffeo to the principal component centered at:

$$f_0^S : \mathbb{C}_1 \sqcup \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \sqcup \mathbb{C}_2, \;\; f_0^S(z) = z^2$$

For a detailed characterization of the construction and properties of the suitable Blaschke-products model spaces, see [M1]. We use the results in the reference to give the needed description of the hyperbolic components in our original parameter space P^Q.

Theorem 2.6.4. *Each hyperbolic component in $U \in P^Q$ is a topological 2-cell which contains a unique post-critically finite point, called its center. Moreover, every bone that intersects such a component does it along a simple arc passing through the center. Subsequently, there could be either one bone crossing* *the component through its center (capture case) or a pair of left-right bones intersecting transversally at the center point (bitransitive and disjoint sinks cases).*

Remark. The theorem above discusses the hyperbolic components in the region U where the boundary of the unit interval is repelling for the map $f_\mu \circ f_\lambda$. The region $P^Q \setminus U = \{(\lambda, \mu) \,/\, \lambda\mu < 1\}$ is itself a hyperbolic component of P^Q, whose maps have all critical points attracted to zero. However, as mentioned in a remark of section 2.5, this component contains no bones.

The theorem completes the proof of the second part of theorem 2.1.1, but also completes the description of the hyperbolic behavior in P^Q, which will be useful for later purposes (see picture 3.2).

Chapter 3

The impossibility of bone-loops

3.1 Introductory remarks

Our plan for this chapter is to prove that bones in the parameter space P^Q can not contain any loops (i.e. simple closed curves). One of the results in the first chapter was that each bone contains a simple arc (that we called B^Q). We have also showed in section 1.1.4 that all possible distinguished kneading data can be found in a certain order along this bone-arc.

We argue by contradiction. Suppose there exists a bone loop L. We will show next that the interior $\mathcal{U}$ of the loop can't contain any hyperbolic maps. This will contradict the genericity of hyperbolicity stated in theorem 3.1.3 and proved in detail in section 3.2.

Remark. The following statements and proofs are given for left bones, but apply by symmetry to right bones.

Lemma 3.1.1. *A left bone loop in P^Q can't contain any distinguished point,* *hence it can't contain any crossing with a right bone.*

Proof. Any distinguished point on the loop L would need to have a kneading-data already achieved along the bone arc (section 1.1.4). Thurston's Theorem shows easily that this is impossible. $\square$

Theorem 3.1.2. *The region enclosed by a left bone loop in P^Q can't contain any hyperbolic maps.*

Proof. We know by theorem 2.6.4 that each hyperbolic component in P^Q is an open topological 2-cell that contains a unique post-critically finite point, called *"center"*. Moreover, the intersection of any bone with a hyperbolic component must be a simple arc passing through the center.

Suppose, by contradiction, that some hyperbolic component $\mathcal{H}$ intersects the region $\mathcal{U}$. We have two cases:

(1) $\mathcal{H} \subset \mathcal{U}$. then there is a bone that passes through the center of $\mathcal{H}$. This can only be a bone arc, as bone loops can't contain distinguished points (by lemma 3.1.1). From the Jordan Curve Theorem, this bone arc has to intersect the bone loop L, contradiction with lemma 3.1.1.

(2) $\mathcal{H}$ intersects the loop L. Then the loop must contain the center point of $\mathcal{H}$, again contradiction. $\square$

Our next goal is to prove density of hyperbolicity in P^Q. In other words:

Theorem 3.1.3. *Consider a polynomial $P = f_\mu \circ f_\lambda \in P^Q$. Then P can be approximated by hyperbolic polynomials in P^Q.*

Remark. The theorem is a modification of the more general version of the Fatou conjecture, showed in [KSvS]. The reference gives a proof that makes use of the following *Rigidity Theorem*:

Theorem 3.1.4. *Let f and f' be two polynomials with real coefficients, real non-degenerate critical points, connected Julia sets and no neutral periodic point. If f and f' are topologically conjugate as dynamical systems on the real line $\mathbb{R}$, then they are quasiconformally conjugate as dynamical systems on the complex plane $\mathbb{C}$.*

We will use ourselves the Rigidity Theorem to prove theorem 3.1.3.

3.2 Preliminaries for proving Theorem 3.1.3.

Recall that a complete characterization of P^Q is the set of degree 4 real polynomials $P : I \to I$ of shape $(+, -, +, -)$, boundary anchored (i.e. $P(0) = P(1) = 0$) and symmetric with respect to $x = \frac{1}{2}$, i.e. $P(x) = P(1-x), \forall x \in I$. We want to emphasize some properties of the maps in our family P^Q, to justify why we are entitled to use the Rigidity Theorem 3.1.4.

(1) For any $P \in P^Q$, the iterates of all critical points are bounded, hence the critical points are all in the filled Julia set $K(P)$. Therefore,the Julia set $J(P)$ is connected (see for example theorem 17.3 in [M4]).

(2) Recall that the three complex critical points of an arbitrary $P \in P^Q$ are C_1, $C_2 = \frac{1}{2}$ and $-C_1$. An equivalent condition to $C_1 \notin \mathbb{R}$ is that:

$$f_\lambda(\frac{1}{2}) < \frac{1}{2} \Leftrightarrow \frac{\lambda}{4} < \frac{1}{2} \Leftrightarrow \lambda < 2$$

For convenience of notation, we prefer to work within this section with a family of normal polynomials (see section 2.4 for definitions) affinely conjugated to our interval maps, hence having the same dynamical behavior. Polynomials Q in this family will be of the form

$$Q : [-1,1] \rightarrow [-1,1],\ Q(-1) = Q(1) = -1,\ Q(-x) = Q(x),\ \forall x \in [-1,1]$$

If we consider the complex extensions of these polynomials, we can define the family $\mathcal{S}_4$ as the set of complex polynomials $Q : \mathbb{C} \rightarrow \mathbb{C}$ of degree 4, even (i.e. $Q(z) = Q(-z), \quad \forall z \in \mathbb{C}$) and "boundary anchored" (i.e. $Q(-1) = Q(1) = -1$). In other words, $\mathcal{S}_4 = \{az^4 + bz^2 + c \,/\, a + b + c = -1\}$.

We define X_s to be the subset of maps in $\mathcal{S}_4$ with the following properties:

- they have real coefficients;
- their three critical points are real and nondegenerate;
- all critical points and values are in [-1,1] (hence their Julia sets are connected);
- the boundary $\{-1, 1\}$ is repelling.

We claim that hyperbolic polynomials are dense in X_s. Then the proof of 3.1.3 follows relatively easily. Indeed, the claim implies directly density of hyperbolicity in the region in P^Q where $\lambda\mu > 1$ and $\lambda \geq 2$. By the symmetry

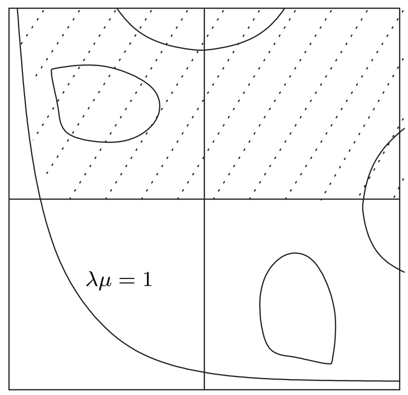


Figure 3.1: *All maps in $\{\lambda\mu < 1\}$ and in $\{\lambda < 2,\ \mu < 2\}$ are hyperbolic. Hyperbolic maps are dense in $\{\lambda\mu > 1,\ \lambda \geq 2\}$. By symmetry, they are dense in the shaded region $\{\lambda\mu > 1,\ \mu \geq 2\}$.*

property (2), the result follows in the region where $\lambda\mu > 1$ and $\mu \geq 2$. In the regions $\{\lambda\mu > 1,\ \lambda < 2,\ \mu < 2\}$ and $\{\lambda\mu < 1\}$ the proof is trivial.

Indeed, if $\lambda\mu < 1$ then all three critical orbits of $f_\mu \circ f_\lambda$ converge to zero, while if $\lambda < 2$, $\mu < 2$ and $\lambda\mu > 1$ then all critical orbits converge to a point in $(0, \frac{1}{2})$.

Next, we aim to prove density of hyperbolicity in X_s.

Lemma 3.2.1. *Consider $P \in X_s$ with one parabolic cycle $\{z_1, ..., z_m\}$. We can approximate P by a polynomial $S \in X_s$ for which the cycle is attracting.*

Proof. Fix $P \in X_s$, hence $P(z) = az^4 + bz^2 + c,\ a, b, c \in \mathbb{R}$, with a¡0 and $a + b + c = -1$.

