Towards classification of laminations associated to quadratic polynomials

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Abstract of the Dissertation

Towards classification of laminations associated to quadratic polynomials

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In this thesis we develop the topological classification of laminations associated to superattracting quadratic polynomials, those are quadratic polynomials with periodic critical point. Such laminations for rational maps were constructed by Lyubich and Minsky. In particular, we prove that the topology of such laminations is determined by the combinatorics of the parameter. We also describe the topology of laminations associated to other types of quadratic polynomials.

To Claudia

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Chapter 1

Introduction

The "natural extension" $\mathcal{N}_f$ (or the "inverse limit") of a rational map f is a very interesting object whose topology and geometry reflects in a intricate way the dynamics of f (see Lyubich and Minsky in [21]). In this thesis, we study the relation between the topology of $\mathcal{N}_f$ and the dynamics of f, focusing on the case of quadratic polynomials $f_c : z \mapsto z^2 + c$.

The natural extension $\mathcal{N}_f$ contains the "regular leaf space" $\mathcal{R}_f$ whose connected components ("leaves") are endowed with a natural conformal structure. When the dynamics of f is simple, the corresponding regular leaf space has a lamination structure that is, there is an atlas of charts, such that the image of every chart is the product of a disk times a Cantor set. Moreover, the leaves of $\mathcal{R}_f$ are simply connected Riemann surfaces conformally isomorphic to the complex plane. This is the case for hyperbolic quadratic polynomials (and in this case $\mathcal{R}_f$ is obtained from $\mathcal{N}_f$ by removing finitely many points). After a suitable refinement of the topology of the inverse limit, this is also the case for quadratic polynomials with non-recurrent critical point on the Julia set. In general, Lyubich and Minsky's construction provides an *orbifold affine lami-*

nation associated to any rational map, which in turn, admits an extension to a *3-dimensional hyperbolic lamination*.

The main goal of this thesis is to prove that, for hyperbolic quadratic polynomials, the topology of the lamination determines the combinatorics of the corresponding parameter. More precisely, if $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ is an orientation preserving homeomorphism between the regular leaf spaces associated to the superattracting parameters c and c', then $c = c'$ (when h is orientation reversing then we have $c' = \bar{c}$).

Let us now give a more detailed description of the contents of the thesis.

In Chapter 2, we summarize the necessary background in the basic holomorphic dynamics. We assume the reader is familiar with the subject, and we highlighten only the facts that are important for understanding the structure of the associated laminations. Most of this theory is readily available in various surveys in complex dynamics, like [3],[6], [9], [19] and [24].

We begin with the definitions of the Fatou and Julia sets, followed by the classification of periodic points and Fatou components.

We then continue with the classical linearization theory near a periodic point. From the lamination point of view, the linearization provides us with a suitable uniformization of the corresponding periodic leaf. We discuss from this point of view the Königs and Böttcher coordinates (near repelling and superattracting points), and then mention some recent results by Tomoki Kawahira [14] concerning the parabolic case.

Although we are focused on the superattracting case, most of our results are set over a broader class of parameters, including those for which the critical point is non-recurrent. According to [21], in this case the corresponding 3-lamination is *convex cocompact*. We describe some basic dynamical features of these convex cocompact parameters.

There are several combinatorial models that describe superattracting quadratic polynomials. For a complete discussion of them see the paper of Henk Bruin and Dierk Schleicher [5]. We describe some of these models, for which we have found topological analogues in the regular leaf space. One of the most informative is the model of *ray portraits* as presented by John Milnor in [25]. For the proof of our main theorem, we will see that the topology of the affine lamination associated with a superattracting quadratic polynomial determines the ray portrait of the corresponding parameter. The interested reader can also find more about the theory of the combinatorics of postcritically finite quadratic polynomials in [5], [8],[9],[4], [18], [25], and [26].

In Chapter 3, we discuss basic definitions and properties of inverse limits. A classical example of an inverse limit is the *dyadic solenoid* $\mathcal{S}^1$, associated to the map $f_0 : z \mapsto z^2$ on the unit circle S^1. As a lamination the dyadic solenoid is one-dimensional; moreover, it is naturally endowed with the structure of a compact topological group.

The map f_0 admits a natural extension $\hat{f}_0$ to the solenoid $\mathcal{S}^1$. It turns out that $\hat{f}_0$-periodic leaves in $\mathcal{S}^1$ are in one-to-one correspondence with periodic points of f_0 in S^1. In turn, the periodic points of f_0 are in one-to-one correspondence with rational angles with odd denominators. These observations are the first step towards the proof of the Main Theorem.

The inverse limit $\varprojlim(f_0, \hat{\mathbb{C}} \setminus \mathbb{D})$ is homeomorphic to a cone, denoted by

$Con\,(\mathcal{S}^1)$, over the dyadic solenoid $\mathcal{S}^1$. Due to the Böttcher's coordinate at infinity, for every quadratic polynomial f_c with connected Julia set, the lift of the basin of infinity to $\varprojlim(f_0, \hat{\mathbb{C}} \setminus \mathbb{D})$ is homeomorphic to the interior of that cone.

We call it the *solenoidal cone at infinity* of f_c. This solenoidal cone admits a foliation by dyadic solenoids $\mathcal{S}^1$ coming from the lifts of the equipotentials in the dynamical plane. Accordingly, we call each leaf of this foliation a *solenoidal equipotential*, in fact, each solenoidal equipotential is canonically identified with $\mathcal{S}^1$.

Let $\hat{f}_c$ be the natural extension of f_c, acting on its solenoidal cone at infinity. Then, the action of $\hat{f}_c$ over a solenoidal equipotential is conjugate to the natural extension of $\hat{f}_0$ on $\mathcal{S}^1$. Hence, periodic leaves of the solenoidal cone at infinity of f_c are in one-to-one correspondence with periodic points in $\mathcal{S}^1$.

In Chapter 4, we describe the topology of the *laminated Julia set*, that is, the lift of the Julia set in the regular leaf space. When f_c is convex cocompact, the laminated Julia set is compact, see [21]. We show that if the postcritical set is not the whole Julia set and the Julia set is locally connected, then f_c is convex cocompact if and only if the laminated Julia set is leafwise connected.

By another result in [21], for a convex cocompact f_c, all leaves of the regular leaf space $\mathcal{R}_c$ are conformally isomorphic to the complex plane. Thus, given a leaf $L \subset R_c$, it makes sense to consider the *unbounded Fatou components* in L. It turns out that, non-periodic leaves of $\mathcal{R}_c$ have no more than 2 unbounded Fatou components. On the other hand, the number of unbounded Fatou components on periodic leaves depends on the valence of the corresponding repelling periodic point.

Once we have described the basic topology of the laminated Julia set, we are ready to give some restrictions to homeormorphisms between regular leaf spaces of superattracting parameters. Because, the regular leaf space is locally compact and the laminated Julia set is compact, we can compute the number of unbounded Fatou components in some leaf L in terms of the topology at infinity of L.

When f_c is superattracting, on the boundary of the attracting basin of the critical cycle there is a repelling cycle, called the *dynamic root cycle* of f_c, which is the fixed point of the first return map of the Fatou component containing the critical value. We prove that the topology at infinity of the leaves containing the lift of the dynamic root cycle is different from all other leaves. Then, any homeomorphism as in the Main Theorem, must send leaves containing the lift of the dynamic root cycle of f_c into the corresponding ones of $f_{c'}$.

In the last section of Chapter 4, we prove that the regular leaf space associated to any hyperbolic quadratic polynomial f_c is homeomorphic to the regular leaf space of the center of the hyperbolic component containing c. Hence, it is enough to describe the superattracting case for all hyperbolic parameters.

In Chapter 5, we prove the Main Theorem. The strategy of the proof is to replace the homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ by another homeomorphism $\tilde{h}$ with special characteristics.

First we show that h is isotopic to a homeomorphism $\tilde{h}$ that sends a solenoidal equipotential of $\mathcal{R}_c$ homeomorphically into a solenoidal equipotential of $\mathcal{R}_{c'}$. Now, using the canonical identifications with $\mathcal{S}^1$ on both solenoidal equipotentials, the map $\tilde{h}$ induces a self-homeomorphism of the dyadic solenoid, which in turn, according to Kwapisz [17] is isotopic to an *affine transformation* of the dyadic solenoid (that is, to the composition of an automorphism of the solenoid and a translation). In order to get $\tilde{h}$, we discuss some isotopic properties of self-homeomorphisms of the dyadic solenoid $\mathcal{S}^1$ and its cone $Con\,(\mathcal{S}^1)$.

From the results of the previous chapter, we can find a homeomorphism between the regular spaces that sends a solenoidal equipotential into a solenoidal equipotential, and the restriction to this solenoidal equipotentials, after the canonical identifications with $\mathcal{S}^1$, is the identity. This implies that the orbit portraits of the dynamic root cycles of c and c' are the same, which leads to the conclusion of the Main Theorem.

In the last section, we discuss some local topological properties of irregular points which make them distinguishable from regular points. In a forthcoming joint work with Avraham Goldstein, we prove that if $h : \mathcal{N}_c \to \mathcal{N}_{c'}$ is a homeomorphism of the natural extensions of any two quadratic polynomials, then h sends the point at infinity in $\mathcal{N}_c$ to the corresponding point in $\mathcal{N}_{c'}$. This generalizes to inverse limits of polynomials of the form $f_c = z^d + c$. Moreover, with Goldstein's idea of "signatures", a sort of combinatorial model of irregular points, we can fully describe the local connectivity properties of a given irregular point.

Chapter 6 is part of a joint project with Yasuhiro Tanaka, where we try to compute the leafwise number of unbounded Fatou components of $\mathcal{F}_c$ using the Monodromy group of the regular leaf space. The case $z^2 - 1$ is easy to carry on and can be generalized to hyperbolic components attached to the Main Cardioid. However, the computation of unbounded components becomes harder, the harder the topology of the Hubbard tree is. Altogether, the point of this chapter is to remark that the computation of the number of Fatou components and their description can be obtained from an algebraic point of view.

When the parameter is postcritically finite, the working tool is a construction of the monodromy group of the regular leaf space, due to Volodymyr Nekrashevych, which allows us to describe the orbits of points on a given fiber. The orbits can be arranged in a graph, the *Monodromy Graph*, which is leafwise connected, and shares the same topology at infinity as the laminated Julia set. Let us remark that, the computation of unbounded Fatou components using the monodromy group, can be done only for convex cocompact polynomials.

In conclusion, we develop the "basillica" case, that is $f_{-1} = z^2 - 1$, counting the number of ends that the monodromy has in each leaf. The only property of f_{-1} that we use is that the corresponding hyperbolic component is attached to the Main Cardioid. Thus, the computation generalizes to all satellite components attached to the Main Cardioid.

Chapter 2

Basic complex dynamics

2.1 Dynamical plane

2.1.1 The Fatou set

Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational function on the Riemann sphere $\bar{\mathbb{C}}$. By $F(f)$, we denote the *Fatou set*, that is, the set of all points in the plane $\mathbb{C}$ for which there is a neighborhood U such that the set of iterates $\{f^n\}$ is a normal family in U.

An *invariant set* A of f is a set such that $f(A) \subset A$. In other hand, the set A is called *completely invariant* if $f^{-1}(A) = A$. By definition, the Fatou set is open and completely invariant.

2.1.2 Periodic points

Let $P = \{p_1, ..., p_m\}$ be a periodic cycle of f of period m. The number $(f^m)'(p_1) = \lambda_{p_1}$ is called the *multiplier* of p_1. By the Chain Rule, the multiplier λ_{p_1} is equal to the product $\prod f'(p_i)$ of the derivatives of the points in P.

Thus, the multiplier only depends on P and not on a particular point in the cycle.

A map f is *locally linearizable* at a periodic point p if there is a function ϕ, defined on a neighborhood of p, satisfying the equation $f(\phi(z)) = \lambda_p \phi(z)$. The map ϕ is called a *linearizing coordinate.* A periodic point p of period m, is called:

- *Attracting* if $0 \leq |\lambda_p| < 1$, when $\lambda_p = 0$, the point is also called *superattracting.*
- *Repelling* if $|\lambda_p| > 1$.
- *Neutral* if $|\lambda_p| = 1$, neutral periodic points can be further separated in three cases:
  - *Parabolic* if $\lambda_p = e^{2\pi i(\frac{p}{q})}$, where $\frac{p}{q}$ is a rational number.
  - *Siegel* if $\lambda_p = e^{2\pi i\theta}$, where θ is irrational and the map f is locally linearizable at p.
  - *Cremer* if $\lambda_p = e^{2\pi i\theta}$, where θ is irrational and f is not locally linearizable at p.

A celebrated Theorem by Denis Sullivan [30], states that every connected component of $F(f)$ is eventually periodic. Thus, following Fatou, the connected components of the Fatou set can be classified upon its associated cycle.

Let U be a periodic Fatou component, then U is called:

- *hyperbolic* if it contains an attracting periodic point p and every orbit in U has p as an accumulation point.

- *parabolic* if it has a parabolic periodic point at the boundary which is an accumulation point of every orbit in U.
- *Siegel disk* if it contains a Siegel periodic point and the set of accumulation points of any orbit in $U \setminus \{p\}$ has more than one point.
- *Herman ring* if it is doubly connected and the set of accumulation points of any orbit on U has more than one point.

Let p be an attracting or superattracting periodic point, the set $A(p)$ denotes the Fatou component containing p, and is called the *immediate basin* of attraction of p. The *immediate basin of attraction* of a periodic cycle P is the union of the immediate basins of the points in P.

Fatou components are either simply, doubly or infinitely connected. When f is a polynomial, the Fatou set has infinitely connected components when at least one finite critical point converges to infinity under dynamics. On the other hand, doubly connected components are either Herman rings or preimages of Herman rings, see Lyubich [19].

2.1.3 The Julia set

The set $J(f) = \mathbb{C} \setminus F(f)$, is called the *Julia set* of the map f. As the complement of a completely invariant open set, the Julia set is closed and completely invariant. A Theorem of Fatou and Julia relates the Julia set with the set of repelling periodic points:

$$J(f) = \overline{\{\text{The set of repelling periodic cycles of } f\}}.$$

In fact, if the degree of $d = \deg(f)$ is at least 2, then all but finitely many periodic points are repelling. So, if $d \geq 2$, then $J(f)$ is non-empty and, moreover, it does not have isolated points.

2.1.4 Local coordinates

One of the important features of holomorphic dynamics is that, besides Cremer periodic points, the local behavior of periodic points can be described in easy terms by locally conjugating the map f to a simple map. In this work, we will only use Königs and Böttcher's coordinates which are associated to attracting and superattracting points. As a consequence of the existence of such coordinates, attracting and superattracting periodic points belong to the Fatou set. For reference, we also include the definition of the Fatou's coordinate which is associated to parabolic periodic points. Parabolic periodic points are of a combined dynamical nature; there are orbits points attracted to them and orbits which are repelled. Hence, parabolic points belong to the Julia set.

When p is a Siegel periodic point there is a linearizing coordinate ϕ conjugating f to a irrational rotation $z \mapsto \lambda_p z$. This implies that Siegel periodic points belong to the Fatou set. Fatou components containing Siegel periodic points are, accordingly, Siegel disks. In other hand, Cremer periodic points do not admit a linearizating coordinate and belong to the Julia set.

2.1.5 Königs coordinate

Let p be an attracting periodic point of f of period m, let $A(p)$ be the immediate basin of attraction of p. Then, a Theorem by Königs states that there is a map $\phi : A(p) \to \mathbb{C}$, called the *Königs coordinate*, such that makes the following diagram commutative:

$$\begin{array}{ccc} A(p) & \xrightarrow{f^m} & A(p) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{C} & \xrightarrow{z \mapsto \lambda_p z} & \mathbb{C} \end{array}$$

the map ϕ is unique up to multiplication by scalars. A classical expression for ϕ is $\phi(z) = \lim\limits_{n\to\infty} \frac{f^n(z)}{\lambda_p^n}$. In the case of repelling periodic points p, we can take the branch of the inverse f^{-1} defined locally on disks around the cycle of p, and such that p is also periodic for f^{-1}. As p is repelling periodic point of f, p is an attracting periodic point of f^{-1}. Thus Königs linearization provides a coordinate chart ϕ for f^{-1}, which conjugates f^{-1} with $z \mapsto \lambda_p^{-1} z$. Since $z \mapsto \lambda_p^{-1} z$ is invertible, ϕ can be used to conjugate f to $z \mapsto \lambda_p z$.

2.1.6 Böttcher's coordinate

Let f be a rational function of degree d, and let p be a superattracting periodic point of f of period m. Then, there is a conformal map $\phi : A(p) \to \mathbb{D}$, called the *Böttcher's coordinate*, that makes the following diagram commutative:

$$\begin{array}{ccc} A(p) & \xrightarrow{f^m} & A(p) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{D} & \xrightarrow{z \mapsto z^d} & \mathbb{D} \end{array}$$

The map ϕ is unique up to multiplication by a $p-1$ root of unity.

2.1.7 Fatou's coordinate

Let p be a parabolic periodic point of period m. Hence, p is a parabolic fixed point for f^m and, there are v_p Fatou components S_i, called the *attracting petals* of p, such that p is a common point in the boundary of all S_i; moreover, if $x \in S_i$, then $f^{(v_p \cdot m)n}(x) \in S_i$ for every n and, the orbit of x under $f^{(v_p \cdot m)}$ tends to p. The number v_p is called the *valence* of p.

Let S be an attracting petal for p, then there is a map $\phi : S \to \mathbb{C}$, called the *Fatou's coordinate* of S, that makes the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{f^{v_p \cdot n}} & S \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{z \mapsto z+1} & \mathbb{C} \end{array}$$

such ϕ is unique.

In analogy with the attracting case, the union of the attracting petals S_i is called the *immediate basin* of attraction of p, and will be also denoted by $A(p)$. Also, the immediate basin of a parabolic cycle P will be the union of the immediate basins of each point in P. A Theorem by Fatou states that there is always at least one critical point in the immediate basins of attracting and parabolic cycles. Thus, all points in attracting and parabolic cycles are accumulation points of the orbit of some critical point.

2.1.8 Dynamics of polynomials

From now on, unless otherwise stated, we restrict our attention to the case of quadratic polynomials Q defined on $\bar{\mathbb{C}}$. Every quadratic polynomial Q is conjugate to a unique polynomial of the form $f_c(z) = z^2 + c$. The number c is called the *parameter* of Q, whenever we refer to a quadratic polynomial we will think of it as already on the normalized form f_c. Objects depending on f_c will be just indexed by c.

The union of the bounded orbits of f_c is called the *filled Julia set* and will be denoted by K_c. As a consequence of the maximum principle, the set of iterates of f_c defined on bounded components of the Fatou set $F(f_c)$ are bounded for every c. So, every bounded Fatou component belongs to the filled Julia set, which justifies the name"filled".

There are no Herman rings for polynomials, thus, all Fatou components are simply connected and are either associated to attracting, parabolic or Siegel cycles. It also can happen that there are no bounded components at all, in this case, the basin of infinity is the whole Fatou set.

Quadratic polynomials of the form $f_c = z^2 + c$ have two critical points, 0 and ∞. The point at infinity ∞ is a superattracting fixed point, its basin of attraction, the *basin of infinity*, will be denoted by $A(\infty)$. Also, the boundary $\partial A(\infty) = J(f_c)$.

Putting aside the fixed point at ∞, the orbit of 0 is called the *critical orbit*, while its closure $P_c = \overline{\cup\{f_c^n(0)\}}$ is called the *postcritical set*. If 0 is a periodic point of f_c, the parameter c is said to be *superattracting* and, accordingly, the cycle of c will be called the *critical cycle*.

2.2 Parameter plane

Most of the dynamical behavior of rational maps can be understood by describing the dynamics of the critical points. A dramatic example of this principle is given by the *quadratic family* $\mathcal{QF} = \{f_c(z) = z^2 + c\,|c \in \mathbb{C}\}$, where the associated Julia set has a clear dichotomy:

Proposition 1 (Julia set dichotomy). *The Julia set J_c of the quadratic polynomial $f_c(z) = z^2+c$ is either connected or totally disconnected, depending on whether the critical orbit is bounded.*

The set $M = \{c \in \mathbb{C}|J_c \text{is disconnected}\}$ is called the *Mandelbrot set*. Another characterization of the Mandelbrot set is as the set of parameters c such that the basin of infinity $A_c(\infty)$ is simply connected in $\bar{\mathbb{C}}$. We refer to the plane of c as the *parameter plane*. In counterpart, the plane where the Julia set lies is called the *dynamical plane*.

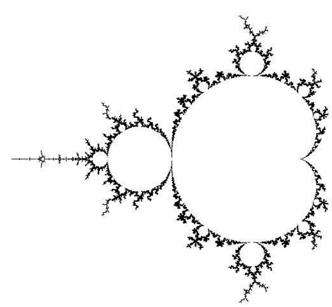


Figure 2.1: The Mandelbrot Set

If the parameter c is such that $|c| > 2$, then the critical orbit of f_c is unbounded. Hence, the set M is bounded by the disk of radius 2 centered at the origin. Having the critical orbit bounded is a closed property, this implies that the Mandelbrot set is closed and therefore it is compact. Also, by a Theorem of Adrien Douady and John Hubbard [9], the set M is connected and simply connected.

The *Böttcher's function* $\phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \mathbb{D}$ conjugates f_c to $z \mapsto z^2$ and is asymptotically the identity at infinity. The function ϕ_c is nothing but the Böttcher coordinate at infinity composed with the map $z \mapsto 1/z$. The Green's function of the complement of the Mandelbrot set $G : \mathbb{C} \setminus M \to \mathbb{R}_+$, is tightly related to ϕ_c. In fact, $G(c) = \log|\phi_c(c)|$ and more, the map $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$, given by $\Phi(c) = \phi_c(c)$, is a conformal map.

Of special interest, is the orthogonal family in $\mathbb{C} \setminus M$ coming from the inverse image under Φ of the orthogonal family of concentric circles centered at the origin, and radial rays in $\mathbb{C} \setminus \mathbb{D}$. In the parameter plane, preimages of concentric circles under Φ are called *external equipotentials*, and preimages of radial rays are called *external rays*. Analogously, induced by the Böttcher's function on $\mathbb{C} \setminus K_c$, we have two foliations by external rays and equipotentials on the basin of infinity. In order to avoid confusion, we will always be specific whether we are dealing with external rays on the parameter plane, or on the dynamical plane.

When c is superattracting, the Böttcher's coordinate around the critical orbit also induces a foliation on the immediate basin of attraction of the critical cycle. Thus, if p is a point in the critical cycle, the basin of attraction $A(p)$, is foliated by rays and equipotentials, coming from concentric circles and radial rays in $\mathbb{D}$. To distinguish them from the ones associated to the critical point at ∞ we will call them *internal equipotentials* and *internal rays*.

We say that the external ray R_θ *lands* at the Mandelbrot set if $\lim_{r\to 1} G^{-1}(re^{2\pi i\theta})$ exists. An important question in holomorphic dynamics is whether every ray lands at M. Nevertheless, Douady and Hubbard proved that every ray with rational angle θ lands at the Mandelbrot set. Furthermore, if $\theta = \frac{p}{q}$ is in reduced form, then the ray $R_{\frac{p}{q}}$ lands in a parabolic parameter when q is odd and in a preperiodic parameter otherwise.

We say that a parameter c is called *hyperbolic* if the map f_c has an attracting cycle different from $\{\infty\}$. The set of hyperbolic parameters is open in the parameter plane. On the dynamical plane, the basin of attraction of any attracting cycle contains a critical point, so if c is hyperbolic, then the critical orbit is bounded and hence c belongs to the Mandelbrot set. A *hyperbolic component* of M is a connected component of the set of hyperbolic parameters. The set of hyperbolic components is dense in the boundary of the Mandelbrot set, in particular M has dense interior.

Let H be a hyperbolic component; for every $c \in H$, let $\lambda_p(c)$ be the multiplier of the corresponding attracting cycle. A Theorem of Douady and Hubbard [9], states that the *multiplier map* Λ, given by $c \mapsto \lambda_p(c)$, is a conformal isomorphism from H to $\mathbb{D}$. In particular, the parameter $c_0(H) = \Lambda^{-1}(0)$, the *center* of H, is a superattracting parameter. By associating hyperbolic components with their center, there is a one-to-one correspondence between superattracting parameters and hyperbolic components. The hyperbolic component containing 0 is known as the *Main Cardioid*.

By a Theorem of Jean-Christophe Yoccoz, the boundary of every hyperbolic component in the Mandelbrot set is locally connected. Hence, the multiplier map extends to ∂H. The point $\Lambda^{-1}(1) \in \partial M$ is a parabolic point called the *root* of H, and it is the only point in ∂H that disconnects H from the Main Cardioid. Hence, hyperbolic components can be identified either with their root or their center. We will see later that the root and the center of a hyperbolic component H share the same combinatorial description than all parameters in H. In particular, the Julia set of any parameter in H is homeomorphic to the Julia set of the root of H.

There are two possibilities for a hyperbolic component H, either the root of H belongs to the boundary of exactly two hyperbolic components, in this case the hyperbolic component H is called *satellite*, or the root of H belongs only to the boundary of H, and then H is called a *primitive* hyperbolic component.

