THE COLLET-ECKMANN CONDITION FOR RATIONAL FUNCTIONS ON THE RIEMANN SPHERE

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Introduction

Let $R(z)$ be a rational function. Assume that there exist constants $C > 0$ and $\gamma > 0$ such that the following holds: For any critical point c, whose forward orbit does not contain any other critical point, we have

$$|(R^n)'(R(c))| \geq Ce^{\gamma n}, \text{ for all } n \geq 0. \tag{0.1}$$

Then we say that the map R is Collet-Eckmann (CE). The number $R(c)$ is often referred to as the *critical value*.

A parameterisation of all rational functions of a fixed degree $d \geq 2$, is given by

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + \ldots + a_d z^d}{b_0 + b_1 z + \ldots + b_d z^d}, \tag{0.2}$$

where $a_i, b_i \in \mathbb{C}$ for $0 \leq i \leq d$. We assume that $P(z) = a_0 + a_1 z + \ldots + a_d z^d$ and $Q(z) = b_0 + b_1 z + \ldots + b_d z^d$ do not have any common zero and at least one of a_d and b_d is non zero. Without loss of generality, we can assume that $b_d = 1$. The case $a_d = 1$ can be treated in exactly the same way. Therefore, the set of rational functions of degree d is a $(2d+1)$-dimensional complex manifold and a subspace of the projective space $\mathbb{P}^{2d+1}(\mathbb{C})$ with two charts corresponding to $a_d = 1$ and $b_d = 1$. We will prove the following.

Theorem A. *The set of Collet-Eckmann maps has positive Lebesgue measure in the parameter space of rational functions for any fixed degree $d \geq 2$.*

The measure is, of course, dependent on the chart which is used. However, the two measures corresponding to $a_d = 1$ and $b_d = 1$ are equivalent. There is also a coordinate independent measure on the projective space $\mathbb{P}^{2d+1}(\mathbb{C})$, induced by the Fubini-Study metric (see [20], pp. 30-31). The Fubini-Study measure and Lebesgue measure on the charts are mutually absolutely continuous.

It will be clear from the proof that the set obtained in Theorem A consists of functions which have the property that for *any* critical point c there is a $k > 0$ such that

$$|(R^n)'(R^k(c))| \geq Ce^{\gamma n}, \text{ for all } n \geq 0. \tag{0.3}$$

The existence of an absolutely continuous invariant probability measure (acip) for a positive measure set of rational functions was first proved by M. Rees in the famous paper [32]. Later we show, using results by Przytycki [30] and Graczyk and Smirnov [18], that the same holds for the set of CE-functions obtained in Theorem A.

0.1. **A brief historical background.** In the early days in complex dynamics, at the end of the 19th century, the local analysis of iteration of analytic functions was developed. A main issue was to describe the local behaviour around fixed points by conjugation. We say that a function $f : U \to U$ is (conformally) conjugate to $g : V \to V$ if there exists a conformal map $\varphi : U \to V$ such that

$$\varphi(f(z)) = g(\varphi(z)). \tag{0.4}$$

What one hopes for, is to find a simple function g conjugating f. Close to fixed points this is especially interesting, since fixed points and their derivatives are preserved under analytic conjugation; the dynamics is translated into another, hopefully easier, coordinate system.

The idea of conformal conjugation was introduced by Schröder in 1871 [35], and equation (0.4) is often referred to as the *Schröder functional equation*. Let z_0 be a fixed point, i.e. $f(z_0) = z_0$. One distinguishes between five types of fixed points:

- Superattracting: $f'(z_0) = 0$,
- Attracting: $0 < |f'(z_0)| < 1$,
- Rationally Neutral: $f'(z_0) = e^{2\pi i\theta}, \theta \in \mathbb{Q}$,
- Irrationally Neutral: $f'(z_0) = e^{2\pi i\theta}, \theta \notin \mathbb{Q}$,
- Repelling: $|f'(z_0)| > 1$.

In 1904 L. E. Böttcher [7] solved the Schröder equation in the superattracting case, more precisely, when

$$f(z) = z_0 + a_p(z - z_0)^p + \dots,$$

where $p \geq 2$, then f is conjugated to $g(z) = z^p$. In 1884 G. Koenigs [23] showed that if

$$f(z) = z_0 + \lambda(z - z_0) + \dots,$$

where $|\lambda| \neq 1, \lambda \neq 0$, then f is conjugated to the linear function $g(z) = \lambda z$. In this case we say that f is *linearisable*. The number λ is called the multiplier of f at z_0. The rationally neutral case was resolved by Fatou in [15] (and earlier by L. Leau, in [25]), where he shows the existence of flower shaped *petals* (around z_0), which are invariant curves for the dynamics. In 1942 C. L. Siegel [36] was the first to find a solution to the Schröder equation in the irrationally neutral case; if $\lambda = e^{2\pi i\theta}$, where θ is *Diophantine*, then f is conjugated to an irrational rotation with angle θ, i.e. $g(z) = e^{2\pi i\theta}z$. That θ is Diophantine means that θ is badly approximable by rational numbers; there are $c > 0$ and $\mu < \infty$ such that

$$\left|\theta - \frac{p}{q}\right| \geq \frac{c}{|q|^\mu},$$

for all integers p and $q \neq 0$. G. A. Pfeiffer discovered in 1917 [29], the first example of a polynomial $f(z) = z\lambda + \dots + z^d$, where $\lambda = e^{2\pi i\theta}$ such that f is not linearisable. H. Cremer [12] continued his work and showed in 1938 that if $|\lambda| = 1$ and $\liminf_{n\to\infty} |\lambda^n - 1|^{1/n} = 0$, then f is not linearisable. For the quadratic polynomial

$$Q(z) = e^{2\pi i\theta}z + z^2,$$

precise conditions for the existence of a linearisation are known. Let p_n/q_n be the rational numbers approximating θ according to the continued fraction expansion.

A conjugation of Q with the function $g(z) = e^{2\pi i\theta} z$ exists if and only if

$$\sum \frac{\log q_{n+1}}{q_n} < \infty. \tag{0.5}$$

The sufficiency of condition (0.5) was proved by A.D. Brjuno in 1965 [9], and the necessity was established by J-C. Yoccoz in 1988 [38]. In [40], Yoccoz gave a new proof of Brjuno's Theorem. In the second part of the same paper he considers the linearisability of the quadratic polynomial $P_\lambda(z) = \lambda z(1-z)$. The main result is that P_λ is linearisable if and only if every holomorphic function $f(z) = \lambda z + \mathcal{O}(z^2)$ is linearisable, that is, if the condition (0.5) is satisfied. It is also shown that if P_λ is not linearisable, then every neighbourhood of the origin contains periodic orbits.

In the beginning of the 20th century, great advances in global complex dynamics was made. A decisive step made by Fatou and Julia ([15], [16], [17] and [22]) was the decomposition of the Riemann sphere into a compact set $\mathcal{J}(f)$ and its complement $\mathcal{F}(f)$, consisting of those points z, for which there is a neighbourhood U of z such that $f^n\big|_U$ is a normal family (that is, there is a subsequence n_k such that $f^{n_k}\big|_K$ converges on compact subsets $K \subset U$ in the spherical metric). The set $\mathcal{F}(f)$ is called the Fatou set of f and its complement $\mathcal{J}(f)$, the Julia set. At the same time Fatou and Julia, with help of Montel's theorem, showed the important theorem that the Julia set is equal to the closure of the repelling periodic orbits, leading to the discovery of the dichotomy between these two sets: on the Julia set the dynamics is chaotic and on the Fatou set it is stable.

The understanding of the structure of the Fatou set was completed in 1985 by D. Sullivan [34], who proved that there can be no wandering domains. A wandering domain is a component U of $\mathcal{F}(f)$ such that $f^i(U) \cap f^j(U) = \emptyset$ whenever $i \neq j$. So, by Sullivan's Theorem, every component of the Fatou set is eventually periodic.

The chaotic behaviour on the Julia is strongly connected to the sensitive dependence on initial conditions, which means that two close points repel each other under iteration. Thus it is interesting to study the derivatives along orbits, especially critical orbits. This makes it natural to introduce Lyapunov exponents, hyperbolicity, Collet-Eckmann maps, invariant measures etc. A function is hyperbolic if every critical point is attracted to an attracting cycle. The famous Fatou conjecture states that the set of hyperbolic maps is open and dense in the space of rational functions. It is still unsolved. It is also a major conjecture that the Julia set of a rational function is either the whole Riemann sphere or has zero area. A little easier problem might be the conjecture that the set of rational maps with non-recurrent critical points and no attractive cycles, (so called Misiurewicz maps) has Lebesgue measure zero in the parameter space.

The existence of rational functions for which the Julia set is the whole Riemann sphere was shown in a nice manner by S. Lattés, [24]. Consider a lattice L in $\mathbb{C}$, consisting of all points $n_1 w_1 + n_2 w_2$, where $n_1, n_2 \in \mathbb{Z}$ and $w_1/w_2 \notin \mathbb{R}$. Since L is invariant under multiplication by an integer, any map $A(z) = \lambda z$ for $\lambda \in \mathbb{Z}$ induces an endomorphism of the complex torus $\mathbb{T}^2 = \mathbb{C}/L$. The Weierstrass $\mathfrak{P}$-function,

$$\mathfrak{P}(z) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

maps $\mathbb{T}^2$ onto $\hat{\mathbb{C}}$. It gives rise to a rational function f on $\hat{\mathbb{C}}$, which completes the commutative diagram below.

$$\begin{array}{ccc} \mathbb{C}/L & \xrightarrow{A} & \mathbb{C}/L \\ \mathfrak{p}\big\downarrow & & \big\downarrow\mathfrak{p} \\ \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \end{array}$$

If $A(z) = 2z$ and $w_1 = 1, w_2 = i$ then f in the above diagram is the rational function

$$f(z) = \frac{(z^2+1)^2}{4z(z^2-1)},$$

where f has its Julia set equal to the whole Riemann sphere. This function is Lattés example from 1918. In [6] J. Bernard studied the family

$$Q_{\alpha,\beta}(z) = \alpha(z + \frac{1}{z}) + \beta,$$

for (α,β) near $(-\frac{1}{2},\sqrt{2})$, which are perturbations of the Lattès example $Q_{-\frac{1}{2},\sqrt{2}}(z)$. He proves that $(\alpha,\beta) = (\frac{1}{2},\sqrt{2})$ is a Lebesgue density point for the set of points such that $Q_{\alpha,\beta}$ has an absolutely continuous invariant probability measure and positive Lyapunov exponent.

Complex dynamics became very popular when B. Mandelbrot made computer experiments, showing beautiful pictures of Julia sets. Even the most simple functions can exhibit complicated and beautiful Julia sets, for example the quadratic map $f_c(z) = z^2 + c$, giving rise to the so called Mandelbrot set, which consists of those parameters c for which $f_c^n(0) \not\to \infty$. The Mandelbrot set M looks very complicated. Also, an equivalent definition of M is the set of parameters c such that the corresponding Julia set J_c is connected. In 1982, A. Douady and J. Hubbard showed in [14] that the set M itself is connected. For c in the complement of M, the corresponding Julia set is a Cantor set. It is still an unsolved and famous problem whether the boundary of the Mandelbrot set is locally connected or not. Extensive work has been going on on this topic.

Let us give a quick glance on the quadratic family $f_a(x) = 1 - ax^2$, $0 < a \leq 2$, $-1 \leq x \leq 1$, and related works. The quadratic family is by now dynamically well understood. In the pioneering work [21] in 1981 by M. Jakobson, it is proved that for a positive measure set of parameters a, f_a admits an acip. M. Benedicks and L. Carleson [4] gave another proof of this in 1985, where they proved subexponential growth of the derivative at the critical value, for a set of parameters a of positive Lebesgue measure. This, in turn, implies the existence of an acip. At the same time, in 1986, M. Rees completed her well-known paper [32], which is a corresponding theorem for rational functions on the Riemann sphere. In [5], L.S. Young and M. Benedicks showed stability of the the acip:s for f_a, with respect to translation invariant small stochastic perturbations. Later, M. Benedicks and L. Carleson also proved that the CE-condition holds for a positive measure set of parameters a, and this was mainly developed for the purpose to complete their fundamental paper [4], where they show that for a set of parameters of positive Lebesgue measure, an attractor exists in families of Hénon maps. L.S Young and Q. Wang made an extensive work on the same subject in [37]. In [28], L. Mora and M. Viana extended the Benedicks-Carleson method to more general perturbations of the quadratic family and Hénon like mappings to prove a conjecture stated by J. Palis.

The famous real Fatou conjecture, states that the set of functions f_a with parameters a such that f_a has attracting periodic orbits, is open and dense in $(0,2)$. This problem was solved by Graczyk and Świątek in [19] and by Lyubich in [26]. Lyubich showed in [27] that the quadratic polynomials f_a fall into two categories: either f_a contains an attractive cycle or it admits an acip. Recently A. Avila and C. Moreira proved in [1] that the set A_{CE} of parameters fulfilling the CE-condition is almost the same as the set A admitting an acip, i.e. $m(A \setminus A_{CE}) = 0$ where m is the Lebesgue measure. D. Sands [34] showed that the set of parameters a for which $f_a(x) = 1 - ax^2$ is Misiurewicz, has measure zero. It is a natural conjecture that the same is true for the rational functions.

In [39], J-C. Yoccoz gave an extensive survey on the subject concernering the quadratic polynomial f_a, families of Hénon maps and rational functions on the Riemann sphere. He states a number of conjectures and finishes with a proof of Jakobson's Theorem on the existence of an acip for a positive measure set of parameters a.

As mentioned in the beginning, in 1986 M. Rees proved in [32] that there exists a positive measure set of ergodic rational functions, which admit an acip. This does not a priori imply that the CE-condition holds. That the Collet-Eckmann condition implies the existence of an acip follows rather easily from results of Przytycki, Smirnov and Graczyk.

Corollary 0.1. *The set of maps with an absolutely continuous invariant probability measure, has positive Lebesgue measure in the space of rational functions for any fixed degree $d \geq 2$. Moreover, the number of ergodic components of the invariant measure is bounded by $2d - 2$.*

To prove Corollary 0.1 we will use Theorem 1 in [18], which among other things states that a rational map, which satisfies the first Collet-Eckmann condition (defined below), can have neither Siegel discs, Herman rings, nor parabolic points (nor Cremer points).