Since P has real coefficients and real critical points, the forward critical orbits are real. The parabolic cycle $\{z_1, ..., z_m\}$ attracts at least one critical

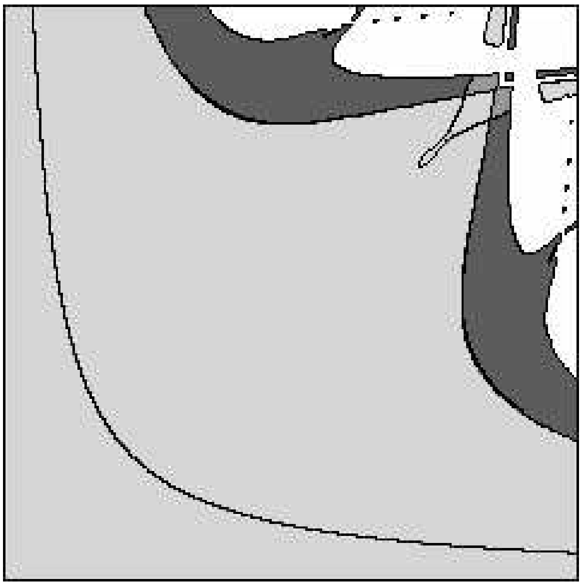


Figure 3.2: *Hyperbolic maps are dense in P^Q. The picture sketches the hyperbolic components with maps whose two critical orbits converge to an attracting fixed point or an attractor of period two (shaded regions). The middle lightly shaded region is the principal component, containing the map $(\lambda, \mu) = (2, 2)$. For the maps in the lower-left region (under the curve $\lambda\mu = 1$), both critical poins are attracted to zero. These two regions have the same corresponding image in the space of normal polynomials (see remark (1) to theorem 2.5.2.)*

point, therefore $z_j \in \mathbb{R}$, $\forall j \in \overline{1, m}$. Also, note that $z_j \neq -z_k$ for any two $z_j \neq z_k$ (otherwise the two points would be mapped to the same value $P(z_j) = P(-z_k)$).

Consider a complex polynomial H such that:

$$H(z_j) = 0, \ \forall j \in \overline{1, m}$$

$$\sum \frac{Re(H'(z_j))}{P'(z_j)} < 0$$

$H'(x) = 0$ when $P'(x) = 0$ (recall P has real critical points)

$H(-1) = H(1) = 0$

By the remark above, we can choose H to be even, i.e. $H(-z) = H(z)$, $\forall z \in \mathbb{C}$. We can change H into a polynomial Q with all properties of H and real coefficients by setting $Q(z) = H(z) + \overline{H(\overline{z})}$. Indeed, for $x \in \mathbb{R}$, $Q(x) = 2Re(H(x))$ and $Q'(x) = 2Re(H'(x))$, hence:

$Q(z_j) = H(z_j) + \overline{H(z_j)} = 2ReH(z_j) = 0$

$Q(-1) = Q(1) = 0$

$Q'(x) = 2Re(H'(x))$, so if $P'(x) = 0$ then $Q'(x) = 0$

$\sum \frac{Q'(z_j)}{P'(z_j)} = \sum \frac{2Re(H'(z_j))}{P'(z_j)} < 0$

Consider the new polynomial $R = P + \epsilon Q$, for small real values of ϵ. R perturbes the neutral cycle of P to an attracting cycle:

$$\sum \log \mid R'(z_j) \mid = \sum \log \mid P'(z_j) \mid + \sum \log \mid 1 + \epsilon \frac{Q'(z_j)}{P'(z_j)} \mid =$$

$$= \epsilon \sum \frac{Q'(z_j)}{P'(z_j)} + o(\epsilon^2) < 0$$

For small enough values of ϵ, R has the following properties:

• the parabolic cycle of P is attracting for R;

• the attracting/repelling cycles of P change to attracting/repelling cycles for R (hence $\{-1\}$ remains a repelling fixed boundary point for R);

• R is even (as Q was chosen even) and $R(-1) = R(1) = -1$, hence $R \in \mathcal{S}^4$;

• R has real coefficients;

• the critical points of R are the same as the critical points of P, hence they are real, nondegenerate; all critical points and values are contained in $[-1, 1]$, hence the Julia set $J(R)$ is connected;

However, in order to satisfy all required conditions, Q (hence R) may have degree larger than 4. We use the Straightening Theorem to obtain a degree 4 polynomial $S \in X_s$ with the same behavior as R (see for example [CG]).

Theorem 3.2.2. ***(Straightening Theorem.)*** *If $f : D_1 \to D_2$ is polynomial-like of degree d, then there are a polynomial g and a quasiconformal map ϕ such that $f = \phi \circ g \circ \phi^{-1}$ on U_1. Moreover, ϕ is unique up to three fixed points.*

Take $\rho > 1$ large. Set $D_2 = \Delta(0, \rho^4)$ and $D_1 = R^{-1}(D_2)$. If ϵ is small enough, then $R \sim P$ near $\{\mid z \mid = \rho\}$, so $(R; D_1, D_2)$ is polynomial-like of degree 4. We may assume that $\{z_1, ..., z_m\} \subset D_1$ and all critical points of P

are also in D_1. By theorem 3.2.2, R is conjugate on D_1 to a polynomial S of degree 4.

Moreover, the quasiconformal map ϕ that conjugates R to S is even and symmetric on D_1 (for details, see the proof of the Straightening Theorem in [CG] and the proof of theorem 2.1.1). We can choose ϕ to fix $1, -1$ and ∞.

Therefore, on the domain D_1:

- $S = \phi^{-1} \circ R \circ \phi$ is symmetric (i.e. $\overline{S(\overline{z})} = S(z)$), hence S has real coefficients;

- S is even, as ϕ is even;

- $S'(x) = 0$ when $R'(\phi(x)) = 0$, so the three critical points of S are in $\phi(D_1)$ and are the images under ϕ of the critical points of R; hence S has real, nondegenerate critical points contained in $[-1, 1]$;

- $S(-1) = S(1) = -1$;

- $J(S)$ is connected, as $J(R)$ is connected.

Hence $S \in X_s$ is close to P and replaced the parabolic cycle of P with an attracting cycle. $\square$

For every $Q \in \mathcal{S}_4$, let $\tau(Q)$ be the number of critical points contained in the attracting basin of a hyperbolic attracting cycle of Q.

Define: $X_s' = \{Q \in X_s \ / \ \tau(Q)$ has a local maximum at $Q\}$.

As τ is uniformly bounded above, X_s' is dense in X_s. Moreover, τ is locally constant at any $P \in X_s'$, hence we have the following:

Proposition 3.2.3. *X_s' is open and dense in X_s.*

Proposition 3.2.4. *No map in X_s' has a neutral cycle.*

Proof. Consider $P \in X_s'$ and Q given by the lemma. By making the perturbation small enough, we can arrange that the other hyperbolic attractors of P do not disappear. Moreover, we can also make sure that the critical points that were attracted to the attracting cycles remain so under the perturbation.

On the other hand, each attracting cycle attracts at least one critical point. Hence introducing a new attractor by perturbing P to Q will change τ as :

$$\tau(Q) \geq \tau(P) + 1$$

contradiction with the local maximality of τ at P. □

We finish by giving a reduced statement, from which theorem 3.1.3 follows now almost immediately. The proof is detailed in section 3.3.

Theorem 3.2.5. *Hyperbolic polynomials are dense in X_s'.*

3.3 Hyperbolic polynomials are dense in X_s'

Recall that two points z_1 and z_2 are in the same foliated equivalence class of a map f if their grand orbits under f have the same closure. For a fixed f, we denote by n_{ac} the number of foliated equivalence classes of acyclic critical points in the Fatou set of f. By [MS], the complex dimension of the Teichmuller space of a map $f : \mathbb{C} \to \mathbb{C}$ is given by:

$\dim(Teich(f)) = n_{ac} + n_{hr} + n_{lf} + n_p$, where:

$n_{ac} = \#$ of foliated equivalence classes of acyclic critical points in the Fatou set $F(f)$;

$n_{hr} = \#$ of Herman rings of f;

$n_{lf} = \#$ invariant line fields;

$n_p = \#$ parabolic cycles.

If $P \in X'_s$, the general theory says that P has no Herman rings and no Siegel discs. By [KSvS] and [S], P does not support an invariant line field in its Julia set. We also proved in section 3.2 that P does not have any parabolic basins. So all connected components of its Fatou set are attracting basins. Hence:

$$n_{hr} = n_{lf} = n_p = 0 \;\Rightarrow\; \dim(Teich(P)) = n_{ac}$$

Hence the set:

$$QC(P) = \{Q \in S_4 \ / \ Q \text{ quasiconformally conjugate to} P\}$$

is covered by countably many complex submanifolds of dimension n_{ac}. Subsequently, the set:

$$QC^{\mathbb{R}}(P) = QC(P) \cap X_s$$

is covered by countably many embedded real analytic submanifolds of X_s with real dimension n_{ac}.