2.2.1 Postcritically non-recurrent parameters

Besides superattracting, hyperbolic and Misiurewicz parameters, there is also a class of parameters such that the orbit of the critical point does not return too close back to itself. More precisely, let $f : X \to X$ be a dynamical system defined in a metric space X. A non-periodic point $x_0 \in X$ is said to be a *recurrent* point of f if there is a sequence $\{n_k\}$ of times such that $x_0 \in \overline{\cup\{f^{n_k}(x_0)\}}$. The action of f on a set A is said to be *non-recurrent* if no point a in A is a recurrent point of f. The omega limit $\omega(x)$-*limit set* of a point $x \in \mathbb{C}$ is the set of accumulation points of the orbit $\{f_c^n(x)\}$ of x.

A stronger notion of recurrence in a set A is when every point of A is an accumulation point of its orbit: Let $f : X \to X$ be a dynamical system defined in a metric space X. A set A is called *minimal* if it is closed invariant under f and no proper subset of A has this property.

Note that every point in a minimal set most be recurrent. A Theorem of Birkhoff shows that every dynamical system contains a minimal set.

A *postcritically non-recurrent* rational map f is a map whose action on the postcritical set is non-recurrent. In the case of quadratic polynomials, it has been proved by Carleson, Jones and Yoccoz [7], that any postcritically non-recurrent quadratic polynomial has locally connected Julia set. Another non-trivial Theorem by Yoccoz states that critically non-recurrent parameters are locally connected in the Mandelbrot set. On [7] these parameters are called *subhyperbolic*, however we will introduce later another term related to the associated lamination.

A Theorem by Fatou states that if the critical points of a rational map f belong to $F(f)$ then, for every x in $J(f)$ there is a number $C > 0$ and $\sigma > 1$ such that $|(f^n)'(x)| > C\sigma^n$. In other words, the map f is expanding on the Julia set. This is the case for all quadratic polynomials with hyperbolic parameter. Now, the following Theorem by Ricardo Mañe [22] describes, in the general case, those points in the Julia set which are expanding.

Theorem 2 (Mañe's Theorem). *Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map. A point $z \in J(f)$ is either a parabolic periodic point, or belongs to the ω-limit set of a recurrent critical point, or for every $\epsilon > 0$, there exists a neighborhood U of x, such that, $\forall n \geq 0$ every connected component of $f^{-n}(U)$ has diameter $\leq \epsilon$.*

A related, and useful, result is the following Lemma:

Lemma 3 (Shrinking Lemma). *Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map. If $K \subset J(f)$ is a compact subset disjoint from parabolic periodic points and ω-limit sets of recurrent critical points, then for every $\epsilon > 0$ there exist $\delta > 0$*

such that for every $k \in K$ and every $n \geq 0$, all connected components of $f^{-n}(B(k, \delta))$ have diameter less than ϵ.

The wilder the postcritical set of a function f is, the more intricate becomes the description of the dynamics of f. In this work, when speaking of critically recurrent quadratic polynomials f_c, we will consider only parameters with locally connected Julia set, and whose postcritical set P_c is either a Cantor set or the action of the map f_c restricted to P_c is minimal.

We will be primarily interested on superattracting quadratic polynomials. The other cases of postcritically finite quadratic polynomials are parameters with preperiodic critical orbit. Preperiodic parameters are also known as *Misiurewicz* parameters. In this case, the corresponding cycle most be contained on the Julia set, also, the Julia sets are locally connected, one dimensional continua with no loops, this type of compact sets are known as *dendrites* see [6].

As we disscused before, superattracting parameters can be associated to the root of their corresponding hyperbolic component. Since parabolic and Misiurewicz parameters are landing points of external rays of rational angle, by taking the minimum of such angles, every postcritically finite parameter is associated to a well defined rational angle. In this way, postcritically finite parameters, can be described by the combinatorics of the corresponding angle. In next section, we treat postcritically finite parameters.

2.3 Combinatorics of postcritically finite polynomials

A natural way to classify postcritically finite polynomials is by describing the different ways the arrangement that the critical orbit may have as a subset in the plane. The combinatorics of this arrangement are reflected in both dynamical and topological properties of the Julia set. In fact, we could say that similar properties are hold for a slightly large set of parameters; namely, parameters postcritically non-recurrent. Any such description is called a *combinatorial model*. Combinatorics also make explicit the relationship between the parameter and the dynamical plane. So, certain combinatorial behaivor of a given parameter c, determines a region in the parameter plane where c must lie.

2.3.1 Hubbard trees

To begin with, let us present a combinatorial model given by Douady and Hubbard; for any postcritically finite parameter c they constructed an abstract graph, called the *Hubbard tree*, describing the dynamical arrangement of the postcritical set. Douady and Hubbard proved that different combinatorics induces different Hubbard trees. The graph is properly embedded in the dynamical plane as a subset of the filled Julia set and can be defined by a finite set of vertices in the Julia set. To see this, note that between any two points z and ζ in the Julia set $J(f_c)$ there is an unique arc γ, embedded in the Julia set, connecting z with ζ. The unicity of γ is subject to the condition that, if the trayectory of γ intersects a Fatou component, then it goes along internal rays. In this way, the *Hubbard tree* of a postcritically finite quadratic polynomial f_c is defined as the smallest collection of arcs, embedded in the Julia set $J(f_c)$ and connecting the entire critical orbit. This tree is finite and forward invariant under the action of f_c.

2.3.2 The doubling map

Let us consider the polynomial $f_0 = z^2$ which is associated to the center of the component inside the Main Cardioid. The orbit of any point inside the unit circle converges to 0 under iteration of f_0; while the orbit of any point outside the unit circle tends to ∞. Hence, the Fatou set consists of only two domains and, the Julia set is just the unit circle $\mathbb{S}^1$ in the complex plane $\mathbb{C}$.

The action of f_0 in the unit circle is $f_0(e^{2\pi i\theta}) = e^{2(2\pi i\theta)}$. That is, it doubles the corresponding angle θ. So, the map $f_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is conjugate to the doubling map $\mathcal{D} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, which sends the angle θ to $2\theta \pmod 1$. For convenience, whenever we refer to a point in the unit circle, we denote it by its angle in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We also assume the standard orientation on the unit circle. The orbit under doubling of every rational angle is finite, and the set of periodic points of $\mathcal{D}$ is exactly the set of rational angles p/q where q is an odd number.

By parameterizing external rays with their corresponding angle in $\mathbb{T}$, and as a consequence of Böttcher conjugacy of f_c to z^2 on the basin of infinity, we have $f_c(R_\theta) = R_{2\theta}$. So, the action of f_c on the set of external rays is conjugate to the action of the doubling map in $\mathbb{T}$. This feature will play a very important role in this work.

2.3.3 Ray portraits

The description by ray portraits of postcritically finite maps is based upon the following proposition:

Proposition 4 (Douady and Hubbard). *Every repelling and parabolic periodic point of a quadratic polynomial f_c is the landing point of an external ray with rational angle. Conversely, every external ray with rational angle lands either at a periodic or preperiodic point in $J(f_c)$.*

From now on, we will follow the exposition of John Milnor in [25], see also the related work of Alfredo Poirier [26] and Dierk Schleicher [29]. By Proposition 4, every parabolic or repelling cycle $P = \{p_1, ..., p_n\}$ is associated to a family of finite sets $\mathcal{O}_P = \{A_1, ..., A_n\}$, where $A_i = \{\theta \in \mathbb{Q}/\mathbb{Z} | R_\theta$ lands in $p_i\}$. The family $\mathcal{O}_P$ is called the *orbit portrait* of the cycle P. The doubling map $\mathcal{D}$ on $\mathbb{T}$ permutes the sets A_i and acts on the angles in A_i in an order preserving way. Moreover, every angle in $\cup A_i$ is periodic under the doubling map, the period of such angle only depends on the cycle. As a consequence, all A_i have the same cardinality. Thus, for $p \in P$, the *valence* v_p of p is the cardinality of any of the sets $A_i \in \mathcal{O}_P$.

Given two angles θ_1 and θ_2 in the unit circle $\mathbb{T}$, by $\widehat{\theta_1\theta_2}$ we denote the *directed arc* in $\mathbb{T}$ from θ_1 to θ_2. Now, if A is a finite set in $\mathbb{T}$, a *complimentary arc* of A is the closure of any connected component of $\mathbb{T} \setminus A$. If the valence is bigger than one, each A_i determines a finite collection of complimentary arcs in the circle. Among the union of all complimentary arcs of all A_i, see Lemmas 2.5

and 2.6 in [25], there is a unique arc of shortest length called the *characteristic arc*. The preimage of the characteristic arc under doubling is also the longest complimentary arc which is called the *critical arc*, also the critical arc is bigger than 1/2. Since the doubling map preserves the order in the complimentary arcs, if $\widehat{\theta_1\theta_2}$ is the characteristic arc of some orbit portrait $\mathcal{O}_P$, then by a straight forward calculation, the critical arc is $\widehat{\eta_1\eta_2}$ where $\eta_1 = \theta_1/2 + 1/2$ and $\eta_2 = \theta_2/2$.

2.3.4 The dynamic root point

Let c be a superattracting parameter of period n, then the first return map of the Fatou component containing c has a unique fixed point on the boundary of the Fatou component containing c. Following Milnor [25] and Schleicher [29], we call r_c the *dynamic root* of the superattracting parameter c. The orbit portrait of r_c is called the *critical portrait* of c. It is a Theorem by Milnor [25] that, the critical portrait characterizes the parameter c. In other words, no two superattracting parameters have the same critical portrait. (See also [26].) Remind that rotations on the unit circle $\mathbb{T}$ are given by maps $r_\theta : \mathbb{T} \to \mathbb{T}$ of the form $r_\theta(\tau) = \tau + \theta$, mod $(\mathbb{Z})$.

Lemma 5. *Let $\mathcal{O}_P$ and $\mathcal{O}_{P'}$ be the ray portraits of the periodic cycles P and P'. If there is a rotation of the circle r_θ, such that $r_\theta(\mathcal{O}_P) = \mathcal{O}_{P'}$, then $\mathcal{O}_P = \mathcal{O}_{P'}$. In particular, if $\mathcal{O}_c$ and $\mathcal{O}_{c'}$ are the critical portraits of two superattracting quadratic polynomials differing by a rotation, then $c = c'$*

Proof. Since the characteristic arc is the minimal complimentary arc; the rotation r_θ must send the characteristic arc of P to the characteristic arc of P'.

Analogously, r_θ must send the critical arcs of P to the critical arc of P'. If $\widehat{\theta_1\theta_2}$ and $\widehat{\theta_1'\theta_2'}$ are the characteristic arcs of P and P' respectively, then the hypothesis yield the following equation

$$\theta_i + \theta = \theta_i'$$

for the characteristic arcs, and

$$\theta_i/2 + 1/2 + \theta = \theta_i'/2 + 1/2$$

for the critical arcs. Thus, $\theta = 0$ and $\theta_i = \theta_i'$ for $i = 1, 2$, then the critical arc of f_c is equal to the critical arc of $f_{c'}$. Since the characteristic arcs generate the orbit portrait, $P = P'$. The second part of the lemma follows from the fact that the critical portrait determines the parameter c. □

2.3.5 Combinatorics in the parameter plane

Now, we briefly discuss how the ubication of certain parameter c on the Mandelbrot set affects combinatorics of periodic orbits in the dynamical plane of f_c. Analogous to Proposition 4, we have:

Proposition 6 (Douady and Hubbard). *In the parameter plane, every parabolic or Misiurewicz parameter is the landing point of at least one external ray of rational angle. Inversely, every external ray with rational angle θ lands at some point c in the boundary of the Mandelbrot set. Moreover, if θ has odd denominator, then c is a parabolic parameter and Misiurewicz otherwise.*

In this way, every rational angle is associated to either a parabolic or a

Misiurewicz parameter. Besides $c = 1/4$, which corresponds to the cusp of the main cardiod, every parabolic parameter c is the landing point of exactly two external rays, say R_{θ_1} and R_{θ_2}. These external rays cut the plane into two parts, one of them contains the Main Cardioid and the other is called the *wake* W_c determined by c. The limb at c is the intersection $W_c \cap M$. When c is Misiurewicz in can be the landing point of several external rays.

On other hand, every parabolic parameter c is the root of some hyperbolic component H_c; if $c \neq 1/4$, then H_c is contained in W_c. Thus, except for the hyperbolic component inside the Main Cardioid, the root of every hyperbolic component H disconnects H from the Main Cardioid. Even more, and here is the one of the most beautiful features of combinatorics of quadratic polynomials, if R_{θ_1} and R_{θ_2} are the rays determining the wakes of the parabolic parameter c, then $\widehat{\theta_1\theta_2}$ corresponds to the characteristic arc of the parabolic cycle of c. Moreover, $\widehat{\theta_1\theta_2}$ is also the characteristic arc of the dynamic root cycle of $H_c(0)$, the center of the hyperbolic component H_c. As for the whole wake W_c, it can be described as the set of parameters in M for which there is a cycle P with orbit portrait $\mathcal{O}_P$, and such that the characteristic arc of $\mathcal{O}_P$ is $\widehat{\theta_1\theta_2}$. To any given hyperbolic H component we associate to H the smallest angle of the external rays landing at the root of H. Another useful fact is that the period of θ_1 is the same as the period of the parabolic cycle of c and the critical cycle of $H_c(0)$.

Given two hyperbolic components or, for this discussion, Misiurewicz points, H_1 and H_2, we said that H_2 is *visible* from H_1 if the rays landing at the root of H_1 separates H_2 from the Main Cardioid. Visibility induces a partial ordering in the hyperbolic components. More precisely, for every pair of hyperbolic components, or Misiurewicz parameters, H_1 and H_2 either one is visible from the other, or there is a hyperbolic component, or Misiurewicz parameter, H_3 from which both H_1 and H_2 are visible.

Given any hyperbolic component H there is a well defined path, called *the combinatorial arc* of H, along the Mandelbrot set connecting the Main Cardioid with H. This imposes a tree structure onto the arrangement of hyperbolic components in the Mandelbrot set. The root of this tree is the Main Cardioid, and the set of vertices is the set of hyperbolic components and Misiurewicz parameters.

Again, by identifying a hyperbolic component with its center, any superattracting quadratic parameter c is associated to a combinatorial arc in the parameter plane.

2.3.6 Internal address

Now, we turn to another combinatorial description of hyperbolic components. The concept of internal address has been introduced by Eik Lau and Schleicher in [18]. Recently, Henk Bruin and Schleicher in [5] completed the combinatorial picture for postcritically finite parameters; they proved the equivalence of the standard combinatorial descriptions that a postcritically finite quadratic polynomial can have.

Given a superattracting parameter c, it belongs to a hyperbolic component H_c, although generally the combinatorial arc of H_c crosses infinitely many hyperbolic components, one can write down, in an increasing sequence of numbers $\{a_n\}$, the periods of the hyperbolic components that the combinatorial path of

H_c crosses. This sequence $\{a_n\}$ is always a finite and is called *internal address* of the component H_c. Following Lau and Schleicher's notation, express the internal address of a superattracting parameter c, by $\{a_n\}$ making $a_1 = 1$ and each element of the sequence connected by arrows.

For example, the internal address of the "*basillica*" map $f_{-1} = z^2 - 1$ is $1 \to 2$, while the "*airplane*" map $f_c(z) = z^2 + c$ where $c = -1.1380006666509645111 + .2403324012620983016 9i$ has internal address given by $1 \to 2 \to 3$, see Appendix. It is a Theorem by Schleicher that if two hyperbolic parameters c and c' have the same internal address then the maps restricted to the filled Julia sets are conjugate.

Let $c \neq 1/4$ be a parabolic parameter, then the multiplier of the corresponding parabolic cycle is a root of unity of the form $e^{\frac{p}{q}\pi i}$, the number p/q is called the *combinatorial rotation number* of c. If, in the internal address, we label each arrow with the combinatorial rotation number of the hyperbolic components on the sequence, then we obtain the *labelled internal address*, is a Theorem of Lau and Schleicher [18], that the labelled internal address of a superattracting parameter c characterizes the parameter.

Chapter 3

Laminations

Following Vadim Kaimanovich and Lyubich in [13], see also Lyubich's notes [19]. A topological space $\mathcal{B}$ is said to have a *product structure* if it is provided with a homeomorphism $\phi : \mathcal{B} \to \mathbb{D}^n \times T$ where $\mathbb{D}^n$ is the open unit disk in $\mathbb{R}^n$, and T is some topological space. Sets of the form $B_x = \phi^{-1}(\mathbb{D}^n \times \{x\})$ are called *local leaves* or, simply *plaques*, while sets of the form $T_z = \phi^{-1}(\{z\} \times T)$ are called *local transversals*. A *laminar map* between product spaces is a continuous map that sends plaques into plaques.

A *lamination* is a Haussdorf topological space $\mathcal{X}$ which is endowed with an atlas of open charts (ϕ, U) where U has a product structure and ϕ is a homeomorphism as above. We also require that the change of coordinates are laminar maps, thus change of coordinates are maps of the form $\gamma_{\alpha\beta} : D \times T \to D' \times T'$ given by $\gamma_{\alpha\beta}(z,t) = (\sigma(z,t), \psi(t))$, where σ and ψ are continuous functions on t. The sets U will be called *flow boxes*.

Different regularity conditions can be impose on laminations. This is done by requiring the corresponding regularity to the map σ along z. For instance, smooth laminations are laminations whose transition maps $\gamma_{\alpha,\beta}$ are smooth in the z variable. Similarly, real and complex analytic laminations can be defined. Laminations are generalizations of the concept of foliations, a foliation is a lamination where $\mathcal{X}$ is a manifold itself.

A lamination is decomposed into a disjoint union of connected n-manifolds $\mathcal{X} = \sqcup L_\alpha$, the sets L_α are called *global leaves*, or just *leaves*. Global leaves can be characterized as the smallest set L with the property that if it intersects a plaque B_x then $B_x \subset L$. The maps ϕ restricted to plaques are, in fact, charts for leaves. Given a point z in $\mathcal{X}$, we will denote by $L(z)$ the leaf in $\mathcal{X}$ which contains z.

We define the *dimension of a lamination* $\mathcal{X}$ as the dimension of any plaque in $\mathcal{X}$. In dimension two, the concept of a conformal lamination is equivalent to the one of a complex analytic lamination, these type of laminations are also called *Riemann surfaces laminations*.

Other categories of laminations, which play an important role, are affine and hyperbolic laminations. For these, we require the change of coordinates on leaves to be affine and hyperbolic isometries, respectively. In this work, we will be interested in affine laminations of dimension two arising from dynamics of quadratic polynomials.

3.1 Inverse limits

Consider $\{f_k : X_k \to X_{k-1}\}$, a sequence of m-to-1 branched covering maps between n-manifolds X_k. Then, define the *inverse limit* $\varprojlim(f_n, X_n)$ as

$$\varprojlim(f_n, X_n) = \{\hat{x} = (x_1, x_2, ...) \in \prod X_n | f_{n+1}(x_{n+1}) = x_n\}.$$

The *natural topology* in $\varprojlim(f_n, X_n)$ is the one induced from the product topology in $\prod X_n$.

We are interested in inverse limits arising from dynamics; these are particular cases where all coverings $f_n \equiv f$ and the manifolds X_n are equivalent to a single phase space X. Such inverse limits are called *solenoids*, or *natural extensions*, we will denote them by $\varprojlim(f, X)$ to make emphasis in X. When f is a rational function and $X = \bar{\mathbb{C}}$ then, following Lyubich and Minsky [21], we will denote $\varprojlim(f, \bar{\mathbb{C}})$ by $\mathcal{N}_f$.

The map $f : X \to X$ has a *natural extension* $\hat{f} : \varprojlim(f, X) \to \varprojlim(f, X)$ defined as

$$\hat{f}(x_0, x_{-1}, ...) = (f(x_0), x_0, x_{-1}, ...).$$

Also, there is a family of natural projections $\pi_{-n} : \varprojlim(f, X) \to X$, given by $\pi_{-n}(\hat{x}) = x_{-n}$. Each of these maps semiconjugates $\hat{f}$ to f, so $\pi_{-n}(\hat{f}(\hat{z})) = f(\pi_{-n}(\hat{z}))$. For simplicity, the subindex of the projection over the first coordinate will be omitted, thus $\pi \equiv \pi_0$. We are interested in studying properties of dynamics of $\hat{f}$ and how they are related to the dynamics of the original map f.

Let A be an invariant set of f, so we have $f(A) \subset A$. The *invariant lift* of A in $\varprojlim(f, \bar{\mathbb{C}})$ is the set $\hat{A}$ of all backward orbits $\hat{z}$ such that $\pi_{-n}(\hat{z}) \in A$ for every n. When f is a rational function, invariant lifts of periodic points are classified upon its corresponding classification in the dynamical plane. Hence, a parabolic periodic point in $\varprojlim(f, \bar{\mathbb{C}})$ is the invariant lift of a parabolic periodic point in $\mathbb{C}$. Note that there is a natural one-to-one identification of the periodic points of f with the periodic points of $\hat{f}$ in $\mathcal{N}_f$. Namely, to every periodic point p of f, the corresponding point $\hat{p}$ is the point in the invariant lift of the cycle of p such that $\pi(\hat{p}) = p$. Vice versa, given a periodic point $\hat{p}$ of $\hat{f}$, the point $\pi(\hat{p})$ is a periodic point of f.

3.1.1 Lamination structure of solenoids

Assume f does not have critical points, so it is a m-to-1 covering map, and let $\hat{x} = (x_0, x_{-1}, x_{-2}, ...)$ be a point in $\varprojlim(f, X)$; each coordinate x_n belongs to the set of m preimages of x_{n-1}. Hence, after a suitable labelling of the branches of f^{-1}, the fiber $\pi^{-1}(x_0)$ can be identified with $\{0, ..., m-1\}^{\mathbb{N}}$; moreover, by taking the product discrete topology on $\{0, ..., m-1\}^{\mathbb{N}}$, this identification is a homeomorphism. Let U be an open neighborhood of x_0, the set $\pi^{-1}(U)$ is homeomorphic to $U \times \{0, ..., m-1\}^{\mathbb{N}}$ and contains $\hat{x}$. This endows $\varprojlim(f, X)$ with a lamination structure. For f with critical points, whenever an open set $U \subset \mathbb{C}$ does not intersect the postcritical set of f, the set $\pi^{-1}(U)$ has a product structure.

Let (U_{-n}) be the *pull-back of* $U = U_0$ *along* $\hat{x}$, where U_{-n} is the connected component of $f^{-n}(U)$ containing x_{-n}. Given a number N, the sets

$$B(U, \hat{z}, N) = \pi_{-N}^{-1}(U_{-N})$$

form a local basis of open sets for $\varprojlim(f, X)$. A plaque, then, can be regarded either as a connected component of a flow box, or as the complete pull back of some open disk U_0 along $\hat{x}$; that is, a sequence of the form $(U_0, U_{-1}, U_{-2}, ...)$.

3.2 Dyadic solenoid

Consider the polynomial $f_0(z) = z^2$ defined on $X = \mathbb{S}^1$, the solenoid $\mathcal{S}^1 = \varprojlim(f_0, \mathbb{S}^1)$ is called the *dyadic solenoid*, see Figure 3.1. As f_0 is a covering map of degree two, for every point $z \in \mathcal{S}^1$, the fiber $\pi^{-1}(\pi(z))$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$. The name "dyadic" comes from the fact that each fiber can be identified with the dyadic numbers $\hat{\mathbb{Z}}_2$. Since $\mathbb{S}^1$ is a compact topological group, the solenoid $\mathcal{S}^1$ is a compact topological group, in which the group multiplication is defined componentwise. The unit $\hat{u}$ is the point $(1, 1, 1, ...)$. The translations of the solenoid, given by left multiplication, will be denoted by $\tau_{\hat{\zeta}}(\hat{z}) = \hat{z} \cdot \hat{\zeta}$.

The dyadic solenoid is isomorphic to the solenoid $\varprojlim(\mathcal{D}, \mathbb{T})$, where $\mathcal{D}$ is the doubling map defined on $\mathbb{T}$. This representation allow us to put the periodic points of $\hat{f}_0 \in \mathcal{S}^1$ in one-to-one correspondence with the reduced rational angles p/q in $\mathbb{Q}/\mathbb{Z}$ with q an odd natural number.

The leaf containing the unit $\hat{u}$ in $\mathcal{S}^1$ is a one-parameter subgroup of the dyadic solenoid, parameterized by the map $\rho : \mathbb{R} \to \mathcal{S}^1$, with $\rho(r) = (e^{2\pi i r}, e^{\pi i r}, e^{\pi i r/2}, ...)$. The image of ρ is dense in the solenoid, that is $\mathcal{S}^1 = \overline{\rho(\mathbb{R})}$, hence by homogeneity of topological groups, every leaf in the solenoid is dense. Let us remark that the dyadic solenoid is connected but not pathwise connected. By transferring the natural order in $\mathbb{R}$ to the leaves in $\mathcal{S}^1$, the map ρ also introduces a leafwise order in $\mathcal{S}^1$, namely, if $\hat{z}$ and $\hat{\zeta}$ are two points in the same leaf, then we say that $z > \zeta$ whenever $\rho^{-1}(z^{-1} \cdot \zeta) > 0$.

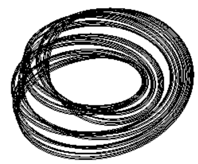


Figure 3.1: The dyadic solenoid $\mathcal{S}^1$.