Definition 0.2 (According to [18]). We say that a rational function f satisfies the *first Collet-Eckmann condition* (CE_1) if there are constants $C_1 > 0$ and $\lambda_1 > 1$ such that for any critical point c whose forward orbit does not contain any other critical point and belongs to or accumulates on the Julia set, the following holds:

$$|(f^n)'(fc)| \geq C_1 \lambda_1^n.$$

In this thesis, in Theorem A it is proved the existence of a positive Lebesgue measure set in the parameter space of CE-maps which also satisfies the first CE-condition. Moreover, the positive measure set obtained consists of functions, which have the property (0.3), that is, R cannot have attracting or superattracting periodic orbits. Hence, such functions have neither Siegel discs, Herman rings, parabolic cycles or (super-) attracting cycles. According to the Classification Theorem, the Fatou set is empty and we have proved the following.

Corollary 0.3. *The set of rational maps which have its Julia set equal to the whole Riemann sphere, has positive measure in the space of rational functions for any fixed degree* $d \geq 2$.

In [30] F. Przytycki showed that if the CE-condition and some additional assumptions are satisfied, then there exists an acip. Before stating the precise result we have to define conformal measure. An *α-conformal measure* μ on the Julia set J is a probability measure which has the property that for every Borel set $B \subset J$ on which f is injective we have

$$\mu(f(B)) = \int_B |f'|^\alpha d\mu.$$

The number α is called the exponent of the conformal measure. Let $\mathcal{C}(f)$ be the set of critical points for f and ν is the maximal multiplicity for f^n at $c \in \mathcal{C}(f)$. The precise statement by Przytycki is as follows (see also Theorem C in [30]):

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function and μ be an α-conformal measure on the Julia set $J = J(f)$ not having atoms at critical points. Assume that there are no parabolic periodic points in J and that there exists $C > 0$ such that for every $n \geq 1$ and for every $c \in \mathcal{C}(f)$

$$\int \frac{d\mu}{\text{dist}(x, f^n(c))^{(1-(1/\nu))\alpha}} < C^{-1}. \tag{0.6}$$

Moreover, assume that there exists $\Lambda > 1$ such that

$$|(f^n)'(f(c))| \geq C\Lambda^n, \text{ for all } n \geq 0, \tag{0.7}$$

whenever $f^k(c)$ is non-critical for all $k > 0$. Then there exists an f-invariant probability measure on J absolutely continuous with respect to μ.

Before we turn to the proof of Corollary 0.1, we say that a rational map f is *expanding* on its Julia set $\mathcal{J}(f)$, if there is a $t \geq 1$ such that $|(f^t)'(z)| > 1$ for all $z \in \mathcal{J}(f)$. An equivalent definition is that there is a metric $\varphi(z)$ smoothly equivalent to the spherical metric, such that

$$\varphi(f(z))|f'(z)| \geq A\varphi(z),$$

for all $z \in \mathcal{J}(f)$ and for some $A > 1$, where f' is the derivative in the spherical metric. Thus the Julia set of an expanding map contains no critical points.

Proof of Corollary 0.1. We have to prove that (0.6) is satisfied for our specific parameters which satisfy the CE-condition. By Corollary 0.3 the Julia set is equal to the whole Riemann sphere.

If $J = \hat{\mathbb{C}}$ then it is easy to construct a conformal measure. It is the standard spherical area measure with exponent $\alpha = 2$. It is easy to verify that the condition (0.6) holds independently of n. This finishes the proof of the first part of Corollary 0.1.

The last statement in Corollary 0.1 follows from Theorem B in F. Przytycki [30]: Let f be a rational map, not expanding on its Julia set, and let μ be an α-conformal measure on J. Assume that for any critical point c, whose forward orbit does not contain any other critical point, we have

$$\sum_{n=0}^{\infty} |(f^n)'(f(c))|^{-1} < \infty.$$

Then the number of ergodic components of μ does not exceed the number of f-critical points in J and of parabolic periodic orbits, (a parabolic cycle is the same as rationally neutral periodic point and an f-critical point is a critical point whose forward orbit does not contain any other critical point).

It is obvious that the functions obtained in Theorem A satisfy the conditions in the Theorem above, since our α-conformal measure is the standard area measure with exponent $\alpha = 2$. Now, by Graczyk and Smirnov [18], as seen above, there are no parabolic cycles. Since the number of critical points is bounded by $2d - 2$ the proof of the corollary is finished. □

Here one should mention the family $F_a(z) = 1 - a/z^2$. In [27], M. Lyubich showed that the Julia set for F_a is the whole Riemann sphere for a positive measure set of parameters a, and moreover, that for the same set of parameters, the critical orbit is dense in $\hat{\mathbb{C}}$.

Other relevant results can be found in [18] if the Fatou set is non empty; Graczyk and Smirnov proved that the CE-condition implies that Fatou components are Hölder domains. A Hölder domain is a simply connected set Ω such that the Riemann mapping $\varphi(z) : \mathbb{D} \to \Omega$ can be extended to a Hölder continuous function in the closed unit disc. Also, the boundary of Fatou components of CE-maps has Hausdorff dimension less than 2. If there is a fully invariant Fatou component, then its boundary coincides with the Julia set (this is a standard result in complex dynamics, see for example [11], p. 57). This fact applies to polynomials, since the basin of attraction at infinity is fully invariant. So the Hausdorff dimension of the Julia set for polynomial CE-maps is less than 2.

In [31] F. Przytycki showed that if a CE-map does not contain parabolic points and has its Julia set not equal to the whole Riemann sphere, then the Julia set has zero measure (see Theorem A in [31]).

In [10], S. van Strien and H. Bruin [10] showed that under the condition that $b_n|Df^n(fc)| \to \infty$ for some sequence b_n, satisfying $\sum_k b_k < 2$ (for f rational on $\hat{\mathbb{C}}$), then the derivative expands for almost every point $x \in \mathcal{J}(f)$ which does not lie in the precritical set (i.e. $\cup_n f^{-n}(Crit(f))$). Their Theorem 1.2 immediately implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 0,$$

for almost every $x \in \mathcal{J}(f)$ not laying in the precritical set for the CE-maps constructed in Theorem A.

0.2. **An outline of the proof of Theorem A.** Theorem A will be a corollary of Theorem B below, where we prove corresponding results for a real one-dimensional perturbation of $R(z)$ in any parameter direction. The starting function $R(z)$ is Misiurewicz-Thurston, which means that all critical points are mapped into a repelling periodic orbit. The perturbation is given as follows. Let $u_0 = (a_0, \ldots, a_d, b_0, \ldots, b_d)$ be the coefficients of the unperturbed function $R(z)$ above.

Define any normalised direction $v = (e_0, \ldots, e_d, f_0, \ldots, f_{d-1})$, where $\{e_i\}_{i=0}^{d}$ and $\{f_i\}_{i=0}^{d-1}$ are complex numbers such that $\sum_i |e_i|^2 + \sum_i |f_i|^2 = 1$. Note that we do not make any perturbation of the coefficient b_d. We make the following parameterisation in the direction v for real a:

$$R(z,a) = \frac{P(z,a)}{Q(z,a)} = \frac{(a_0 + e_0 a) + \ldots + (a_{d-1} + e_{d-1}a)z^{d-1} + (a_d + e_d a)z^d}{(b_0 + f_0 a) + \ldots + (b_{d-1} + f_{d-1}a)z^{d-1} + z^d}. \tag{0.8}$$

Thus $R(z,0)$ is the unperturbed function. This parameterisation can be viewed as a real line in $\mathbb{C}^{2d+1} = \mathbb{R}^{4d+2}$. For each parameter a we get a point in $\mathbb{R}^{4d+2}$, i.e we have a map

$$\mathbb{R} \ni a \mapsto u_0 + va = u \in \mathcal{R}_d \subset \mathbb{R}^{4d+2}, \tag{0.9}$$

where $\mathcal{R}_d$ is the space of all rational functions of degree d. The following theorem is the main result of this thesis. Theorem A then follows by Fubini's Theorem.

Theorem B. *For any direction v define $R(z,a)$ as in (0.8). Assume that all critical points for $R(z,0)$ are strictly preperiodic. Then the set of parameters a for which $R(z,a)$ is Collet-Eckmann, has positive Lebesgue measure.*

The approach to prove Theorem B is the method used by Benedicks-Carleson (see [3] and [4]), where they prove corresponding results for the quadratic family $f_a(x) = 1 - ax^2$ and families of Hénon maps. We will start with a given rational function R with all critical points strictly preperiodic; in particular the postcritical set is finite. These maps cannot have attracting periodic orbits, since the basin of attraction for an attractive cycle always contains a critical point, which would imply that the postcritical set is infinite. Rotation domains (Siegel discs and Herman rings) are ruled out by the fact that the boundary of a rotation domain is contained in the postcritical set (see eg. [11], p. 82). So each critical point must end up in a repelling orbit. Obviously such functions satisfy the CE-condition. Also, R can have no superattracting periodic points, and we conclude that the Julia set of R is the whole Riemann sphere. We then show that in a neighbourhood of R there is a set of parameters of positive measure such that the corresponding functions also satisfy the CE-condition. We allow recurrence of the critical points according to the following approach rate condition:

$$\mathrm{dist}(R^n(c), \mathcal{C}(R)) \geq e^{-\alpha n}, \text{ for all } n \geq 0, \tag{0.10}$$

for some small $\alpha > 0$, and all $c \in \mathcal{C}(R)$, where $\mathcal{C}(R)$ is the set of critical points for R. We will consider a small parameter interval $\omega_0 \subset [0, a_0]$, for some small $a_0 > 0$, and iterate the corresponding functions simultaneously. To get control of the geometry we use strong distortion estimates of the derivatives, not only for the absolute value, but also for the argument. An important result (Proposition 7.3) is that as long as the derivative grows exponentially with at least some "lower exponent" γ_0, the geometry is under good control. Then induction will be used to show the existence of a set of positive measure such that (0.10) holds. Only deleting parameters not satisfying (0.10) will slowly lower the expansion of the derivative, i.e. the exponent γ in (0.1) will tend to zero. Therefore we will use induction over time intervals of the type $(n, 2n)$. First, we assume that at time n, (0.1) holds for some $\overline{\gamma} > 0$. After that we delete parameters not satisfying (0.10) in the interval

$(n, 2n)$. We then end up with a set for which the derivative grows with exponent $\gamma_0 = \overline{\gamma}/2$. To restore $\overline{\gamma}$, a large deviation argument of the type developed in [4] will be adopted to get positive Lyapunov exponent, i.e. $\overline{\gamma} > 0$ in (0.1). To handle the problem with (finitely) many critical points, we use a method described in [2]. This is explained in the last section. The idea is roughly that we consider parameter intervals which are "good" for a single critical point up to time n and "good" for the other critical points up to time $\alpha_0 n$ (where α_0 is a small fraction), so that we may use the binding information of the other critical points. To continue a single critical orbit, we must delete parameters so that the binding information of the other critical points can be used longer. The main idea is that the sets deleted in this way differ very much in size compared to the partition elements of the actual single critical orbit considered. Therefore we delete only whole partition elements and in this way overcome the problem of destroying the partitions.

In [32], an example of a rational Misiurewicz-Thurston-map of degree $d \geq 2$ is given by

$$f_\lambda(z) = \lambda \frac{(z-2)^d}{z^d}, \tag{0.11}$$

where 2 and 0 are the critical points which are iterated as below:

$$2 \mapsto 0 \mapsto \infty \mapsto \lambda.$$

Here λ is a fixed point, if $\lambda = 2/(1-t)$, where $t^d = 1$. Also, $f_\lambda'(\lambda) = dt^{d-1}(1-t)$, so $f_\lambda(z)$ above is Collet-Eckmann, if $d|1-t| > 1$, which is true if $t \neq 1$, $t^d = 1$.

1. Notations and definitions

1.1. The spherical metric. The spherical distance between z and w on $\hat{\mathbb{C}}$ is defined by

$$|z-w|_\sigma = d_\sigma(z,w) = \inf_\gamma \int_\gamma \frac{|dt|}{1+|t|^2},$$

where the infimum is taken over all (continuous) curves γ from z to w. It is natural to define the spherical derivative of a holomorphic function f to be

$$D_\sigma f(z) = \frac{\partial_\sigma f(z)}{\partial z} = f'(z)\frac{1+|z|^2}{1+|f(z)|^2},$$

which coincides with expansion of f measured in the spherical metric. Note that the spherical derivative satisfies the Chain Rule! Also, define the spherical derivative w.r.t. the parameter a as

$$\frac{\partial_\sigma f(z,a)}{\partial a} = \frac{\partial f(z,a)}{\partial a}\frac{1}{1+|f(z,a)|^2}.$$

With these definitions we have, since the Riemann sphere is compact, a bound on both the z-derivative and the a-derivative: there exist constants B and Γ such that

$$\left|\frac{\partial_\sigma R(z,a)}{\partial a}\right| \leq B, \tag{1.1}$$

$$\left|\frac{\partial_\sigma R(z,a)}{\partial z}\right| \leq e^\Gamma, \tag{1.2}$$

for all $z \in \hat{\mathbb{C}}$. It follows that $D_\sigma R(z) = 0$ if and only if z is a critical point.

In this thesis most constants will be denoted C. These constants can differ from time to time. For convention C means just "a constant", so for example equalities like $2C = C$ can appear.

Let $\mathcal{C}(f)$ be the set of critical points for a function f and define the *postcritical set* $P = P(f)$ of f to be the closure of the union of all strict forward orbits of the critical points:

$$P(f) = \overline{\bigcup_{c \in \mathcal{C}(f), n>0} f^n(c)}.$$

Now, let $R(z) = P(z)/Q(z)$ be our starting rational CE-map, with strictly preperiodic critical points. The degree d of R is defined by $d = \max(\deg(P), \deg(Q))$. Write

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + \ldots + a_d z^d}{b_0 + b_1 z + \ldots + b_d z^d},$$

as in (0.2), where $b_d = 1$. The procedure which will be used is to iterate different parameters simultaneously. The set of parameters iterated will be a real parameterised line as in (0.8). We sometimes use the notation $R(z,a) = R_a(z)$.