We will also use the following([dMvS], pp 93):

Definition 3.3.1. If the 3-modal maps $P, Q : [-1, 1] \to [-1, 1]$ are such that

$h_P^Q : \bigcup_{n,i} P^n(c_i(P)) \to \bigcup_{n,i} Q^n(c_i(Q)) \ i = \overline{1, 2, 3}$

defined by *:*

$h_P^Q(P^n(c_i(P))) = Q^n(c_i(Q)), \ \forall i = \overline{1, 2, 3}, \ \forall n \in \mathbb{N}$

is an order-preserving bijection, then we say that P and Q are combinatorially equivalent as 3-modal maps of the interval.

The relationship between combinatorial equivalence and topological conjugacy in our space X_s is given by ([dMvS]):

Theorem 3.3.2. *Call $\mathcal{F}$ the family of maps f of the interval satisfying the following:*

(1) they are of class $\mathcal{C}^3$;

(2) they have nonflat critical points (i.e. $D^2 f(c) \neq 0, \ \forall c$ such that $Df(c) = 0$);

(3) they have negative Schwartzian derivative: $Sf < 0$;

(4) the boundary of the interval is repelling (in other words $\mid Df(x) \mid > 1$, if $x \in \{-1, 1\}$);

(5) they have no one-sided periodic attractors.

Two maps $f, g \in \mathcal{F}$ are topologically conjugate ($f \overset{top}{\sim}_{\mathbb{R}} g$) if and only if they are combinatorially equivalent ($f \overset{c.e.}{\sim}_{\mathbb{R}} g$).

Remark. If P and Q are maps in X_s' restricted to the interval $[-1, 1]$, then both the conditions of theorem 3.3.2 and the Rigidity Theorem are satisfied, hence we have the following implications:

$$P \overset{c.e.}{\sim}_{\mathbb{R}} Q \Leftrightarrow P \overset{top}{\sim}_{\mathbb{R}} Q \Rightarrow P \overset{qc.}{\sim}_{\mathbb{C}} Q$$

Proof of theorem 3.2.5. Fix $P \in X_s'$.

We think of $\mathcal{S}_4 \subset \mathbb{C}^2$ and we consider the three holomorphic functions $c_i : \mathcal{U} \to \mathbb{C}$, $i = \overline{1, 2, 3}$ that give the three critical points of each map $Q \in \mathcal{U}$. By taking $\mathcal{B} \subset \mathcal{U} \subset \mathcal{S}^4$ to be a small ball around P, we can arrange to have $c_1(Q) < c_2(Q) < c_3(Q) = -c_1(Q)$, for any $Q \in \mathcal{B} \cap X_s$. Take $\mathcal{B}$ small enough for τ to be constant: $\tau = \tau(Q)$, $\forall Q \in \mathcal{B} \cap X_s$ (recall τ is locally constant at each $P \in X_s'$).

We want to prove (by contradiction) that $\mathcal{B} \cap X_s$ contains hyperbolic maps. Suppose the maps in $\mathcal{U} \cap X_s$ are not hyperbolic, hence $\tau < 3$. There are two cases that remain for analysis:

(1) $\tau = 1$ (only C_2 is attracted) or $\tau = 2$ (only C_1 and C_3 are attracted). Either way there is only one foliated equivalent class of critical points in the Fatou set, hence $n_{ac} \leq 1$ (note that the critical points are not necessarily acyclic). Hence $QC^{\mathbb{R}}(Q)$ is in this case at most a countable union of lines in X_s, for any $Q \in \mathcal{B} \cap X_s$.

(2) $\tau = 0$ (no critical points are attracted). Hence $n_{ac} = 0$, so $QC^{\mathbb{R}}(Q)$ is a countable union of points in X_s, for any $Q \in \mathcal{B} \cap X_s$.

A. Suppose first there are no bones crossing the neighbourhood $\mathcal{B}$.

If there are no other "critical relations" in $\mathcal{B}$ (i.e. there are no $m, n \in \mathbb{N}$ such that $Q^m(c_1(Q)) = Q^n(c_2(Q))$ for some $Q \in \mathcal{B}$), then for any arbitrary $Q \in \mathcal{B}$ the map h_P^Q defined in 3.3.1 is order preserving.(Note that we do not consider $Q(c_1(Q)) = Q(c_3(Q))$ a critical relation.) Indeed: Suppose that h reverses the order of two elements:

$P^k(c_i(P)) < P^l(c_j(P))$ and

$Q^k(c_i(Q)) > Q^l(c_j(Q))$

By continuity, there exists a $T \in \mathcal{B}$ such that:

$T^k(c_i(T)) = T^l(c_j(T))$, contradiction.

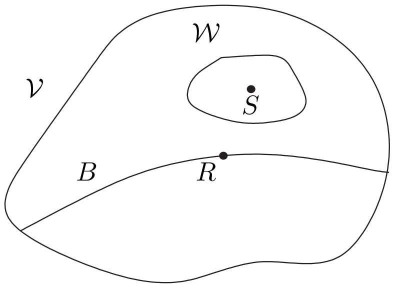
Since h_P^Q is order-preserving for any $Q \in \mathcal{B} \cap X_s$, it follows that P is combinatorially equivalent to any $Q \in \mathcal{B} \cap X_s$, hence P is quasiconformally conjugate to any $Q \in \mathcal{B} \cap X_s$. This contradicts the fact that $QC^{\mathbb{R}}(P)$ is at most a union of countably many lines in X_s.

Clearly, the "no critical relations" condition applies in the case $\tau = 1$ or $\tau = 2$.

If $\tau = 0$, it could happen that all neibourhoods of τ, arbitrarily small, contain critical relations. In other words, there exists a map R arbitrarily close to P that has a critical relation, say $R^m(c_1(R)) = R^n(c_2(R))$.

Consider $\Sigma = \{Q \in \mathcal{B} \cap X_s \,/\, Q^m(c_1(Q)) = Q^n(c_2(Q))\}$. This is a 1-dim curve in $\mathcal{B} \cap X_s$. There clearly are no other critical relations on Σ, hence the map h_R^Q is order-preserving for any $Q \in \Sigma$. Subsequently, all maps in Σ are combinatorially equivalent to R, hence quasiconformally conjugate to R. This contradicts the fact that $QC^{\mathbb{R}}(R)$ is a collection of countably many points in X_s, as $\tau = 0$.

B. If $\mathcal{B} \cap X_s$ is crossed by a bone B, let $R \in B \cap \mathcal{B} \cap X_s$.



Bones can't accumulate at R, or R would be hyperbolic. So there exists a neighbourhood $\mathcal{V}$ of R, $\mathcal{V} \subset \mathcal{B} \cap X_s$ that intersects no other bones than B. Take $S \in \mathcal{V} \backslash B$ and take $\mathcal{W}$ a neighbourhood of S in $\mathcal{V} \backslash B$. Then the argument at **A.** applies for $\mathcal{W}$ and leads us to a contradiction.

The proof of theorem 3.2.5 is now finished. $\square$

Chapter 4

Topological conclusions

4.1 The entropy and the bones

Recall some notations and results from the general theory of m-modal maps of the interval.

If $f : I \to I$ is an m-modal map with folding points $c_1 \leq c_2 \leq ... \leq c_m$, we define the sign of the fixed point x of $f^{\circ k}$ with itinerary $\Im(x) = (A_0, A_1, ..., A_{k-1})$ as the number:

$$sign(x) = \epsilon(A_0)\epsilon(A_1)...\epsilon(A_{k-1})$$

where $\epsilon(A_j) = +1,\ -1$ or 0 according to A_j being an increasing/decreasing lap of f or a folding point $c_1, ..., c_m$. If $sign(x) = -1$ we say that x is a fixed point of negative type of $f^{\circ k}$.

We define $Neg(f^{\circ k})$ as the number of fixed points of negative type of $f^{\circ k}$.

Theorem 4.1.1. *(*[MT], page 22*) If f is an interval m-modal map, then its topological entropy is:*

$$h(f) = \overline{\lim_{k\to\infty}} \frac{1}{k} \log^+(Neg(f^{\circ k}))$$

where $\log^+ s = max(\log(s), 0)$.

Remark: $Neg(f^{\circ k})$ is an integer ≥ 1 unless $f^{\circ k}$ has no fixed points of negative type; in that case, $\log^+(Neg(f^{\circ k})) = 0$.

Lemma 4.1.2. *Suppose that for two sequences* $(a_k)_{k\in\mathbb{N}}$ *and* $(b_k)_{k\in\mathbb{N}}$ *of positive integers, there exists a* $C > 0$ *such that:*

$$\mid a_k - b_k \mid < C,\ \forall k \in \mathbb{N}$$

Then the sequence $\mid \log^+ a_k - \log^+ b_k \mid$ *is bounded.*

Proof. We analyze three cases:

(1) $a_k, b_k \neq 0$. We can assume WLOG that $a_k \geq b_k$. Then :

$$\mid \log^+ a_k - \log^+ b_k \mid = \log(a_k) - \log(b_k) = \log \frac{a_k}{b_k} =$$

$$= \log(\frac{a_k - b_k}{b_k} + 1) < \log(\frac{C}{b_k} + 1) \leq \log(C+1)$$

(2) $a_k \neq 0,\ b_k = 0$. Then $1 \leq a_k < C,\ \forall k \in \mathbb{N}$.

$$\mid \log^+ a_k - \log^+ b_k \mid = \mid \log(a_k) \mid = \log(a_k) < \log(C)$$

Same for $a_k = 0$ and $b_k \neq 0$.