3.2.1 The adding machine

The projection $\pi : \varprojlim(f_0, \mathbb{S}^1) \to \mathbb{S}^1$ is a fibration map, the monodromy group of this covering is isomorphic to $\mathbb{Z}$, and its action on the fiber is equivalent to the action of the *adding machine* on the group of dyadic numbers, i.e., it is generated by the operation of adding one modulo 2. Let F be the fiber of π over 1, the adding machine action on F can be extended continuously to the whole solenoid. In fact, its generator is given explicitly by the map $\sigma : \mathcal{S}^1 \to \mathcal{S}^1$, defined as $\sigma(\hat{z}) = \rho(1) \cdot \hat{z}$. We refer to the action of the group $< \sigma >$ as the *adding machine action*. The solenoid $\mathcal{S}^1$ can be described as the quotient of the product space $S = [0, 1] \times F$ by the relation $(1, f) \sim (0, \sigma(f))$.

Two points $\hat{z}$ and $\hat{\zeta}$ in F belong to the same leaf if and only if $\hat{z}$ and $\hat{\zeta}$ belong to the same orbit under the adding machine action. The fiber F is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ which is uncountable, but each orbit under the adding machine action has a countable number of points in F, so the dyadic solenoid has uncountable many leaves.

3.3 Solenoidal cones

The solenoids $\varprojlim(f_0, \mathbb{D}^*)$ and $\varprojlim(f_0, \mathbb{C} \setminus \mathbb{D})$, are both homeomorphic to $\mathcal{S}^1 \times (0,1)$. Since $\mathcal{S}^1 \times (0,1)$ is homotopic to $\mathcal{S}^1$, the action over any fiber of the monodromy group of the projection $\pi : \mathcal{S}^1 \times (0,1) \to \mathbb{S}^1 \times (0,1)$ is just the diagonal action of the adding machine times the identity on (0,1). Now, because ∞ is a superattracting fixed point of f_0, the inverse limit $\varprojlim(f_0, \bar{\mathbb{C}} \setminus \mathbb{D})$ is homeomorphic to the cone over the dyadic solenoid $\mathcal{S}^1$, defined as $\mathcal{S}^1 \times [0,1]/\{(s,1) \sim (s',1)\}$ for all $s, s' \in \mathcal{S}^1$. The vertex of this cone corresponds to the point $\hat{\infty} = (\infty, \infty, \infty, ...)$. A *closed solenoidal cone* is a space homeomorphic to $\varprojlim(f_0, \bar{\mathbb{C}} \setminus \mathbb{D})$. In particular, the solenoidal cone $\varprojlim(f_0, \bar{\mathbb{C}} \setminus \mathbb{D})$ will be denoted by $Con\,(\mathcal{S}^1)$. The dyadic solenoid $\mathcal{S}^1$ is contained in $Con\,(\mathcal{S}^1)$, and we regard $\mathcal{S}^1$ as the boundary of $Con\,(\mathcal{S}^1)$. See figure 3.2. Since there is no local product structure on $\hat{\infty}$, the solenoid $Con\,(\mathcal{S}^1)$ is not a lamination. In general, for dynamical systems with critical points, the situation where critical points occur infinitely many times in the coordinates of a given point, is one of the possible obstructions for the inverse limit to be a lamination.

It is important to note that $\varprojlim(f_0, \mathbb{C} \setminus \mathbb{D}) \simeq \mathcal{S}^1 \times (0,1)$ is not pathwise connected, whereas $Con\,(\mathcal{S}^1)$ is pathwise connected. Later we will see that, locally, path connectivity properties characterizes those points in the inverse limits with lack of local product structure.

Let $f_c(z) = z^2 + c$ be a quadratic polynomial with c on M, then f_c has a solenoidal cone associated to it. This is a consequence of Böttcher's Theorem; indeed, the Böttcher's coordinate $\phi_c : (A(\infty), \infty) \to (\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}, \infty)$ conjugates f_c

to f_0. The map ϕ_c naturally lifts to a homeomorphism $\hat{\phi}_c : \varprojlim(f_c, A(\infty)) \to Con\,(\mathcal{S}^1) \setminus \mathcal{S}^1$ given by $\hat{\phi}_c(z_0, z_{-1}, ...) = (\phi_c(z_0), \phi_c(z_{-1}), ...)$ which conjugates $\hat{f}_c$ to $\hat{f}_0$. We will call the solenoid $\varprojlim(f_c, A(\infty)) \subset \mathcal{N}_{f_c}$ the *solenoidal cone at infinity* of f_c.

Another solenoidal cone is $\varprojlim(f_0, \bar{\mathbb{D}})$, the two solenoids $Con\,(\mathcal{S}^1)$ and $\varprojlim(f_0, \bar{\mathbb{D}})$ share the common boundary $\mathcal{S}^1$. The natural extension $\mathcal{N}_{f_0}$ is obtained by gluing $Con\,(\mathcal{S}^1)$ and $\varprojlim(f_0, \bar{\mathbb{D}})$ along $\mathcal{S}^1$. Hence, $\mathcal{N}_{f_0}$ is homeomorphic to the double cone over the solenoid $\mathcal{S}^1$.

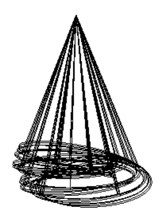


Figure 3.2: The solenoidal cone $Con\,(\mathcal{S}^1)$.

The solenoid $\varprojlim(f_0, \mathbb{D}^*)$ is homeomorphic to the cylinder $\mathcal{S}^1 \times (0, 1)$.

Lemma 7. *The solenoids $\mathcal{S}^1 \times (0, 1)$ and $Con\,(\mathcal{S}^1)$ and are connected, and every leaf is dense.*

Proof. It follows from the fact that the one-dimensional solenoid $\mathcal{S}^1$ is connected and every leaf in it is dense. So, the same properties hold in it's cylinder and its cone $Con\,(\mathcal{S}^1)$. $\square$

3.3.1 Subsolenoidal cones

For $r > 1$, let $\mathbb{D}_r = \{z \in \mathbb{C} | |z| < r\}$. Then, the *canonical homeomorphism* between $\varprojlim(f_0, \mathbb{C} \setminus \mathbb{D}_r)$ and $\mathcal{S}^1 \times [0, 1)$ is given by

$$(z_0, z_{-1}, ...) \mapsto (\frac{z_0}{|z_0|}, \frac{z_{-1}}{|z_{-1}|}, ...) \times (1 - \frac{r}{|z_0|})$$

extends to a homeomorphism between $\varprojlim(f_0, \bar{\mathbb{C}} \setminus \mathbb{D}_r)$ and $Con\,(\mathcal{S}^1)$, fixing $\hat{\infty}$. This implies that, for the quadratic polynomial $f_c(z) = z^2 + c$ and the space $A_r = \{z \in A_c(\infty) | |\phi_c(z)| \geq r\}$ outside the equipotential E_r in $A_c(\infty)$, the solenoid $\varprojlim(f_c, A_r)$ is also homeomorphic to $Con\,(\mathcal{S}^1)$. In particular, the fiber over $\pi^{-1}(E_r)$ of any equipotential E_r is homeomorphic to the dyadic solenoid $\mathcal{S}^1$. We call such space, the *solenoidal equipotential* over E_r, and denote it by $\mathcal{S}_r$. Solenoidal equipotentials form a foliation of the solenoidal cone at infinity in $\mathcal{N}_{f_c}$. Similarly, external rays induce a foliation in $\pi^{-1}(A(\infty))$ of *solenoidal external rays*, each solenoidal external ray is homeomorphic to a cone over a Cantor set. Periodic external rays under f_c can be lifted to periodic external rays of $\hat{f}_c$. Hence, periodic external rays are parameterized by reduced rational numbers $\frac{p}{q} \in \mathbb{T}$ with q odd. Since Böttcher's coordinate ϕ_c conjugates f_c to f_0, every periodic external ray intersects $\mathcal{S}^1_r$ in the corresponding periodic point in $\mathcal{S}^1$. So, periodic external rays keep track of the periodic points along the solenoidal equipotential foliation. Moreover, every periodic solenoidal ray "lands" in a periodic point of $\hat{f}_c$ in $\mathcal{N}_c$. Hence, any periodic point which is landing point of periodic solenoidal external rays $\hat{p}$ of $\hat{f}_c$ can be associated to the orbit portrait of $\pi(\hat{p})$.

We summarize the previous discussion in the following lemma:

Lemma 8. *Let L be a leaf containing a periodic point $\hat{p}$ such that $\pi(\hat{p})$ belongs to the Julia set J_c, then L intersects the solenoidal equipotential on the periodic leaves associated to the periodic rays landing at p.*

3.4 Lyubich-Minsky laminations

Finally, we get to the object of study of this work. Inverse limits are also defined for branching mappings, however, as we saw in $Con\,(\mathcal{S}^1)$, these solenoids may not have a local lamination structure. The case for rational maps on the sphere was addressed by Lyubich and Minsky in [21]. In this setting, there are two obstructions for the natural extension to be a lamination. The first, as we have noted before, is structural; there are some points which fail to have local product structure. The second, comes from the fact that the geometric structure on the leaves may not vary continuously on the transverse direction. Thus, it is necessary to refine the topology in an appropriate way. The first problem is easy to carry over, simply by removing all points that don't have local product charts. The second is of more delicate nature, in fact, for many rational functions the new topology is, in general, hard to describe by intrinsic properties of the natural extension. The way out found by Lyubich and Minsky was to embed the lamination into an universal space.

First let us introduce the regular space. To keep things simple, we return to the assumption that f_c is a quadratic polynomial defined on the Riemann sphere.

Definition. *A point $\hat{z} \in \varprojlim(\bar{\mathbb{C}}, f_c)$ is called regular if there exists a neighborhood U_0 of $\pi(\hat{z})$ such that the pull back of U_0 along $\hat{z}$ is eventually univalent.*

The set $\mathcal{R}_c \subset \varprojlim(\bar{\mathbb{C}}, f_c)$ *of regular points is called the regular part of* f_c*.*

The regular part is a disjoint union of Riemann surfaces. To check this, regard the above definition in terms of plaques $\hat{U} = (U_0, U_{-1}, U_{-2}, ...)$, thus a point $\hat{z}$ is regular if there is a number $N \geq 0$ such that U_{-n} does not contain critical points for $n \geq N$. Then, naturally, we can take as a conformal chart for the plaque containing $\hat{z}$ any of the maps $\pi_{-n} : \hat{U} \to U_{-n}$. Plaques glue together to form a Riemann surface L. The set L is called a *leaf* of the regular part. Later, we will see in Proposition 51 that this conformal structure on plaques of regular points is inherent of regularity.

The conformal structure on leaves makes the natural extension $\hat{f}$ a conformal biholomorphism sending $L(\hat{z})$ to $L(\hat{f}(\hat{z}))$.

Another result in [21] states that all leaves in $\mathcal{R}_c$ are simply connected. This is not true for general rational functions where invariant lifts of Herman rings are doubly connected in $\mathcal{R}_c$. Lyubich and Minsky also proved that there are no compact leaves in $\mathcal{R}_c$, consequence to the fact that the projections π_n restricted on leaves are branched coverings and $\pi_n \circ \hat{f} = f \circ \pi_{n+1}$.

Therefore by the Uniformization Theorem, conformally, leaves are either disks or planes. The union $\mathcal{A}_c$ of the leaves in $\mathcal{R}_c$ conformally isomorphic to the complex plane $\mathbb{C}$ is called the *affine part* of $\mathcal{R}_c$. We call any such leaf an *affine leaf.*

The conformal structure of the leaves is related to how dense the postcritical set is. A leaf conformally isomorphic to the disk is called a *hyperbolic leaf.* The easiest cases where hyperbolic leaves arise are invariant lifts of Siegel disks. Examples of non-rotational hyperbolic leaves have been constructed by

Jeremy Khan and Juan Rivera-Letteliér.

Nevertheless, there are always infinitely many affine leaves. (See Proposition 4.5 in [21]). For a point $\hat{z} \in \mathcal{R}_c$, the set $\alpha(\hat{z})$ is the set of accumulation points of $\{z_{-n} = \pi_{-n}(\hat{z})\}$. If a point $\hat{z}$ does not have all of its coordinates on the postcritical set of recurrent critical points, and is neither parabolic nor attracting periodic point, then the leaf $L(\hat{z})$ containing $\hat{z}$ is affine.

In particular, leaves containing repelling periodic points are affine. Moreover, for such leaves L Königs coordinate provides the uniformization coordinate from L to $\mathbb{C}$.

The natural topology on $\mathcal{R}_c$ as a subspace of $\varprojlim(\bar{\mathbb{C}}, f)$ may not be enough to supply each point of $\mathcal{R}_c$ with a local lamination structure. The issue is that local degree of plaques might not be continuous in the transversal direction. However, for hyperbolic parameters we have:

Proposition 9 (Lyubich-Minsky). *If c is hyperbolic, the associated regular part $\mathcal{R}_c$ is a lamination with the topology induced by the natural topology.*

In Lyubich-Minsky [21] the concept of convex cocompactness is introduced in terms of the compactness of the quotient of the laminated Julia set under dynamics. This justifies the terminology, however, we use as a definition a proposition in the same paper which characterizes convex cocompactness.

Definition. *A parameter c is called convex cocompact if the critical point is not recurrent and does not converge to a parabolic cycle.*

In particular, attracting and superattracting parameters are convex cocompact. If c is convex cocompact, leaves in the regular part $\mathcal{R}_c$ are all affine, so $\mathcal{A}_c = \mathcal{R}_c$; in this case, the affine part has also a simple description (see Proposition 4.5 in [21]), $\mathcal{A}_c = \mathcal{R}_c = \mathcal{N}_c \setminus \{\text{attracting and parabolics}\}$.

Let $\mathcal{J}_c = \pi^{-1}(J_c) \cap \mathcal{R}_c$ be the lift of the Julia set on the regular part. We call $\mathcal{J}_c$ the *laminated Julia set* associated to f_c. Of c is a convex co-compact parameter, by a result of Lyubich and Minsky, $\mathcal{J}_c$ is compact inside $\mathcal{N}_c$ with the natural topology. When c is convex cocompact, there are no irregular points on the laminated Julia set $\mathcal{J}_c$.

Chapter 4

Topology of inverse limits

4.1 The laminated Julia set

In this section we discuss the topological properties of $\mathcal{J}_c$. Given a repelling periodic point q in the dynamical plane, let us denote by $\hat{q}$ the periodic point in the regular part which satisfies $\pi(\hat{q}) = q$. Let $\mathcal{P}$ be the set of repelling periodic points in $\mathcal{R}_c$.

Lemma 10. *The set $\mathcal{J}_c$ is a closed and perfect set in $\mathcal{R}_c$. Every periodic point in $\mathcal{R}_c$ either belongs to $\mathcal{P}$ or is a Siegel periodic point. Moreover, $\mathcal{J}_c = \bar{\mathcal{P}}$.*

Proof. The projection π is a continuous function from $\mathcal{R}_c$ to $\mathbb{C}$. Since $J(f_c)$ is closed and perfect, the set $\mathcal{J}_c$ inherits these properties from the dynamical plane.

The fact that lifts of Siegel periodic points in the dynamical plane belong to the regular part is a consequence of the existence of linearizing coordinates around Siegel periodic points. By a Theorem by Fatou, parabolic and attracting cycles are the accumulation set of some critical orbit, hence the pull back

$\{U_{-n}\}$ of every neighborhood U_0 contains critical points at infinitely many times n, so parabolic and attracting cycles lift to irregular points.

Now, let us prove that Cremer cycles also lift to irregular points. This is, actually, a consequence of the Shrinking Lemma. To illustrate it's use, we will follow the proof given by Lyubich and Minsky [21], see also Proposition 1.10 in Lyubich's survey [19].

Suppose on the contrary, that a Cremer cycle lifts to a periodic regular point. By considering an appropriate iterate of f_c, we can assume that the cycle is a Cremer fixed point a_0. As $\hat{a}_0$ is regular, there exist an open neighborhood U_0 of a_0 such that no component U_{-n} of the pull back of U_0 along $\hat{a}_0$ contains critical points. Without loss of generality we can assume that U_0 is a small disk around a_0, then U_{-n} is also a topological disk, since $f : U_{-n} \to U_{-n+1}$ is a conformal isomorphism. Hence, there is a sequence of Riemann maps $\phi_n : \mathbb{D} \to U_{-n}$ with $\phi_n(0) = a_0$. Put $\rho_n = f^n \circ \phi_n : \mathbb{D} \to U_0$, then by Montel's Theorem, the sequence $\{\rho_n\}$ is a normal family. Let $P = \{\alpha_1, \alpha_2, ..., \alpha_p\}$ be any cycle of f_c of period $p \geq 3$. By normality there is a disk $D' \subset\subset \mathbb{D}$ around 0 such that $\phi_n(D')$ does not intersect P for every n. Therefore, $\{\phi_n|_{D'}\}$ is also a normal family.

Let $B(a_0, \delta)$ denote the ball around a_0 of radius δ, since ρ_n is a conformal isomorphism, for δ small enough $\text{mod}(\rho_n^{-1}(U_0 \setminus B(a_0, \delta)) = \text{mod}(U_0 \setminus B(a_0, \delta))$, so the modulus of $\rho_n^{-1}(U_0 \setminus B(a_0, \delta))$ depends only on δ. Because $\text{mod}(U_0 \setminus B(a_0, \delta)) \to \infty$ when $\delta \to 0$, there exist a δ such that $B = B(a_0, \delta) \subset \rho_n(D')$ for all n.

It follows that $\text{diam}(\phi_n^{-1}(B)) \to 0$ uniformly when n tends to infinity, otherwise by taking a converging subsequence from $\{\phi_n\}$ and by normality there would be a limiting open set B_∞ containing a_0 such that $f^{n_k}(B_\infty) \subset U_0$ for all k, contradicting the fact that every Cremer periodic point belongs to the Julia set $J(f_c)$. If the diameters of B_n tend to zero, there is an m such that $B_m \subset B$ and $f^m(B_m) = B$, but this would imply that f is repelling at a_0, again a contradiction.

Take $\hat{z} \in \mathcal{J}_c$, by continuity of f_c and the density of the set of repelling periodic points in $J(f_c)$, there is a periodic point p in the δ neighborhood of z_{-n} such that $|f_c^j(p) - z_{-n+j}| < \epsilon$ for $j = 0, ..., n$. Then, $\widehat{f_c^n(p)}$ is a repelling periodic point in $B(D(z_0, \epsilon), \hat{z}, n)$. □

4.2 Leafwise connectivity of $\mathcal{J}_c$

Lemma 11. *Let c be a parameter with locally connected Julia set, $\overline{P(f_c)} \neq J_c$, and such that $\pi^{-1}(J(f_c))$ contains an irregular point in the natural extension $\mathcal{N}_c$. Then, there is a leaf L in the regular part $\mathcal{R}_c$, such that $\mathcal{J}_c \cap L$ is disconnected.*

Proof. Let $\hat{z}$ be an irregular point in the Julia set in $\mathcal{N}_c$, thus $\pi_{-j}(\hat{z}) \in J(f_c)$ for every j. Since the Julia set is locally connected, for every j there is an external ray $R(j)$ landing at z_{-j}, let r_0 be a point in $R(0) \cap A(\infty)$, by pulling back r_0 along the backward orbit determined by $\hat{z}$, there is a point $\hat{r} \in \mathcal{R}_c$ such that $\pi(\hat{r}) = r_0$ and $\pi_{-j}(\hat{r}) \subset R(j)$, now by moving r_0 along the ray $R(0)$, we construct a line $\hat{R}$ in the regular part, such that $\pi_{-j}(\hat{R}) = R(j)$. Let L be the leaf in the regular part containing $\hat{R}$. Since by construction the endpoints of $\hat{R}$ are the irregular points $\hat{z}$ and $\hat{\infty}$, this line can not have accumulation points in $\mathcal{R}_c$ when r_0 either tends to z_0 or to ∞. So, $\hat{R}$ is a line escaping to infinity in both directions and separates L in two pieces.

Fix r_0 in the external ray $R(0)$, let $a \in J(f_c) \setminus \overline{P(f_c)}$, and choose two paths, σ_1 and σ_2, from r_0 to a starting at different directions with respect to the ray $R(0)$, and such that none of them crosses $R(0)$ again. These two paths lifts to paths in L, joining $\hat{r}$ with points in $\mathcal{J}_c \cap L$, by construction the points lie on different sides of the line $\hat{R}$. Thus $\mathcal{J}_c \cap L$ is disconnected. $\square$

As interesting examples we have: parameters with parabolic cycles (see Tomoki Kawahira [14]), the Feigenbaum parameter. For parameters with parabolic cycles, such leaves only occur as the periodic leaves corresponding to the parabolic cycle. As in the case of repelling cycles the linearizing coordinate of parabolic cycles, Fatou's coordinate, provides a global picture for such leaves described by Kawahira in [14]. In particular, the set $\mathcal{J}_c \cap L$ consists of finitely many components. For Feigembaum, however, we know that the critical point has 4 rays landing. So, it makes sense to think that the set $\mathcal{J}_c \cap L$ consists of at most 4 components. In the worst case we could have, due to branching leaves, 2^n components of the set $\mathcal{J}_c \cap L$, with arbitrary n. Which, may be reflecting the fact that the parameter is infinitely renormalizable with period 2.

As a consequence of Lemma 11, given that the dynamical Julia set J_c is locally connected and $J_c \neq \overline{P(f_c)}$, if the laminated Julia set $\mathcal{J}_c$ is leafwise connected, then there are no irregular points in $\mathcal{J}_c$ but this implies that c must be convex cocompact. The following Lemma shows that these are all the cases:

Lemma 12. *If c is a convex cocompact parameter, then the set $\mathcal{J}_c$ is leafwise* *connected.*

Proof. Let us first consider the case when c is postcritically finite. Let L be a leaf in the regular part, and let $\hat{z}$ and $\hat{\zeta}$ be two points in $\mathcal{J}_c \cap L$. Take paths γ_1 and γ_2 outside the Julia set $J(f_c)$ joining $\pi(\hat{z}) = z_0$ and $\pi(\hat{\zeta})$ with the α fixed point of f_c, respectively. We can lift γ_1 and γ_2 to paths $\hat{\gamma_1}$ and $\hat{\gamma_2}$ in L starting at $\hat{z}$ and $\hat{\zeta}$, respectively, ending at points in $\pi^{-1}(\alpha) \cap L$. Since $\hat{z}$ and $\hat{\zeta}$ belong to L there exist a path σ in L joining $\hat{z}$ with $\hat{\zeta}$. Then $\hat{\gamma_2} \circ \sigma \circ \hat{\gamma_1}^{-1}$ it is a path with end points at $\pi^{-1}(\alpha)$, thus it is homotopic $rel\{\pi^{-1}(\alpha)\}$ to a path in the Julia set $\mathcal{J}_c$. By reparameterizing this path, we can assume that $\hat{z}$ and $\hat{\zeta}$ are fixed by the corresponding homotopy. In such a way, σ is homotopic $rel\{0, 1\}$ to a path in $\mathcal{J}_c$ joining $\hat{z}$ and $\hat{\zeta}$.

If c is postcritically infinite convex cocompact, let L be a leaf in the regular part. By analytic continuation on the Caratheodory loop at the Julia set, π projects every component of $\mathcal{J}_c \cap L$ surjectively on the Julia set $J(f_c)$. Take two points $\hat{z}$ and $\hat{\zeta}$ in $\mathcal{J}_c \cap L$, and let $\hat{\gamma}$ be a path joining these points. We can also assume that $\hat{\gamma}$ neither auto intersects nor intersects the Julia set at other points but its endpoints.

Let $\gamma = \pi(\hat{\gamma})$, consider a finite open covering $\{U_0, ..., U_k\}$. By Mañé's Theorem, for $\epsilon > 0$ there is an $N > 0$ such that for $n > N$ the diameter of every component of $f_c^{-n}(\cup U_i)$ is less than, say, $\epsilon/10$. Now consider the shortest path σ, in the Julia set $J(f_c)$, joining the points $\pi_{-n}(\hat{z})$ and $\pi_{-n}(\zeta)$. Then $\pi_{-n}(\hat{\gamma})$ is contained in the ϵ neighborhood V of σ. Again by Mañé's Theorem, there is an $M > 0$ such that for every $m > M$, every component of $f^{-m}(V)$ branches at most K times. Thus, for large enough m, the component of $f^{-m}(V)$ containing

$\pi_{-m-n}(\hat{\gamma})$ does not contain critical values. Since $\pi(\hat{\gamma})$ is contained in $\mathcal{A}(\infty)$ except at its end points. We can use the homotopy induced by moving along external rays, from $\pi_{-m-n}(\hat{\gamma})$ to a path in the Julia set, without crossing any critical values. Hence, this homotopy lifts to a homotopy in L from $\hat{\gamma}$ to a path in the Julia set. Therefore, $\mathcal{J}_c \cap L$ is connected. $\square$

Together, Lemma 11 and Lemma 12 imply:

Proposition 13. *A quadratic parameter c with locally connected Julia set $J(f_c)$ and $\overline{P(f_c)} \neq J_c$ has leafwise connected Julia set $\mathcal{J}_c$ in its regular part $\mathcal{R}_c$, if and only if c is convex cocompact.*

The proof for the case of non-postcritically finite convex cocompact parameters of Lemma 12 applies also for the postcritically finite, however, the proof we presented in these cases gives a hint the relation between the monodromy group over the fiber $\pi^{-1}(a)$ and the Julia set $\mathcal{J}_c$ in the regular part in Chapter 6 we will discuss monodromy groups for postcritically finite parameters. Such relationship should also be present in the case of convex cocompact quadratic polynomials. By modifying the last homotopy argument in the proof of Lemma 12, the lemma can be extended to convex cocompact rational functions. Unfortunately, Lemma 11 may not be true in this setting.