Denote by c_i the critical points of R and let $k_i > 0$ be the smallest integer such that

$$R^{k_i}(c_i) = v_i, \tag{1.3}$$

where v_i lies on a repelling periodic orbit.

1.2. **Acceleration of R.** To make computations easier, we will replace R with an appropriate iterate R^n such that the repelling cycles become repelling fixed points. In addition, we can choose n such that $k_i = 1, 2$ for all indices i in (1.3).

If we can prove that the CE-condition holds for R^n, clearly it also holds for R. However, the coefficient b_d for the accelerated function may not be 1 anymore and the coefficients $a_i + e_i a$ and $b_i + f_i a$ will not be linear for the accelerated function R^n. They will look like $a_i + e_i(a)$ and $b_i + f_i(a)$ instead, where $e_i(a)$ and $f_i(a)$ are polynomials in a. The coefficient $f_d(a)$ is not necessarily zero anymore.

Definition 1.1 (Redefinition of $R(z,a)$). Let $c_i(a)$ be the critical points of the perturbed accelerated function R_a^n and let $c_i = c_i(0)$. We will construct neighbourhoods U_i' (see Definition 1.2) around each c_i and for $z \in U_i'$ we redefine R_a such that $R_a(z)$ always means $R_a^{nk_i}(z)$, where $k_i = 1, 2$. For $z \notin U' = \cup_i U_i'$, by $R_a(z)$ we always mean $R_a^n(z)$. Thus, with this definition, $R(c_i(0), 0)$ is a repelling fixed point for all i.

Define

$$v_l(a) = R(c_l(a), a).$$

With this in mind we may drop the constant C in (0.1) and prove that there is a constant $\gamma > 0$ such that

$$|(R^n)'(v_l(a), a)| \geq e^{\gamma n}, \text{ for all } n \geq 0 \text{ and all } l,$$

for a positive measure set of parameters a.

Let d_i be the degree of R at c_i and define $K = \max(d_i)$. Choose one critical point $c_l(a)$, and define

$$\xi_{n,l}(a) = R^n(c_l(a), a). \tag{1.4}$$

We shall often drop the index l in (1.4) to make notations simpler. By $\xi_n(a)$ we mean $\xi_{n,l}(a)$ for some critical point $c_l(a) \in \mathcal{C}(R_a)$.

Note that the newly defined R is of course not meromorphic on the boundary of those U_i' which has that $k_i = 2$. Here we make the following convention, namely that whenever $\xi_n(\omega)$ crosses $\partial U_i'$ we define $\xi_{n+1}(a) = R^{k_i}(\xi_n(a), a)$.

1.3. **Splitting of the critical points. Reparameterisation.** If the multiplicity of a critical point c_i is higher than one, it may split into several critical points under perturbation. Assume that every c_i is split into the critical points $c_{ij}(a)$, $1 \leq j \leq N_i$, under the perturbation (0.8). The critical points for R_a in a neighbourhood of $c_i \in \mathcal{C}(R)$ are the zeros of $R'(z, a) = 0$. Hence the numerator $N(z, a)$ of $R'(z, a)$ (in local coordinates, see Subsection 1.4), which is a polynomial in z and a, has to satisfy

$$N(z, a) = 0.$$

Assume without loss of generality that $(z, a) = (0, 0)$ is a solution. Let $F(z, a)$ be an irreducible factor of $N(z, a)$. We now use the standard theory of the structure of zero sets of irreducible polynomials, see eg. Theorem 1 in [8], p. 386. The solutions set to $F(z, a) = 0$ is parameterised as below:

$$\begin{aligned} z &= g(t), \\ a &= t^m, \end{aligned}$$

where $g(t)$ is analytic, for t in some neighbourhood U of 0. The number $m \geq 0$ is order of $N(z, a)$ in the variable z. Since g is analytic, we have a convergent power series

$$z = g(t) = \sum_{j=0}^{\infty} c_j t^j.$$

Now, taking any branch $t = a^{1/m}$ we get

$$z = g(a^{1/m}) = \sum_{j=0}^{\infty} c_j a^{j/m}, \tag{1.5}$$

where $a^{j/m}$ is to be interpreted as $(a^{1/m})^j$. The series (1.5) is a so called Puiseux expansion of the curve with equation $F(z, a) = 0$.

For two different irreducible factors we get two corresponding solutions sets z_1 and z_2 as in (1.5);

$$\begin{aligned} z_1(a) &= \sum_{j=0}^{\infty} c_j a^{j/m}, \\ z_2(a) &= \sum_{j=0}^{\infty} c_j a^{j/l}. \end{aligned}$$

It follows that either the two series are equal or

$$z_1(a) - z_2(a) = ca^{p/q} + \mathcal{O}(|a|^{(p+1)/q}),$$

for some positive integers p and q, and $c \neq 0$.

Reparameterising the line $a^{1/m}$ for positive a with $\gamma(a) = ae^{i\theta}$ for $\theta = \arg(a^{1/m})$, we see that $g(ae^{i\theta})$ is an analytic curve in a. Thus, for a suitable reparameterisation of the perturbation $R(z, a)$, replacing a with a^n for some suitable n, it follows that all curves $c_{ij}(a)$ are analytic and of constant multiplicity for $0 < |a| \leq a_0$, for sufficiently small $a_0 > 0$. The sum of the multiplicities of $c_{ij}(a)$ is the same as the multiplicity of c_i (of course). Define

$$\mathcal{C}_i(a) = \bigcup_{j=1}^{N_i} c_{ij}(a). \tag{1.6}$$

It follows from above that the set $\mathcal{C}_i(a)$ is a union of solution sets called *critical stars*, each one corresponding to an irreducible factor of $N(z, a)$, (see Figure 1).

PSfrag replacements

c_i

$c_{ij}(a_0)$

$c_{ik}(a_0)$

FIGURE 1. The critical points emerging from c_i under perturbation.

Convention 1. Sometimes we shall drop the index i or ij, writing only $c(a)$ for some $c(a) \in \mathcal{C}(R_a)$. Also, recall that the points $c_i(a)$ are the critical points of R_a.

1.4. **Local coordinates.** We will now introduce the notion of local coordinates on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The reason for this is that we want to view $\xi_n(a)$ for $a \in \omega$ as a curve in a coordinate system where the metric is equivalent to the spherical metric. Here ω is a parameter interval in $[0, a_0]$. We now introduce a number S such that if the spherical length of $\xi_n(\omega)$ reaches S then the curve is immediately cut into pieces with equal lengths at most S, where S is a number to be defined later. We also cut the parameter interval ω correspondingly. Later we define how the partition is carried out in more detail. To ensure that we can use one coordinate system for every whole curve $\xi_n(\omega)$, for any partition element ω, the charts have to overlap each other. To do this, cover the Riemann sphere with the two charts $\mathcal{U}_1 = (z, B_1)$ and $\mathcal{U}_2 = (z^{-1}, B_2)$ where $B_1 = \{z : |z| < \tilde{\rho} + \tilde{S}\}$ and $B_2 = \{z : |z| > \tilde{\rho} - \tilde{S}\} \cup \{\infty\}$, for some suitable $\tilde{S} > 0$ chosen such that $\xi_n(\omega)$ cannot cross both $B(0, \tilde{\rho} + \tilde{S})$ and $B(0, \tilde{\rho} - \tilde{S})$, provided the spherical length of $\xi_n(\omega)$ is less than S. Note that the spherical length is comparable to the length in the local coordinate system where the curve segment $\xi_n(\omega)$ lies. We let $\varphi : \hat{\mathbb{C}} \to \mathbb{C}$, be the local coordinates so that $\varphi(z) = \tilde{z}$, (so $\tilde{z}$ is z viewed in local coodinates). If $z \in B_1^c$ then $\varphi(z) = 1/z$ and if $z \in B_2^c$ then $\varphi(z) = z$. In the overlapping set

$B_1 \cap B_2$, there is an ambiguity of how to define φ. However, we will let the curve $\xi_n(\omega)$ define φ here (see below). Also, we associate to each $\tilde{z} = \tilde{z}(z)$ an "indicator" telling in what chart z lies, so that φ can be inverted. Define

$$\varphi(\xi_n(a)) = \tilde{\xi}_n(a) = \begin{cases} \xi_n(a) & \text{if } \xi_n(\omega) \cap B_1^c = \emptyset, \\ \xi_n(a)^{-1} & \text{if } \xi_n(\omega) \cap B_1^c \neq \emptyset. \end{cases}$$

For $\tilde{z} = \tilde{\xi}_n(a)$, where $a \in \omega$ for some interval $\omega \subset [0, a_0]$, define

$$\widetilde{R}(\tilde{z}, a) = \widetilde{R}_a(\tilde{z}) = \varphi \circ R_a \circ \varphi^{-1}(\tilde{z}).$$

This gives the recursion formula

$$\tilde{\xi}_n(a) = \widetilde{R}(\tilde{\xi}_{n-1}(a), a).$$

Thus $\widetilde{R}$ is R viewed in local coordinates, so $\widetilde{R}$ can (locally) be viewed as an analytic function from $\mathbb{C}$ into itself. It is easy to verify that $\tilde{R}'$ satisfies the Chain Rule. We use the same chart for an entire partition element ω, i.e. for all $a, b \in \omega$ we consider

$$\tilde{R}(\tilde{\xi}_n(a), b) = \varphi \circ R_b \circ \varphi^{-1}(\tilde{\xi}_n(a)),$$

where the coordinates φ and φ^{-1} are constant for all $a, b \in \omega$. The curve $\tilde{\xi}_{n+1}(a)$, $a \in \omega$ determines the coordinates uniquely.

It follows that $\widetilde{R}'$ is equivalent to the spherical derivative, that is, $\widetilde{R}'(\tilde{z}, a) = 0$ if and only if $\tilde{z}$ is a critical point. Also, $\partial_a \widetilde{R}(\tilde{z}, a)$ is equivalent to the spherical a-derivative.

Convention 2. Since we only deal with spherical derivatives or equivalent derivatives, from now on R' and R'' always mean the first and second derivative respectively in local coordinates and $|z - w|$ always means the distance in the proper coordinate system, unless otherwise stated. We also assume that the bounds (1.1) and (1.2) are valid also when we refer to R' or $\partial_a R$. Obviously, there is also a bound on R'' and $\partial_a R'$ by the same reasoning. Choose the number $\tilde{\rho}$ above such that

$$\operatorname{dist}(\mathcal{C}(R), \{|z| = \tilde{\rho}\}) \geq 10\rho,$$

where ρ will be defined precisely in Subsection 1.7. The definition will ensure that if the curve $\xi_n(\omega)$ with length at most S crosses the circle of radius $\tilde{\rho}$, then the curve is far from a critical point.

We define neighbourhoods of the postcritical set $P = P(R)$ by

$$P_{\delta^2} = \bigcup_{i=1}^{N} R(B(c_i, \delta^2)), \quad P_\delta = \bigcup_{i=1}^{N} R(B(c_i, \delta)) \text{ and } P_{\delta'} = \bigcup_{i=1}^{N} R(B(c_i, \delta')),$$

where R is the accelerated function as in Definition 1.1. So P_η is a neighbourhood of the repelling fixed points for R, $\eta = \delta^2, \delta, \delta'$.

With some for the whole thesis fixed $0 < \tau < 1$, we determine fixed constants $\alpha, \beta, \gamma_0, \overline{\gamma}$ with the following relations (remember that $K = \max(d_i)$):

$$\begin{aligned} &\tau < \log\lambda/(24K + \log\lambda) \qquad \overline{\gamma} = (1-\tau)\log\lambda \qquad \gamma_0 = \overline{\gamma}/2 \qquad \underline{\gamma} = \overline{\gamma}/4 \\ &\alpha \leq \min(\underline{\gamma}/(1000K^2), \underline{\gamma}^2/(4000K^2\Gamma)), \qquad \beta = 5K\alpha. \end{aligned} \tag{1.7}$$

The number λ in (1.7) is the minimum of the expansion in Lemma 3.6, and all multipliers $\mu = \inf_{a\in[0,a_0]} = |R'(p(a),a)|$, where $p(a)$ is a repelling fixed point. The quantity

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log |(R^n)'(v(a),a)|$$

is usually referred to as the *Lyapunov exponent* at $v(a)$, and we will show that $\gamma \geq (1-\tau)\log\lambda - K\alpha > 0$ in (0.1), for a positive measure set of parameter a in an interval $\omega_0 \subset [0,a_0]$. Hence Theorem A implies that the Lyapunov exponent is positive for a positive measure set of parameters. We will in the inductive proof of Theorem B need to have (0.1) satisfied with exponent $\underline{\gamma}$.

Now we make some definitions about neighbourhoods of the critical points.

Definition 1.2 (Critical neighbourhoods). For integers $\Delta' < \Delta$ set $\delta' = e^{-\Delta'}, \delta = e^{-\Delta}$. Let $a_0 > 0$. Let D_i be the diameter of $\cup_{a\in[0,a_0]}\mathcal{C}_i(a)$ and let $\tilde{\delta} = 10\max(D_i)$. Choose δ such that $\delta^2 \gg \tilde{\delta}$ and $\tilde{r}$ such that $\tilde{\delta} = e^{-\tilde{r}}$. Define

$$U_i' = B(c_i,\delta'), \quad U_i = B(c_i,\delta), \quad U_i^2 = B(c_i,\delta^2), \quad \tilde{U}_i = B(c_i,\tilde{\delta}),$$

and put

$$U = \cup_i U_i, \quad U' = \cup_i U_i', \quad U^2 = \cup_i U_i^2, \quad \tilde{U} = \cup_i \tilde{U}_i.$$

Thus the parameter dependence is very small compared to the size of U, U' and U^2. The neighbourhood $\tilde{U}$ should be thought of as a border around the critical stars $\mathcal{C}_i(a)$ such that outside $\tilde{U}$ the distance to the critical points $\mathcal{C}_i(a)$ are almost equal and we view them a single one, and inside $\tilde{U}$ we have to deal with each critical point in $\mathcal{C}_i(a)$ separately.