(3) $a_k = b_k = 0$. Then $| \log^+ a_k - \log^+ b_k |= 0$.

We found an upper bound for our sequence:

$$| \log^+ a_k - \log^+ b_k |\leq \log(C+1),\ \forall k \in \mathbb{N}$$

$\square$

Lemma 4.1.3. *If for two m-modal interval maps f and g the topological entropies $h(f) \neq h(g)$, then the sequence $| Neg(f^{\circ k}) - Neg(g^{\circ k}) |$ must be unbounded as $k \to \infty$.*

Proof. Suppose :

$$| Neg(f^{\circ k}) - Neg(g^{\circ k}) |< C\ ,\ \forall k \in \mathbb{N}$$

WLOG, assume $h(f) = h(g) + \epsilon\ ,\ \epsilon > 0$. Then:

$$\epsilon = h(f) - h(g) = \limsup \frac{1}{k} \log^+ Neg(f^{\circ k}) - \limsup \frac{1}{k} \log^+ Neg(g^{\circ k}) \leq$$

$$\leq \limsup \frac{1}{k} | \log^+ Neg(f^{\circ k}) - \log^+ Neg(g^{\circ k}) |\leq$$

$$\leq \limsup \frac{1}{k} \log(C+1) = 0\ ,$$

contradiction. $\square$

Lemma 4.1.4. *Consider $p_1 = (\lambda_1, \mu_1)$ and $p_2 = (\lambda_2, \mu_2)$ in P^Q such that*

$$h(f_{\mu_1} \circ f_{\lambda_1}) \neq h(f_{\mu_2} \circ f_{\lambda_2})$$

Then any path in P^Q from p_1 to p_2 crosses infinitely many bones.

Proof. To simplify notation, for $p = (\lambda, \mu) \in P^Q$ call $g_p = f_\mu \circ f_\lambda$

By the previous lemma, $\mid Neg(g_{p_1}^{\circ k}) - Neg(g_{p_2}^{\circ k}) \mid$ is unbounded as $k \to \infty$.

Consider an arbitrary path in P^Q from p_1 to p_2:

$$p : [0,1] \to P^Q \ , \ p(t) = (\lambda(t), \mu(t))$$
$$p(0) = p_1 = (\lambda_1, \mu_1) \ , \ p(1) = p_2 = (\lambda_2, \mu_2)$$

For a fixed $k \in \mathbb{N}$, as t goes from 0 to 1, $Neg(g_{p(t)}^{\circ k})$ changes where a fixed point of $g_{p(t)}^{\circ k}$ (i.e. a periodic point of $g_{p(t)}$ of period dividing k) of negative type appears or disappears. An existing negative-type fixed point of $g_{p(t)}^{\circ k}$ can be lost under continuous deformations of the map by becoming a positive-type fixed point. Conversely, a such fixed point can appear by a reverse process. Both changes imply the existence of an intermediate state, corresponding to some $t^* \in [0,1]$, in which the respective fixed point is a critical point of $g_{p(t^*)}^{\circ k}$. We want to see what a such critical point x of $g_{p(t^*)}^{\circ k}$ signifies for $g_{p(t^*)}$:

$$(g_{p(t^*)}^{\circ k})'(x) = \prod_{j=0}^{j=k-1} g'_{p(t^*)}(g_{p(t^*)}^{\circ j}(x)) = 0$$

In other words, $g_{p(t^*)}^{\circ j}(x)$ is a critical point for $g_{p(t^*)}$, for some $j \in \overline{0, k-1}$. But $y_j = g_{p(t^*)}^{\circ j}(x)$ is a periodic point of period dividing k under $g_{p(t^*)}$:

$$g_{p(t^*)}^{\circ k}(y_j) = g_{p(t^*)}^{\circ j}(g_{p(t^*)}^{\circ k}(x)) = y_j$$

In conclusion, a critical point of $g_{p(t^*)}$ has to be periodic of period dividing k. This implies that $p(t^*) = (\lambda(t^*), \mu(t^*)) \in P^Q$ is on either a left or a right bone of period $2n \mid 2k$.

So if the integer $Neg\,((f_{\mu(t)} \circ f_{\lambda(t)})^{\circ k})$ has an actual change at $t = t^*$, then the path $p(t)$ crosses a bone at $t = t^*$.

To end the proof of the lemma, suppose that the path $p(t)$ only crosses N bones. Then, for all $k \in \mathbb{N}$,

$$\mid Neg(g_{p_1}^{\circ k}) - Neg(g_{p_2}^{\circ k}) \mid$$

would be bounded by N, contradiction with lemma 4.1.3. □

4.2 The entropy and the cellular structure

To continue, we will make use of two nontrivial results on topological entropy for both families of stunted sawtooth maps and polynomials.

Assertion 1: ([MT] - pp 28) *The topological entropy of a stunted sawtooth map depends continuously of its parameter.*

Assertion 2: ([MT] - pp 16) *The topological entropy is continuous as a function*

$$h : \mathcal{C}^\infty(I, I) \to [0, \infty)$$

.

Remark: In the quadratic family, the function $f_\mu \circ f_\lambda$ changes continuously in $\mathcal{C}^\infty(I, I)$ as the pair (λ, μ) moves continuously in P^Q.

Most results in this section stand for both our families of maps, so the notation P for the space of parameters is to be understood as P^{ST} or P^Q, as the case requires. For a fixed n, P_n will stand for the cell complex defined by the bones in either parameter space. Depending on the case, a parameter $p \in P$ is a pair:

- $p = (a, b) \in P^{ST}$ and the corresponding map $g_p^{ST} = f^b \circ f^a$
- $p = (\lambda, \mu) \in P^Q$ with corresponding polynomial $g_p^Q = f_\mu \circ f_\lambda$

We will omit the superscript ST or Q as long as it doesn't allow ambiguity.

Lemma 4.2.1. *For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that, if p and p' belong to the same closed cell in P^n, then the corresponding maps satisfy:*

$$\mid h(g_p) - h(g_{p'}) \mid < \epsilon$$

Proof. Suppose the contrary: there exists $\epsilon > 0$ such that, for all $n \in \mathbb{N}$, there are two parameters p_n and p'_n in some common cell of P_n with:

$$| h(g_{p_n}) - h(g_{p'_n}) | \geq \epsilon$$

By the compactness of P, we can choose a subsequence $(k_n)_n \subset \mathbb{N}$ such that both $(p_{k_n})_n$ and $(p'_{k_n})_n$ converge in P:

$$p_{k_n} \longrightarrow p \quad \text{as } n \to \infty$$

$$p'_{k_n} \longrightarrow p' \quad \text{as } n \to \infty$$

The two assertions deliver the continuity of the entropy function of parameters in either family. Using it and passing to the limit:

$$| h(g_{p_{k_n}}) - h(g_{p'_{k_n}}) | \geq \epsilon \quad \Rightarrow \quad | h(g_p) - h(g_{p'}) | \geq \epsilon$$

Moreover, the closed cells of P_n are nested as n increases (in other words, the cell complex gets *"finer"* with larger values of n).

Fix an arbitrary $N \in \mathbb{N}$. For all $k_n \geq N$, p_{k_n} and p'_{k_n} are in the same closed cell of P_{k_n}, hence in the same closed cell of P_N.

In conclusion, for any arbitrary $N \in \mathbb{N}$, p and p' are in the same closed cell of P_N, yet:

$$| h(g_p) - h(g_{p'}) | \geq \epsilon > 0$$

contradiction with lemma 4.2.1. □

Theorem 4.2.2. *In P^{ST}, the entropy function is a monotone function of either coordinate.*

Proof. See proof in Appendix A. □

Lemma 4.2.3. *Fix $n \in \mathbb{N}$. In either parameter space P, the entropy function:*

$$P_n \longrightarrow [0, log4]$$

$$p \longrightarrow h(g_p)$$

restricted to any closed cell in P_n takes its maximum and minimum values on the boundary of the cell (more precisely on the boundary vertexes).

Proof. In the case $P_n = P_n^{ST}$, the proof is a simple corollary of lemma 4.2.2. We have to prove the identical statement for $P_n = P_n^Q$.

For the fixed $n \in \mathbb{N}$, suppose the lemma is not true for some closed cell $C_n^Q \in P_n^Q$, that is : there exists $p^* = (\lambda^*, \mu^*) \in int(C_n^Q)$ such that

$$h(g_{p^*}) = h(f_{\mu^*} \circ f_{\lambda^*}) > h_{max} \, ,$$

where h_{max} is the maximum value of the entropy on the boundary $\delta(C_n^Q)$.