4.3 Unbounded Fatou components

Let $\mathcal{F}_c = \pi^{-1}(F(f_c)) \cap \mathcal{R}_c$ be the lift of the Fatou set to the regular part. Given an affine leaf L in $\mathcal{R}_c$, consider the uniformization $\phi : L \to \mathbb{C}$. We call a subset A in L *bounded*, if the corresponding set $\phi(A)$ is bounded in $\mathbb{C}$.

Lemma 14. *Let f_c be a convex cocompact quadratic polynomial, and let A be a Fatou component of $\mathcal{F}_c$ inside an affine leaf L. Then A is bounded if and only if the restriction $\pi|_A$ has finite degree.*

Proof. If the dynamical Fatou set consist only of the basin of infinity, then all Fatou components in $\mathcal{F}_c$ are unbounded, this is because the lift of Böttcher's coordinate maps $\pi^{-1}(A(\infty))$ to an open solenoidal cone, and each leaf of this cone has $\hat{\infty}$ on the boundary. Moreover, since the fiber of z^2 on the solenoidal cone intersects each leaf infinitely many times, the restriction of $\pi|A$ over every leaf of the solenoidal cone has infinite degree.

The remaining case is when f_c is hyperbolic. By the same argument above, every Fatou component A in $\pi^{-1}(A(\infty))$ is unbounded and the restriction $\pi|_A$ has infinite degree. So let A a Fatou component in $\mathcal{F}_c$ such that $\pi(A)$ is bounded in the dynamical plane. Since f_c is hyperbolic, $\pi(A)$ is actually simply connected and the boundary $\partial(\pi(A))$ is a Jordan closed curve. The set A is bounded if and only if $\partial(A)$ is bounded. The lift of $\partial(\pi(A))$ is also a Jordan curve, say C, so C bounded if and only if $\pi|_C$ has finite degree, but $\pi|_C$ corresponds to the degree of $\pi|_A$. □

We want to count how many unbounded Fatou components there are in L.

Remind that the valence v_p, of a repelling periodic point p, is the number of external rays landing at p. When p is the dynamic root point r_c and c belongs to a satellite hyperbolic component, there are v_p Fatou components touching at r_c. If c belongs to a primitive hyperbolic component, r_c only touches one Fatou component and $v_{r_c} = 2$, see Milnor's [25]. As noted in Lyubich and Minsky [21], the Königs's coordinate ϕ around a repelling periodic point p

lifts to the uniformization Φ of $L(\hat{p})$. Moreover, if m is the period of p, then Φ conjugates $\hat{f}_c^m|_{L(\hat{p})}$ to the affine map $z \mapsto \lambda_p z$, where λ_p is the multiplier of p. By means of Φ, local properties of p are reflected in global properties in $L(\hat{p})$. This is the idea behind the proof of the following proposition.

Proposition 15. *Let f_c be a quadratic polynomial and let L be a periodic affine leaf in the regular part $\mathcal{R}_{f_c}$ containing a periodic point $\hat{p}$. Then, the number of unbounded Fatou components of $L \setminus \mathcal{J}_c$ is either $2v_p$, if the corresponding periodic point p belongs to the dynamic root cycle and c belongs to a satellite hyperbolic component; 3 if p belongs to the dynamic root cycle and c is in a primitive hyperbolic component, finally is v_p otherwise.*

Proof. Since Siegel periodic points lift into hyperbolic leaves in $\mathcal{R}_c$, by Lemma 10, p most be a repelling periodic point. In the dynamical plane, there are v_p rays landing at p, so, these rays cut $\mathbb{C}$ in v_p sectors. Let S be one of these sectors, then S is invariant under an appropriate iterate of f_c, say f_c^k, where k is a multiple of the period m.

Now, every ray landing at $p = \pi(\hat{p})$ lifts to a landing ray in $L(\hat{p})$. As in the dynamical plane, landing rays cut the leaf $L(\hat{p})$ also in v_p sectors. We will check that when p is not in the dynamic root cycle, for each sector $\hat{S}$ in L there is an unbounded subset $\hat{E}$ of the Julia set $\mathcal{J}_c$. In fact, suffices to find a subset $\hat{E}$ in $\mathcal{J}_c \cap L$ invariant under the action of $\hat{f}_c^k$. Since $\hat{f}_c^k$ is a similarity, the set $\hat{E}$ would be unbounded.

To do that, let us construct a fundamental piece to the action of f_c^k in $\mathbb{C}$. Every sector S contains a subset E of the Julia set. Let us take a point b in $J(f_c) \cap S$, very close to p, and such that there is a pair of rays R_b and R'_b landing at b, whose images $R_{f_c^k(b)}$ and $R'_{f_c^k(b)}$ belong to the wake W_S determined by R_b and R'_b. For instance, b can be a preimage of the dynamic root point, since the set of such preimages is dense in the Julia set. Fix an equipotential E_r and join consecutive landing rays by arcs of this equipotential. We obtain a region P around p. The image $f_c^k(P)$ is a region around p and complactly contains P. The annulus $A = \overline{P \setminus f_c^k(P)}$ is the fundamental piece we are looking for, see Figure 4.1.

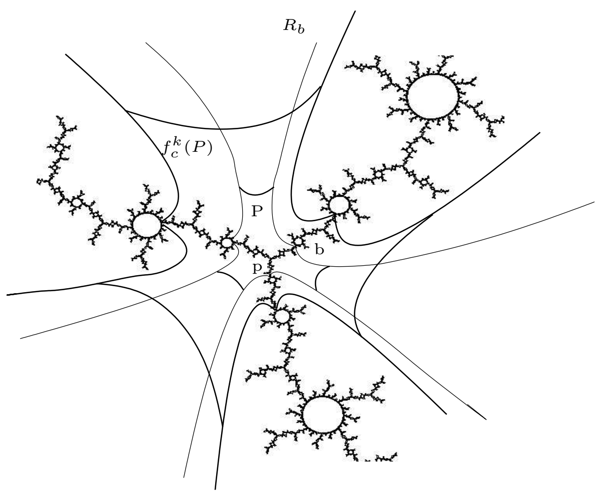


Figure 4.1: The annulus A when p is not in the dynamic root cycle.

The Julia set intersects the annulus A at the wake W_S defined by the rays R_b and R'_b, thus for every S, we can enclose the subset of the Julia set in $A \cap W_S$, in a simply connected open set V_S contained in $A \cap W_S$.

Now, A lifts to an annulus $\hat{A}$ in L, which by construction, is a fundamental region for the action of $\hat{f}_c^k$. By iterating $\hat{f}_c^k$ on $\hat{V}_S$, the lifts of the sets V_S, we obtain a set in L, similar to a pearl beds string, which contains the subset of the Julia set $\hat{E}$ in $\mathcal{J}_c$ that we claim before. By construction, this pearl beds string is unbounded and, there is one for each sector S. The conclusion follows.

When p is the dynamic root point r_c, each Fatou component attached to p, lifts to another unbounded Fatou component in $L(\hat{p})$. In this case, let S be a sector that contains a Fatou component, say F_S, having p in the boundary. Instead of the point b as above, we consider a pair of points q and q' on ∂F_S and on opposite sides of p. There is two pairs of landing rays, $\{R_q, I_q\}$ and $\{R_{q'}, I_{q'}\}$ landing at q and q', the names R stand for external and I for internal rays. On the basin of infinity we follow the same construction as in the case before, whereas in F_S connect the internal rays I_q with I'_q by a internal equipotential. So we obtain again a puzzle piece P around p, see Figure 4.2. From now on, the argument above goes through either for S or sectors that do not contain Fatou components attached to p. $\square$

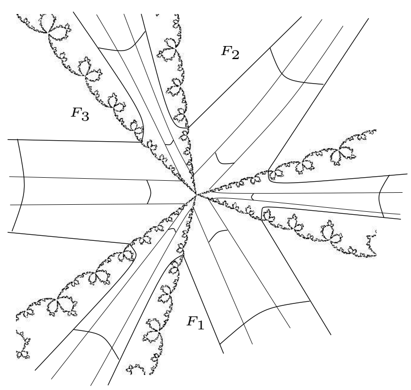


Figure 4.2: The annulus A when p belongs to the dynamic root cycle.

The following proposition gives another restriction that combinatorics impose in the leaf structure of the Julia set. However, it works in a more weak generality.

Proposition 16. *Let f_c be a postcritically finite quadratic polynomial, and* *let L be a non-periodic affine leaf. Then, the number of unbounded Fatou components of $L \setminus \mathcal{J}_c$ is either* 1 *or* 2*.*

Proof. Let R and R' be two external rays landing at the Julia set such that the wake W determined by R and R' does not contain postcritical points. If $\hat{R}$ and $\hat{R}'$ are lifts of the rays R and $\hat{R}$ landing at the same point in a leaf L in $\mathcal{R}_c$, then every arc connecting R and R' along W must lift to an arc joining $\hat{R}$ and $\hat{R}'$. Such arc, for instance, can be taken to be part of an equipotential. Thus, if $\hat{W}$ is the wake determined by $\hat{R}$ and $\hat{R}'$, then $\hat{W} \cap \mathcal{J}_c$ is a bounded set in L. So, $\hat{R}$ and $\hat{R}'$ belong to the same unbounded Fatou component.

Now, let L be a non-periodic affine leaf. Since the Julia set $J(f_c)$ is locally connected, the set $\mathcal{J}_c$ is leafwise locally connected and every point in $\mathcal{J}_c$ is the landing point of some external ray in $\mathcal{R}_c$. If L has more than 3 unbounded Fatou components, by Lemma 12, $\mathcal{J}_c \cap L$ is path connected, so there is a point $\hat{z}$ in $\mathcal{J}_c \cap L$ on the boundary of at least 3 unbounded Fatou components in L, this implies that $\hat{z}$ is the landing point of, at least, 3 rays, each ray in a different unbounded Fatou component. Hence, each coordinate $\pi_{-n}(\hat{z}) \in J(f_c)$ is also the landing point of at least 3 rays. By the argument above, the rays landing at $\pi_{-n}(\hat{z})$ cut the postcritical set in three disjoint pieces. This implies that $\pi_{-n}(\hat{z})$ must be a vertex of the Hubbard tree of $J(f_c)$ for each n. Since L is non-periodic, $\hat{z}$ is non-periodic either, and the set of coordinates $\pi_{-n}(\hat{z})$ is an infinite set in the dynamical plane. But, this contradicts the fact that the Hubbard tree is a finite graph. $\square$

Let us note that when L is a non-periodic affine leaf, the cases where L has 1 or 2 unbounded Fatou components can both happen. To get leaves with two unbounded Fatou components, consider, for instance, the set of *biaccesible* points $\hat{z}$ in $\mathcal{J}_c$, i.e., points where at least 2 rays land, with the property that exactly 2 wakes determined by the rays landing at z_{-n} contain postcritical points. The case where L has only 1 unbounded component is the most common inside regular parts of postcritically finite parameters c, that is because almost all repelling periodic points in $J(f_c)$ are the landing point of exactly one ray. Non-periodic leaves with one unbounded Fatou components can be also constructed.

In regular parts of postcritically finite parameters, leaves with more than three unbounded components are in one-to-one correspondence with the vertexes of the Hubbard tree with degree greater than 2. All other leaves either have one or two unbounded components. So, there are only finitely many leaves with more than three unbounded components. This is how the number of unbounded Fatou components is related to the combinatorics of the parameter c. The following proposition describes the cycle of leaves with more than 3 unbounded components using internal addresses.

Proposition 17. *Let c be a superattracting parameter with internal address $1 \to n_1 \to n_2 \to \ldots \to n_k$. If $n_{j-1} \mid n_j$, for $j < k$ then there is a cycle of n_j periodic leaves such that each leaf has $\frac{n_j}{n_{j-1}}$ unbounded Fatou components. When $j = k$ there are n_k leaves with $2\frac{n_k}{n_{k-1}}$ unbounded Fatou components. If $n_{j-1} \nmid n_j$, there is a cycle of n_j periodic leaves with 2 unbounded Fatou components for $j < k$, and 3 unbounded Fatou components if $j = k$.*

Proof. The condition of whether n_{j-1} divides n_j or not reflects the fact that the combinatorial arc of c crosses a satellite or a primitive hyperbolic component in the parameter plane. Let $j = k$, the number n_k corresponds to the period of the critical orbit, which is equal to the period of the dynamic root point; hence by Proposition 15, if $n_{k-1} \mid n_k$ the parameter belongs to a satellite component, then the valence of the dynamic root point is $\frac{n_k}{n_{k-1}}$, and there are $2\frac{n_k}{n_{k-1}}$ unbounded Fatou components. If $n_{k-1} \nmid n_k$, then c is the center of a primitive hyperbolic component and there are n_k leaves with 3 unbounded Fatou components.

Now, let $j < k$, if $n_{j-1} \mid n_j$, then there is a repelling cycle P of f_c with period n_j and valence $\frac{n_j}{n_{j-1}}$, by Proposition 15, the lift $\hat{P}$ belongs to a cycle of n_j periodic affine leaves with $\frac{n_j}{n_{j-1}}$ unbounded Fatou components in $\mathcal{R}_{f_c}$. If $n_{j-1} \nmid n_j$, the corresponding ray portrait has valence 2 and period n_j. □

Leaves containing the lift of the dynamic root cycle of primitive parameters have 3 unbounded Fatou components. When the parameter c crosses to a satellite hyperbolic component, one of the unbounded Fatou components collapses to an infinite number of *bounded* Fatou components. So, the leaves in the corresponding cycle only have 2 unbounded Fatou components, each of them on the lift of the basin of infinity $\pi^{-1}A_c(\infty)$, see Figure 4.3.

Proposition 18. *Let c_1 and c_2 be two superattracting parameters. If $h : \mathcal{R}_{c_1} \to \mathcal{R}_{c_2}$ is a homeomorphism conjugating $\hat{f}_{c_1}$ with $\hat{f}_{c_2}$, then $c_1 = c_2$.*

Proof. Any such conjugation sends periodic points into periodic points. Hence, by Lemma 10, h sends the Julia set $\mathcal{J}_{c_1}$ into the Julia set $\mathcal{J}_{c_2}$. Since h is a homeomorphism it has to leave invariant the number of unbounded Fatou components, and as h is a conjugacy of dynamics, it also leave invariant the combinatorial rotation numbers among the unbounded Fatou components on

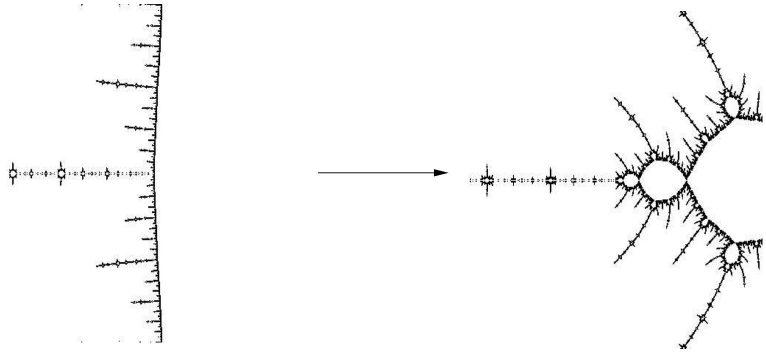


Figure 4.3: Collapsing of unbounded Fatou components by bifurcation.

periodic leaves. This means that c_1 and c_2 must have the same labelled internal address. But by Lau and Schleicher this implies $c_1 = c_2$. $\square$

4.4 Topology of ends

In this section, we will deal with regular parts which are always locally compact with the natural topology. This is not the case for affine laminations endowed with Lyubich-Minsky topology, Lyubich and Lasse Rempe [Lyu-Rem] recently found some examples of affine laminations which are not locally compact.

Consider the one point compactification $\hat{\mathcal{R}}_c$ of the regular part of f_c, let $*$ be the point at infinity. A path $\gamma : [0,1) \to \mathcal{R}_c$ *escapes to infinity* if eventually leaves every compact set $K \subset \mathcal{R}_c$. Equivalently, γ escapes to infinity if admits an extension $\hat{\gamma} : [0,1] \to \hat{\mathcal{R}}_c$ with $\hat{\gamma}(1) = *$. Two paths, γ_1 and γ_2, escaping to infinity are *homotopic at infinity* if for every compact set $K \subset \mathcal{R}_c$ there is an $r \geq 0$ such that the subpaths $\gamma_1|_r : [r,1) \to \mathcal{R}_c$ and $\gamma_2|_r : [r,1) \to \mathcal{R}_c$ are homotopic in $\mathcal{R}_c \setminus K$.

Definition. *Given a leaf L in $\mathcal{R}_c$, the number $E(L)$ denotes the number of* *non-homotopic paths escaping to infinity.*

Now, for a convex cocompact parameter c, we can describe the unbounded Fatou components in $\mathcal{R}_c$ from a homotopic point of view.

Lemma 19. *If c is a convex compact parameter, then for every leaf L in $\mathcal{R}_c$, the number $E(L)$ is equal to the number of unbounded Fatou components in L.*

Proof. This is a direct consequence to the fact, due to Lyubich and Minsky, that when c is a convex cocompact parameter the laminated Julia set $\mathcal{J}_c$ is compact in $\mathcal{R}_c$, so every path escaping to infinity must leave eventually the Julia set $\mathcal{J}_c$ then it most escape through an unbounded Fatou component. □

Let LU_n be the set of leaves with exactly n unbounded Fatou components.

Corollary 20. *The cardinality of LU_n is a topological invariant.*

Proof. The number of non-homotopic paths escaping to infinity is a topological invariant. So, if $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ is a homeomorphism between regular parts, then $E(L) = E(h(L))$ for every leaf L in $\mathcal{R}_c$. □

Corollary 21. *Let f_c be a convex cocompact quadratic polynomial, then $E(L(\hat{\beta})) = 1$.*

Proof. The β fixed point is, by definition, a repelling fixed point where exactly 1 external ray lands. So, by Proposition 15, the leaf $L(\hat{\beta})$ has exactly one unbounded Fatou component. □

Corollary 22. *If f_c is a convex cocompact quadratic polynomial, then $\hat{\infty}$ is the only disconnectivity point of $\mathcal{N}_c$.*

Proof. By going through external rays, all leaves in the regular part have, at least, one access to $\hat{\infty}$. But, leaves with one unbounded Fatou component have access only to $\hat{\infty}$. By Lemma 21, at least, the leaf corresponding to the β fixed point has only one unbounded Fatou component. So, $\hat{\infty}$ is the only irregular point that can be accessed from from every leaf in $\mathcal{N}_c$. □

Thus, the solenoidal cone at infinity can be characterized as the only solenoidal cone in $\mathcal{N}_c$ which connects all of the leaves of the regular part in the natural extension.

Lemma 23. *If c is convex cocompact, then every periodic leaf has exactly one periodic point.*

Proof. It is clear that every periodic point in $\mathcal{R}_c$ lies in a periodic leaf. Now, let L be a periodic leaf in $\mathcal{R}$ of period n, so $\hat{f}_c^n(L) = L$. Because, in the case of convex cocompact parameters, the affine part coincides with the regular part, for every leaf L in the regular part there is a uniformization $\psi : L \to \mathbb{C}$ which conjugates the map $\hat{f}_c^n : L \to L$ to an affine map $\hat{f}_c^n(z) = az + b$ where a is a complex number. We claim that $a \neq 1$. If on the contrary $a = 1$, as the Julia set upstairs is compact, there is a finite covering of small flow boxes, with the property that the derivative of ϕ is bounded away from zero on the Julia set $\mathcal{J}_c$.

Let $\hat{z} \in \pi^{-1}(A(\infty))$ any point on the lift of the basin of infinity. Take W around $z_0 = \pi(\hat{z})$ as in the Shrinking Lemma 3 and let W' the plaque containing $\hat{z}$ in the fiber of W. By assumption, $\hat{f}_c^{-nm}(W')$ has the same diameter for all m, since translations are isometries, on the other hand the diameters of $f_c^{-nm}(W)$ are shrinking to 0 by the Shrinking Lemma 3. Moreover, for every neighborhood V around the dynamical Julia set $J(f_c)$ we have $f_c^{-nm}(W) \subset V$ for large enough n. This means that the derivative of π, under uniformization ϕ, shrinks to 0 which is a contradiction. So, the map $\hat{f}_c^n$ can not be conjugated to a translation in L. Therefore $a \neq 1$, which implies the existence of a periodic point in L. $\square$

At every periodic point in the dynamic root cycle there are, at least, 2 landing external rays. Also, by definition, any point in the dynamic root cycle is on the boundary of, at least, 1 Fatou component. Let L be one of the leaves containing a periodic point in the lift of the dynamic root cycle. Then, by Proposition 15, $E(L) \geq 3$.

If c is the center of a primitive hyperbolic component, then any leaf containing a periodic point in the dynamic root cycle, has 3 unbounded Fatou components, however, two of the unbounded Fatou components are associated to the basin of infinity, whereas the other is associated to a Fatou component in the basin of attraction of the critical cycle. By Proposition 15 and Lemma 19, we obtain the following characterization of regular parts of primitive superattracting parameters:

Corollary 24. *A superattracting parameter c is primitive if and only if there is a leaf L in $\mathcal{R}_c$ such that $E(L) = 3$, and the classes of paths non-homotopic at infinity in L belong to different solenoidal cones.*

In order to make the previous corollary more precise, let us introduce the concept of ends of laminated sets. Let $\mathcal{L}$ be a locally compact laminated set, and consider the one point compactification $\hat{\mathcal{L}} = \mathcal{L} \cup \{*\}$. Two paths, $\sigma : [0,1) \to \mathcal{L}$ and $\tau : [0,1) \to \mathcal{L}$ escaping to infinity, are said to be *equivalent* *at infinity*, if for every compact set $K \subset \mathcal{L}$ there is a number $r > 0$ such that $\sigma([r, 1))$ and $\tau([r, 1))$ belong to the same connected component of $\mathcal{L} \setminus K$. This is an equivalence relationship in the set of paths escaping to infinity.

Definition. *An end of a locally compact laminated set $\mathcal{L}$, is an equivalence class of the relationship above described. Let $End(\mathcal{L})$ denote the set of ends of $\mathcal{L}$, then $\mathcal{L} \cup End(\mathcal{L})$ is the end compactification of $\mathcal{L}$.*

By definition, each end contains the homotopic class of each of its elements, so equivalence at infinity is a weaker relationship than the equivalence relationship of being homotopic at infinity.