Definition 1.3 (Annular neighbourhoods). For any integer $r \geq \Delta'$ and $a \in \omega_0 \subset [0,a_0]$ we make the following definitions. If $\tilde{r} \geq r \geq \Delta'$ define

$$J_r^i = \{z : e^{-r} \leq |z-c_i| < e^{-r+1}\}.$$

For $r = \tilde{r}+1$ define

$$J_r^i(a) = \{z : e^{-r} \leq \inf_j |z - c_{ij}(a)|, |z-c_i| < e^{-r+1}\}.$$

For $r > \tilde{r}+1$ define

$$J_r^i(a) = \{z : e^{-r} \leq \inf_j |z-c_{ij}(a)| < e^{-r+1}\}.$$

For simplicity, we will often drop the explicit parameter dependence and simply write $J_r^i = J_r^i(a)$ even if $r > \tilde{r}$. Define $J_r = \cup_i J_r^i$ in any such case.

Assume that $A(x)$ and $B(x)$ are two expressions depending on some variable x. By $A \sim B$ we mean that there is a (universal) constant C not depending on x, such that

$$C^{-1}B(x) \leq A(x) \leq CB(x),$$

for all x considered.

The classical definition of bound period, as in [3] and [4], is as follows:

Definition 1.4 (Bound period). Assume $z \in J_r(a) \subset U'$. Choose $c(a) \in \mathcal{C}(R_a)$ nearest to z and put $R_a(z) = z_0$. Define $p = p(r, a)$ to be the greatest integer such that

$$|R^j(z_0, a) - R^j(v(a), a)| \le e^{-\beta j},$$

for $j = 0, \ldots, p-1$, where $v(a) = R(c(a), a)$.

Given a return $z = \xi_n(a)$, the number p is called the *bound period* for the actual return.

1.5. **Free orbits, deep and shallow returns.** After the bound period the free orbit begins and stops when $\xi_{j,l}(a)$ returns to U. This return is either essential or inessential depending on the length of the curve $\xi_{j,l}(\omega)$, under the assumption that the geometry is "good".

Given a parameter a, we define the free return times of every single critical orbit $\xi_i(a) = \xi_{i,l}(a)$ where $1 \le l \le 2d-2$, to be $\nu_0(a), \nu_1(a), \ldots$ and its corresponding bound periods $p_0(a), p_1(a), \ldots$.

Definition 1.5. If $\xi_{\nu_i}(a)$ is a free return, and p_i its bound period, then $\xi_j(a)$ for $j = p_i + 1, \ldots, \nu_{i+1}$ is the called the *free orbit*. The length of the free orbits are denoted by $q_i(a) = \nu_{i+1}(a) - (\nu_i(a) + p_i(a))$.

Definition 1.6. If $\xi_n(a)$ returns into $U \setminus U^2$ then we speak of a *shallow return* and if $\xi_n(a)$ returns into U^2 we speak of a *deep return*, with corresponding bound orbits as above. The length of the free orbit following a deep return until the next deep return is denoted by $\mu(a)$. For example, if $\nu_j(a)$ is the first deep return after a deep return $\nu_i(a)$, then $\mu_i(a) = \nu_j(a) - (\nu_i(a) + p_i(a))$.

1.6. **The partition of parameter intervals.** In Subsection 5.2, we show that if $\xi_{n,l}(\omega)$ returns into U, where ω is an interval in $\omega_0 = [k_0 a_0, a_0]$, for some $k_0 < 1$ close to 1 (defined in Subsection 5.1), then the length of $\xi_{n,l}(\omega)$ is exponentially much larger than the length of $c_i(\omega)$. This means that the set $J_r(a)$ depend insignificantly on the parameter a. In the following, we will have this in mind.

In the definitions below, we assume that the curves $\xi_{n,l}(\omega)$ satisfy the following condition where n is a return time, i.e. $\xi_{n,l}(\omega) \subset U$:

Definition 1.7 (Good geometry control). For an interval $\omega \subset [0, a_0]$, we say that $\omega \in \mathcal{G}_{n,l}$ if

$$\left| \frac{\xi'_{n,l}(a)}{\xi'_{n,l}(b)} - 1 \right| \le 1/100, \text{ for all } a, b \in \omega. \tag{1.8}$$

The above condition implies that the curve $\xi_{n,l}(t)$, $t \in \omega$, is indeed very straight, and $\mathcal{G}_{n,l}$ stands for "good" geometry control.

Each critical orbit $\xi_{n,l}(\omega)$ gives rise to its own partition. We consider one critical point at a time. For simplicity, let us drop the index l. Also, in the following we assume that $\omega \subset \omega_0$.

The partition for free returns $\xi_n(\omega)$ is carried out as follows, (no partition is made during bound periods). Let $\omega = [a, b]$ and assume that $\xi_n(\omega) \subset U$. We have that $\xi_n(a) \subset J_{r_0}(a)$ for some $r_0 \ge \Delta$. Take the smallest $a_1 > a$ such that $\xi_n(a_1) \in J_{r_1}(a_1)$ where $|r_1 - r_0| = 1$ (if there is no smallest such a_1, take the infimum instead). Cut ω at a_1. Take the smallest $a_2 > a_1$ such that $\xi_n(a_2) \in J_{r_2}(a_2)$ where $|r_2 - r_1| = 1$. Cut ω at a_2 and so on. Either the sequence a_k terminates at $a_n = b$ for some n or not. If not then it means that $\xi_n(a') = c_i(a')$ for some $a' \in \omega$. In this case we get an infinite sequence a_k terminating at a'. If $a' = b$ then do nothing more. There are at most finitely many parameters a'_k such that $\xi_n(a'_k) = c_i(a'_k)$. In all the intervals (a'_k, a'_{k+1}) do the same procedure to determine a partition of (a'_k, a'_{k+1}). This gives a partition of ω into disjoint intervals ω_k such that $\xi_n(\omega_k) \subset J_{r_k}(a)$ for all $a \in \omega_k$.

There are some special cases here, namely that the curve $\xi_n(a_k)$ crosses the boundary $J_r(a_k)$ for many consequtive a_k for some fixed r (i.e. it is almost a tangent to $J_r(a_k)$). For instance, there is a sequence ω_k such that $\xi_n(\omega_k)$ has length less than $e^{-r_k}/(2r_k^2)$, where $r_k = r$. If this is the case adjoin the set ω_k to an adjacent set $\omega_{k\pm 1}$ until the length of $\xi_n(\omega_k \cup \omega_{k+1} \cup \ldots \cup \omega_l)$ exceeds $e^{-r}/(2r^2)$ and view the curve $\xi_n(\omega_k \cup \omega_{k+1} \cup \ldots \cup \omega_l)$ as a return into $J_r(a'')$, where a'' is the midpoint of $\omega_k \cup \omega_{k+1} \cup \ldots \cup \omega_l$.

Now, we are ready to refine our partition. For simplicity, we drop the parameter dependence for the sets $J_r(a)$ and write instead only J_r.

Definition 1.8. A free return $\xi_n(\omega) \subset J_r$ is *essential* if the length of the curve is larger than or equal to e^{-r}/r^2. Otherwise it is *inessential*.

Definition 1.9 (Generic partition of ω). Assume that $\xi_n(\omega) \subset J_r$ is an essential (free) return for some $r \geq \Delta$. Cut the curve $\xi_n(\omega)$ into smaller curves of equal length at most e^{-r}/r^2, and as close to e^{-r}/r^2 as possible. Cut the parameter interval ω according to the partition of $\xi_n(\omega)$. For inessential free returns and bound returns make no partition.

This defines a partition of ω corresponding to the orbit of the critical point $c_i(\omega)$ for free returns $\xi_{n,l}(\omega) \subset U$, under the assumption that the curvature is small. We will show in Section 7 that this is the fact for "good" returns. In particular, the length of $\xi_{n,l}(\omega) \subset J_r$, after partitioning ω, is between $e^{-r}/(2r^2)$ and e^{-r}/r^2.

Now, we simply divide the parameter interval ω according to the partition of $\xi_{n,l}(\omega)$ and then continue to iterate these smaller intervals. This procedure implies that the partition becomes finer as the iterate n grows. We say that $\xi_{n,l}(a)$ and $\xi_{n,l}(b)$ have the same history if $\xi_{j,l}(a)$ and $\xi_{j,l}(b)$ belongs to the same undivided curve for $j \leq n$.

We may now also speak of bound periods for a whole parameter interval $\omega \subset [0, a_0]$. Let A and B be two sets and define

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|$$

and let the Hausdorff distance between A and B be

$$\text{HD-dist}(A, B) = \max(d(A, B), d(B, A)).$$

Definition 1.10 (Bound period for essential returns). Let $\xi_{n,l}(\omega) \subset J_r$ be an essential return. Choose k such that $\text{HD-dist}(c_k(\omega), \xi_{n,l}(\omega))$ is minimal. The bound period $p = p(r, \omega)$ is defined to be the greatest integer p such that

$$|R^j(z, a) - R^j(w, b)| \leq e^{-\beta j}$$

for $j = 0, \ldots, p-1$, for all $a, b \in \omega$, and for all $z, w \in \mathcal{K}(\xi_{n+1,l}(\omega) \cup v_k(\omega))$, where $\mathcal{K}(E)$ is the convex hull of E and $v_k(a) = R(c_k(a), a)$.

In the case of inessential returns, we will use the notion of *host curve* from [4]:

Definition 1.11 (Bound period for inessential returns). If $\xi_{n,l}(\omega) \subset J_r$ is an inessential return, it means that $\xi_{n,l}(\omega)$ is contained in a larger curve which has length between $e^{-r}/(2r^2)$ and e^{-r}/r^2. Also, it is possible to enlarge $\xi_n(\omega)$ in a way so that it is is almost straight, i.e. such that (1.8) is satisfied. To do this, draw a straight line L starting at $\xi_{n,l}(b)$ and going through $\xi_{n,l}(a)$, so that the length of L is equal to $e^{-r}/(2r^2)$. This divides L into two parts L_1 and L_2, where L_1 has $\xi_{n,l}(a)$ and $\xi_{n,l}(b)$ as its endpoints. The curve $\overline{\xi}_n(\omega) = \xi_{n,l}(\omega) \cup L_2$ is called the *host curve* for $\xi_{n,l}(\omega)$. Define $\overline{\xi}_{n+1,l}(\omega) = \xi_{n+1,l}(\omega) \cup R_a(L_2)$. The bound period for $\xi_{n,l}(\omega)$ is then defined to the greatest integer p such that

$$|R^j(z,a) - R^j(w,b)| \leq e^{-\beta j}$$

for $j = 0, \ldots, p-1$, for all $a, b \in \omega$, and for all $z, w \in \mathcal{K}(\overline{\xi}_{n+1,l}(\omega) \cup v_k(\omega))$.

Remark 1.12. We shall write $\mathcal{K}(\overline{\xi}_{n+1,l}, v_k(\omega))$ and never $\mathcal{K}(\xi_{n+1,l}, v_k(\omega))$ in the Definitions 1.10 and 1.11, that is, with $\overline{\xi}_{n+1,l}(\omega)$, we mean the host curve in the case of inessential returns, or the original curve in the case of essential returns.

We also need an additional condition on the partition elements, namely that

$$|\xi_{k,l}(a) - \xi_{k,l}(b)| \leq S, \tag{1.9}$$

for all $a, b \in \omega$, all $k \leq n$, where $S = \delta/C_2$. The only exception from this rule is the very first iterates and during bound periods, (see Section 5 and Subsection 5.1). This, since during bound periods we have still very good distortion estimates (see the Main Distortion Lemma). We determine the number $C_2 \geq 1$ in the proof of the Main Distortion Lemma (there is also a condition in Lemma 4.3 on C_2, not depending on δ). It will be clear that C_2 depends on the unperturbed function R_0 and not on δ. Therefore, we may choose δ so that $S/e^\Gamma \geq \delta/\Delta^2 \gg \delta^2$, where $|R'(z,a)| \leq e^\Gamma$ for all $(z,a) \subset \hat{\mathbb{C}} \times [0, a_0]$. During free periods, we have to cut the curve $\xi_{n,l}(\omega)$ *before* it exceeds S in length to have control of the geometry. To achieve this, we cut the curve, just before it exceeds S in equal number of pieces and make a corresponding partition of the interval ω. If the length of $\xi_n(\omega)$ has exceeded S during the bound period, then when the bound period has ended we immediately cut the curve into equal pieces, such that (1.9) holds again for every partition element.

Definition 1.13 (The partition). We say that an interval $\omega \in \mathcal{P}_{n,l}$ if a and b has the same history up to iterate n for all $a, b \in \omega$ and such that $\omega \in \mathcal{G}_{k,l}$ ("good geometry", see Definition 1.7) for all return times $k \leq n$ and such that the length in local coordinates of $\xi_{k,l}(\omega)$ is at most S for all $k \leq n$, where k belongs to a free time period or a return time.

In fact, we show inductively (Proposition 7.3) that the geometry is under good control at all times as long as the derivative of R^n grows exponentially with at least γ_0 in (0.1). Also, since each critical orbit gives rise to its own partition we have to be careful. However, in most lemmas we consider only one orbit at a time and assume that we can use the binding information of all the others. In Section 9 we give an inductive method to handle this in detail.