Let

$$\epsilon = \frac{h(g_{p^*}) - h_{max}}{2} \geq 0$$

By lemma 4.2.1, there exists $m \in \mathbb{N}$ such that the entropy variation on all closed cells of P_m^Q is less than ϵ. WLOG, we can take $m > n$. Call C_m^Q the closed cell in P_m^Q such that $p^* \in C_m^Q \subset C_n^Q$ and consider any arbitrary vertex $p_m = (\lambda_m, \mu_m)$ of C_m^Q.

As p^*, $p_m \in C_m^q$, we automatically have:

$$\mid h(g_{p^*}) - h(g_{p_m}) \mid < \epsilon$$

But $\quad h_{max} + 2\epsilon \leq h(g_{p^*}) \quad$, so:

$$h(g_{p_m}) > h_{max}$$

The homeomorphism of complexes $\eta_m^{-1} : P_m^Q \longrightarrow P_m^{ST}$ carries vertexes to vertexes with the same entropy, edge to edge with the same interval of entropies and 2-cells to 2-cells. So $C_m^{ST} = \eta_m^{-1}(C_m^Q)$ will be a 2-cell in P_m^{ST} and $q_m = \eta_m^{-1}(p_m)$ will be a vertex of C_m^{ST}. Also, $\eta_n^{-1}(\delta C_n^Q) = \delta(C_n^Q) = \delta C_n^{ST}$, so the maximum value $h_{max}(\delta C_n^Q)$ of the entropy on δC_n^Q is the same as the maximum value $h_{max}(\delta C_n^{ST})$ on C_n^{ST}. Hence, in the stunted family:

$$h(g_{q_m}^{ST}) = h(g_{p_m}^Q) > h_{max}(\delta C_n^Q) = h_{max}(\delta C_n^{ST}) \, ,$$

contradiction, since the result has already been proved for P^{ST}. □

Corollary 4.2.4. *For a fixed $n \in \mathbb{N}$, the interval of entropy values realized by any cell in P_n^Q is the same as the interval of values for the corresponding cell in P_N^{ST}.*

4.3 Connectedness of isentropes

Definitions and notations:

For $h_0 \in [0, log4]$ we call the h_0-isentrope in either family the set of parameters:

$i^{ST}(h_0) = \{(a, b) \ / \ h(f^b \circ f^a) = h_0\}$

$$i^Q(h_0) = \{(\lambda, \mu) \ / \ h(f_\mu \circ f_\lambda) = h_0\}$$

For a fixed $n \in \mathbb{N}^*$, we also use the following notations:

$$\mathcal{U}_n^{ST}(h_0) = \{C_n^{ST} \in P_n^{ST} \text{ closed cell } / \ C_n^{ST} \cap i^{ST}(h_0) \neq \Phi\}$$

$$\mathcal{U}_n^{Q}(h_0) = \{C_n^{Q} \in P_n^{Q} \text{ closed cell } / \ C_n^{Q} \cap i^{Q}(h_0) \neq \Phi\}$$

for the collection of closed cells in either cell complex that touch the respective h_0-isentrope.

The unions of such closed cells will be denoted by :

$$N_n^{ST}(h_0) = \bigcup \{C_n^{ST} \ / \ C_n^{ST} \in \mathcal{U}_n^{ST}(h_0)\}$$

$$N_n^{Q}(h_0) = \bigcup \{C_n^{Q} \ / \ C_n^{Q} \in \mathcal{U}_n^{Q}(h_0)\}$$

Remarks : (1) Clearly: $i^{ST}(h_0) \subset N_n^{ST}(h_0)$ and $i^Q(h_0) \subset N_n^Q(h_0)$.

(2) Recall that for fixed n we have the homeomorphism of cell complexes:

$$\eta_n : P_n^{ST} \to P_n^Q$$

If C_n^{ST} is a cell in P_n^{ST} that touches $i^{ST}(h_0)$, then the corresponding cell $C_n^Q = \eta_n(C_n^{ST})$ will touch $i^Q(h_0)$ and conversely. This follows from corollary 4.2.4 , which states that the interval of entropy values is the same in the two closed cells C_n^{ST} and C_n^Q.

Hence:

$$\mathcal{U}_n^Q(h_0) = \{\eta_n(C_n^{ST}) \ / \ C_n^{ST} \in \mathcal{U}_n^{ST}(h_0)\}$$

Subsequently:

$$N_n^Q(h_0) = \bigcup \{\eta_n(C_n^{ST}) \ / \ C_n^{ST} \in \mathcal{U}_n^{ST}(h_0)\}$$

Theorem 4.3.1. *For any value $h_0 \in [0, log4]$, the h_0-isentrope $i^{ST}(h_0)$ is contractible.*

We will prove Theorem 4.3.1 in two steps.

Lemma 4.3.2. *For each $h_0 \in [0, \log 4]$, the isentrope $i^{ST}(h_0) = \{h = h_0\}$ is a deformation retract of $\{h \leq h_0\}$ in P^{ST}.*

Proof. Fix an arbitrary point $p = (a, b) \in \{h \leq h_0\} \subset P^{ST}$.

Construct the path:

$$q_p : [0, 1] \ \rightarrow \ P^{ST}, \ q_p(t) = (1 + t(a - 1) \, , \, 1 + t(b - 1))$$

This is the continuous, entropy-increasing path in P^{ST} connecting the fixed $p \in \{h \leq h_0\}$ with the upper-right corner of P^{ST} (of maximal entropy $\log 4$). Hence there exists (Mean Value Theorem) a smallest $t^* = t^*(p) \in [0, 1]$ such that $q_p(t^*(p)) \in \{h = h_0\}$.

We define the homotopy:

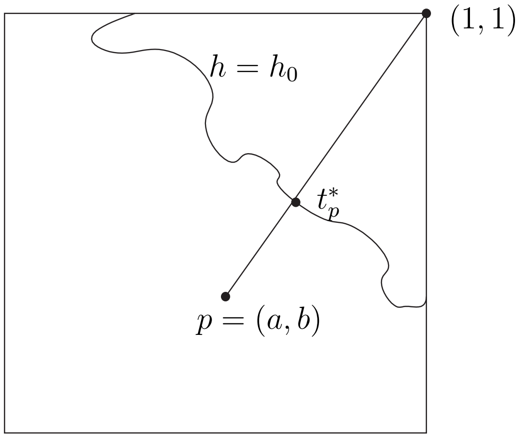


Figure 4.1: *The path $q_p(t) = t(a,b) + (1-t)(1,1)$ has to cross $h = h_0$ at some point. Call t_p^* the first crossing.*

$H : [0,1] \times \{h \leq h_0\} \rightarrow \{h \leq h_0\}$

$$H(t,p) = \begin{cases} q_p(t) \ , \ t \leq t^*(p) \\ q_p(t^*(p)) \ , \ t > t^*(p) \end{cases}$$

We have that for $p_1, p_2 \in \{h \leq h_0\}$:

$$\| q_{p_1}(t) - q_{p_2}(t) \| = t \| p_1 - p_2 \|$$

It follows easily that the map H is continuous in both arguments.

Clearly:

$H(t,p) \in \{h \leq h_0\}$, $\forall t \in [0,1]$, $p \in \{h \leq h_0\}$

$H(1,p) \in \{h = h_0\}$, $\forall p \in \{h \leq h_0\}$

$H(0,p) = q_p(0) = p, \ \forall p \in \{h \leq h_0\}$

Hence H is a deformation retract from $\{h \leq h_0\}$ to $\{h = h_0\}$. $\square$

Lemma 4.3.3. *The region $\{h \leq h_0\}$ is contractible in P^{ST}.*

Proof. For a fixed $p \in \{h \leq h_0\}$, construct the straight segment joining $r_p(0) = p$ and $r_p(1) = (0,0)$: $r_p(t) = (1-t)p, \ t \in [0,1]$.

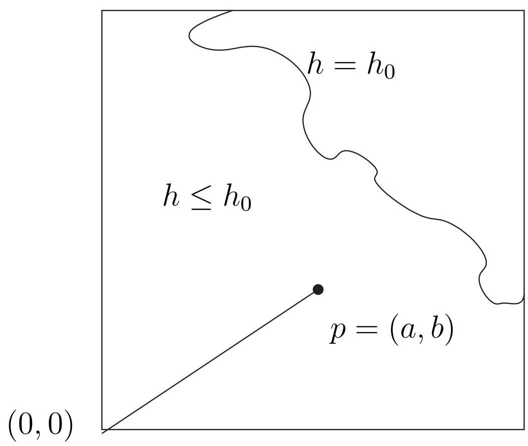


Figure 4.2: *The region $h \leq h_0$ contracts to the origin by straight lines*

The homotopy $H(t,p) = r_p(t)$, $t \in [0,1]$, $p \in \{h \leq h_0\}$ contracts our region to the lower-left point $(0,0) \in P^{ST}$ of entropy zero. $\square$

Proof of theorem 4.3.1. The isentrope $i^{ST}(h_0)$ is a deformation retract of the contractible region $\{h \leq h_0\}$, hence it is contractible. $\square$

Remark. In [MT] the same result is shown in the space V of classical parameters for m-modal stunted sawtooth maps. For each $h_0 \in [0, log(m+1)]$, the h_0-isentrope $\{w \in V \ / \ h(f_w) = h_0\}$ is contractible.