Lemma 25. *If the Julia set $J(f_c)$ is locally connected, for every irregular point $\hat{I}$ in $\mathcal{N}_c$ there is an end in $End(\mathcal{R}_c)$ associated to $\hat{I}$.*

Proof. Since the coordinates of $\hat{I}$ belong to the postcritical set, the coordinates of $\hat{I}$ belong either to the Julia set or to an attracting or superattracting cycle. In any case, by the local connectivity of $J(f_c)$, there is a point z_0 in the Fatou set $F(f_c)$ and a path γ from z_0 to $i_0 = \pi(\hat{I})$ such that the trajectory of the path γ intersects the postcritical set, or the Julia set, exactly at i_0. So, the pullbacks $\{\gamma_n\}$ of γ are well defined, and altogether define a path $\hat{\gamma} : [0, 1) \to \mathcal{R}_c$ that escapes to infinity in the regular part. Hence, we associate the irregular point $\hat{I}$ with the end $[\hat{\gamma}]$. Let $\hat{\gamma}'$ be any other path defined as $\hat{\gamma}$, we want to check that $\hat{\gamma}'$ is equivalent at infinity with $\hat{\gamma}$. By definition, $\hat{\gamma}$ and $\hat{\gamma}'$ extend to paths from $[0, 1]$ to $\mathcal{N}_c$ satisfying $\hat{\gamma}(1) = \hat{\gamma}'(1) = \hat{I}$, so the trajectories of $\hat{\gamma}$ and $\hat{\gamma}'$ eventually belong to any neighborhood of $\hat{I}$ in $\mathcal{N}_c$. Since for every $t \in [0, 1)$ the points $\hat{\gamma}(t)$ and $\hat{\gamma}'(t)$ belong to the regular part, for every compact set $K \subset \mathcal{R}_c$

the paths γ and γ' eventually belong to the same connected component in $\mathcal{R}_c \setminus K$. ☐

Lemma 26. *Let c be a parameter in the Mandelbrot set, then there is one and only one end $E_{\hat{\infty}} \in End(\mathcal{R}_c)$ associated to $\hat{\infty}$ in $\mathcal{R}_c$. Furthermore, if c is superattracting, then every end of $\mathcal{R}_c$ is associated to a unique irregular point.*

Proof. Consider the equipotential E_r in the dynamical plane, since E_r is compact and does not contain postcritical points, the corresponding solenoidal equipotential $\mathcal{S}_r = \pi^{-1}(E_r)$ is compact in $\mathcal{R}_c$. Now, let R_θ be any external ray in the dynamical plane, then the end $[\hat{R}]$ of any lift $\hat{R}$ of R_θ in $\mathcal{R}_c$ is associated to $\hat{\infty}$. If $\hat{\gamma}$ is equivalent to infinity to $\hat{R}$ then $\hat{\gamma}$ must eventually lie in the same connected component of $\mathcal{R}_c \setminus \mathcal{S}_r$ as $\hat{R}$. This implies that $\pi(\hat{\gamma})$ converges to ∞ in the dynamical plane, and so $\hat{\gamma}$ must converge to $\hat{\infty}$ in $\mathcal{N}_c$. If c is superattracting, then instead of the equipotential E_r, we can consider to be an internal equipotential inside the corresponding Fatou component on the basin of attraction of the critical cycle. The argument goes on using the corresponding internal solenoidal equipotential. ☐

Lemma 27. *Let f_c be a quadratic polynomial, every end of $\mathcal{R}_c$ is associated to an irregular point in $\mathcal{N}_c$.*

Proof. Let $[\gamma]$ be an end of $\mathcal{R}_c$, with γ a representative of this end in $\mathcal{R}_c$. Let A_n denote the accumulation set of $\gamma_n = \pi_{-n}(\gamma)$. Let us check that $f_c(A_n) = A_{n-1}$, by continuity $f_c(A_n) \subset A_{n-1}$, now let $y \in A_{n-1}$ then there is a sequence $t_m \nearrow 1$ such that $\gamma_{n-1}(t_m)$ converges to y. That means that $\gamma_n(t_m)$ is as close as we want to a point in $f_c^{-1}(y)$, since f_c has finite degree, $\gamma_n(t_m)$ must actually converge to a point in $f_c^{-1}(y)$. So, $f_c(A_n) = A_{n-1}$ as we claimed, this implies that we can construct a backward orbit $\hat{y} \in \mathcal{N}_c$, such that $\pi_{-n}(\hat{y}) \in A_n$. Let us check that $\hat{y}$ must be irregular. If, on the contrary, $\hat{y}$ is regular, then there is a N such that y_n is outside the postcritical set of f_c for $n > N$, since the postcritical set is closed there is a neighborhood U of y_n such that U is outside $P(f_c)$. Let $K \subset U$ be a compact neighborhood of y_n then $\pi_{-n}^{-1}(K)$ is a compact neighborhood of $\hat{y}$, but $\gamma_n(t_m)$ converges to y_n so $\gamma(t_m)$ is contained in K for m large. This contradicts the fact that γ is escaping to infinity. □

Lemma 25 does not rule out the possibility that several irregular points are associated to the same end. It is not clear whether there is a one-to-one correspondence between the irregular points and the ends of the regular part. This would imply that the natural extension of every quadratic parameter c corresponds to the end compactification of $\mathcal{R}_c$. However, we have a positive answer for certain parameters.

Proposition 28. *Let c be a parameter such that $J(f_c)$ is locally connected, and c is either convex cocompact or the postcritical set of f_c is a Cantor set, then the set of ends corresponds to the set of irregular points, and the end compactification of $\mathcal{R}_c$ is $\mathcal{N}_c$.*

Proof. If f_c is convex cocompact then, by Lyubich and Minsky, the Julia set $\mathcal{J}_c$ is compact. Hence, the only irregular points in $\mathcal{N}_c$ correspond to attracting cycles. By Proposition 34 and Lemma 26, attracting cycles correspond to vertices of solenoidal cones, and vertices of solenoidal cones correspond to ends of the regular part. Thus, if c is convex cocompact, the end compactification of $\mathcal{R}_c$ is homeomorphic to the natural extension $\mathcal{N}_c$.

Now, assume that the postcritical set is a Cantor set, in this case, there are no bounded Fatou components in the dynamical plane. By Lemma 25 and Lemma 27 it is enough to prove that different irregular points are associated to different ends. Let $\hat{I}$ and $\hat{I}'$ be two irregular points in $\mathcal{N}_c$, without loss of generality we can assume that $\pi(I) = i_0 \neq i_0' = \pi(I')$, also by 25 assume that both i_0 and i_0' belong to the Julia set $J(f_c)$.

By the local connectivity of $J(f_c)$, there is a path γ embedded in $J(f_c)$ connecting i_0 with i_0'. Let U and U' neighborhoods around i_0 and i_0' small enough that $\{t \in [0,1] | \gamma(t) \notin U \cup U'\}$ contains an interval (t_1, t_2). Since $P(f_c)$ is a Cantor set, there is a $t' \in (t_1, t_2)$ and an open neighborhood V around $\gamma(t')$ and not intersecting $P(f_c)$. Since V is open, there are two external rays R and R' landing at both sides of γ in V, say at z_1 and z_2, and such that the path τ embedded in $J(f_c)$ from z_1 to z_2 lies completely in V.

Let T the image of τ in the dynamical plane. By construction, the curve whose trajectory is $\sigma = R \cup T \cup R'$ separates U from V. Finally, take any equipotential E_r, the set that consists of the union of E_r and the part of σ inside E_r is a compact set K. By construction K does not intersect the postcritical set, so $\pi^{-1}(K)$ is a compact set in $\mathcal{R}_c$ such that I and I' lies in different connected components in $\mathcal{N}_c$. See Figure 4.4. □

Corollary 29. *Let c be any parameter as in Proposition 28, then every homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ between regular parts extends to a homeomorphism of the natural extensions $\tilde{h} : \mathcal{N}_c \to \mathcal{N}_{c'}$. Moreover, $\tilde{h}(\hat{\infty}) = \hat{\infty}$*

Proof. By Proposition 28, if c is superattracting the end compactification of $\mathcal{R}_c$ is homeomorphic to the natural extension $\mathcal{N}_c$, since any homeomorphism $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ extends to the end compactification. By Corollary 22, $\hat{\infty}$ is the

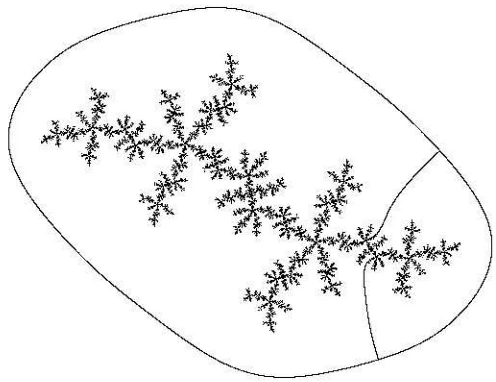


Figure 4.4: To the proof of Proposition 28.

only disconnection point among the irregular points, and the second part of the corollary follows. □

Corollary 30. *Let* $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ *be a homeomorphism between the regular parts of two superattracting parameters* c *and* c'*, then the periods of* c *and* c' *are equal.*

Proof. By Corollary 29, $h : \mathcal{R}_c \to \mathcal{R}_{c'}$ extends to a homeomorphism of the natural extensions sending irregular points into irregular point. If p is the period of c, there are $p+1$ irregular points in $\mathcal{N}_c$. □

Lemma 31. *Let* c_1 *and* c_2 *be two superattracting parameters, and* $h : \mathcal{R}_{c_1} \to \mathcal{R}_{c_2}$ *be a homeomorphism, then* h *sends the leaves containing the dynamic root cycle of* c_1 *into the leaves of the dynamic root cycle of* c_2*.*

Proof. By Proposition 15 there are, at least, three unbounded Fatou components associated to the leaves containing the dynamic root cycle. On the other hand, these unbounded Fatou components correspond to, at least, two different ends in the regular part. So, one of the ends is associated to at least two unbounded Fatou components, this end most be $\hat{\infty}$ and no other periodic point in the regular part can have access to different ends and multiple access to $\hat{\infty}$. □

Given a superattracting parameter c, let v_c be the valence of the dynamic root point.

Corollary 32. *Let c_1 and c_2 be two superattracting parameters with the same period if $v_{c_1} \neq v_{c_2}$, then the corresponding regular parts R_{c_1} and R_{c_2} are not homeomorphic.*

Proof. Having different valence, the corresponding leaves containing the lift of the dynamic root cycle on each regular part must have different number of unbounded Fatou components, which implies that the corresponding leaves have different number of access to infinity. □

4.5 Hyperbolics components

As we pointed before, combinatorially, the Julia sets associated to parameters within a hyperbolic component H and the root of H, are indistinguishable. Here, we present a proposition that topologically ties the regular parts of parameters inside hyperbolic components with the regular part of their center.

Although it was not explicitly stated the proof of next proposition follows immediately from Lemma 11.1 in Lyubich and Minsky's. We include the settings and the statement of that lemma for reference. The interested reader can find the proof on [21].

Let U and V two open sets with $\bar{U} \subset V$ and let $f : U \to V$ be an analytic branched covering. Let us remark that when U and V are disks with the property $U \subset\subset V$ the map f is called *polynomial-like* or *quadratic-like* if the degree of f is 2, let $\mathcal{N}_f$ denote the set of backward orbits of f. In this setting, iterations of the map $\hat{f}$ on $\mathcal{N}_f$ may not be defined, since f is not defined on $V \setminus U$. For $m = 1, 2, ...$ let $\mathcal{N}_m \subset \mathcal{N}_f$ be the set of backward orbits $\hat{z}$ that can be iterated under $\hat{f}$ *at most* m times. So, we have inclusions $\mathcal{N}_m \subset \mathcal{N}_{m+1}$ for $m = 1, 2, ...$

The map $\hat{f}^{-m} : \mathcal{N}_f \to \mathcal{N}_f$ is an immersion, that maps $\mathcal{N}_f$ onto $\mathcal{N}^m$ for $m = 1, 2, ...$ So, by composing with the inclusions $\mathcal{N}^m \hookrightarrow \mathcal{N}_f$ consider $\mathcal{N}_f$ as an extension of $\mathcal{N}^m$ and let us denote these extensions by $\mathcal{N}^m$. Make $\mathcal{N}_f = \mathcal{N}_0$ and identify any point $\hat{z} \in \mathcal{N}^m$ with $\hat{f}^{-1}(\hat{z})$, so the map $\hat{f}^{-1}$ induces the following increasing sequence of sets

$$\mathcal{N}^0 \hookrightarrow \mathcal{N}^1 \hookrightarrow \mathcal{N}^2 \hookrightarrow ...$$

let $\mathcal{D}_f = \cup \mathcal{N}^m$, a set W in $\mathcal{D}_f$ is said to be open if $W \cap \mathcal{N}^m$ is open for every m. The set $\mathcal{D}_f$ is called the *direct limit* of the increasing sequence above. The natural extension $\hat{f}$ of f in $\mathcal{N}_f$ induces a homeomorphism of $\mathcal{D}_f$ into itself. Now, we can state the following Lemma:

Lemma 33 (Lyubich and Minsky). *Assume that a branched covering $f : U \to V$ is the restriction of a rational endomorphism $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\mathbb{C} \setminus V$ is contained in the basin of attraction of a finite attracting set A. Then*

$\hat{f} : \mathcal{D}_f \to \mathcal{D}_f$ *is naturally conjugate to* $\hat{R} : \mathcal{N}_R \setminus \hat{A} \to \mathcal{N}_R \setminus \hat{A}$.

Proposition 34. *The regular part of a quadratic hyperbolic map is homeomorphic to the regular part of its corresponding superattracting map.*

Proof. First, let us discuss parameters inside the Main Cardioid.

Let $f_\epsilon(z) = z^2 + \epsilon$ a quadratic polynomial with ϵ a parameter inside the Main Cardioid. Thus f_ϵ has an attracting fixed point a_ϵ. The Fatou set consists of two open sets corresponding to the basins of infinity $A(\infty)$ and $A(a_\epsilon)$. The Julia set $J(f_\epsilon)$ is a quasicircle, so that the conjugating map $\phi : J(f_\epsilon) \to \mathbb{S}^1$ can be quasiconformally extended to a neighborhood of $J(f_\epsilon)$. Since f_ϵ is expanding on the Julia set, the map $\phi \circ f_\epsilon \circ \phi^{-1}$ is an expanding circle map of degree 2.

By a Theorem of Shub the map $\phi \circ f_\epsilon \circ \phi^{-1}$ is topologically conjugate to z^2, that is, there is a map $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_0 = h \circ \phi \circ f_\epsilon \circ \phi^{-1} \circ h^{-1}$. Also, h admits an equivariant extension to a neighborhood of $\mathbb{S}^1$. Actually, Böttcher's coordinate in the basin of infinity extends h to the whole basin of infinity. So we obtain a conjugacy of f_ϵ to f_0 defined on a simply connected neighborhood U containing the basin of infinity and the Julia set $J(f_\epsilon)$. We can choose U small enough such that the map $f_\epsilon : U \to V$ is quadratic-like. This implies that the map $f_\epsilon : U \to V$ is topologically equivalent to the map $f_0 : \phi(U) \to \phi(V)$. By construction, $\mathbb{C} \setminus V$ is contained in the basin of attraction of a_ϵ.

By Lemma 33, the map $\hat{f}_\epsilon : \mathcal{D}_{f_\epsilon} \to \mathcal{D}_{f_\epsilon}$ is conjugate to $\hat{f}_\epsilon : \mathcal{N}_\epsilon \setminus \hat{a}_\epsilon \to \mathcal{N}_\epsilon \setminus \hat{a}_\epsilon$, also the map $\hat{f}_0 : \mathcal{D}_{f_0} \to \mathcal{D}_{f_0}$ is conjugate to $\hat{f}_0 : \mathcal{N}_0 \setminus \hat{0} \to \mathcal{N}_0 \setminus \hat{0}$. But, the conjugacy ϕ from f_ϵ to f_0 lifts to a conjugacy from $\hat{f}_\epsilon : \mathcal{D}_{f_\epsilon} \to \mathcal{D}_{f_\epsilon}$ to $\hat{f}_0 : \mathcal{D}_{f_0} \to \mathcal{D}_{f_0}$. So, $\mathcal{R}_\epsilon = \mathcal{N}_\epsilon \setminus \{\hat{\infty}, \hat{a}_\epsilon\}$ is homeomorphic to $\mathcal{R}_0 = \mathcal{N}_0 \setminus \{\hat{\infty}, \hat{0}\}$.

It follows that $\mathcal{N}_\epsilon$ is homeomorphic to the double cone over $\mathcal{S}^1$. Let us remark that since the homeomorphism above is a conjugacy, it sends the Julia set $\mathcal{J}_\epsilon$ onto $\mathcal{S}^1 = \mathcal{J}_0$, so we can restrict such homeomorphism to the lift of the basin of attraction of a_ϵ, therefore $\varprojlim(f_\epsilon, A(a_\epsilon)) \cup J(f_\epsilon)$ is homeomorphic to $Con\,(\mathcal{S}^1)$.

Let $f_c(z) = z^2 + c$ be a quadratic polynomial with c in a hyperbolic component H, with attracting cycle $P = \{p_1, ..., p_n\}$ of period n. Let $U_1, U_2, ..., U_n$ be the Fatou components of the basin of attraction of P.

The regular part $\mathcal{R}_c$ can be decomposed in several parts, by cutting along the Julia set $\mathcal{J}_c$ which are:

- The Julia set $\mathcal{J}_c$.
- The lift of the basin at infinity $\pi^{-1}(A(\infty))$ in $\mathcal{R}_c$.

  This set is homeomorphic to $\mathcal{S}^1 \times (0,1)$ by means of the lift of Böttcher's coordinate.

- Fatou components of finite branching.

  This set is homeomorphic to a countable union of sets $\{\mathcal{V}_n\}$, where each $\mathcal{V}_n$ is homeomorphic to $\mathbb{D} \times \{0,1\}^{\mathbb{N}}$. Each point $\hat{z} \in \mathcal{V}_n$ has at most finitely many coordinates in the basin of attraction of the critical cycle $\cup U_j$. Also, each component in $\mathcal{V}_n$ projects onto some U_i for i fixed.

- The invariant lift $\hat{U}$ of the basin of attraction of P.

  Let $\hat{U}_i$ be the of points $\hat{z} \in \hat{U}$ such that $\pi(\hat{z}) \in U_i$, with the lift of Köning's coordinate on $\hat{U}_i$, $\hat{f}_c^n$ has the form of $z \mapsto \lambda z$, where λ is the multiplier of the cycle P, so $\hat{f}_c^n$ in $\hat{U}_i$ is topologically equivalent to $\hat{f}_\lambda$ in $\pi^{-1}(A(a_\lambda))$. By the discussion above, $\hat{U}_i$ is homeomorphic to $\mathcal{S}^1 \times (0,1)$.

Let $c_0 = H(0)$ be the center of H. Now, the regular part $\mathcal{R}_{c_0}$ has the same decomposition as $\mathcal{R}_c$, the decomposition is such that the corresponding components are homeomorphic. These homeomorphisms glue together to a homeomorphism from $\mathcal{R}_c$ to $\mathcal{R}_{c_0}$. □

We call the decomposition of $\mathcal{R}_c$ above, the *laminated decomposition* of the regular part associated to the hyperbolic parameter c. Tomoki Kawahira independently proved Proposition 34 in the more general setting of hyperbolic rational maps and in the quasiconformal level. That is, hyperbolic affine laminations are stable in Lyubich-Minsky setting. See [15]. Let H be a hyperbolic component, and $c_0 = H(0)$ be the center of H. For any hyperbolic parameter c in the boundary of H, the path from c_0 induces a transformation h_c from $\mathcal{R}_{c_0}$ to $\mathcal{R}_c$. If correspondingly c_1 denotes the root of H, then f_{c_1} is a parabolic quadratic polynomial. Let $\mathcal{L}$ be the regular part of $\mathcal{R}_{c_0}$ with the leaves containing the dynamic root cycle removed. Analogously, let $\mathcal{L}'$ be set obtained by removing from $\mathcal{R}_{c_1}$ the periodic leaves associated to Fatou's coordinate. Then, another result of Kawahira, see [14], states that the map $h_{c_1} = \lim h_c$, as c tends to c_1 in H, is a laminar homeomorphism between $\mathcal{L}$ and $\mathcal{L}'$, and moreover, the map h_{c_1} semi-conjugates $\hat{f}_{c_0}|_{\mathcal{L}}$ to $\hat{f}_{c_1}|_{\mathcal{L}'}$.

Chapter 5

Classification of regular parts of superattracting quadratic polynomials

5.1 Isotopies

A *self-embedding* of a compact topological space X is a continuous injective map of X into itself. A homeomorphism $h : X \to X$ is called *isotopic to the identity*, if there exist a continuous map $\Phi : [0,1] \times X \to X$ such that $\Phi(0,x) = x$, $\Phi(1,x) = h(x)$ and the restriction $\Phi_t(x) = \Phi(t,x)$ is a homeomorphism of X for every $t \in [0,1]$. In general, two maps $h_1 : X \to Y$ and $h_2 : X \to Y$ are called *isotopic* if there is a homeomorphism $\phi : Y \to Y$ isotopic to the identity, such that $\phi \circ h_1 = h_2$. In this section, we will prove that every self-embedding of the solenoidal cone $Con\,(\mathcal{S}^1)$ is isotopic to a self-embedding of $Con\,(\mathcal{S}^1)$ sending $\mathcal{S}^1$ to a solenoid of the form $\mathcal{S}^1 \times \{r\}$ in $Con\,(\mathcal{S}^1)$.

5.1.1 Isotopies of the dyadic solenoid

The group $H(\mathcal{S}^1, \mathcal{S}^1)$ of self-homeomorphisms of the solenoid, endowed with the uniform topology, is a topological group. As the dyadic solenoid $\mathcal{S}^1$ is also a topological group, an *automorphism* of $\mathcal{S}^1$ is an element in $H(\mathcal{S}^1, \mathcal{S}^1)$ that preserves the group structure of $\mathcal{S}^1$. The set of automorphisms of the solenoid is denoted by $Aut(\mathcal{S}^1)$. Let $\tau \in \mathcal{S}^1$ the map $\zeta \mapsto \tau \cdot \zeta$ is called a *left translation* of $\mathcal{S}^1$, abusing notation, we identify the map with the element $\tau \in \mathcal{S}^1$. The set $Aff(\mathcal{S}^1)$ of *affine maps* of the solenoid, is a transformation in $\mathcal{S}^1 \times Aut(\mathcal{S}^1)$, where the first factor corresponds to the set of left translations in $\mathcal{S}^1$.

Let ℓ^2 be the standard Hilbert space. The following result, due to James Keesling [16], describes the topological embedding of $Aut(\mathcal{S}^1)$ inside $H(\mathcal{S}^1, \mathcal{S}^1)$.

Proposition 35 (Keesling). *The group $H(\mathcal{S}^1, \mathcal{S}^1)$ is homeomorphic to $\ell_2 \times \mathcal{S}^1 \times Aut(\mathcal{S}^1)$.*

Actually, Keesling's proof shows that $H(\mathcal{S}^1, \mathcal{S}^1)/Aff(\mathcal{S}^1)$ is homeomorphic to ℓ_2. Since ℓ_2 is a vector space, this implies that every self-homeomorphism of the solenoid is isotopic to an affine map. According to Jaroslaw Kwapisz [17], the group $Aut(\mathcal{S}^1)$ has a simple set of generators:

Proposition 36 (Kwapisz). *The group $Aut(\mathcal{S}^1)$ is the infinite dihedral group generated by $\hat{f}_0$ and the inversion $s \mapsto \bar{s}$.*

This proposition was proved in [17] in the more general setting of P-adic solenoids, where P is an arbitrary sequence of prime numbers. Together these propositions yield the following corollary:

Corollary 37 (Kwapisz). *Every homeomorphism of the dyadic solenoid onto itself is isotopic to an affine map of the form* $\tau \circ \hat{f}^n \circ r$*, where* r *is the identity if the map is orientation preserving, or the inversion* $s \mapsto \bar{s}$ *otherwise.*

The following is a known topological property of the dyadic solenoid, see [17].

Lemma 38. *The image of very continuous map* $\phi : \mathcal{S}^1 \to \mathcal{S}^1$*, of the solenoid into itself, is either a point, a closed interval or onto.*

Proof. The solenoid is a connected, compact metric Haussdorf space. So, it is $\phi(\mathcal{S}^1)$ by continuity. Now consider a leaf $L \subset \mathcal{S}^1$, then its image $\phi(L)$ is contained in a leaf L'. We claim that, if $\phi(L)$ is unbounded in L' then $\overline{\phi(L)} = \mathcal{S}^1$. If the image of L is a complete leaf then is clear because density of leaves, now assume that $\phi(L)$ is a half line in the solenoid. Recall that the leaf containing the unit is a one parameter subgroup of the solenoid. By homogeneity, assume that the unit belongs to $\phi(L)$ and, after identifying $\phi(L)$ with the real numbers $\mathbb{R}$, that $\phi(L)$ covers the positive numbers. Now let $-M$ be some negative number. Since the numbers 2^m transversally converge to 0 as m goes to infinity, the numbers $2^m - M$ converge to $-M$ as m goes to infinity, so the closure of $\phi(L)$ contains the whole leaf containing the unit and our claim follows.

Assume that $\phi(L)$ is bounded, then by connectivity $\phi(\mathcal{S}^1)$ is on the connected component of a bounded set, therefore it should be an interval or a point. □

5.1.2 Isotopies of solenoidal cones

Remind that we can regard the solenoid $\mathcal{S}^1$ as the quotient of $S = I \times F$ by the map σ, which is the generator of the adding machine action. Here, $I = [0,1]$ and F denotes the fiber over 1 of the projection $\pi : \mathcal{S}^1 \to \mathbb{S}^1$; F is homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$. Thus, the cylinder $I \times \mathcal{S}^1$ can be expressed as the quotient of $I \times S$ by the adding machine action. For $x \in F$, let $R_x = I \times (0,x) \subset I \times S$, then $R = \cup R_x$ over all $x \in F$ is just the trivial one dimensional lamination $I \times F$. The goal of this section is to prove:

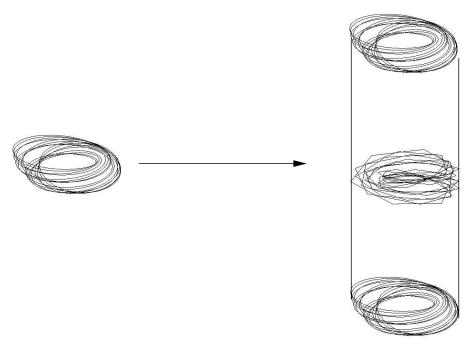


Figure 5.1: Embedding of $\mathcal{S}^1$ into $[0,1] \times \mathcal{S}^1$.

Proposition 39. *Let $\phi : Con\,(\mathcal{S}^1) \to Con\,(\mathcal{S}^1)$ be an orientation preserving self-embedding with $\phi(\mathcal{S}^1) \cap \mathcal{S}^1 = \emptyset$, then $Con\,(\mathcal{S}^1) \setminus \phi(Con\,(\mathcal{S}^1))$ is homeomorphic to $\mathcal{S}^1 \times I$.*

First, we say that two 1-dimensional laminations embedded into a third lamination of dimension 2 *intersect transversally* if they intersect leafwise transversally. Transversality is a smooth notion, so we need an isotopy that regularizes the embedding ϕ on $\mathcal{S}^1$, the existence of such isotopy is given by a generalization of the corresponding theorem about surfaces. Namely, if $\gamma : I \to I \times I$ is a curve in the unit square, such that $\gamma(I) \cap \partial(I \times I)$, then there is a map $h : I \times I \to I \times I$ isotopic to the identity, *rel* the endpoints of γ, such that $h \circ \gamma$ is piecewise linear, and h leaves the extremes of γ fixed. See [11].