1.7. **The choice of δ'. Linearisation.** The definitions of d_i and v_i mean that we have a Taylor expansion around c_i which looks like

$$R(z) = v_i + t_i(z-c_i)^{d_i} + \mathcal{O}(|z-c_i|^{d_i+1}).$$

We will now choose δ' according to the rule that the Taylor expansion of R at c_i is valid up to a multiplicative constant say $1/100$ in $U' = \cup_i B(c_i, \delta')$, i.e.

$$\frac{99}{100} \leq \frac{|R(z)|}{|T_i(z)|} \leq \frac{101}{100},$$

if $|z-c_i| \leq \delta'$, where $T_i(z)$ is the first order Taylor expansion of R at c_i. We will always assume that $\delta < \delta'$. We need more conditions on δ'. First, define a so called "expansive" neighbourhood $\mathcal{N}_\rho$ of the union of the repelling fixed points v_i as

$$\mathcal{N}_\rho = \bigcup_i \{z : |z-v_i| \leq \rho\}$$

where $\rho = (\delta')^{\frac{\beta}{\Gamma}}$ and such that $|R'(z,a)| \geq |\lambda_i| - \varepsilon$ for all $(z,a) \in \mathcal{N}_\rho \times \omega$, where $\varepsilon \leq (|\lambda_i|-1)/1000$. Assume now that δ' is chosen according to the rule above. A return into $U' \setminus U$ is called a *pseudo return.*

Moreover, in the neighbourhood $\mathcal{N}_\rho$ the Schröder functional equation shall be valid; i.e. that there exists conformal maps $\varphi_t : \mathcal{N}_\rho \to \mathcal{N}'_\rho$, where $\mathcal{N}'_\rho$ is a neighbourhood of $\varphi_t(p(t))$, such that

$$g_t \circ \varphi_t(z) = \varphi_t \circ R_t(z), \tag{1.10}$$

where $g_t(z) = \lambda_t(z - \varphi_t(p(t))) + \varphi_t(p(t))$, and $\lambda_t = R'_t(p(t))$. Also, $\varphi_t(z)$ is analytic in t which can be shown by identifying coefficients in the power series on each side of (1.10) and noting that $|\lambda_t| \geq |\lambda| > 1$ for all $t \in [0, a_0]$, (see eg. [11], pp. 31-32). We may normalise $\varphi_t(z)$ so as to fulfil $\varphi'_t(p(t)) = 1$ and $\varphi_t(p(t)) = p(t)$. So $g_t(w)$ becomes

$$g_t(w) = \lambda_t(w - p(t)) + p(t).$$

1.8. **Exponential growth of derivatives, the basic assumption and the free assumption.** We need to show Theorem B for all critical points simultaneously, for even if we choose one critical point, we need to use the expanding orbits during the bound periods for the other critical points. To do this we make the following setup where $\alpha_0 = 2K\alpha/\gamma_0$.

Definition 1.14 (Exponential growth of derivatives). Let $A_n(\gamma, l)$ be the set of parameters a such that

$$|(R^k)'(v_l(a), a)| \geq e^{(\gamma - K\alpha)k} \text{ for all } k \leq n \text{ and}$$

$$|(R^k)'(v_l(a), a)| \geq e^{\gamma k} \text{ for all } k \text{ such that } \nu_j + p_j \leq k \leq \min(\nu_{j+1}, n),$$

where $\nu_j = \nu_j(a)$ are the returns times for a and p_j the corresponding bound periods. Recall $v_l(a) = R(c_l(a), a)$. Define

$$\mathcal{E}_n(\gamma, l) = A_n(\gamma, l) \bigcap \Bigg(\bigcap_{k \neq l} A_{\alpha_0 n}(\gamma, k) \Bigg).$$

In the the following definition we give the important approach rate condition, called the basic assumption.

Definition 1.15 (Basic assumption). Let $\mathcal{B}'_{n,l}$ be the set of parameters such that

$$|\xi_{k,l}(a) - c(a)| \geq e^{-\alpha k}, \quad \text{for all } k \text{ such that } 0 < k \leq n, \tag{1.11}$$

for all critical points $c(a) \in \mathcal{C}(R_a)$. Similar to the above definition, let

$$\mathcal{B}_{n,l} = \mathcal{B}'_{n,l} \bigcap \Big(\bigcap_{k \neq l} \mathcal{B}'_{\alpha_0 n,k} \Big).$$

This condition $\mathcal{B}_{n,l}$ or $\mathcal{B}'_{n,l}$ is called the *basic assumption*, and we will prove inductively that the set $\cap_{n,l}\mathcal{B}_{n,l}$ has positive measure.

Also, to get positive Lyapunov exponent, it is crucial that the frequency of the deep returns is bounded from below. The following definition reflects this.

Definition 1.16 (Free assumption). Set $F_{n,l}(a) = \sum_{i=0}^{s} \mu_i(a)$, where $\mu_i(a)$ are the deep free periods, for some critical point $c_l(a)$. Let $\mathcal{F}_{n,l}$ be the set of parameters a such that

$$F_{k,l}(a) \geq k(1-\tau), \quad 0 \leq k \leq n, \tag{1.12}$$

where $0 < \tau < 1$ is the constant in (1.7).

To be able to use induction in the proof of Theorem B, we will need to delete parameters from the sets $\mathcal{E}_n(\gamma, l)$ and $\mathcal{B}_{n,l}$ respectively, so that, for a single critical orbit, we may use the binding information from the other critical orbits for a long time. For this purpose, we make the following definition:

Definition 1.17. Define

$$\mathcal{B}_{n,l,*} = \mathcal{B}'_{n,l} \bigcap \Big(\bigcap_{k \neq l} \mathcal{B}'_{2\alpha_0,k} \Big), \text{ and}$$

$$\mathcal{E}_n(\gamma, l, *) = A_n(\gamma, l) \bigcap \Big(\bigcap_{k \neq l} A_{2\alpha_0}(\gamma, k) \Big).$$

The choice of α_0 comes from Lemma 6.3, where by the basic assumption $p \leq d_i r/\gamma \leq K\alpha n/\gamma$ if n was a return time. Thus $\mathcal{E}_n(\gamma, l)$ and $\mathcal{B}_{n,l}$ are good parameters for the orbit $\xi_{n,l}(a)$ and we can use the binding information of all other critical points in the next step, if n was a return time. In particular, the fact that $a \in \cap_{n,l}\mathcal{E}_n(\gamma, l)$ means that R_a is Collet-Eckmann, if $\gamma > 0$.

Convention 3. To make notations simpler, by $\mathcal{E}_n(\gamma)$, we mean $\mathcal{E}_n(\gamma, l)$ for some critical point $c_l(a)$. Also, by $\mathcal{P}_n$, $\mathcal{F}_n$, $\mathcal{B}_n$ and $\mathcal{B}'_n$, we mean $\mathcal{P}_{n,l}$, $\mathcal{F}_{n,l}$, $\mathcal{B}_{n,l}$ and $\mathcal{B}'_{n,l}$ respectively, for some index l. In the statements of all sublemmas, lemmas and propositions, we keep the indices, but in the proofs we shall drop the indices.

2. Some lemmas

We begin with the following lemma.

Lemma 2.1. *Let $u_n \in \mathbb{C}$ be complex numbers for $1 \le n \le N$. Then*

$$\left|\prod_{n=1}^{N}(1+u_n)-1\right| \le \exp\left(\sum_{n=1}^{N}|u_n|\right)-1. \tag{2.1}$$

Proof. We will prove (2.1) by induction. The case $N=1$ is trivial. Let

$$a_N=\prod_{i=1}^{N}(1+u_n) \quad \text{and} \quad b_N=\exp\sum_{n=1}^{N}|u_n|.$$

Note that $1+x\le e^x$, for all $x\in\mathbb{R}$, so $|a_n|\le b_n$. Now, assume that (2.1) holds for $N=m$, i.e. $|a_m-1|\le b_m-1$. Then

$$\begin{aligned}|a_{m+1}-1| &= |a_m(1+u_{m+1})-1| \le |a_m-1|+|u_{m+1}||a_m| \\ &\le b_m-1+|u_{m+1}||a_m| \le (1+|u_{m+1}|)b_m-1 \le b_{m+1}-1.\end{aligned}$$

□

The following lemma deals with the behaviour of R'_a near the critical points. Recall that we consider R_a in local coordinates. In a neighbourhood of c_i, the first and second derivatives of R_a can be written

$$R'_a(z)=\varphi(z)\prod_{k=1}^{d_i-1}(z-c_k(a)), \tag{2.2}$$

and

$$R''_a(z)=\varphi(z)\sum_{j=1}^{d_i-1}\prod_{k\neq j}(z-c_k(a))+\varphi'(z)\prod_{j=1}^{d_i-1}(z-c_j(a)),$$

where $1/A\le|\varphi(z)|\le A$ for $z\in U'_i$, for some $A>0$.

Lemma 2.2. *Assume that $z\in U'_i$. Let $c(a)\in\mathcal{C}_i(a)$ be the nearest critical point to z and put $v(a)=R_a(c(a))$. Then*

$$|R(z,a)-v(a)|\le C_1|z-c(a)||R'(z,a)|, \tag{2.3}$$

for some $C_1>0$, depending only on δ'. Moreover,

$$C_1^{-1}|z-c(a)|^{\tilde{d}_i-1}\le|R'(z,a)|\le C_1|z-c(a)|^{\tilde{d}_i-1} \tag{2.4}$$

and

$$|R''(z,a)|\le C_1|z-c(a)|^{\tilde{d}_i-2}, \tag{2.5}$$

where $2\le\tilde{d}_i=\tilde{d}_i(z)\le d_i$.

Proof. Define $z(t) = c(a) + t(z - c(a))$, $0 \le t \le 1$. Since $c(a)$ is the nearest critical point to z, we have $|z(t) - c_j(a)| \le 2|z - c_j(a)|$ for all other critical points $c_j(a) \ne c(a)$, where $c_j(a) \in \mathcal{C}_i(a)$. Thus,

$$\begin{aligned}|R(z,a) - R(c(a),a)| &\le |z - c(a)||R'(z(t),a)| \\ &= |z - c(a)||\varphi(z(t))| \prod_j |z(t) - c_j(a)| \\ &\le |z - c(a)| 2^{d_i - 1} A \prod_j |z - c_j(a)| \le C|z - c(a)||R'(z,a)|,\end{aligned}$$

which proves (2.3).

To prove (2.4), since $c(a)$ is nearest to z we can define $\tilde{d}_i$ so as to fulfil

$$\prod_k |z - c_k(a)| = |z - c(a)|^{\tilde{d}_i - 1}.$$

It it obvious that $2 \le \tilde{d}_i \le d_i$, which proves (2.4), since $1/A \le |\varphi(z)| \le A$. To show (2.5), again assume that z is nearest to $c(a)$. Then we get

$$\begin{aligned}|z - c(a)||R_a''(z)| &\le C|z - c(a)| \left[\sum_{j=1}^{d_i - 1} \prod_{k \ne j} |z - c_k(a)| + \prod_{j=1}^{d_i - 1} |z - c_j(a)| \right] \\ &\le C|z - c(a)|^{\tilde{d}_i - 1},\end{aligned}$$

for some constant $C > 0$, which proves (2.5). Now take C_1 as the largest constant obtained for (2.3), (2.4) and (2.5). The lemma is proved. □

Remark 2.3 (Generic degree). In particular, $|R_a'(z)| \sim \prod_j e^{-r_j}$, where $|z - c_j(a)| = e^{-r_j}$ whenever $z \in U$. If $z = \xi_n(a)$ is a return into J_r^i (or J_r, $J_{r,r'}^i$ etc), then by Definition 1.3 and Lemma 2.2 we get

$$|R_a'(z)| \sim e^{-(\tilde{d}_i - 1)r},$$

where $2 \le \tilde{d}_i \le d_i$. This fact will be used frequently in the rest of this thesis. Of course, $\tilde{d}_i$ depends on z. However, we let $\tilde{d}_i(z)$ be constant on each partition element.

The following lemma shows that we can use the expansion of the first bound steps of the critical value if a (large) iterate returns into a neighbourhood of a critical point. There is an analogue to this in [4].

Lemma 2.4 (Bound period distortion)**.** *Assume that $z \in J_r^i(a)$ and $a \in \mathcal{B}'_{p,k}$. Let $c_k(a) \in \mathcal{C}(R_a)$ be nearest to z. Put $z_0 = R_a(z)$. Then, for $0 \le j \le p - 1$,*

$$\left| \frac{(R^j)'(z_0,a)}{(R^j)'(v_k(a),a)} - 1 \right| \le \varepsilon_0, \tag{2.6}$$

where p is the bound period, and $v_k(a) = R_a(c_k(a))$. The number $\varepsilon_0 \to 0$ as $\delta \to 0$.

In particular, there is a constant C_0 close to 1 such that, during the bound period we have

$$C_0^{-1} \le \frac{|(R_a^j)'(z_0)|}{|(R_a^j)'(v_k(a))|} \le C_0.$$

Proof. We use the notation $R(z,a) = R_a(z)$. Set $w_0 = v(a) = R_a(c(a))$, $z_n = R_a^n(z_0)$ and $w_n = R_a^n(w_0)$. By Lemma 2.1 and the Chain Rule, (2.6) follows if

$$\sum_{j=0}^{p-1}\left|\frac{R_a'(z_j)}{R_a'(w_j)} - 1\right| = \sum_{j=0}^{p-1}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|} \leq \log(1+\varepsilon_0),$$

independently of p. Since $z \in J_r^i(a)$ we have $|z - c_k(a)| \sim e^{-r}$. By Lemma 2.2, $|z_0 - w_0| \leq C_1 e^{-\tilde{d}_i r}$. Now define $J = \tilde{d}_i r/10(K\alpha + \Gamma)$, where Γ is as in (1.2) and $K = \max(d_i)$. We divide the sum into two parts:

$$\sum_{j=0}^{p-1}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|} \leq \sum_{j=0}^{J}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|} + \sum_{j=J+1}^{p-1}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|}.$$

In the first sum, we use that $|z_j - w_j| \leq e^{\Gamma j}|z_0 - w_0|$. Moreover, $|R_a'(z_j) - R_a'(w_j)| \leq C|z_j - w_j| \leq Ce^{\Gamma j}|z_0 - w_0|$, since $|R''(z)|$ is bounded. The definition of bound period together with (1.11) gives an estimate to the first sum, namely

$$\begin{aligned}\sum_{j=0}^{J}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|} &\leq C\sum_{j=0}^{J} e^{(K-1)\alpha j}e^{\Gamma j}|z_0 - w_0| \\ &\leq C\sum_{j=0}^{J} e^{(\alpha(K-1)+\Gamma)j - \tilde{d}_i r} \leq Ce^{-\frac{9}{10}\tilde{d}_i r} \leq Ce^{-\frac{9}{10}\Delta}.\end{aligned}$$

In the second sum we use that $|z_j - w_j| \leq e^{-\beta j}$:

$$\sum_{j=J}^{p-1}\frac{|R_a'(z_j) - R_a'(w_j)|}{|R_a'(w_j)|} \leq \sum_{j=J}^{p-1} Ce^{-(\beta-(K-1)\alpha)j} \leq \sum_{j=J}^{\infty} Ce^{-\alpha K j} \leq Ce^{-\frac{9}{10}\Delta}.$$

The lemma is proved. □

The following corollary shows that we have very good distortion estimates of $(R_a^j)'$ for fixed a, during the bound period. In fact, Lemma 2.4 implies that R_a^j, $j \leq p-1$ is almost affine on the convex set $\mathcal{K}(\overline{\xi}_{n+1}(\omega), v_k(\omega))$ (see below), for a *fixed* parameter a.