Fix an entropy value $h_0 \in [0, log4]$ and an $n \in \mathbb{N}^*$.

Since $N_n^{ST}(h_0)$ and $N_n^Q(h_0)$ are both unions of closed cells, they are compact subsets of P^{ST} and P^Q, respectively. By the previous theorem, $N_n^{ST}(h_0)$ is connected, so its image $N_n^Q(h_0) = \eta_n(N_n^{ST}(h_0)$ is also connected. Hence we have the following:

Summary. *For any $n \in \mathbb{N}^*$, the set $N_n^Q(h_0)$ is compact, connected and contains $i^Q(h_0)$.*

We have now a quite comprehensive description of the sets $N_n^Q(h_0)$. To obtain topological properties of $i^Q(h_0)$, we try to relate it to the collection $\{N_n^Q(h_0)\}_{n\in\mathbb{N}}$.

Lemma 4.3.4. $\bigcap N_n^Q(h_0) = i^Q(h_0)$

Proof. Since $i^Q(h_0) \subset N_n^Q(h_0)$ for all $n \in \mathbb{N}^*$, the inclusion $i^Q(h_0) \subset \bigcap N_n^Q(h_0)$ is trivial.

For the converse, suppose there exists $(\lambda, \mu) \in \bigcap N_n^Q(h_0) \backslash i^Q(h_0)$. In other words: for any arbitrary $n \in \mathbb{N}^*$, (λ, μ) is contained in a closed cell $C_n^Q \subset P_n^Q$ that touches $i^Q(h_0)$, but such that $(\lambda, \mu) \notin i^Q(h_0)$. For any such closed cell C_n^Q, there exists $(\lambda_n^*, \mu_n^*) \in i^Q(h_0) \cap C_n^Q$.

The sequence $(\lambda_n^*, \mu_n^*)_{n\in\mathbb{N}^*}$ satisfies in particular:

(1) $(\lambda_n^*, \mu_n^*) \neq (\lambda, \mu),\ \forall n \in \mathbb{N}^*$

(2) $h(f_{\mu_n^*} \circ f_{\lambda_n^*}) = h_0$

We calculate:

$$| \, h(f_{\mu^*} \circ f_{\lambda^*}) - h(f_\mu \circ f_\lambda) \, | = | \, h_0 - h(f_\mu \circ f_\lambda) \, |$$

This contradicts the statement of lemma 5.2.1: the maximal variation of the entropy over cells in P_n^Q can be made arbitrarily small by increasing n. $\square$

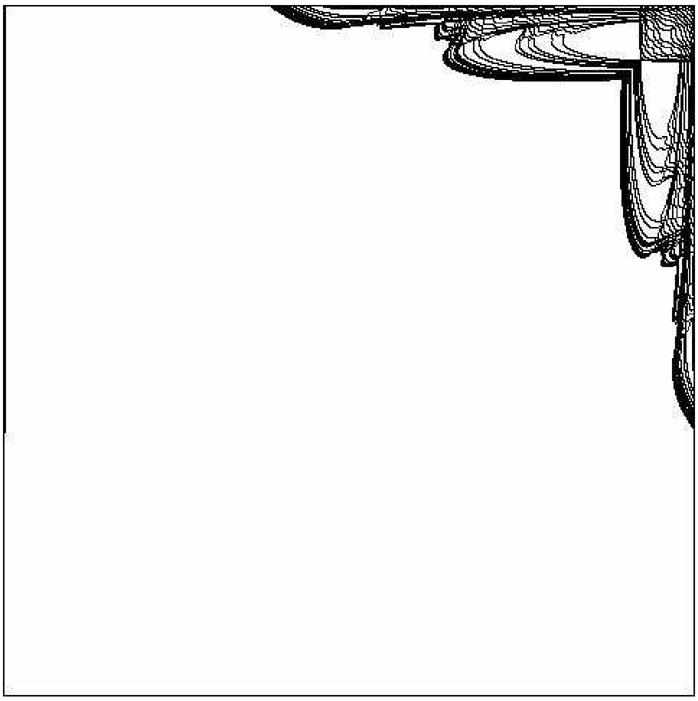


Figure 4.3: *The isentropes in P^Q appear to be either arcs joining two points in ∂P^Q, or connected regions between such arcs, or a single point (the case $(\lambda, \mu) = (4, 4)$ of entropy $\log 4$. Appendix E shows a magnified upper corner.*

Theorem 4.3.5. *$i^Q(h_0)$, the h_0-isentrope in P^Q, is connected.*

Proof. $i^Q(h_0)$ is an intersection of compact, connected sets in P^Q, therefore it is compact and connected. □

Appendix A

Lemma A1. *Suppose a complex polynomial P of degree 4 and with real coefficients has critical points $C_2 = 0$, $C_1 \neq C_3$ such that the critical values $P(C_1) = P(C_3)$. Then P is an even complex function.*

Proof. WLOG, assume P' is monic.

$$P'(z) = z(z - C_1)(z - C_3) = z^3 - (C_1 + C_3)z^2 + C_1C_3z$$

Hence:

$$P(z) = \tfrac{z^4}{4} - \tfrac{C_1+C_3}{3}z^3 + C_1C_3z^2 + k$$

$$P(C_1) = \tfrac{2C_1^3C_3 - C_1^4}{12} + k$$

$$P(C_3) = \tfrac{2C_1C_3^3 - C_3^4}{12} + k$$

As $P(C_1) = P(C_3)$, we have :

$$2C_1^3C_3 - C_1^4 = 2C_1C_3^3 - C_3^4 \;\Leftrightarrow\; (C_1 - C_3)^3(C_1 + C_3) = 0 \;\Rightarrow\; C_3 = -C_1$$

So:

$$P'(z) = z(z - C_1)(z + C_1) = z^3 - C_1^2z$$

$P(z) = \frac{z^4}{4} - C_1^2 \frac{z^2}{2} + k$

which makes P an even polynomial: $P(-z) = P(z),\ \forall z \in \mathbb{C}$. $\square$

Remark. If $C_2 \neq 0$, by an affine conjugation we obtain that $P(C_2 + z) = P(C_2 - z)$, for all $z \in \mathbb{C}$, i.e. P is "symmetric with respect to C_2".

Lemma A2. *Consider a real polynomial $P : I = [0,1] \longrightarrow I$ with real nondegenerate critical points $0 < C_1 < C_2 < C_3 < 1$, boundary anchored (i.e. $P(0) = P(1) = 0$). P is a composition of quadratic polynomials in the logistic family if and only if P is symmetric with respect to $\frac{1}{2}$ (i.e. $P(1-x) = P(x),\ \forall x \in \mathbb{R}$).*

Proof. "$\Rightarrow$" $P = f_\mu \circ f_\lambda \Rightarrow P(1-x) = (f_\mu \circ f_\lambda)(1-x) = P(x)$

"$\Leftarrow$" Suppose $P(x) = P(1-x),\ \forall x \in \mathbb{R}$. Let $Q(x) = P(\frac{1}{2} + x)$, hence $Q(x) = Q(-x),\ \forall x \in \mathbb{R}$

so:

$$Q(x) = ax^4 + bx^2 + c,\ a \leq 0$$

As P is boundary anchored, we have:

$$Q(\frac{1}{2}) = Q(-\frac{1}{2}) = 0 \ \Rightarrow \ \frac{a}{16} + \frac{b}{4} + c = 0$$

Q has nondegenerated critical points in $[-\frac{1}{2}, \frac{1}{2}]$, so $a < 0$.

We want to find $(\lambda, \mu) \in \mathbb{R}^2$ such that:

$$P(x) = (f_\mu \circ f_\lambda)(x) \ \Leftrightarrow Q(x) = (f_\mu \circ f_\lambda)(\frac{1}{2} - x) = -\lambda\mu(\lambda x^4 + \frac{\lambda - 2}{2}x^2 + \frac{4-\lambda}{16})$$

We solve: $a = -\lambda^2\mu$, $b = \frac{\lambda(\lambda-2)\mu}{2}$, $c = \frac{\lambda(4-\lambda)\mu}{16}$

As $a < 0$, we can write:

$$\frac{\lambda - 2}{-2\lambda} = \frac{b}{a} \ \Rightarrow \ \frac{1}{\lambda} = \frac{b}{a} + \frac{1}{2}$$

Suppose $a + 2b = 0$. Then $b + 8c = 0$, so $Q(x) = c(16x^4 - 8x^2 + 1) = c(4x^2 - 1)^2$, which has critical points $-\frac{1}{2}, 0, \frac{1}{2}$, hence does not satisfy our original requirements.