Moreover, the corresponding map from the set of embeddings $\text{Emb}(I, I \times I)$ to the set of self-homeomorphisms of the unit square isotopic to identity can be chosen to depend continuously on parameters. That is, if X is a topological space, and the family of maps $\phi_x : I \to I \times I$ in $Top(I, I \times I)$ depends continuously on x, then there is a family of maps h_x in $Top(I \times I, I \times I)$ depending continuously on x such that $h_x \circ \gamma$ is piecewise linear. So, making $X = F$, we have the following lemma:

Lemma 40. *Let $S \to I \times S$ be a laminar embedding, such that $\phi(S) \cap \partial(I \times S) = \emptyset$. Then, there is a homeomorphism $h : I \times S \to I \times S$, isotopic to the identity, such that $h \circ \phi$ is piecewise linear.*

This immediately implies:

Lemma 41. *Given an embedding $\phi : \mathcal{S}^1 \to I \times \mathcal{S}^1$ such that $\phi(\mathcal{S}^1) \cap I \times \mathcal{S}^1 = \emptyset$, then, there exist a homeomorphism $h : I \times \mathcal{S}^1 \to I \times \mathcal{S}^1$, isotopic to the identity, such that $h \circ \phi$ is piecewise linear.*

Proof. Let $\{B_1, B_2, ..., B_k\}$ be a partition by square flow boxes in $\mathcal{S}^1$ such that each plaque of B_i intersects at most one vertical segment R_x. So, we can apply Lemma 40 to each B_i. □

By further local isotopies, we can assure that $\phi(\mathcal{S}^1)$ *does not contain vertical segments and that* $\phi(\mathcal{S}^1)$ *intersects* R *transversally.*

Lemma 42. *For every* $x \in F$*, the intersection* $\phi(\mathcal{S}^1) \cap R_x$ *consist of a finite number of points.*

Proof. By transversality, every intersection is leafwise isolated. Since every R_x is compact, there are finitely many intersections in every R_x. ☐

We can identify R_x with I for each $x \in F$.

Lemma 43. *Let* $m(x) = \min\{r | r \in R_x \cap \phi(\mathcal{S}^1)\}$*. Then, the function* $m : F \to I$ *depends continuously in* x*.*

Proof. By compactness and Lemma 42, there is a covering of $\phi(\mathcal{S}^1) \cap R$, by flow boxes $\{B_1, B_2, ..., B_k\}$ in $I \times S$ and such that each plaque of B_i contains a single point of $\phi(\mathcal{S}^1) \cap R$ for every i.

The intersection $\phi(\mathcal{S}^1) \cap R \cap B_i$ is a transversal for B_i. The points where m attains the minimums are arranged into these transversals. By definition, transversals depend continuously on the fiber. ☐

Fix $x \in F$, by the action of σ, the segments $\{R_{\sigma^n(x)}\}$ with $n \in \mathbb{Z}$ are precisely the segments in R that belong to the same leaf in $I \times S$, let L be the corresponding leaf in $\mathcal{S}^1$ that contains $\phi^{-1}(x)$. By a suitable parametrization, identify L with $\mathbb{R}$ in an order preserving way.

Lemma 44. *Assume that* ϕ *is orientation preserving and let* $t_x = \phi^{-1}(m(x))$*, then* $t_x < t_{\sigma(x)}$*.*

Proof. Every leaf S in $\mathcal{S}^1 \times [0,1]$ is an infinite horizontal strip. By Lemma 38, composition of ϕ with the vertical projection over the solenoid is onto. This implies that of every $x \in F$, there is a first time t_0 such that $\phi(t_0) \in R_x$ and a last time t_1 such that $\phi(t_1) \in R_{\sigma(x)}$. Suppose, on the contrary, that $t_{\sigma(x)} < t_x$, since ϕ is orientation preserving, $t_0 < t_{\sigma(x)}$, also we have $t_1 > t_{\sigma(x)}$. By definition $\phi(t_0) > m(x)$, and $\phi(t_1) > m(\sigma(x))$ in R_x and $R_{\sigma(x)}$ respectively. Now, from $m(x)$ there is no way that the trajectory of ϕ gets to $\phi(t_1)$ without self-intersecting or crossing $R_{\sigma(x)}$ in a lower point than $m(\sigma(x))$, see Figure 5.2. Therefore $t_x < t_{\sigma(x)}$. □

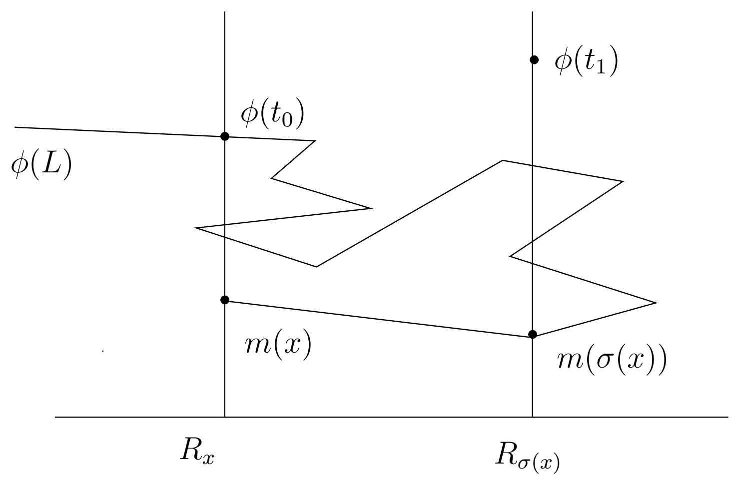


Figure 5.2: To the proof of Lemma 44.

Proof of proposition 39. Let $(t_x, t_{\sigma(x)})$ be the arc joining t_x with $t_{\sigma(x)}$ in $\mathcal{S}^1$, by definition all of these arcs are disjoint. Moreover, $\mathcal{S}^1 = \bigcup_{x \in F} \overline{(t_x, t_{\sigma(x)})}$. By Lemma 43 the extremes of these arcs depend continuously on F. Now, the arcs $(m(x),(x,0))$ along R_x, also depend continuously on F. So, for every $x \in F$ we have a quadrilateral Q_x in $I \times S$, with vertices $(m(x), m(\sigma(x)), (\sigma(x), 0), (x, 0))$, see Figure 5.3. For every $x \in F$, consider homeomorphisms ψ_x from Q_x to the plaque $I \times I \times x$ in $I \times S$ such that $\psi_x = \psi_{\sigma(x)}$ on the side $(m(\sigma(x)), (\sigma(x), 0)$. The homeomorphisms ψ_x paste together to form the desired homeomorphism from $Con\,(\mathcal{S}^1) \setminus \phi(Con\,(\mathcal{S}^1))$ to $I \times \mathcal{S}^1$. □

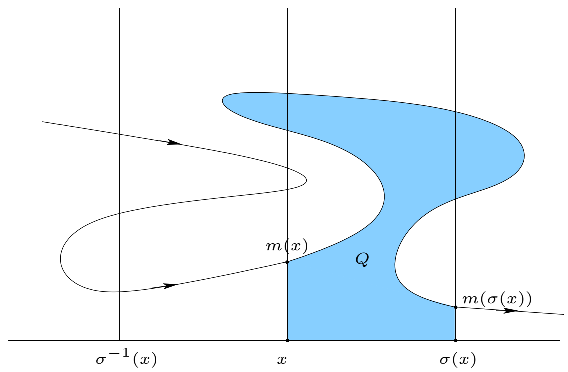


Figure 5.3: The quadrilateral Q_x.

Corollary 45. *Let $\phi : Con\,(\mathcal{S}^1) \to Con\,(\mathcal{S}^1)$ be an embedding with $\phi(\mathcal{S}^1) \cap \mathcal{S}^1 = \emptyset$, then there is an isotopy of ϕ to a map that sends the boundary onto the boundary.*

Proof. The external ray foliation in S gives the track of the desired isotopy. □

Lemma 46. *A map $h : \mathcal{S}_R \to \mathcal{S}_R$ isotopic to the identity, extends to map $\tilde{h} : Con\,(\mathcal{S}^1) \to Con\,(\mathcal{S}^1)$ isotopic to the identity. This extension can be done in such a way that $\tilde{h}$ restricted to the complement $Con\,(\mathcal{S}^1) \setminus N$ of some neighborhood N of S_R is the identity.*

Proof. Consider $N = \cup_{t \in (R-\epsilon, R+\epsilon)} \mathcal{S}_t$, clearly N is a neighborhood of $\mathcal{S}_{E_R}$. Let $\Phi : I \times \mathcal{S}_{E_T} \to \mathcal{S}_{E_R}$ be an isotopy of h to the identity. Define $b : I \to I$ by

$b(t) = max\{1, |\frac{\epsilon}{2}(t - R)|\}$, then the map $\tilde{h} : Con\,(\mathcal{S}^1) \to Con\,(\mathcal{S}^1)$ given by $\tilde{h}(t, s) = (t, \Phi(b(t), s))$ satisfies the conditions of the lemma. □

Corollary 47. *Let $\phi : \mathcal{S}^1 \to I \times \mathcal{S}^1$ be an embedding. Assume that ϕ admits an extension to a map $\mathcal{S}^1 \times (-\epsilon, \epsilon) \to I \times \mathcal{S}^1$ for some positive ϵ. Then, the image of ϕ is isotopic to $\mathcal{S}^1 \times 0$.*

Proposition 48. *Assume that $h : \mathcal{N}_c \to \mathcal{N}_{c'}$ is a homeomorphism such that $h(\hat{\infty}) = \hat{\infty}$. Then h is isotopic to a homeomorphism sending a solenoidal equipotential $\mathcal{S}_R(c)$ onto $\mathcal{S}_R(c')$.*

Proof. The solenoidal cone at infinity admits a foliation by solenoidal equipotentials. These solenoidal equipotentials define a local base of neighborhoods homeomorphic to solenoidal cones around $\hat{\infty}$. So, any homeomorphisms send any of this solenoidal local base of neighborhoods into a local base of neighborhoods around $\hat{\infty}$ in $\mathcal{N}_{c'}$. Thus, there is a solenoidal cone at $\mathcal{S}_R(c)$ in $\mathcal{N}_c$ which is embedded by h into some solenoidal cone at $\mathcal{S}_R(c')$ in $\mathcal{N}_{c'}$. Now, the proposition follows from Proposition 39. □

5.2 Classification of laminations associated to superattracting parameters

In this section, we will prove the central theorem of this work, the remaining of the chapter will be dedicated to generalizations to larger class of parameters. The idea is to recognize the topological impressions that combinatorics of parameters impose over the regular parts. We will prove the following:

Theorem 49. *Let $h : \mathcal{R}_{f_c} \to \mathcal{R}_{f_{c'}}$ be a orientation preserving homeomorphism between the regular parts of two superattracting quadratic polynomials, f_c and $f_{c'}$. Then, $f_c = f_{c'}$.*

Proof. By Corollary 29, the homeomorphism h admits an extension $\tilde{h} : \mathcal{N}_c \to \mathcal{N}_{c'}$ such that $\tilde{h}(\hat{\infty}) = \hat{\infty}$. By Proposition 48, $\tilde{h}$ is isotopic to a homeomorphism that sends a solenoidal equipotential $\mathcal{S}_R$ in $\mathcal{N}_c$ onto a solenoidal equipotential $\mathcal{S}'_R$ in $\mathcal{N}_{c'}$. Using the lift of Böttcher's coordinate at the solenoidal cones at infinity in $\mathcal{N}_c$ and $\mathcal{N}_{c'}$ the map $\tilde{h}$ restricted to $\mathcal{S}_R$, becomes a homeomorphism of $\mathcal{S}^1$ into itself. Now, Corollary 37 there exist a map, say $\psi : \mathcal{S}^1 \to \mathcal{S}^1$, isotopic to the identity such that $\psi \circ \tilde{h}$ is an affine transformation of $\mathcal{S}^1$ of the form $\tau \circ \hat{f}_0^n$. By Lemma 46, ψ extends to a map $\tilde{\psi}$ isotopic to the identity, defined on a neighborhood N of $\mathcal{S}_R$, so that $\tilde{\psi} \circ \tilde{h}$ coincides with $\tilde{h}$ outside N. The map $\hat{f}_{c'}^{-n} \circ \tilde{\psi} \circ \tilde{h}$ is conjugate to τ in $\mathcal{S}_R$. So, by means of the previous normalizations we can assume that $\tilde{h}$ restricted to $\mathcal{S}_R$ is already the translation τ. On the other hand, by Lemma 31, $\tilde{h}$ must send the leaves containing the dynamic root cycle of $\hat{f}_c$ into the leaves containing the dynamic root cycle of $\hat{f}_{c'}$. We do not know whether the dynamic root lift $\hat{r}_c$ is actually map to $\hat{r}_{c'}$. But at least on the level of solenoidal equipotentials, by Lemma 8, the map $\tilde{h}$ maps the corresponding periodic leaves in the solenoid $\mathcal{S}_R$ (under doubling), into periodic leaves of $\mathcal{S}'_R$. Remind that there is a one-to-one correspondence between periodic leaves and periodic points on $\mathcal{S}^1$. Thus, if necessary after another isotopic deformation of $\tilde{h}$, we can assume that $\tilde{h}|_{\mathcal{S}_R}$ sends a periodic point in $\mathcal{S}^1$ to a periodic point in $\mathcal{S}^1$ but if a translation τ in $\mathcal{S}^1$ sends a periodic point of $\mathcal{S}^1$ into a periodic point in $\mathcal{S}^1$, then τ itself must be periodic.

By Corollary 30, the periods of c and c' must be the same.

Therefore, τ sends every periodic point in $\mathcal{S}_R \cap L(\hat{r}_c)$ into every periodic point in $\mathcal{S}'_R \cap L(\hat{r}_{c'})$. That happens in every leaf containing the lift of the dynamic root cycle. So, by projecting onto $\mathbb{S}^1$ by π, the action of τ becomes a rotation in $\mathbb{S}^1$ that sends the ray portrait of r_c onto the ray portrait of r'_c. By Lemma 5, the dynamic root cycle must be the same and then $c = c'$. □

5.3 Irregular points.

Proposition 50. *Let f_c be a quadratic polynomial then $\varprojlim(f_c, \bar{\mathbb{C}})$ is pathwise connected.*

Proof. Let $\hat{z}$ be any point in $\varprojlim(f_c, \mathbb{C})$ then consider a path γ in $\mathbb{C}$ joining $z_0 = \pi(\hat{z})$ with ∞. The lift of γ to $\hat{z}$ connects $\hat{z}$ to $\hat{\infty}$, since the fiber of infinity $\pi^{-1}(\infty)$ consists only of $\hat{\infty}$. □

Proposition 51. *Let V be a local pathwise component of an irregular point $\hat{I}$, then $\hat{I}$ is a path disconnectivity point of V.*

In particular, if there are irregular points in $\mathcal{N}_f$, the pathwise connected components of $\mathcal{N}_f$ can not be Riemann surfaces. Remind that two sequences $\{p_k\}$ and $\{q_k\}$ are called *cofinal* if there exist a number N such that $q_n = p_n$ for $n > N$.

Proof. The proof is divided in two cases:

Case 1: We first assume that only finitely many of the coordinates of $\hat{I}$ are critical points. In fact, without loss of generality by considering a cofinal subsequence of $\mathcal{N}_f$, we can assume that none of the points in the backward orbit of $\hat{I}$ is a critical point. Let $\pi_j(\hat{I}) = i_j$, and let $\phi : (U_1, i_1) \to (\mathbb{D}, 0)$ be a local chart around i_1 in $\mathbb{C}$. Let U_n be the pullback of U_1 along the orbit of $\hat{I}$ and, let n_1 the first time such that π_{n_1} is not univalent. Then, there is a critical point $c_1 \in U_{n_1} \subset \mathbb{C}$, let $b_1 = \pi_{n_1}(c_1)$. Now let be B_1 some small ball around 0 such that $\phi^{-1}(B_1)$ does not contain b_1; for safeness take the ball with radius $\frac{1}{2}d(\phi(b_1), 0)$. Again there is a first moment n_2 where the pullback of $\phi^{-1}(B_1)$ by π_{n_2} is not univalent. If $n_2 = n_1$ we take the closest critical point to i_{n_1} and define that critical point to be c_1 otherwise let c_2 the corresponding critical point in $\mathbb{C}$. Since at every level there are only finitely many critical points, this process defines: An increasing sequence $\{n_k\}$ of times, a sequence of critical points c_k in $\mathbb{C}$ respectively, and a sequence of postcritical points b_k converging to i_1 in $\mathbb{C}$. Let us emphasize that, by construction, for every k there is a ball around i_1 such that $\phi^{-1}(B_k)$ lifts univalently along $\hat{I}$ up to the $k - 1$ coordinate.

Cover the sequence $\{\phi(b_k)\}$ by disjoint balls V_k of small radii, say of size $1/10$ of the distance between $\phi(b_k)$ and the rest of the sequence. There might be another critical points inside this balls, since we only care for the critical points in lower levels, we can take the balls small enough such that, for every k, there are no critical points in $\phi^{-1}(V_k)$ for π_j and $j < k$. Let $\gamma_k = \phi^{-1}(\partial V_k)$, be the loop in U_1 that projects to the boundary of the ball corresponding to $\phi(b_k)$. For every k, let $p_k, q_k \in \gamma_k$, such that $\phi(p_k)$ and $\phi(q_k)$ are the points with largest and smallest modulus in V_k. The points p_k and q_k separates γ_k into two paths say σ_k and σ_k'. We will label the paths with the primes according the parity of k. For k odd σ_k is the path that connects p_k with q_k and has b_k on the left; for k even σ_k is the path connecting p_k with q_k having b_k to the right. Then connect q_k with p_{k+1} by a path l_k whose is as short as possible. The closure of the union the loops γ_k and the paths ρ_k contains the point i_1 and defines two paths σ and σ', both starting at i_1 and ending at p_1 in the following way: Every time σ crosses q_k goes along σ_k to p_k and goes along l_k from p_k to q_{k-1}. On the other hand, define σ' as the path by always chooses the path σ'_k from q_k to p_k. By the election of σ_k each path σ is 'snaking' along the sequence $\{b_k\}$. So we have a picture like in Figure 5.4. This picture lifts univalently until $n_1 - 1$ where the first loop γ_1 encloses the critical value of c_1. If we lift one more time we get a picture like in Figure 5.5.

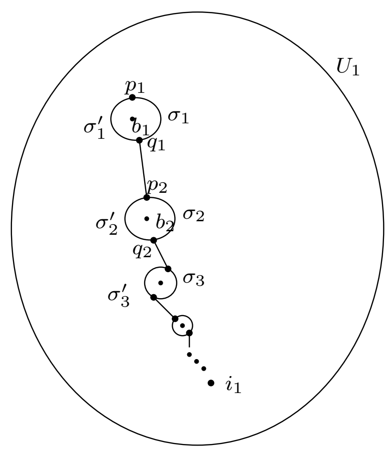


Figure 5.4: A sequence of postcritical points.

Since there are no other critical points closer to i_1. The portion of σ and σ' from i_1 to q_1 lift univalently to unique paths in $\mathbb{C}$ starting at i_2 and ending at some point $\tilde{q}_1$ over the π_1 fiber of q_1. Now having $\tilde{q}_1$, the paths σ_1 and σ'_1 lift to unique paths starting at $\tilde{q}_1$ and ending at $\tilde{p}_1$ and $\tilde{p}'_1$, respectively. Which project onto p_1 under π_2. This give a procedure to subsequently lift the paths, namely; each time we get to a critical point c_k consider the point $\tilde{q}_{n_k}$ uniquely

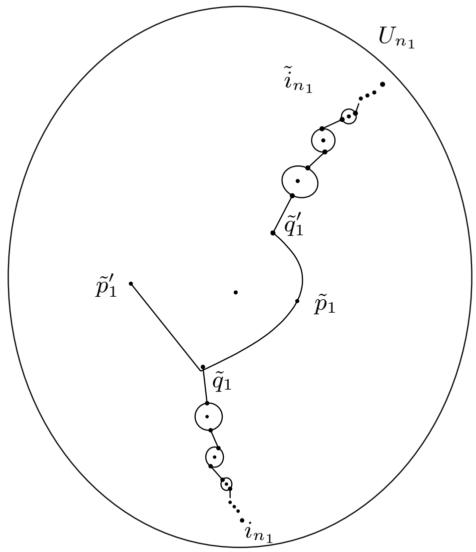


Figure 5.5: A sequence of postcritical points.

defined by univalent lifting of portions of σ (or σ'), from i_1 to q_{n_k}. And then take the lifts of σ_k and σ'_k defining two points in the fiber of p_k, we could have different choices for the lift of the path from p_k to p_1, since it could happen that contains critical points. However, any choice of these paths would work for us. Following this for all points b_k we can construct two points P_1 and P_2 in $\pi^{-1}(p_1) \subset \mathcal{N}_f$, that belong to the path connected component of $\hat{I}$.

Now connect P_1 and P_2 with any path $\hat{s}$, this path projects to a loop s in $\mathbb{C}$ which must be non-trivial in every level k. The only way that this can happen is if s is homotopic to $\sigma \circ \sigma'$ relative to a cofinal sequence of b_k. But, then the π_k image of $\hat{s}$ over $\mathbb{C}$ most cross i_k for all k. Thus $\hat{s}$ must cross $\hat{I}$.

Case 2: There are infinitely many critical points in the coordinates of $\hat{I}$. This case is more simple than the previous. Consider a local chart $U_1 \subset \mathbb{C}$ around i_1. Take a point p_1 not in the postcritical set, and joint it by a path σ outside the postcritical set. For every time k that there is a critical point in the coordinate of $\hat{I}$, we can choose two paths in the π_k fiber of γ. Such that if we join the end points of these lifting, we get a path that π_1 projects onto a loop in $\mathbb{C}$ based at p_1 and with winding number bigger than 2^{k-1}. So, again can construct two points P_1 and P_2 in the π fiber of p_1. Such that any path $\hat{s}$ connecting P_1 and P_2 in $\mathcal{N}_f$ have to pass through $\hat{I}$. $\square$

From the proof of the previous proposition, we have the following corollaries:

Corollary 52. *Let $\phi : Con(\mathcal{S}^1) \to Con(\mathcal{S}^1)$ be a self-embedding of the solenoidal cone, then $\phi(\hat{\infty}) = \hat{\infty}$.*

Corollary 53. *Every coordinate of an irregular point belongs to the postcritical set.*

Lemma 54. *Let f_c be a quadratic polynomial with minimal postcritical set. Then, the point $\hat{\infty}$ is the only isolated irregular point in the natural extension of $\mathcal{N}_c$.*

Proof. Let $\hat{i}$ be an irregular point in $\varprojlim(f_c, \bar{\mathbb{C}}) \setminus \{\infty\}$, then each coordinate of $\hat{i}$ belongs to the postcritical set $P_{f_c}(0)$, since it is minimal with respect to f_c, we have $0 \in \overline{\{f^n(i_0)\}}$, where $i_0 = \pi(\hat{i})$. Let $\mathcal{U}$ be any neighborhood around $\hat{i}$ and U_0 be an open set in $\mathbb{C}$, evenly covered by the connected component of $\mathcal{U}$ containing $\hat{i}$. This implies that there exist $m > 0$ such that $0 \in U_m$. Since $0 \in \overline{f^n(i_0)}$, there exist $r > m$ such that $\hat{f}^{r+m}(\hat{i}) \in \mathcal{U}$. $\square$

Proposition 55. *Let f_c be a quadratic polynomial that is either convex cocompact or the postcritical set $P(f_c)$ is a minimal set, then any homeomorphism $h : \mathcal{N}_{f_c} \to \mathcal{N}_{f_{c'}}$ fixes $\hat{\infty}$.*

Proof. An irregular point is a local disconnectivity point by Proposition 51, hence irregular points are topologically different than regular points. Now, we have to check that under the proposition's hypothesis, the irregular point at infinity is topologically distinguishable. The case of convex cocompact is clear; since 0 is not recurrent then, by Mañe's Theorem, there are no irregular points associated to 0. Hence, in this case, $\hat{\infty}$ is the only irregular point. When c is either parabolic or hyperbolic, $\hat{\infty}$ is the only disconnectivity point of $\mathcal{N}_c$. Finally, when the postcritical set is a minimal set, $\hat{\infty}$ is the only isolated irregular point by Lemma 54. □

Chapter 6

Monodromy groups

6.1 Construction of monodromy

Let f be a postcritically finite rational map defined on the Riemann sphere $\bar{\mathbb{C}}$ of degree d. Given a path $\gamma : [0,1] \to \bar{\mathbb{C}} \setminus P_f$, defined on the complement of the postcritical set P_f of f, and any point $\hat{z} \in \pi^{-1}(\gamma(0))$ there is a lift $\hat{\gamma}$ in $\varprojlim(f, \bar{\mathbb{C}} \setminus P_f)$ such that $\hat{\gamma}(0) = \hat{z}$ and whose end point $\hat{\gamma}(1) \in \pi^{-1}(\gamma(1))$. This defines a *holonomy* map H_γ from $\pi^{-1}(\gamma(0))$ to $\pi^{-1}(\gamma(1))$, which is actually a homeomorphism of fibers. In fact, H_γ depends only on the homotopy class of γ.