Corollary 2.5. *Assume that $\xi_{n,l}(\omega) \subset J_r$, $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{B}_{n,l}$ and let k be chosen such that HD-dist$(\xi_{n,l}(\omega), c_k(\omega))$ is minimal. Take any $z_0, w_0 \in \mathcal{K}(\overline{\xi}_{n+1,l}(\omega), v_k(\omega))$ and any $a \in \omega$. Put $w_0 = v_k(a) = R_a(c_k(a))$, $z_j = R_a^j(z_0)$ and $w_j = R_a^j(w_0)$. Then*

$$C_0'^{-1}|z_j - w_j| \leq |z_0 - w_0||(R_a^j)'(u)| \leq C_0'|z_j - w_j|, \tag{2.7}$$

whenever $j \leq p-1$, $u \in \gamma(t) = z_0 + t(w_0 - z_0)$.

Proof. We write

$$z_j - w_j = (z_0 - w_0)\int_0^1 (R_a^j)'(z_0 + t(w_0 - z_0))dt,$$

where the integral is the mean value of the derivatives of R_a^j. Note that the line $\gamma(t) = z_0 + t(w_0 - z_0)$ is contained in $\mathcal{K}(\overline{\xi}_{n+1}(\omega) \cup v_k(\omega))$, which means that the orbit $R_a^j(\gamma(t))$ of any point on $\gamma(t)$ is bound to the orbit w_j, $j \leq p-1$. By Lemma 2.4, $(R_a^j)'$ changes arbitrarily little on $\gamma(t)$ and we conclude that (2.7) holds.

□

Remark 2.6. The number C_0' in Corollary 2.5 is easily seen to be close to 1. Therefore, from now on, we replace C_0' with C_0 so that the constant C_0 can be used in both Corollary 2.5 and in Lemma 2.4.

The following lemma will be used in connection with pseudo returns (see Section 1.7), and it will be used in Lemma 4.2, Lemma 4.3 and Lemma 6.5.

Lemma 2.7. *Assume that $z_1 \in P_{\delta'}$ and $z_2 \in P_{\delta'} \setminus P_\delta$. For $a, b \in [0, a_0]$ we have*

$$\left| \frac{(R^j)'(z_1, a)}{(R^j)'(z_2, b)} - 1 \right| \leq 1/1000, \tag{2.8}$$

whenever $R^k(z_1, a), R^k(z_2, b) \in \mathcal{N}_\rho$, for $0 \leq k \leq j$.

Moreover, assume that $z \in J_r^i \subset U_i'$, and put $v_i = R_0(c_i)$. Then

$$|R_0^j(R_0(z)) - v_i| \leq C|z - c_i|^{d_i} |(R_0^j)'(v_i)|. \tag{2.9}$$

whenever $R_0^k(z) \in \mathcal{N}_\rho$, for $0 \leq k \leq j$, where C only depends on ρ and the unperturbed function $R(z) = R_0(z)$.

Proof. Let a_0 be such that

$$|\xi_j(a) - \xi_j(0)| \leq \delta^{4K},$$

for all $j \leq N$ and all $a \in [0, a_0]$, where N is the upper bound on the number of iterates such that $R^N(z, a) \in \mathcal{N}_\rho$ if $z \in P_{\delta'} \setminus P_\delta$ and $a \in [0, a_0]$. Indeed, let $|\lambda|$ be the minimal multiplier of $R(z, a)$, i.e. $|\lambda| = \min |R'(p_i(a), a)|$ where $p_i(a)$ is the repelling fixed point. Then

$$1 \geq \rho \geq (|\lambda| - \varepsilon)^N |z - p_i(a)| \geq (|\lambda| - \varepsilon)^N e^{-\Delta},$$

where $\varepsilon \leq (|\lambda| - 1)/1000$. So $N \leq C\Delta$, where C only depends on $|\lambda|$.

We use the linearisation of R_t around $p(t)$ according to Subsection 1.7 to get

$$(R_t^n)'(z) = (\varphi_t^{-1})'(g_t^n(\varphi_t(z)))(g_t^n)'(\varphi_t(z))\varphi_t'(z), \tag{2.10}$$

where

$$g_t(w) = \lambda_t(w - p(t)) + p(t).$$

The multipliers $\lambda_t = R_t'(p(t))$ are analytic, so

$$\lambda_a = \lambda_0 + c\lambda_0 a^l + \mathcal{O}(a^{l+1}).$$

For small perturbations we get

$$\lambda_t^n = (\lambda_0 + c\lambda_0 a^l + \mathcal{O}(a^{l+1}))^n = \lambda_0^n (1 + ca^l + \mathcal{O}(a^{l+1}))^n \sim \lambda_0^n (1 + nca^l).$$

We get that $nca^l \leq NC\Delta ca^l$, which is very small if $a \leq a_0$ and a_0 is very small compared to δ. Therefore,

$$\left| \frac{\lambda_t^n}{\lambda_s^n} - 1 \right| \leq C \left| \frac{1 + nct^l}{1 + ncs^l} - 1 \right| \leq \varepsilon$$

can be fulfilled if a_0 is small enough. Since φ_t is conformal, by continuity and (2.10) we can get (2.8) if ρ is chosen appropriately.

To prove (2.9), now note that $z \in J_r$. By the linearisation we get immediately that $g_0^n(w) = \lambda_0^n(w - p(0)) + p(0)$. By the continuity of φ_0 and Lemma 2.2, this implies

$$|R_0^j(R_0(z)) - v_i| \leq C|(R_0^j)'(v_i)||R_0(z) - v_i| \leq CC_1|(R_0^j)'(v_i)||z - c_i|^{d_i},$$

where $C = C(\rho)$. This completes the proof of the lemma. □

3. The hyperbolic metric

In this section we introduce the hyperbolic metric and show how it leads to the important so called Outside Expansion Lemma. The hyperbolic (or Poincaré) metric on the unit disc D, is defined by

$$\rho_D(z)|dz| = \frac{2|dz|}{1 - |z|^2},$$

for $z \in D$.

Lemma 3.1. *Let f be a meromorphic function of degree at least 2. Assume that f has finite postcritical set P, which consists of at least 3 points. Then there exists an expanding metric $\varphi(z)$ on $\hat{\mathbb{C}} \setminus P$ such that*

$$\varphi(f(z))|f'(z)| > \varphi(z),$$

for all $z \in \hat{\mathbb{C}} \setminus (P \cup f^{-1}(P))$.

Proof. Let $\pi : D \to \hat{\mathbb{C}} \setminus \mathrm{P}$ be the universal covering map. Since P consists of at least 3 points, $\hat{\mathbb{C}} \setminus P$ is a hyperbolic domain and D is conformally equivalent to the open unit disc. Now, consider a lift h^{-1} of f^{-1} to D induced by π, see below.

$$\begin{array}{ccc} D & \xrightarrow{h^{-1}} & D \\ \pi\big\downarrow & & \big\downarrow\pi \\ \hat{\mathbb{C}} \setminus P & \xrightarrow{f^{-1}} & \hat{\mathbb{C}} \setminus f^{-1}(P) \end{array}$$

Since $f(P) \subset P$ we get $f^{-1}(\hat{\mathbb{C}} \setminus P) \subset \hat{\mathbb{C}} \setminus P$. Since $\{f^{-k}(P)\}_{k=0}^{\infty}$ is dense in the Julia set the last inclusion is strict. Thus, $h^{-1}(D) \subset D$ is also strict inclusion. Hence, h^{-1} is strictly contractive w.r.t. the hyperbolic metric on D and thus h is strictly expansive on D, by Schwarz Lemma. Hence f is strictly expansive on the induced hyperbolic metric on $\hat{\mathbb{C}} \setminus (P \cup f^{-1}(P))$. □

Remark 3.2. The induced hyperbolic (or Poincaré) metric will be denoted by $\varphi(z)$ from now on. There is no obstruction to assume that P consists of at least 3 points as can be seen from the examples (0.11), where the critical points 2 and 0 are mapped as shown below:

$$2 \mapsto 0 \mapsto \infty \mapsto \lambda,$$

where λ is a repelling fixed point, and the postcritical set is $P = \{0, \infty, \lambda\}$.

Using the metric φ on $\hat{\mathbb{C}} \setminus P$ we will now construct a metric which is expansive for the function $R(z, u)$ for u sufficiently close to u_0 (see (0.9)).

Definition 3.3. Let $R(z,u_0)$ be the unperturbed function $R(z)$ and let u be a perturbation in any parameter direction defined in (0.9). Let *the Jacobian in the metric* φ be defined by

$$J_\varphi(z,u) = |R'(z,u)|\frac{\varphi(R(z,u))}{\varphi(z)}, \qquad J_\varphi(z) = J_\varphi(z,u_0).$$

Define

$$J^n_\varphi(z,u) = \prod_{j=0}^{n-1} J_\varphi(R^j(z,u),u), \qquad J^n_\varphi(z) = J^n_\varphi(z,u_0).$$

In particular,

$$J^n_\varphi(z,u) = |(R^n)'(z,u)|\frac{\varphi(R^n(z,u))}{\varphi(z)}.$$

Lemma 3.4. *Let D be a hyperbolic domain and let $\delta(z)$ be the distance from ∂D to z. Then the induced hyperbolic metric $\varphi(z)$ satisfies*

$$\frac{1+o(1)}{\delta(z)\log 1/\delta(z)} \leq \varphi(z) \leq \frac{2}{\delta(z)},$$

as $z \to \partial D$.

Proof. See [11], p. 13. □

In the following lemma we consider the unperturbed accelerated redefined function described in Definition 1.1. We distinguish between two types of critical points: Denote by c'_j the critical points not intersecting P, and for those intersecting P, by c''_j. By the construction of the metric φ, and Lemma 3.4, it follows that φ has a singularity at c''_j but not on c'_j. The idea is to remove the singularity at c''_j, replacing φ with a suitable constant there and redefine φ in the repelling neighbourhood $P_{\delta'}$. Recall that the function R in Definition 1.1 maps *all* critical points onto repelling fixed points.

Lemma 3.5. *There is a metric $\psi(z)$ which satisfies*

$$J_\psi(z) \geq \lambda,$$

for all $z \notin P \cup R^{-1}(P)$ and some $\lambda > 1$, (where λ does not depends on δ). Moreover,

$$\lim_{z\to v} J_\psi(z) = \mu^{1/d},$$

where v is a repelling fixed point, $|R'(v)| = \mu$, and d is the maximal degree of R at some critical point c mapped onto v.

Proof. The Taylor expansion of R near c is

$$R(z) = v + t(z-c)^d + \mathcal{O}((z-c)^{d+1}),$$

where we assume that d is the maximal degree of all such expansions for all critical points c mapped onto v under R. In the neighbourhood $\mathcal{N}_\rho$ of v we have

$$R(z) = v + \mu(z-v) + \mathcal{O}((z-v)^2),$$

where $|R'(v)| = \mu > 1$. Define

$$\begin{aligned} A_1 &= P_{\delta'} \setminus R^{-1}(P_{\delta'}), \\ A_2 &= R(P_{\delta'}) \setminus P_{\delta'}, \\ A_3 &= R^2(P_{\delta'}) \setminus R(P_{\delta'}). \end{aligned}$$

Then $R(A_1) = A_2$ and $R(A_2) = A_3$. Let U_j'' be those components of $R^{-1}(P_{\delta'})$ which have the property that there is a critical point $c_j'' \in U_j''$, such that c_j'' also lies in the postcritical set. Let A_j'' be the annular components of $R^{-1}(A_2)$ which touch U_j'', and put $U'' = \cup_j U_j''$ and $A'' = \cup_j A_j''$. Also, assume that all the sets U_j' are the preimages of $P_{\delta'}$ under R in this proof.

We shall construct a continuous metric $\psi(z)$, defined by

$$\psi(z) = \begin{cases} \varphi(z), \text{ if } z \notin R(P_{\delta'}) \cup U'' \cup A'', \\ g(z), \text{ if } z \in A_2, \\ C'|z-v|^{\frac{1-d}{d}}, \text{ if } z \in P_{\delta'}, \\ C'', \text{ if } z \in U'', \\ h(z), \text{ if } z \in A'', \end{cases}$$

for some suitable C' and C'', where

$$g(z) = \varphi(z) + \theta(z)(C'|z-v|^{\frac{1-d}{d}} - \varphi(z)), \quad \text{and} \quad h(z) = \varphi(z) + \theta_1(z)(C'' - \varphi(z)),$$

for some continuous $\theta(z)$ and $\theta_1(z)$, defined on A_2 and A'' respectively. Moreover, $0 \le \theta(z) \le 1$, for all $z \in A_2$ and $0 \le \theta_1(z) \le 1$, for all $z \in A''$. The functions $\theta(z)$ and $\theta_1(z)$ shall satisfy the following boundary conditions:

$$\theta(z) = \begin{cases} 0, \text{ if } z \in \partial R(P_{\delta'}), \\ 1, \text{ if } z \in \partial P_{\delta'}. \end{cases} \qquad \theta_1(z) = \begin{cases} 0, \text{ if } z \in \partial(R^{-1}(R(P_{\delta'})) \cap A''), \\ 1, \text{ if } z \in \partial(R^{-1}(P_{\delta'}) \cap A''). \end{cases}$$

The existence of $\theta(z)$ and $\theta_1(z)$ follows from Urysohn's Lemma (see for example [33], p. 39). Let

$$J_\psi(z) = |R'(z)| \frac{\psi(R(z))}{\psi(z)}.$$

We have to show that $J_\psi(z) \ge \lambda > 1$ for some λ only depending on δ'.

First, we consider $J_\varphi(z)$ on $P_{\delta'} \cup R^{-1}(P_{\delta'})$. Clearly, $J_\psi(z) \ge \lambda > 1$ for $z \notin P_{\delta'} \cup R^{-1}(P_{\delta'})$, since $(P_{\delta'} \cup R^{-1}(P_{\delta'}))^c$ is compact and both sides of the inequality

$$\varphi(R(z))|R'(z)| > \varphi(z)$$

are continuous. So,

$$J_\varphi(z) \ge \lambda, \tag{3.1}$$

for all $z \in (P_{\delta'} \cup R^{-1}(P_{\delta'}))^c$, for some $\lambda > 1$, only depending on δ'.