Hence $a + 2b \neq 0$, so we have:
$\lambda = \dfrac{2a}{a+2b}$ and $\mu = -\dfrac{(a+2b)^2}{4a} \geq 0$.

We need to show that the pair (λ, μ) obtained above is in $[0, 4]^2$.

$$P(\frac{1}{2}) = f_\mu(f_\lambda(\frac{1}{2})) = f_\mu(\frac{\lambda}{4}) = \frac{\mu\lambda(4-\lambda)}{16} \in [0, 1]$$

We ask that $P(\frac{1}{2}) \in [0, 1]$. As $\mu \geq 0$ it follows that $\lambda(4 - \lambda) \geq 0$, hence

$\lambda \in [0,4]$.

Moreover, if we ask that the critical value $P(C_1) = f_\mu(\frac{1}{2}) = \frac{\mu}{4} \leq 1$, then we get that $\mu \leq 4$.

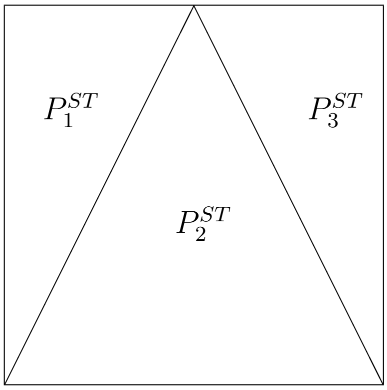
Theorem A3. *Let $P : I \longrightarrow I$ be real polynomial of degree 4 and shape (+,-,+,-), boundary anchored and with real critical points $C_1 \leq C_2 \leq C_3$. Then there exists $(\lambda, \mu) \in [0,4]^2$ such that $P = f_\mu \circ f_\lambda$ if and only if $P(C_1) = P(C_3)$.*

Proof. "$\Rightarrow$" If $P(x) = (f_\mu \circ f_\lambda)(x)$, then $P'(x) = f'_\mu(f_\lambda(x))f'_\lambda(x)$, so $f_\lambda(C_1) = f_\lambda(c_3) = \frac{1}{2}$ and $P(C_1) = P(C_3) = f_\mu(\frac{1}{2})$.

"$\Leftarrow$" If $C_1 < C_2 < C_3$, it follows from lemma 1 that P is symmetric. If $C_1 = C_2$, then also $C_3 = C_2$ (from $P(C_1) = P(C_3)$), so $C_1 = C_2 = C_3 = \frac{1}{2}$, hence P is again symmetric. In both cases, lemma 2 implies that P is composition of two quadratic polynomials in our family. $\square$

Appendix B

Let f^a and f^b be two maps in the stunted tent family, i.e. $(a, b) \in P^{ST} = [0, 1]^2$. We can write explicitly what the composition $f^b \circ f^a$ looks like, for all parameter values.



I Let $P_1^{ST} = \{(a, b) \setminus b \geq 2a\}$. For $(a, b) \in P_1^{ST}$:

$$(f^b \circ f^a)(x) = \begin{cases} 4x\,, & x < \frac{a}{2} \\ 2a\,, & \frac{a}{2} \leq x < 1 - \frac{a}{2} \\ 4 - 4x\,, & 1 - \frac{a}{2} \leq x < 1 \end{cases}$$

II Let $P_2^{ST} = \{(a, b) \setminus b \leq 2a \leq 2 - b\}$. For $(a, b) \in P_2^{ST}$:

$$(f^b \circ f^a)(x) = \begin{cases} 4x\,, & x < \frac{b}{4} \\ b\,, & \frac{b}{4} \leq x < 1 - \frac{b}{4} \\ 4 - 4x\,, & 1 - \frac{b}{4} \leq x < 1 \end{cases}$$

III Let $P_3^{ST} = \{(a,b) \setminus b \geq 2-2a\}$. For $(a,b) \in P_3^{ST}$:

$$(f^b \circ f^a)(x) = \begin{cases} 4x\,, & x < \frac{b}{4} \\ b\,, & \frac{b}{4} \leq x < \frac{1}{2} - \frac{b}{4} \\ 2-4x\,, & \frac{1}{2} - \frac{b}{4} \leq x < \frac{a}{2} \\ 2-2a\,, & \frac{a}{2} \leq x < 1 - \frac{a}{2} \\ 4x-2\,, & 1 - \frac{a}{2} \leq x < \frac{1}{2} + \frac{b}{4} \\ b\,, & \frac{1}{2} + \frac{b}{4} \leq x < 1 - \frac{b}{4} \\ 4-4x\,, & 1 - \frac{b}{4} \leq x < 1 \end{cases}$$

Recall [MT, page 28]: Let f and g be two m=modal maps of shape $(+,-,+...)$, folding vectors $v(f) = (v_f^1, ..., v_f^m)$, $v(g) = (v_g^1, ..., v_g^m)$ and kneading data $\mathbf{K}(f)$, $\mathbf{K}(g)$. We say that:

(1) $v(f) << v(g)$ iff $(-1)^j v_f^j \geq (-1)^j v_g^j$

(2) $\mathbf{K}(f) << \mathbf{K}(g)$ iff $\sigma_{i-1}\mathcal{K}_i(f) \leq \sigma_{i-1}\mathcal{K}_i(g)$, $\forall 1 \leq i \leq m$

With these definitions, we have:

$$v(f) << v(g) \;\Rightarrow\; \mathbf{K}(f) << \mathbf{K}(g) \;\Rightarrow\; h(f) \leq h(g)$$

Lemma B1. *The entropy function* $(a,b) \in P^{ST} \longrightarrow h(f^b \circ f^a) \in [0, \log 4]$ *increases with either parameter.*

Proof. Fix $a \in [0,1]$. We want to show that:

$$b_1 \leq b_2 \;\Rightarrow\; \mathbf{K}_1 = \mathbf{K}(f^{b_1} \circ f^a) << \mathbf{K}_2 = \mathbf{K}(f^{b_2} \circ f^a)$$

We have the following cases:

(1) $(a, b_1), (a, b_2) \in P_1^{ST}$. Then $f^{b_1} \circ f^a = f^{b_2} \circ f^a$, hence $\mathbf{K_1} = \mathbf{K_2}$.

(2) $(a, b_1), (a, b_2) \in P_2^{ST}$. Then both composed maps can be considered unimodal with folding vectors $(b_1) << (b_2)$, hence $\mathbf{K_1} << \mathbf{K_2}$.

(3) $(a, b_1) \in P_2^{ST}, (a, b_2) \in P_1^{ST}$. Then the folding vectors $(b_1) << (2a)$, hence $\mathbf{K_1} << \mathbf{K_2}$.

(4) $(a, b_1), (a, b_2) \in P_3^{ST}$. Then the two maps are 3-modal, and $(b_1, 2 - 2a, b_1) << (b_2, 2 - 2a, b_2) \;\Rightarrow\; \mathbf{K_1} << \mathbf{K_2}$.

(5) $(a, b_1) \in P_2^{ST}, (a, b_2) \in P_3^{ST}$. Then: $(b_1) << (b_2)$ and $(b_2, b_2, b_2) << (b_2, 2 - 2a, b_2)$.

In all cases it follows that $h(f^{b_1} \circ f^a) \leq h(f^{b_2} \circ f^a)$.

Similarly, we fix $b \in [0, 1]$ and $a_1 \leq a_2$.

(1) $(a_1, b), (a_2, b) \in P_1^{ST} \;\Rightarrow\; (2a_1) << (2a_2)$

(2) $(a_1, b) \in P_1^{ST}, (a_2, b) \in P_2^{ST} \;\Rightarrow\; (2a_1) << (b)$

(3) $(a_1, b), (a_2, b) \in P_2^{ST} \;\Rightarrow\; \mathbf{K_1} = \mathbf{K_2}$.

(4) $(a_1, b) \in P_2^{ST}, (a_2, b) \in P_3^{ST} \;\Rightarrow\; (b, b, b) << (b, 2 - 2a_2, b)$

(5) $(a_1, b), (a_2, b) \in P_3^{ST} \;\Rightarrow\; (b, 2 - 2a_1, b) << (b, 2 - 2a_2, b)$

In all cases, it follows that $h(f^b \circ f^{a_1}) \leq h(f^b \circ f^{a_2})$. $\square$

Appendix C

Consider $U = \{(\lambda,\mu) \in [0,4] \,/\, \lambda\mu > 1\}$ and the map $\phi : U \to \mathbb{R}^2$ given by $\phi(\lambda,\mu) = (A(\lambda,\mu), B(\lambda,\mu))$, where:

$$A(\lambda,\mu) = \frac{\lambda\mu(2-\lambda)}{2-\sqrt[3]{\lambda^2\mu}}$$

$$B(\lambda,\mu) = \frac{1}{16}[8-\lambda\mu(4-\lambda)]\sqrt[3]{\lambda^2\mu}$$

We claim and will prove that ϕ is a diffeomorphism from U onto its image in $\mathbb{R}^2$.