6.1.1 Encoding trees and their automorphisms

A *tree* is a planar graph without cycles. Given a finite set $X = \{x_1, ..., x_d\}$, we consider X as an *alphabet* in d symbols. Denote by X^ω the set of all finite words made with the alphabet X. We regard the empty set $\{\emptyset\}$, as the word of length zero in X^ω. The set X^ω is endowed with a *rooted tree* structure, denoted by $T(X)$ in the following way. The *root* of the tree is the empty set

$\{\emptyset\}$, two words ω_1 and ω_2 have an edge in common if one is obtained by adding a letter at end of the other. The tree $T(X)$ is also a *graded tree*, the n-th level of the $T(X)$ consists of the words with exactly n-letters. Except for the root of the tree, every vertex on $T(X)$ is attached to $d+1$ symbols, one "ancestor" and d "childs". We refer to the tree $T(X)$, as the *encoding tree* in d symbols.

Every automorphism of the tree $T(X)$ most leave invariant each n-th level. Moreover, the restriction on the n-level of γ is a permutation of d^n symbols. Let $Aut(T(X))$ denote the set of automorphisms of $T(X)$. The subtree $T_\omega(X)$ of $T(X)$ consisting of the words that start with the word ω is canonically isomorphic to $T(X)$. In particular, the set of automorphisms of $T(X)$ fixing the one letter word (x_i) is isomorphic to $Aut(T(X))$. By postcomposing γ with the inverse of the associated permutation on the first level, γ induces d automorphisms of $T(X)$ each corresponding to every symbol x_i. If S^d denotes the group of permutations in d symbols, then we have the following representation $Aut(T(X)) \simeq Aut(T(X))^d \rtimes S^d$.

6.1.2 Preimage trees

Given a fixed point $* \in \mathbb{C} \setminus P_f$, the set $B(*) = \bigcup f^{-n}(*)$ of all backward images of $*$, also has a natural rooted tree structure. Simply, the root of the tree is $*$ itself, and a point $x \in B(*)$ is a child of $y \in B(*)$ if and only if $y = f(x)$. The n-th level of the tree is precisely the set $f^{-n}(*)$. Let us denote by $T(f,*)$ the set $B(*)$ endowed with the rooted tree structure, $T(f,*)$ will be called the *preimage tree* of $*$ under f. As rooted trees, the tree $T(f,*)$ is isomorphic to the encoding tree $T(X)$ in d symbols. However, any isomorphism of trees would break the dynamical information of the preimage tree.

Following Bartholdi, Grigorchuk and Nekrashevich, we build a one-to-one correspondence between the *vertices* of $T(X)$ and $T(f,*)$. To make this bijection, first identify X with $f^{-1}(*)$ so we can select d paths inside $\mathbb{C} \setminus P_f$ such that for every $x \in X$, l_x will denote a path joining $*$ with x.

Given l_x, for every $y \in X$ there is a preimage l of the path l_x starting at y. Identify the end point of l with the word yx. This procedure, induces the desired bijection between the vertices of the encoding tree $T(X)$, with the vertices of $T(f,*)$. When $\deg f = 2$, Figure 6.1 sketches the first three levels of this morphism. Let us remark that the bijection above depends on the homotopy classes of the paths l_x in $\mathbb{C} \setminus P_f$.

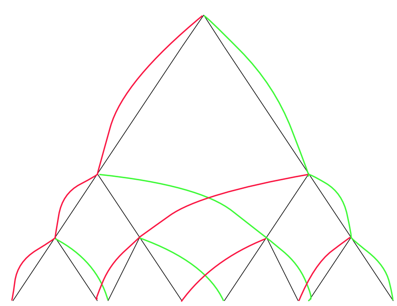


Figure 6.1: Here, we put together $T(X)$ and $T(f,*)$, the last is drawn with straight lines.

6.1.3 Iterated Monodromy groups

Every loop γ based on $*$ induces an automorphism of the tree $T(f,*)$ as follows, given $x \in X$ let γ_x be the preimage of γ starting at x, the end point $\gamma_x(1)$ is again an element in the first level of $T(f,*)$. As γ_x is uniquely defined, the preimages of γ to the first level of $T(f,*)$ induces a permutation of the symbols X. We can repeat this procedure to the subsequent levels of $T(f,*)$, so altogether γ induces a transformation of $T(f,*)$ onto itself which, it turns out, it is also a transformation of $T(X)$ onto itself by means of the bijection above. Once again, the corresponding automorphism of $T(X)$ only depends on the homotopy class of γ, and up to conjugacy, does not depend on the choice of the base point $*$.

Thus, we have a representation of the fundamental group $\pi_1(\mathbb{C} \setminus P_f, *)$ into $Aut(X)$. The image of this group is called the *iterated monodromy group* of $(\mathbb{C} \setminus P_f, *)$. By the discussion on Subsection 6.1.1, the action of γ admits a representation of the form $Aut(T(X))^d \rtimes S^d$. The permutation factor is given by the pullback of γ to the first level X of $T(f,*)$, and for each symbol $x \in X$ the automorphism corresponding to the subtree starting at x is the automorphism of $T(X)$ induced by the loop $l_x \cdot \gamma_x \cdot l_{\gamma(1)}^{-1}$. Where $\cdot$ represents the concatenation of paths, read from left to right.

Note that the fiber $\pi^{-1}(*) \subset \varprojlim(f, \mathbb{C} \setminus P_f)$ is naturally identified with the set of infinite simple paths on $T(f,*)$. In this way, the Iterated Monodromy Group of $(\mathbb{C} \setminus P_f, *)$ becomes a representation of the monodromy group of the fiber $\pi^{-1}(*)$ in $\varprojlim(f, \mathbb{C} \setminus P_f)$.

When f is postcritically finite, the fundamental group $\pi_1(\mathbb{C} \setminus P_f, *)$ is finitely generated. In this case, the action of the fundamental group on $T(f,*)$ is conveniently represented by an *automaton* called the *Moore diagram*. That is, a labelled directed graph whose nodes represents generators of the fundamental group $\pi_1(\mathbb{C} \setminus P_f)$, labels are a directed pair of elements in X, so that if a edge is going from the vertex γ to γ' the labelling pair is $(x, \gamma_x(1))$.

Let us consider the situation $f_0 = z^2$, then $P_{f_0} = \{0\}$ and $\mathbb{C} \setminus P_{f_0} = \mathbb{C}^*$ the punctured plane. Take the fixed point 1 for the base point $*$ in $\mathbb{C}^*$. The set $f^{-1}(1) = \{-1, 1\} \subset \mathbb{C}^*$ is identified with the set of symbols $X = \{0, 1\}$ taking 1 to "0" and -1 to "1". To encode the preimage tree $T(f_0, 1)$ into $T(\{0, 1\})$, let l_0 be a simple loop based on 1 homotopic to 0 rel 1 in $\mathbb{C}^*$, and l_1 be the semi-arc on the unit circle going from 1 to -1 in the standard orientation.

The fundamental group $\pi_1(\mathbb{C}^*, 1) \simeq \mathbb{Z}$ is generated by any simple loop, say γ, around 0. The action of γ on X is the trivial transposition τ, whereas the transition maps $l_0 \cdot \gamma_0 \cdot l_1^{-1} = Id$ and $l_1 \cdot \gamma_1 \cdot l_0^{-1} = \gamma$. Thus, the action of γ is expressed in terms of $A(0, 1)^2 \rtimes S^2$ as $(Id, \gamma)\tau$. Then the automaton contains two vertices the Identity Id and σ. If we have a sequence of 0's and 1's, the automaton works as a "reader" of the sequence from left to right. Applies the action of a vertex to the first coordinate, say θ_0, and applies the action of the end vertex of the corresponding arrow. For instance $\sigma(0, 0) = (1, 0)$ whereas $\sigma(1, 0) = (0, 1)$, so the action of σ on sequences of 0's and 1's corresponds precisely to the action of the generator of the Adding Machine σ, (hence the abuse of notation). So, the extension of this action to the set of infinite sequences is the Adding Machine action in $\{0, 1\}^{\mathbb{N}}$. The Moore Diagram for f_0 is depicted on Figure 6.2.

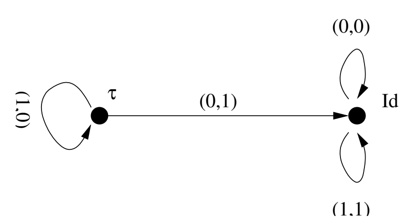


Figure 6.2: The Moore diagram of the Adding Machine

6.2 The basillica, $f_{-1} = z^2 - 1$

Now let us describe the case $f_{-1}(z) = z^2 - 1$. Here the postcritical set $P_{f_{-1}} = \{-1, 0\}$. As the base point $*$, let us take the α fixed point.

Since $\deg f_{-1} = 2$ we identify the set of preimages of α, that is $\{\alpha, -\alpha\}$ with $X = \{0, 1\}$ taking α to the symbol "0" and $-\alpha$ to the symbol "1". Let l_0 be a simple loop based at α, and let l_1 be the path from α to $-\alpha$ as on Figure 6.3. The paths l_0 and l_1 are the auxiliary paths that allow us to identify the preimage tree of $T(f_{-1}, \alpha)$ with the coding tree $T \equiv T(\{0, 1\})$.

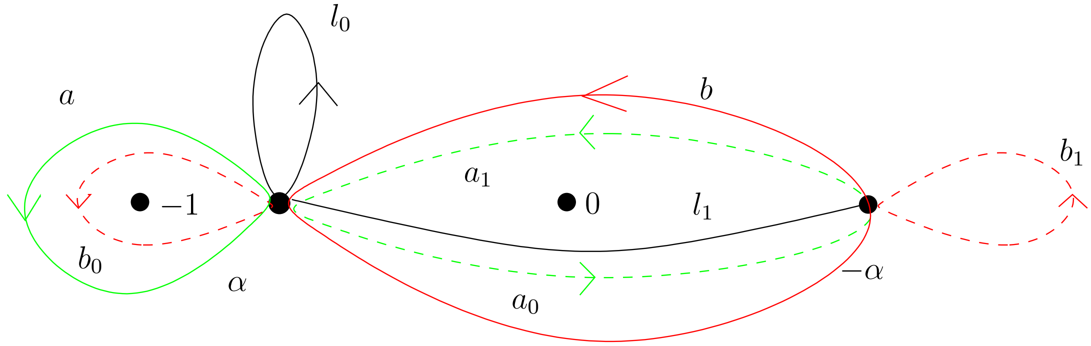


Figure 6.3: To the construction of the IMG for $z^2 - 1$.

Let a and b be simple loops around -1 and 0, such that a and b generate the fundamental group $\pi_1(\mathbb{C} \setminus P, \alpha)$ (see Figure 6.3). The preimages of a under f_{-1} are the paths a_0 connecting α with $-\alpha$ and a_1 connecting $-\alpha$ with α. Thus the action of a on the first level of $T(f_{-1}, \alpha)$ is just the transposition. Moreover, the loop $l_0 \cdot a_0 \cdot l_1^{-1}$ represents the identity Id in $\pi_1(\mathbb{C} \setminus \{-1, 0\}, \alpha)$, whereas the loop $l_1 \cdot a_1 \cdot l_0^{-1}$ is homotopic to b. Thus, the representation of a in terms as an element of $Aut(T)^2 \rtimes S^2$ is $a = (Id, b)\tau$.

The preimages of b are the loops b_0 and b_1 based on α and $-\alpha$, respectively (see Figure 6.3). The loop $l_0 \cdot b_0 \cdot l_0^{-1}$ is homotopic to a, and the loop $l_1 \cdot b_1 \cdot l_1^{-1}$ is trivial. Therefore, the action of b in terms of $Aut(T)$ is $b = (a, Id)Id$. The Moore Diagram of the corresponding Iterated Monodromy Group is given in Figure 6.4.

The action of the Iterated Monodromy Group of f_{-1} extends to $\{0,1\}^{\mathbb{N}}$, which is identified with $\pi^{-1}(\alpha)$ by encoding. We have the following expressions:

$$a(\theta_1, \theta_2, \theta_3, ...) = \begin{cases} (1, Id(\theta_2, \theta_3, ...)) & \text{if } \theta_1 = 0 \\ (0, b(\theta_2, \theta_3, ...)) & \phantom{\text{if }} \theta_1 = 1 \end{cases},$$

and

$$b(\theta_1, \theta_2, \theta_3, ...) = \begin{cases} (0, a(\theta_2, \theta_3, ...)) & \text{if } \theta_1 = 0 \\ (1, Id(\theta_2, \theta_3, ...)) & \phantom{\text{if }} \theta_1 = 1 \end{cases}.$$

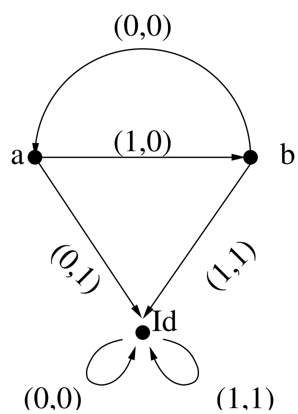


Figure 6.4: The Moore diagram for $f_{-1} = z^2 - 1$.

Let $f_c = z^2 + c$ be a quadratic polynomial whose postcritical set P_c is finite. Consider the fiber $\pi^{-1}(\alpha)$ over the α fixed point of f_c. Now, we introduce an object in $\varprojlim(f_{-1}, \mathbb{C} \setminus P_c)$ that will help us to keep track of the orbits of points in $\pi^{-1}(\alpha)$ under the action of the Iterated Monodromy Group of f_c.

Let $\{a_1, ..., a_n\}$ be a set of simple loops representatives of a generating set of the fundamental group $\pi_1(\mathbb{C} \setminus P_c, \alpha)$, and let $\mathcal{G}_c = \pi^{-1}(\bigcup a_i) \subset \varprojlim(f_c, \mathbb{C} \setminus P_C)$, then for every leaf $L \subset \varprojlim(f_c, \mathbb{C} \setminus P_c)$ the set $\mathcal{G}_c(L) = \mathcal{G}_c \cap L$ is an infinite graph embedded in L. By construction $\mathcal{G}_c$ belongs to the regular part of f_c.

Definition. *The set $\mathcal{G}_c$ is called the* monodromy graph *of $\mathcal{R}_c$.*

The monodromy graph is related to Schrerier's graph as defined in [2]. The π_{-n}-projection of the monodromy graph corresponds to the n-level of Schrerier's graph.

Starting from any vertex $v \in \pi^{-1}(\alpha) \cap \mathcal{G}_c(L)$, the *number of ends* of $\mathcal{G}_c(L)$ is the number of disjoint infinite simple paths along $\mathcal{G}_c(L)$ starting from v. If $\mathcal{G}_c(L)$ is connected, then the number of ends of $\mathcal{G}_c(L)$ is independent of the starting point v.

The relevance of the monodromy graph to the topology of regular parts of postcritically finite quadratic polynomials is that topologically resembles the Julia set $\mathcal{J}_C \cap L$, in next lemma we prove it for $c = -1$.

Lemma 56. *The monodromy graph $\mathcal{G}_{-1}$ is homotopic to $\mathcal{J}_{-1}$ relative to $\pi^{-1}(\alpha)$.*

Proof. The first thing to notice is that the loops a and b can be taken to be homotopic to the Julia set $J(f_{-1})$ rel $\{\alpha\}$. Because f_{-1} is hyperbolic, the Julia set of a f_{-1} is locally connected, this implies that the Böttcher coordinate ϕ_{-1} extends continuously to $J(f_{-1})$. Consider any equipotential E_r on the basin of infinity, by deforming E_r along the external ray foliation, in such way that the points in E_r over the external rays $R_{\frac{1}{3}}$ and $R_{\frac{2}{3}}$ landing at α go to α. Then we obtain a and b loops based on α and homotopic to $J(f_{-1})$ rel α. Since the homotopy, say H, is done along external rays which do not contain critical points. The homotopy H lifts to a homotopy from $\mathcal{G}_{-1}$ to $\mathcal{J}_{-1}$. $\square$

Hence we have the following corollary:

Corollary 57. *Given a leaf L in $\mathcal{R}_{-1}$, the number of ends of $\mathcal{G}_{-1}(L)$ is equal to the number of unbounded Fatou components of L.*

Later on, we will discuss how to generalize the previous corollary and lemma to postcritically finite quadratic polynomials. First, let us count the number of ends of $\mathcal{G}_{-1}$ using only the algebraic properties of the Iterated Monodromy Group for f_{-1}. To do so, we have to describe the action of a and b in terms of infinite sequences in $\{0,1\}^{\mathbb{N}}$.

Lemma 58. *Let $\theta \in \{0,1\}^{\mathbb{N}}$, then*

- *a acts invariantly on the even coordinates of θ, i.e., $(a(\theta))_{2n} = \theta_{2n}$, $\forall n \in \mathbb{N}$.*
- *b acts invariantly on the odd coordinates of θ, i.e., $(b(\theta))_{2n-1} = \theta_{2n-1}$, $\forall n \in \mathbb{N}$.*

Furthermore, both elements act finitely on the coordinates of any point in $\{0,1\}^{\mathbb{N}}$, with the exception of one point for each generator.

Proof. It is immediate from the definition of a and b. The action of both generators is an alternating game between them that ends with the Identity. Whenever, a acts it interchanges the first coordinate and switches either to b or Id, when b acts it leaves invariant the first coordinate, and either switches to a over the remaining coordinates or to Id. The only way that this game never stops is when at every time that a acts there is a 1 on the first coordinate, and at every time that b acts there is a 0. This is only the case for $(\overline{10})$, starting with a, or for $(\overline{01})$ when b is the starting element. Note that $a(\overline{10}) = b(\overline{01}) = (\bar{0})$. □

Given any sequence $\theta \in \{0,1\}^{\omega}$, where ω can be either finite or infinite, if $\theta \neq (\bar{0})$ let $m(\theta)$ be the smallest number j such that $\theta_j = 1$. Remind that two points in the same fiber are *leaf equivalent* if they belong to the same leaf L. The following lemma allow us to use the encoding of $\pi^{-1}(\alpha)$ into $\{0,1\}^{\mathbb{N}}$ to describe the leaf equivalence on $\pi^{-1}(\alpha)$. We say that two sequences θ and θ' in $\{0,1\}^{\mathbb{N}}$ are *cofinal equivalent* if there exist an N such that $\theta_n = \theta'_n$ for all $n \geq N$. Two points in $\pi^{-1}(\alpha)$ belong to the same leaf if and only if they belong to the same orbit of the Iterated Monodromy Group. The following lemma is an algebraic proof that every leaf L is dense on every fiber, and therefore every the closure of every leaf is the whole regular part.

Lemma 59. *The action of the holonomy for $z^2 - 1$ is transitive on cylinders.*

Proof. It is enough to check that any finite cylinder $\theta \in \{0,1\}^n$, can be taken to the cylinder $(\bar{0}) \in \{0,1\}^n$. If $m(\theta)$ is odd, then $a^{-\frac{m+1}{2}}(\theta)$ is a cylinder with the first $m(\theta)$ digits equal to 0. If, now $m(\theta)$ is even then $b^{-\frac{m}{2}}([\theta_1, \theta_2, ..., \theta_n])$ also increases $m(\theta)$. Repeating this procedure a finite number of times, at the end, we get by composition an element in the Iterated Monodromy Group of f_{-1} such that sends θ to $(\bar{0})$. □

Corollary 60. *With the exception of points in $L(\hat{\alpha}) \cap \pi^{-1}(\alpha)$, the leaf equivalence in $\pi^{-1}(\alpha)$ coincides with cofinal equivalence.*

Proof. By last remark in the proof of Lemma 58, the points $(\overline{01})$, $(\overline{10})$ and $(\bar{0})$ belong to the same leaf. Actually, the point $(\bar{0})$ corresponds to the point $\hat{\alpha}$ in

$\pi^{-1}(\alpha)$. Now, by definition, if two points two points $\theta, \theta' \in \{0,1\}^{\mathbb{N}}$ are cofinal, the coordinates of θ and θ' are different at most on a cylinder of size N. By Lemma 59, there is an element γ in the Iterated Monodromy Group of f_{-1} such that $\gamma(\theta) = \theta'$. Conversely, if two points θ and θ' belong to the same leaf, they are related by the Iterated Monodromy Group and if they are not, and do neither $(\overline{01})$, $(\overline{10})$ or $(\bar{0})$, then by Lemma 58 the sequences θ and θ'. □

Let $\theta \in \{0,1\}^{\mathbb{N}}$, let $e(\theta)$ the smallest number j such that $\theta_{2j} = 1$, we call $e(\theta)$ the *a-obstruction* of θ. Analogously, $o(\theta)$ is the smallest number j such that $\theta_{2j-1} = 1$ and we call $o(\theta)$ the *b-obstruction* of θ. If, in any case, no such 1 exist, we said that there is no obstruction of the kind.

For instance, $e(0100...) = 1$, and $o(0001101...) = 3$. Let $\theta \in \{0,1\}^{\mathbb{N}}$, the order of the orbit θ under the cyclic group $< a >$ is related to $e(\theta)$ in fact, θ is periodic under $< a >$ if and only if $e(\theta)$ is finite. Analogously for the orbit of θ under b. More precisely:

Lemma 61. *Let $\theta \in \{0,1\}^{\mathbb{N}}$, then*

i) the orbit of θ under $< a >$ has order $2^{e(\theta)}$, or infinity if there is no a-obstruction; and,

ii) the orbit of θ under $< b >$ has order $2^{o(\theta)}$, or infinity if there is no b-obstruction.

Proof. By Lemma 58, a acts only on the odd coordinates of θ, so $< a >$ will only change the odd coordinates of θ by the definition of b if there is 0 at the even coordinates then the action of a can go on. But if it finds a 1 it switches to the Identity and the automaton stops. As for the orbit of b, it behaves as a but in even places. $\square$

We could say that each group, $< a >$ and $< b >$, acts as an "alternating" Adding Machine on $\{0,1\}^{\mathbb{N}}$ with possible obstructions to the "carrying" one operation. If θ belongs to the leaf L in $\mathcal{R}_{-1}$ and the orbit of θ under a is finite, we have an *a-loop* attached to θ on $\mathcal{G}_{-1}(L)$, the size of this loop is precisely $2^{e(\theta)}$. Similarly, if θ has finite orbit under $< b >$ there is a *b-loop* attached to θ in $\mathcal{G}_{-1}(L)$.

Proposition 62. *Let L be a leaf in $\mathcal{R}_{-1}$, then the monodronomy graph $\mathcal{G}_{-1}(L)$ has either 4, 2 or 1 ends.*

Proof. By Lemma 58 and Lemma 61, the action of $< a >$ creates obstructions to the action of $< b >$ and viceversa. If a point θ has no obstruction, say of the kind a, then there an "infinite" loop attached to θ in $\mathcal{G}_{-1}(L)$ infinite loops create at least to ends of the monodromy graph. Again by Lemma 58, $L(\hat{\alpha})$ is the only leaf that contains a point, namely $(\bar{0})$, with infinite loops of both kinds attached to it. Thus $L(\hat{\alpha})$ has at least 4 ends. See Figure 6.5.

Given a finite loop, say an a-loop A, among the vertices of A there is a vertex with maximum b-obstruction, actually such vertex θ is the only vertex in A such that $e(\theta) < o(\theta)$. The b-loop attached to θ also contains a unique vertex with maximum a obstruction. This features creates an order in the way a-loops and b-loops are glued: For every loop C there is a unique vertex in C which has a loop attached of bigger size than C. Unless the size of C is already infinite. So, ends of $\mathcal{G}_{-1}(L)$ either go along infinite loops, or if there

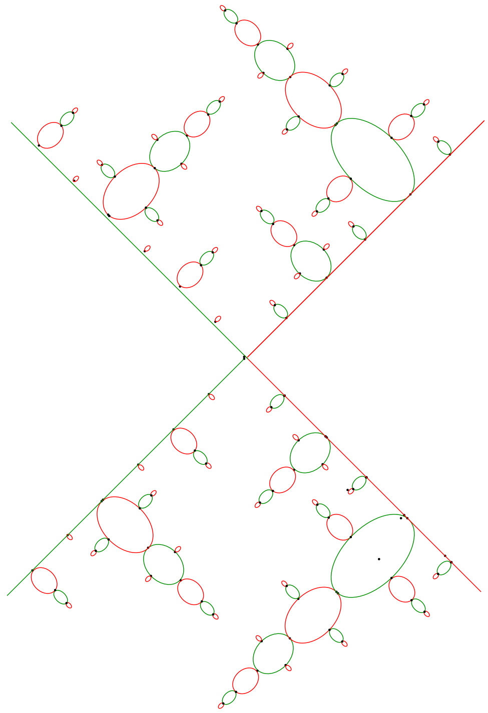


Figure 6.5: The Monodromy graph in the invariant leaf $L(\hat{\alpha})$.

is no infinite loops on the leaf, an end arise as a sequence of increasing loops, in this case there is only one end in $\mathcal{G}_{-1}(L)$. See Figure 6.6

If there is an infinite loop C, on every vertex of it there is attached a finite sequence of loops. So, if there is an infinite loop the number of ends is either 2 or 4, which only happens in the leaf $L(\hat{\alpha})$. See Figure 6.7. $\square$

Using Proposition 62 and Corollary 57, we obtain the same result of Proposition 15 for f_{-1}:

Lemma 63. *Given any leaf $L \in \mathcal{R}_{-1}$ the number of unbounded Fatou components of L is either 1, 2, or 4. Moreover, the number is 4 only in the case of the invariant leaf $L(\hat{\alpha})$.*

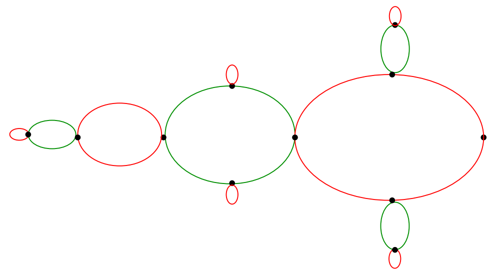


Figure 6.6: The monodromy graph, with one end, bigger and bigger loops are attached.