Let us show that ψ is expansive on U'. First, take any $z \in U_j'$ containing a critical point $c \notin P$. Then $\varphi(c)$ is well defined, in fact $\varphi(z) \neq 0$ if $z \in \hat{\mathbb{C}} \setminus P$. We have, by definition, that $R(z) \in P_{\delta'}$. By the definition of ψ we get

$$J_\psi(z) = |R'(z)| \frac{\psi(R(z))}{\psi(z)} \ge C|z-c|^{d-1} \frac{C'|z-c|^{1-d}}{\varphi(z)} \ge \frac{CC'}{\varphi(c)} \ge \lambda > 1, \tag{3.2}$$

if C' is chosen suitable (large enough). Note that C' only depends on the unperturbed function and the hyperbolic metric φ. Taking another critical point $c_1 \neq c$ mapped onto v under R, the Taylor expansion of R at c_1 is

$$R(z) = v + t_1(z - c_1)^{d_1} + \mathcal{O}((z - c_1)^{d_1+1}),$$

where $d_1 \leq d$. If $c_1'' \in P$ then by the definition of ψ it is easy to check that (3.2) holds if $z \in U_1''$, where $U_1'' \subset U''$ is the neighbourhood of c_1''. If c_1' is a critical point outside P and $c_1' \neq c$ then by the fact that $d_1 \leq d$, it is again easy to conclude that (3.2) holds in U_1' containing c_1'.

Assume that A_2' is a component of $R^{-1}(A_2)$, so that A_2' is the annular neighbourhood around a critical point $c \in U'$. If $z \in A_2'$ then $R(z) \in A_2$ and we get, again by the definition of ψ,

$$J_\psi(z) = |R'(z)|\frac{g(R(z))}{\psi(z)} \geq C|z - c|^{d-1}\frac{\min(C'|R(z) - v|^{1-d}, \varphi(R(z))}{\max(C'', \varphi(z))} \geq \lambda > 1,$$

since we can choose δ' so that $C'' \leq \varphi(z)$ for $z \in U'' \cup A''$, (this is possible since $\varphi(z) \to \infty$ as $z \to P$).

If we take $z \in W \cup A'''$, where W is some component of $R^{-1}(U_j'')$ not containing any critical point and A''' is the component of $R^{-1}(A_j'')$ which touches W, then $|R'(z)| \geq C_{\delta'}$ for $z \in W \cup A'''$, and $R(z) \in U'' \cup A''$, where $C_{\delta'}$ only depends on δ'. By the definition of ψ we may choose C'' such that

$$J_\psi(z) = |R'(z)|\frac{\psi(R(z))}{\psi(z)} \geq C_{\delta'}\frac{\min(C'', \varphi(R(z)))}{\varphi(z)} \geq \lambda > 1,$$

since $\varphi(z) \geq C > 0$ on $W \cup A'''$. Here C'' only depends on δ', and (3.2) follows directly.

Now, we prove that $J_\psi(w) \to \mu^{1/d}$, as $w \to v$. Indeed, using the Taylor expansion, we get for small z

$$J_\psi(v + z) = |R'(v + z)|\frac{\psi(v + \mu z + \mathcal{O}(z^2))}{\psi(v + z)} \to \mu\frac{C'(\mu|z|)^{\frac{-d+1}{d}}}{C'|z|^{\frac{-d+1}{d}}} = \mu \cdot \mu^{\frac{-d+1}{d}} = \mu^{1/d}.$$

So, for example, $J_\psi(z) \geq (\mu - \varepsilon)^{1/d}$, for $z \in P_{\delta'} \setminus A_1$, where $\varepsilon \leq (\mu - 1)/1000$.

So far, we have proved that $J_\psi(z) \geq \lambda > 1$, if $z \notin (A_1 \cup A_2) \cup (P \cup R^{-1}(P))$, where λ only depends on δ'.

Let us show that ψ is expansive on the annular neighbourhood A_2. Take any z such that $w = v + z \in A_2$. Since $R(A_2) = A_3 \subset R(P_{\delta'})^c$ we get

$$J_\psi(w) = |R'(w)|\frac{\psi(R(w))}{\psi(w)} = |R'(w)|\frac{\varphi(R(w))}{g(w)} \geq |R'(w)|\frac{\varphi(R(w))}{\max(\varphi(w), C'|w - v|^{\frac{1-d}{d}})}.$$

We have two cases. If

$$\varphi(w) \geq C'|w - v|^{\frac{1-d}{d}}$$

then we get $J_\psi(w) \geq \lambda$ directly from (3.1). If

$$\varphi(w) \leq C'|w - v|^{\frac{1-d}{d}},$$

then we use Lemma 3.4 and the Taylor expansion around the fixed point v;

$$J_\psi(w) \geq |R'(v+z)| \frac{\varphi(R(v+z))}{C'|z|^{\frac{1-d}{d}}} \geq \frac{1}{2}|R'(v+z)| \frac{(-\mu|z|\log(\mu|z|))^{-1}}{C'|z|^{\frac{1-d}{d}}}$$
$$= \frac{1}{2C'} \frac{-1}{|z|^{1/d}\log(\mu|z|)} \geq \lambda > 1,$$

since $|z| \leq \delta'$. Here δ' depends on C', so λ depends only on δ'.

Finally, if $w \in A_1$ then, from the above calculations,

$$J_\psi(w) = |R'(w)| \frac{g(R(w))}{C'|w-v|^{\frac{1-d}{d}}} \geq |R'(w)| \frac{\min(\varphi(R(w)), C'|R(w)-v|^{\frac{1-d}{d}})}{C'|w-v|^{\frac{1-d}{d}}} \geq \lambda.$$

So, indeed, $J_\psi(z) \geq \lambda > 1$ for all $z \notin P \cup R^{-1}(P)$. □

Lemma 3.6. *There exists a neighbourhood V of u_0 and a $\lambda > 1$ such that*

$$J_\psi(z,u) \geq \lambda,$$

for $u \in V$ and for all $z \notin P_{\delta^2} \cup R^{-1}(P_{\delta^2})$.

Proof. Let $S(a_0) = \{u : |u - u_0| \leq a_0\}$ be the closed ball in the parameter space $\mathbb{R}^{4d+2}$ of radius a_0 (see (0.9)).

By Lemma 3.5

$$\psi(R(z,u))|R'(z,u)| \geq \lambda\psi(z), \tag{3.3}$$

holds if $u = u_0$ and $z \notin P \cup R^{-1}(P)$. By the continuity of both sides of (3.3), for any $z \in (P_{\delta^2} \cup R^{-1}(P_{\delta^2}))^c$, there is a perturbation $a_1 = a_1(z) > 0$ such that (3.3) holds whenever $|u - u_0| \leq a_1(z)$, for some slightly smaller λ. Of course, $a_1(z)$ is continuous. Therefore, by the compactness of $(P_{\delta^2} \cup R^{-1}(P_{\delta^2}))^c$, there is a perturbation $0 < a_0 \leq a_1(z)$ for all $z \in (P_{\delta^2} \cup R^{-1}(P_{\delta^2}))^c$ such that (3.3) holds for all $(z,u) \in (P_{\delta^2} \cup R^{-1}(P_{\delta^2}))^c \times S(a_0)$, for some $\lambda > 1$, (This is possible if a_0 is sufficiently small compared to δ^2, the radius of U^2). The lemma is proved. □

Now, we are ready to prove the main lemma of this section. It will give uniform expansion outside U^2. We will often use the lemma with U^2 replaced by U instead. Of course, the lemma still holds in that case.

Lemma 3.7 (The Outside Expansion Lemma)**.** *If $R^k(z,u) \notin U^2$ (or U) for $k = 0, \ldots, j-1$ and $R^j(z,u) \in P^c_{\delta'}$ then*

$$|(R^j)'(z,u)| \geq C_\sigma \lambda^j,$$

for some constant $C_\sigma > 0$, for $u \in V$, where V is the same as in Lemma 3.6. The numbers C_σ and λ depend only on δ', i.e. the unperturbed function.

Proof. It follows from Lemma 3.4 that

$$|(R^j)'(z,u)| = J^j_\psi(z,u) \frac{\psi(z)}{\psi(R^j(z,u))} \geq C_\sigma J^j_\psi(z,u),$$

for some constant C_σ, only depending on δ', since $R^j(z,u) \in P^c_{\delta'}$. If $R^k(z,u) \notin P_{\delta^2} \cup R^{-1}(P_{\delta^2})$, for $0 \leq k \leq j-1$, Lemma 3.6 and Lemma 3.5 implies that

$$J^j_\psi(z,u) \geq \lambda^j,$$

for some $\lambda > 1$, only depending on δ', the perturbation a_0 and $R(z)$.

It may happen that $w = R^k(z,u)$ belongs to some component of $R^{-1}(P_{\delta^2})$ not intersecting U^2 or P_{δ^2}. Then $R(w,u) \in P_{\delta^2}$. If $R^k(w,u) \in P_{\delta'}$, for $0 < k \leq l-1$, and $R^l(w,u) \notin P_{\delta'}$, then $l \geq N = N(\delta^2/\delta')$. This is easily seen from the linearisation around the fixed point $p(u)$. Put

$$\mu = \min_{u \in V} |R'(p(u),u)| > 1,$$

where $p(u)$ is a repelling fixed point, ($\varepsilon \leq (\mu-1)/1000$). Put $R_u(z) = R(z,u)$. For some $\lambda > 1$, we get

$$|(R^l)'(w,u)| = |R'(w,u)||(R^{l-1})'(R_u(w),u)| \geq C(\mu-\varepsilon)^{l-1} \geq C'(\mu-\varepsilon)^l \geq \lambda^l,$$

if δ^2/δ' is small enough, since $|R'(w,u)| \geq C > 0$ in any component of $R^{-1}(P_{\delta^2})$ not intersecting U^2. Apply this argument for all intervals (k_j, k_j+l_j), where $R^{k_j}(z,u) \in P_{\delta^2}$, $R^k(z,u) \in P_{\delta'}$ for $k_j < k \leq k_j + l_j - 1$ and $R^{k_j+l_j}(z,u) \notin P_{\delta'}$.

Obviously, $R^k(z,u) \notin R^{-1}(P_{\delta^2}) \cup P_{\delta^2}$ for $k_j + l_j \leq k < k_{j+1}$. Since $\varphi(z)$ tends to ∞ as $z \to P$ we can make the neighbourhood $P_{\delta'}$ so small so that $z \in R(P_{\delta'})$ and $w \notin R(P_{\delta'})$ implies $\varphi(z) > \varphi(w)$. Note that this condition on δ' comes only from the unperturbed function R under sufficiently small perturbations. Since $R^{k_j+l_j}(z,u) \in R(P_{\delta'})$ and $R^{k_{j+1}}(z,u) \notin R(P_{\delta'})$, we get

$$|(R^{k_{j+1}-(k_j+l_j)})'(R^{k_j+l_j}(z,u))| \geq \lambda^{k_{j+1}-(k_j+l_j)}.$$

Putting all these cases together, and using the Chain Rule it follows that

$$|(R^j)'(z,u)| \geq C_\sigma \lambda^j,$$

for some $\lambda > 1$ and $C_\sigma > 0$. □

Lemma 3.8. *If $R^k(z) \notin U^2$ (or U), for $0 \leq k \leq j-1$ then*

$$|(R^j)'(z,u)| \geq C\lambda^{j-1} \inf_{0 \leq k \leq j-1} |R'(R^k(z,u),u)|,$$

for all $u \in V$. The constant $C > 0$ does not depend on δ.

Proof. Put $R_u(z) = R(z,u)$. Let $l \leq j$ be the largest integer such that $R_u^l(z) \in U' \setminus U^2$. Then $R_u^{l+1}(z) \in P_{\delta'}$. If $R_u^j(z) \in P_{\delta'}^c$, The Outside Expansion Lemma gives

$$|(R_u^{j-l-1})'(R_u^{l+1}(z))| \geq C_\sigma \lambda^{j-l-1}.$$

If $R_u^j(z) \in P_{\delta'}$ then $\varphi(R_u^j(z))$ blows up and we use the derivative instead. We have

$$|(R_u^{j-l-1})'(R_u^{l+1}(z))| \geq (\mu-\varepsilon)^{j-l-1},$$

where $\mu = \min_{u \in V} |R_u'(p(u),u)| > 1$ and $\varepsilon \leq (\mu-1)/1000$. The Outside Expansion Lemma can be used to estimate $|(R_u^l)'(z)|$:

$$|(R_u^l)'(z)| \geq C_\sigma \lambda^l.$$

Finally,

$$\begin{aligned} |(R_u^j)'(z)| &= |(R_u^{j-l-1})'(R_u^{l+1}(z))||R_u'(R_u^l(z))||(R_u^l)'(z)| \\ &\geq C\mu^{j-l-1}\lambda^l \inf_{0 \leq k \leq j-1} |R_u'(R_u^k(z))| \geq C\lambda^{j-1} \inf_{0 \leq k \leq j-1} |R_u'(R_u^k(z))|, \end{aligned}$$

for some $\lambda > 1$, since $\mu > 1$. □

4. Basic geometry

First, we give a general formula to estimate the curvature of a parameterised curve (or, rather a point on the curve) $\gamma_0(s)$ iterated n times under a holomorphic function f. Set $f^n(\gamma_0(s)) = \gamma_n(s)$ and let $\kappa_n(s)$ be the curvature of $\gamma_n(s)$ at the point s.