Define:

$\xi : U \to \mathbb{R}^2, \quad \xi(\lambda,\mu) = (u,v) = (\sqrt[3]{\lambda^2\mu}, \lambda\mu(2-\lambda))$ and

$\psi : \xi(U) \to \mathbb{R}^2, \quad \psi(u,v) = (A,B) = (\frac{v}{2u}, \frac{1}{16}[8-u^3-2v]u)$

Clearly $\phi = \psi \circ \xi$. We will prove separately that ξ and ψ are $\mathcal{C}^1$ diffeomorphisms.

(1) We consider $\xi(\lambda,\mu) = (u,v)$, where:

$$u = \sqrt[3]{\lambda^2\mu}, \quad v = \lambda\mu(2-\lambda)$$

ξ is a $\mathcal{C}^1$ injective map from $U = \{(\lambda,\mu) \,/\, \lambda\mu > 1\}$ onto its image. We note that if $(u,v) \in \xi(U)$ then $u > 0$ and $u^3 + v = 2\lambda\mu > 2$.

We calculate the inverse map $\xi^{-1} : \xi(U) \to U, \quad \xi^{-1}(u,v) = (\lambda,\mu)$:

$$\lambda = \frac{2u^3}{u^3+v}, \quad \mu = \frac{(u^3+v)^2}{4u^3}$$

It is clear that ξ^{-1} is a $\mathcal{C}^1$ map on $\xi(U)$, hence ξ is a diffeo.

(2) We consider $\psi(u,v)=(A,B)$, where:

$$A=\frac{v}{2u},\quad 16B=(8-u^3-2v)u$$

ψ is $\mathcal{C}^1$ on $\psi(U)$. We want to show that it is injective and that its inverse $\psi^{-1}:\phi(U)\to\psi(U)$ is $\mathcal{C}^1$.

Suppose $\psi(u_1,v_1)=\psi(u_2,v_2)$. Then:

$u_1^4+4Au_1^2-8u_1=u_2^4+4Au_2^2-8u_2 \Rightarrow$

$(u_1-u_2)[(u_1+u_2)(u_1^2+u_2^2)+4A(u_1+u_2)-8]=0$

But :

$u_1^3+2Au_1=u_1^3+v_1>2 \Rightarrow u_1^2+2A>\frac{2}{u_1}$

$u_2^3+2Au_2=u_2^3+v_2>2 \Rightarrow u_2^2+2A>\frac{2}{u_2}$

Hence:

$(u_1+u_2)(u_1^2+u_2^2)+4A(u_1+u_2)-8>(u_1+u_2)(\frac{2}{u_1}+\frac{2}{u_2})-8=$

$=4+2(\frac{u_1}{u_2}+\frac{u_2}{u_1})-8>0$

Hence $u_1=u_2$, and so $(u_1,v_1)=(u_2,v_2)$.

This proves that ψ is injective. We show that ψ^{-1} is continuous. Indeed:

$\psi(u,v)=(A,B)=(\dfrac{v}{2u},\dfrac{1}{16}[8-u^4-2vu^2])$

Hence, for $(u,v)\in\xi(U)$:

$\frac{\partial}{\partial u}\psi(u,v)=-\frac{1}{16}(4u^3+4v)=-\frac{1}{4}(u^3+v)<0$

Therefore ψ is a homeo on $\xi(U)$. By the Implicit Function Theorem, ψ is a $\mathcal{C}^1$ diffeo.

Appendix D

Let $X \neq \Phi$ be a compact space and $\mathcal{C}$ a cover of X. Let $f : X \to X$ be a continuous map and k a positive integer. Define the cover:

$$\mathcal{C}_f^k = \{D_0 \cup f^{-1}(D_1) \cup ... \cup f^{-(k-1)}(D_{k-1}) \,/\, D_j \in \mathcal{C},\, \forall j = \overline{0, k-1}\}$$

Call $n(\mathcal{C}_f^k)$ the smallest cardinality of a finite subcover of $\mathcal{C}_f^k$. We define the entropy of f for the cover $\mathcal{C}$ to be:

$$h(f, \mathcal{C}) = \lim_{k \to \infty} \frac{1}{k} \log n(\mathcal{C}_f^k)$$

The limit exists and $0 \leq h(f, \mathcal{C}) \leq \infty$. We define the topological entropy of f as:

$$h(f) = sup\{h(f, \mathcal{C}) \,/\, \mathcal{C} \text{ open cover of } X\}$$

We want to outline some basic results on the topological entropy of m-modal maps of the interval; proofs and references could be found in [MT].

Theorem 4.3.6. ***(Misiurewicz, Yomdin)*** *Let I be the unit interval. The topological entropy function $h : \mathcal{C}^\infty(I, I) \to [0, \infty)$ is continuous.*

Corollary 4.3.7. *For $d \geq 1$, the entropy function is continuous on the compact space of polynomials $P : I \to I$ of degree $\leq d$.*

Proposition 4.3.8. ***(Misiurewicz, Szlenk, Rothschild)*** *If $f : I \to I$ is an m-modal map, then:*

$$h(f) = \lim_{k\to\infty} \frac{\log l(f^{\circ k})}{k} \leq \log(m+1),$$

where $l(f^{\circ k})$ is the minimum number of intervals of monotonicity of $f^{\circ k}$.

The proposition above gives an easy computation method for the entropy in the case the orbits of the folding points give a Markov partition of I.

Another useful alternate definition involves the notion of *negative orbit complexity*:

$$h(f) = \overline{\lim_{k\to\infty}} \frac{1}{k} \log^{+}(Neg(f^{\circ k})),$$

where $Neg(f^{\circ k})$ is the number of fixed points of negative type of $f^{\circ k}$.

Finally, we state the following results to illustrate the relation between the topological entropy of an m-modal map and its kneading-data.

Theorem 4.3.9. *The topological entropy $h(f)$ of an m-modal map f is determined by its kneading-data $\mathbf{K}(f)$ and depends continuously on it.*

Moreover, if $\mathbf{K}(f) >> \mathbf{K}(g)$ then $h(f) \geq h(g)$.

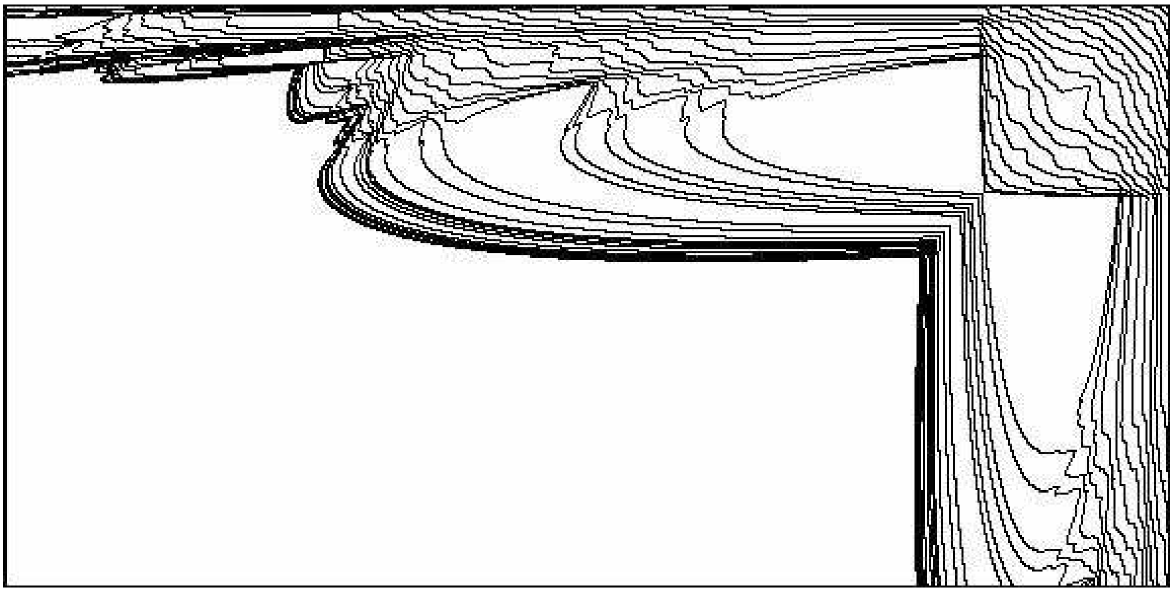


Figure 4.4: *The figure shows the region in* P^Q *with* $\lambda > \frac{1}{2}$ *and* $\mu > \frac{3}{4}$*. The two large white arcs correspond to the regions with entropy* $\log 2$*. The points on the diagonal of* P^Q *should have the same corresponding entropy values as the points in the real Mandelbrot set, described in [D].*

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