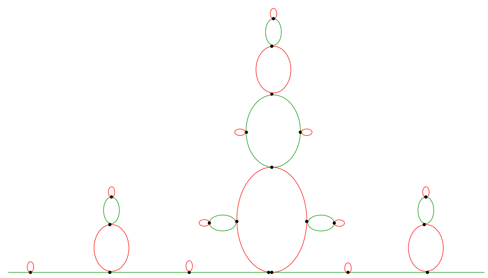


Figure 6.7: The Monodronomy graph with two ends.

6.3 Other parameters

In this section we discuss how to generalize the results of the previous section to other parameters. The easiest cases are *first bifurcation* superattracting parameters, which are centers of hyperbolic components attached to the Main Cardioid of the Mandelbrot set M. The point $c = -1$ is a first bifurcation superattracting parameter. First bifurcation parameters can be classified as the parameters whose Hubbard trees are star shaped with common vertex at the α fixed point.

We denote the first bifurcation parameter with critical cycle of size q and combinatorial rotation number p/q by $c_{\frac{p}{q}}$. Now, let us calculate the Iterated Monodromy group for $f_{c_{\frac{p}{q}}}$. The group $\pi_1(\mathbb{C}\setminus P_{c_{\frac{p}{q}}}, \alpha)$ is generated by q elements.

Let a be a simple loop around the critical value c, and let $b^i = f^i_{c_{\frac{p}{q}}}(a)$ for $i = 1, ..., q-1$. Hence, the loop b^{q-1} contains 0. These simple loops represent a set of generators of the fundamental group $\pi_1(\mathbb{C} \setminus P_{c_{\frac{p}{q}}}, \alpha)$. We assume that a, and therefore all b^i, are oriented counter clockwise.

By construction, $f^{-1}_{c_{\frac{p}{q}}}(a)$ consist of two paths connecting α with $-\alpha$ in both senses. So, the permutation associated to a is also the transition τ. On the other hand, $f^{-1}_{c_{\frac{p}{q}}}(b^i)$ are either loops based at α or at $-\alpha$. Thus, the corresponding transition maps for the actions of a and b^i on the tree $T(\{0,1\}^{\mathbb{N}})$ are

$$\begin{array}{ll} l_1^{-1} a_0 l_0 \approx Id & l_0^{-1} b_0^1 l_0 \approx a \\ l_0^{-1} a_1 l_1 \approx b^{q-1} & l_1^{-1} b_1^1 l_1 \approx Id \end{array}$$

and

$$l_0^{-1} b_0^i l_0 \approx b^{i-1} \quad l_1^{-1} b_1^i l_1 \approx Id$$

for $i = 2, ..., q-1$. So, the generators of the Iterated Monodromy group for $f_{c_{\frac{p}{q}}}$ are $a = (Id, b^{q-1})\tau$, $b^{q-1} = (a, Id)$ and $b^i = (b^{i-1}, Id)$ for $i = 2, ..., q-1$.

In terms of functions defined in $\{0,1\}^{\mathbb{N}}$, for $(\theta_1, \theta_2, ...) \in \{0,1\}^{\mathbb{N}}$, we have

$$a(\theta_1, \theta_2, \theta_3, ...) = \begin{cases} (1, Id(\theta_2, \theta_3, ...)) & \text{if } \theta_1 = 0 \\ (0, b^1(\theta_2, \theta_3, ...)) & \phantom{\text{if }} \theta_1 = 1 \end{cases},$$

for the critical loop:

$$b^1(\theta_1, \theta_2, \theta_3, ...) = \begin{cases} (0, a(\theta_2, \theta_3, ...)) & \text{if } \theta_1 = 0 \\ (1, Id(\theta_2, \theta_3, ...)) & \phantom{\text{if }} \theta_1 = 1 \end{cases},$$

and, finally

$$b^i(\theta_1, \theta_2, \theta_3, ...) = \begin{cases} (0, b^{i-1}(\theta_2, \theta_3, ...)) & \text{if } \theta_1 = 0 \\ (1, Id(\theta_2, \theta_3, ...)) & \theta_1 = 1 \end{cases}.$$

For $i = 2, ..., q-1$. The Moore diagram is given in Figure 6.8. All results from the previous section generalize to the Iterated Monodromy Group by just changing the discussion of even and odd coordinates of θ by positions modulo q. So, analogous to Proposition 62, we have:

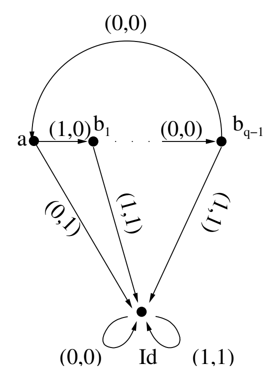


Figure 6.8: Moore diagram of the parameter $c_{\frac{p}{q}}$

Proposition 64. *Let L be a leaf in $\mathcal{R}_{c_{\frac{p}{q}}}$, then the holonomy graph $\mathcal{G}_{c_{\frac{p}{q}}}(L)$ either has:*

i) 2q ends.

ii) 2 ends.

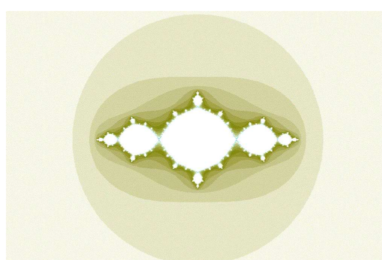
iii) 1 end.

Appendix A

Examples of Iterated Monodromy groups.

A.1 Period 2.

A.1.1 The basillic, c=-1



Internal address $1 \to 2$

Wake $1/3, 2/3$

Kneading Sequence $[1/3, \{down, (\bar{1})\}, \{up, (\overline{1,0})\}], [2/3, \{down, (\overline{1,0})\}, \{up, (\bar{1})\}]$

Binary $[1/3, (\overline{0,1})], [2/3, (\overline{1,0})]$

Holonomy:

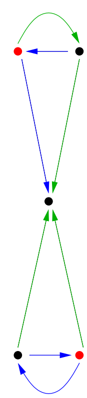
$$a_1 = (Id, a_2)\sigma$$

$$a_2 = (a_1, Id)$$

$$a_1^{-1} = (a_2^{-1}, Id)\sigma$$

$$a_2^{-1} = (a_1^{-1}, Id)$$

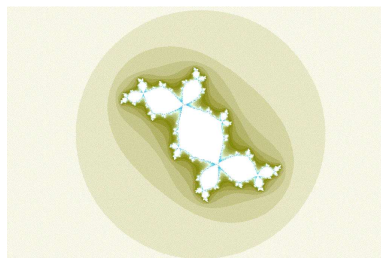
Moore Diagram



A.2 Period 3.

A.2.1 First bifurcation.

The rabbit, c=-.12256116687665361998+.74486176661974423660I



Internal address $1 \to 3$

Wake $1/7, 2/7$

Kneading Sequence $[1/7, \{down, (\bar{1})\}, \{up, (\overline{1, 1, 0})\}], [2/7, \{down, (\overline{1, 1, 0})\}, \{up, (\bar{1})\}]$

Binary $[1/3, (\overline{0, 0, 1})], [2/3, (\overline{0, 1, 0})]$

Holonomy:

$a_1 = (Id, a_3)\sigma$

$a_2 = (a_1, Id)$

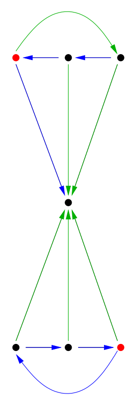
$a_3 = (a_2, Id)$

$a_1^{-1} = (a_2^{-1}, Id)\sigma$

$a_2^{-1} = (a_1^{-1}, Id)$

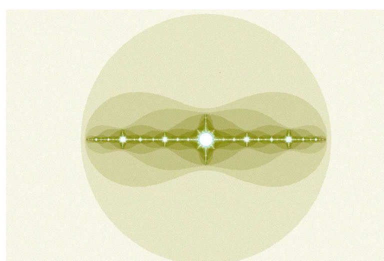
$a_3^{-1} = (a_2^{-1}, Id)$

Moore Diagram



A.2.2 Primitive.

The airplane, c=-1.7548776662466927601



Internal address $1 \to 2 \to 3$

Wake $3/7, 4/7$

Kneading Sequence $[3/7, \{down, (\overline{1,0,1})\}, \{up, (\overline{1,0,0})\}], [4/7, \{down, (\overline{1,0,0})\},$
$\{up, (\overline{1,0,1})\}]$

Binary $[3/7, (\overline{0,1,1})], [4/7, (\overline{1,0,0})]$

Holonomy:

$$a_1 = (Id, a_3)\sigma$$

$$a_2 = (a_1, Id)$$

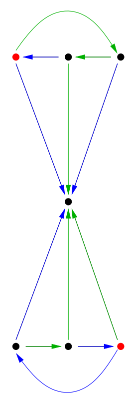
$$a_3 = (Id, a_2)$$

$$a_1^{-1} = (a_2^{-1}, Id)\sigma$$

$$a_2^{-1} = (a_1^{-1}, Id)$$

$$a_3^{-1} = (Id, a_2^{-1})$$

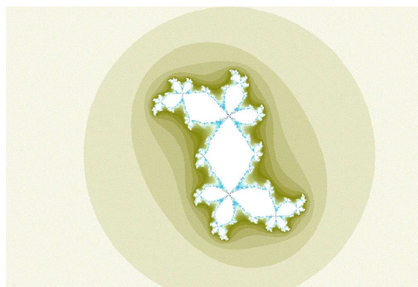
Moore Diagram



A.3 Period 4.

A.3.1 First bifurcation.

c=.28227139076691387970+.53006061757852529949I



Internal address $1 \to 4$

Wake $1/15, 2/15$

Kneading Sequence $[1/15, \{down, (\bar{1})\}, \{up, (\overline{1,1,1,0})\}], [2/15, \{down, (\overline{1,1,1,0})\},$

$\{up, (\bar{1})\}]$

Binary $[1/15, (\overline{0,0,0,1})], [2/15, (\overline{0,0,1,0})]$

Holonomy:

$a_1 = (Id, a_3)\sigma$

$a_2 = (a_1, Id)$

$a_3 = (a_2, Id)$

$a_4 = (a_3, Id)$

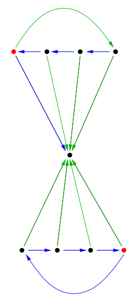
$a_1^{-1} = (a_2^{-1}, Id)\sigma$

$a_2^{-1} = (a_1^{-1}, Id)$

$a_3^{-1} = (a_2^{-1}, Id)$

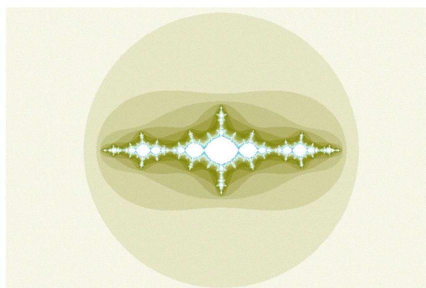
$a_4^{-1} = (a_3^{-1}, Id)$

Moore Diagram



A.3.2 Second bifurcation.

c=-1.31070264133368328836



Internal address $1 \to 2 \to 4$

Wake $2/5, 3/5$

Kneading Sequence $[2/5, \{down, (\overline{1,0})\}, \{up, (\overline{1,0,1,1})\}], [3/5, \{down, (\overline{1,0,1,1})\},$

$\{up, (\overline{1,0})\}]$

Binary $[2/5, (\overline{0,1,1,0})], [3/5, (\overline{1,0,0,1})]$

Holonomy:

$$a_1 = (a_2^{-1}, a_4a_2)\sigma \qquad a_4a_2 = (a_3a_1, Id)$$

$$a_2 = (a_1, Id) \qquad a_3a_1 = (Id, a_4a_2)\sigma$$

$$a_3 = (Id, a_2)$$

$$a_4 = (a_3, Id)$$

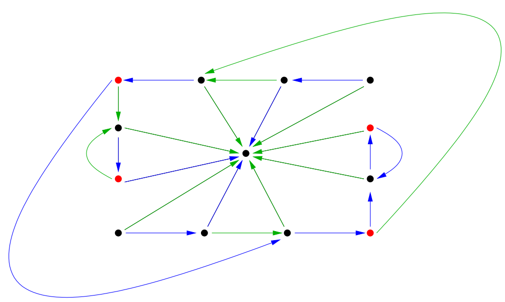
$$a_1^{-1} = ((a_4a_2)^{-1}, a_2)\sigma \qquad (a_4a_2)^{-1} = (a_3a_1, Id)$$

$$a_2^{-1} = (a_1^{-1}, Id) \qquad (a_3a_1)^{-1} = ((a_4a_2)^{-1}, Id)\sigma$$

$$a_3^{-1} = (Id, a_2^{-1})$$

$$a_4^{-1} = (a_3^{-1}, Id)$$

Moore Diagram :



A.4 Period 6.

A.4.1 First Bifurcation

c=.38900684056977123544+.21585065087081910777I

Internal address

$$1 \to 6$$

The holonomy

$a_1 = (Id, a_6)\sigma$

$a_2 = (a_1, Id)$

$a_3 = (a_2, Id)$

$a_4 = (a_3, Id)$

$a_5 = (a_4, Id)$

$a_6 = (a_5, Id)$

$a_1^{-1} = (a_6^{-1}, Id)\sigma$

$a_2^{-1} = (a_1^{-1}, Id)$

$a_3^{-1} = (a_2^{-1}, Id)$

$a_4^{-1} = (a_3^{-1}, Id)$

$a_5^{-1} = (a_4^{-1}, Id)$

$a_6^{-1} = (a_5^{-1}, Id)$

c=-1.4760146427284298975

Internal address $1 \to 2 \to 4 \to 6$

Wake $26/63, 37/63$

Kneading Sequence $[26/63, \{down, (\overline{1, 0, 1, 1, 1, 0})\}, \{up, (\overline{1, 0, 1, 1, 1, 1})\}],$

$[37/63, \{down, (\overline{1,0,1,1,1,1})\}, \{up, (\overline{1,0,1,1,1,0})\}]$

Binary $[26/63, (\overline{0,1,1,0,1,0})]$, $[37/63, (\overline{1,0,0,1,0,1})]$

Real Kneading Sequence L, R, L, L, L.

Real Combinatorics $x_1, x_5, x_3, x_4, x_0, x_2$. Holonomy

$$a_1 = (a_2, a_6 a_2)\sigma \qquad a_6 a_2 = (a_5 a_1, a_2)$$

$$a_2 = (a_1, Id) \qquad a_5 a_1 = (a_2, a_4 a_6 a_2)\sigma$$

$$a_3 = (Id, a_2) \qquad a_4 a_6 a_2 = (a_1 a_5 a_3, Id)$$

$$a_4 = (a_3, Id) \qquad a_1 a_5 a_3 = (Id, a_4 a_6 a_2)\sigma$$

$$a_5 = (a_4, Id)$$

$$a_6 = (a_5, Id)$$

$$a_1^{-1} = ((a_6 a_2)^{-1}, a_2^{-1})\sigma \qquad (a_6 a_2)^{-1} = ((a_5 a_1)^{-1}, Id)$$

$$a_2^{-1} = (a_1^{-1}, Id) \qquad (a_5 a_1)^{-1} = ((a_4 a_6 a_2)^{-1}, a_2^{-1})\sigma$$

$$a_3^{-1} = (Id, a_2^{-1}) \qquad (a_4 a_6 a_2)^{-1} = ((a_1 a_5 a_3)^{-1}, Id)$$

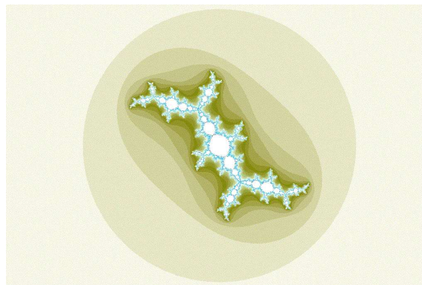
$$a_4^{-1} = (a_3^{-1}, Id) \qquad (a_1 a_5 a_3)^{-1} = ((a_4 a_6 a_2)^{-1}, Id)\sigma$$

$$a_5^{-1} = (a_4^{-1}, Id)$$

$$a_6^{-1} = (a_5^{-1}, Id)$$

A.4.2 Second Bifurcation

c=-.11341865594943657219+.86056947250157305512I



Internal address $1 \to 3 \to 6$

Wake $10/63, 17/63$

Kneading Sequence $[10/63, \{down, (\overline{1,1,0})\},$

$\{up, (\overline{1,1,0,1,1,1})\}], [17/63, \{down, (\overline{1,1,0,1,1,1})\}, \{up, (\overline{1,1,0})\}]$

Binary $[10/63, (\overline{0,0,1,0,1,0})], [17/63, (\overline{0,1,0,0,0,1})]$

Holonomy

$a_1 = (a_3^{-1}, a_6a_3)\sigma$ $\quad a_6a_3 = (a_5a_2, Id)$

$a_2 = (a_1, Id)$ $\quad a_5a_2 = (a_4a_1, Id)$

$a_3 = (a_2, Id)$ $\quad a_4a_1 = (Id, a_6a_3)\sigma$

$a_4 = (Id, a_3)$

$a_5 = (a_4, Id)$

$a_6 = (a_5, Id)$

Moore Diagram

$a_1^{-1} = ((a_6a_3)^{-1}, a_3)\sigma$ $\quad (a_6a_3)^{-1} = ((a_5a_2)^{-1}, Id)$

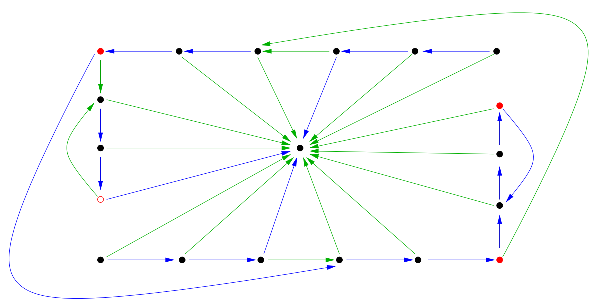
$a_2^{-1} = (a_1^{-1}, Id)$ $\quad (a_5a_2)^{-1} = ((a_4a_1)^{-1}, Id)$

$a_3^{-1} = (a_2^{-1}, Id)$ $\quad (a_4a_1)^{-1} = ((a_6a_3)^{-1}, Id)\sigma$

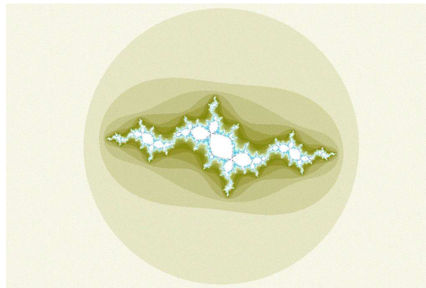
$a_4^{-1} = (Id, a_3^{-1})$

$a_5^{-1} = (a_4^{-1}, Id)$

$a_6^{-1} = (a_5^{-1}, Id)$



c=-1.13800066665096451l1+.24033240126209830169I



Internal address $1 \to 2 \to 6$

Wake $22/63, 25/63$

Kneading Sequence $[22/63, \{down, (\overline{1,0})\},$

$\{up, (\overline{1,0,1,0,1,1})\}], [25/63, \{down, (\overline{1,0,1,0,1,1})\}, \{up, (\overline{1,0})\}]$

Binary $[22/63, (\overline{0,1,0,1,1,0})], [25/63, (\overline{0,1,1,0,0,1})]$

Holonomy

$$a_1 = ((a_4a_2)^{-1}, a_6a_4a_2)\sigma \qquad a_4a_2 = (a_3a_1, Id)$$

$$a_2 = (a_1, Id) \qquad a_3a_1 = (a_4^{-1}, a_6a_4a_2)\sigma$$

$$a_3 = (a_2, Id) \qquad (a_4a_2)^{-1} = ((a_3a_1)^{-1}, Id)$$

$$a_4 = (a_3, Id) \qquad (a_3a_1)^{-1} = ((a_6a_4a_2)^{-1}, a_4)\sigma$$

$$a_5 = (a_4, Id)$$

$$a_6 = (a_5, Id)$$

$$a_1^{-1} = ((a_6a_4a_2)^{-1}, a_4a_2)\sigma \qquad a_6a_4a_2 = (a_5a_3a_1, Id)$$

$$a_2^{-1} = (a_1^{-1}, Id) \qquad a_5a_3a_1 = (Id, a_6a_4a_2)\sigma$$

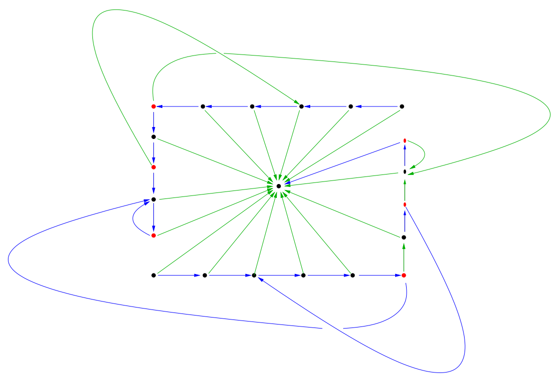
$$a_3^{-1} = (a_2^{-1}, Id) \qquad (a_6a_4a_2)^{-1} = ((a_5a_3a_1)^{-1}, Id)$$

$$a_4^{-1} = (a_3^{-1}, Id) \qquad (a_5a_3a_1)^{-1} = ((a_6a_4a_2)^{-1}, Id)\sigma$$

$$a_5^{-1} = (a_4^{-1}, Id)$$

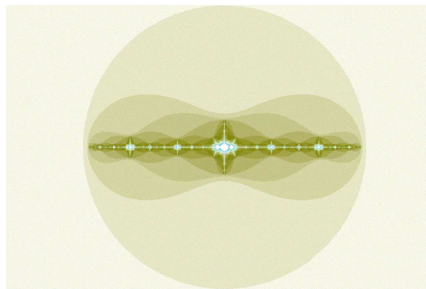
$$a_6^{-1} = (a_5^{-1}, Id)$$

Moore Diagram :



A.4.3 Primitive, first bifurcation.

c=-1.7728929033816237994



Internal address $1 \to 2 \to 3 \to 6$

Wake $4/9, 5/9$

Kneading Sequence $[4/9, \{down, (\overline{1,0,0})\},$

$\{up, (\overline{1,0,0,1,0,1})\}], [5/9, \{down, (\overline{1,0,0,1,0,1})\}, \{up, (\overline{1,0,0})\}]$

Binary $[4/9, (\overline{0,1,1,1,0,0})], [5/9, (\overline{1,0,0,0,1,1})]$

Holonomy

$a_1 = (a_3^{-1}, a_6a_3)\sigma$

$a_2 = (a_1, Id)$

$a_3 = (Id, a_2)$

$a_4 = (Id, a_3)$

$a_5 = (a_4, Id)$

$a_6 = (Id, a_5)$

$a_6a_3 = (Id, a_5a_2)$

$a_5a_2 = (a_4a_1, Id)$

$a_4a_1 = (a_6a_3, Id)\sigma$

$a_1^{-1} = ((a_6a_3)^{-1}, a_3)\sigma$

$a_2^{-1} = (a_1^{-1}, Id)$

$a_3^{-1} = (Id, a_2^{-1})$

$a_4^{-1} = (Id, a_3^{-1})$

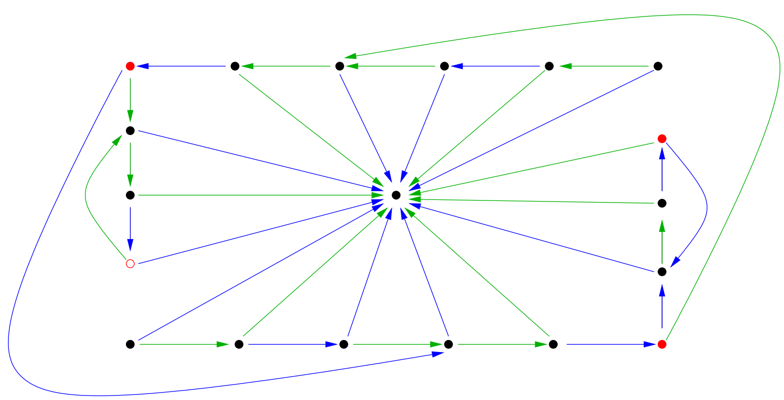
$a_5^{-1} = (a_4^{-1}, Id)$

$a_6^{-1} = (Id, a_5^{-1})$

$(a_6a_3)^{-1} = ((a_5a_2)^{-1}, Id)$

$(a_5a_2)^{-1} = ((a_4a_1)^{-1}, Id)$

$(a_4a_1)^{-1} = (Id, (a_6a_3)^{-1})\sigma$



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