Lemma 4.1. *For $n \geq 1$,*

$$\kappa_n(s) \leq \sum_{i=1}^{n} \frac{1}{|(f^{n-i})'(\gamma_i(s))|} \frac{|f''(\gamma_{i-1}(s))|}{|f'(\gamma_{i-1}(s))|^2} + \frac{1}{|(f^n)'(\gamma_0(s))|} \frac{|\gamma_0''(s)|}{|\gamma_0'(s)|^2}.$$

Proof. By the Chain Rule

$$\gamma_n'(s) = \frac{d}{ds} f^n(\gamma_0(s)) = f'(\gamma_{n-1}(s)) f'(\gamma_{n-2}(s)) \cdot \ldots \cdot f'(\gamma_0(s))\gamma_0'(s). \tag{4.1}$$

Hence,

$$\gamma_n''(s) = \sum_{i=1}^{n} f''(\gamma_{n-i}(s))\gamma_{n-i}'(s) \prod_{j \neq i} f'(\gamma_{n-j}(s)) + \gamma_0''(s) \prod_{j=1}^{n} f'(\gamma_{n-j}(s)). \tag{4.2}$$

Substituting (4.1) into (4.2) gives

$$\begin{aligned}\gamma_n''(s) = \sum_{i=1}^{n} f''(\gamma_{n-i}(s)) \prod_{j=1}^{i-1} f'(\gamma_{n-j}(s)) \Big[\gamma_0'(s) \prod_{k=i+1}^{n} f'(\gamma_{n-k}(s))\Big]^2 \\ + f'(\gamma_{n-1}(s)) \cdot \ldots \cdot f'(\gamma_0(s))\gamma_0''(s).\end{aligned} \tag{4.3}$$

For $\gamma_n = x_n + iy_n$ the curvature κ_n satisfies

$$\kappa_n = \frac{|x_n'' y_n' - x_n' y_n''|}{|(x_n')^2 + (y_n')^2|^{3/2}} = \frac{|\mathrm{Im}(\gamma_n'' \overline{\gamma_n'})|}{|\gamma_n'|^3} \leq \frac{|\gamma_n''|}{|\gamma_n'|^2}. \tag{4.4}$$

Substituting (4.1) and (4.3) into (4.4) we finally get the desired formula. □

Next, we give a uniform estimate of the curvature of $\gamma_n(t)$ outside U. The following lemma will be used in Lemma 4.3 to give expansion between two close points iterated under R_a for fixed a.

Lemma 4.2. *Let $\gamma_0(t)$ be a curve in $\hat{\mathbb{C}}$ and put $\gamma_n(t) = R_a^n(\gamma_0(t))$. Assume that $\gamma_k(t) \notin U$ for $0 \leq k \leq n$ and let $\kappa_j(t)$ be the curvature at the point t of $\gamma_j(t)$. If $\gamma_j(t) \in P_{\delta'}^c$ then*

$$\kappa_j(t) \leq C_3 + \frac{\kappa(t)}{|(R_a^j)'(\gamma_0(t))|},$$

if $\delta \gg a_0$, where $\kappa(t) = |\gamma_0''(t)|/|\gamma_0'(t)|^2$, i.e. the upper bound for $\kappa_0(t)$. The constant C_3 does not depend on δ, (recall that we always have $\delta \ll \delta'$).

Proof. Since we look at the curvature at a specific point t, let us fix t for the moment. The estimate of the curvature is partially a sum of terms of the form

$$\kappa_{j,k} = \frac{1}{|(R_a^{j-k})'(R_a(\gamma_k(t)))|} \frac{|R_a''(\gamma_{k-1}(t))|}{|R_a'(\gamma_{k-1}(t))|^2}. \tag{4.5}$$

We will estimate the sum of all terms $\kappa_{j,k}$ for $k = 1, \ldots, j$. For $z \notin U'$ we have an obvious bound on

$$\frac{|R_a''(z)|}{|R_a'(z)|^2} \leq C_p,$$

depending on δ' but not on δ.

So let us assume that $\delta \leq |z - c_i| = e^{-r} \leq \delta'$, i.e. z is a pseudo return and $\Delta' \leq r \leq \Delta$. Put $z = \gamma_{l-1}(t)$. We may use the Taylor expansion of R at a pseudo return. Let T_i be the first order Taylor expansion of R around c_i;

$$T_i(z) = v_i + t_i(z - c_i)^{d_i}.$$

Replace $R_a(z)$ with $T_i(z)$ in the expression (4.5). Lemma 2.2 gives

$$\frac{1}{|(R_a^k)'(R_a z)|}\frac{|R_a''(z)|}{|R_a'(z)|^2} \leq C_1^3 \frac{1}{|(R_a^k)'(R_a z)||z - c_i|^{d_i}}.$$

Put $R_a(z) = z_0$. Let N be the greatest integer such that $R_a^j(z_0) \in \mathcal{N}_\rho$, for $0 \leq j \leq N$. Since $z_0 \in P_{\delta'} \setminus P_\delta$ we have, by an argument identical to that in the beginning of the proof of Lemma 2.7, that $N \leq C\Delta$. Also, the difference

$$|R^N(z_0, a) - R^N(z_0, 0)| \leq B^N a \leq B^{C\Delta} a_0$$

(recall $B = \sup |\partial_a R(z)|$), can be made arbitrary small if a_0 is sufficiently small compared to δ. Now Lemma 2.7 implies

$$\begin{aligned} |(R_a^N)'(R_a z)|e^{-d_i r} &\geq (3/4)|(R_0^N)'(z_0)|e^{-d_i r} \geq \frac{1}{C}|R_0^N(z_0) - v_i| \\ &\geq \frac{1}{2C}|R_a^N(z_0) - v_i| \geq \frac{\rho}{4C} = C'. \end{aligned}$$

Thus for each pseudo return, we get that

$$\begin{aligned} \kappa_{j,l} &= \frac{1}{|(R_a^{j-l-N})'(R_a(\gamma_{l+N}(t)))|}\frac{1}{|(R_a^N)'(\gamma_l(t))|}\frac{|R_a''(\gamma_{l-1}(t))|}{|R_a'(\gamma_{l-1}(t))|^2} \\ &\leq \frac{C'}{|(R_a^{j-l-N})'(R_a(\gamma_{l+N}(t)))|}. \end{aligned}$$

Now, put $z_k = R_a^k(\gamma(t), t)$. Summing over all terms which correspond to pseudo returns z_{k_i-1} and their corresponding periods $N = N_i$ as above, we get by the Outside Expansion Lemma,

$$\begin{aligned} \kappa_{pseudo}(t) &\leq \sum_i \frac{1}{|(R_a^{j-k_i})'(z_{k_i})|}\frac{|R_a''(z_{k_i-1})|}{|R_a'(z_{k_i-1})|^2} \leq C' \sum_i \frac{1}{|(R_a^{j-N_i-k_i})'(z_{N_i+k_i})|} \\ &\leq \frac{C'}{C_\sigma}\sum_i \lambda^{-(j-N_i-k_i)}. \end{aligned}$$

Finally, summing the terms corresponding to pseudo returns (m) and the other terms (m') separately, according to Lemma 4.1, the total curvature becomes

$$\begin{aligned}\kappa_j(t) &\leq \sum_{l\in(m)\cup(m')} \frac{1}{|(R_a^{j-l})'(z_l)|}\frac{|R_a''(z_{l-1})|}{|R_a'(z_{l-1})|^2} + \frac{\kappa(t)}{|(R_a^j)'(\gamma_0(t))|} \\ &\leq C_p\sum_m \lambda^{-m} + \frac{C'}{C_\sigma}\sum_{m'}\lambda^{-m'} + \frac{\kappa(t)}{|(R_a^j)'(\gamma_0(t))|} \\ &\leq C_3 + \frac{\kappa(t)}{|(R_a^j)'(\gamma_0(t))|},\end{aligned}$$

where C_3 depends on δ' but not on δ. □

Now, choose δ such that $\delta^{-1} \gg C_3$, and so that $S = \delta/C_2 \ll C_3$. This relation between S and C_3 will be used in the following lemma where we prove that two points repel each other under iteration of R_a for a *fixed* parameter a. The following lemma is only used in Lemma 6.7.

Lemma 4.3. *Let $\gamma(t) = z_0 + t(w_0 - z_0)$, $0 \leq t \leq 1$, and assume that $R_a^k(\gamma([0,1])) \cap U = \emptyset$ for $k = 0, 1, \ldots, j-1$, $|R_a^k(z_0) - R_a^k(w_0)| \leq S$ for $k = 0, 1, \ldots, j$ and $R_a^j(\gamma([0,1])) \subset P_{\delta'}^c$. Then*

$$\left|\frac{(R_a^j)'(\gamma(s))}{(R_a^j)'(\gamma(t))} - 1\right| \leq C\frac{|R_a^{j-1}(\gamma(s)) - R_a^{j-1}(\gamma(t))|}{\delta} \leq 1/20, \tag{4.6}$$

for any $s, t \in [0,1]$, where C does not depend on δ.

Moreover, if $R_a^k(\gamma([0,1])) \cap U = \emptyset$ for $k = 0, 1, \ldots, j-1$ and $|R_a^k(z_0) - R_a^k(w_0)| \leq S$ for $k = 0, 1, \ldots, j$, then

$$|R_a^j(z_0) - R_a^j(w_0)| \geq (15/16)|(R_a^j)'(\gamma(t))||z_0 - w_0|, \tag{4.7}$$

for all $t \in [0,1]$.

Proof. Put $\zeta_k = R_a^k(\gamma(s))$ and $\eta_k = R_a^k(\gamma(t))$ for arbitrary $s, t \in [0,1]$ and let $\gamma_k(u) = \zeta_k + u(\eta_k - \zeta_k)$, $0 \leq u \leq 1$. Using Lemma 2.1, we estimate

$$\sum_{k=0}^{j-1}\frac{|R_a'(\zeta_k) - R_a'(\eta_k)|}{|R_a'(\eta_k)|}. \tag{4.8}$$

Let $\kappa_{j,k}(u)$ be the curvature of the curve $R_a^{j-k}(\gamma_k(u))$. By Lemma 4.2, we have $\kappa_{j,k}(u) \leq C_3$, if $R_a^j(\gamma(t)) \in P_{\delta'}^c$, since the curvature of $\gamma_k(u)$ is zero. Since the length of the curve $R_a^{j-k}(\gamma_k(u))$ is less than $S \ll C_3$, this means that

$$|\zeta_j - \eta_j| \geq C|(R_a^{j-k})'(\gamma_k(u'))||\zeta_k - \eta_k|, \tag{4.9}$$

for some $C = C(\delta', \delta) \to 1$ as $\delta/\delta' \to 0$ and $u' \in [0,1]$. So C is very close to 1 here, take $C = 31/32$. By the Outside Expansion Lemma

$$|\zeta_j - \eta_j| \geq (31/32)|(R_a^{j-k})'(\gamma_k(u'))||\zeta_k - \eta_k| \geq (31/32)C_\sigma\lambda^{j-k}|\zeta_k - \eta_k|.$$

Let us see how the terms look like near the critical points. By Lemma 2.2,

$$|R'(\zeta_k, a) - R'(\eta_k, a)| \leq |R''(\gamma_k(t), a)||\zeta_k - \eta_k| \leq 2C_1 e^{-r(\tilde{d}_i - 2)}|\zeta_k - \eta_k|.$$

The denominator in (4.8) is estimated from below, using Lemma 2.2, by

$$C|\eta_k - c(a)|^{\tilde{d}_i - 1} \geq Ce^{-(\tilde{d}_i-1)r}.$$

So, for $\zeta_k, \eta_k \in U' \setminus U$ we get

$$\frac{|R_a'(\zeta_k) - R_a'(\eta_k)|}{|R_a'(\eta_k)|} \leq C\frac{|\zeta_k - \eta_k|}{e^{-r}} \leq \frac{|\zeta_k - \eta_k|}{\delta}.$$

For $\zeta_k, \eta_k \notin U'$ we have a trivial estimate;

$$\frac{|R_a'(\zeta_k) - R_a'(\eta_k)|}{|R_a'(\eta_k)|} \leq C_{\delta'}|\zeta_k - \eta_k|.$$

Equation (4.8) becomes

$$\begin{aligned}\sum_{k=0}^{j-1}\frac{|R_a'(\zeta_k) - R_a'(\eta_k)|}{|R_a'(\eta_k)|} &\leq C\sum_{k=0}^{j-1}\frac{|\zeta_k - \eta_k|}{\delta} \leq \sum_{k=0}^{j-1}\frac{|\zeta_{j-1} - \eta_{j-1}|\lambda^{-(j-k)}}{\delta} \\ &\leq C\frac{|\zeta_{j-1} - \eta_{j-1}|}{\delta} \leq C\frac{S}{\delta} \leq 1/20,\end{aligned}$$

if C_2 is chosen suitable, since $S = \delta/C_2$. This proves (4.6). By (4.9) with $k = 0$, (4.7) follows if $R_a^j(\gamma([0,1])) \subset P_{\delta'}^c$.

We have to deal with the case $R_a^j(\gamma([0,1])) \subset P_{\delta'}$. Then there is a largest integer $\tilde{n} \leq j$ such that

$$R_a^{\tilde{n}}(\gamma([0,1])) \cap U' \neq \emptyset,$$

i.e. the last pseudo return. By the previous argument we have

$$|R_a^{\tilde{n}}(z_0) - R_a^{\tilde{n}}(w_0)| \geq C|(R_a^{\tilde{n}})'(\gamma(t))||z_0 - w_0|,$$

for $C = C(\delta/\delta') \to 1$ as $\delta/\delta' \to 0$. Remember that $|R_a^{\tilde{n}}(z_0) - R_a^{\tilde{n}}(w_0)| \leq S = \delta/C_2$. Put $\hat{\gamma}(t) = R_a^{\tilde{n}}(z_0) + t(R_a^{\tilde{n}}(w_0) - R_a^{\tilde{n}}(z_0))$. Since $\hat{\gamma}(t) \cap U = \emptyset$, by the definition on S, both the argument and absolute value of $R_a'(\hat{\gamma}(t))$ changes not more than, eg. 1/20 if C_2 is chosen appropriately. The curve $\hat{\gamma}(t)$ is also very close to the curve $R_a^{\tilde{n}}(\gamma(t))$ since the curvature of $R_a^{\tilde{n}}(\gamma(t))$ is at most C_3. Therefore,

$$\begin{aligned}|R_a^{\tilde{n}+1}(z_0) - R_a^{\tilde{n}+1}(w_0)| &\geq C|R_a^{\tilde{n}}(z_0) - R_a^{\tilde{n}}(w_0)||R_a'(\hat{\gamma}(t))| \\ &\geq C'|R_a^{\tilde{n}}(z_0) - R_a^{\tilde{n}}(w_0)||R_a'(R_a^{\tilde{n}}(\gamma(t))|,\end{aligned}$$

for all $t \in [0,1]$, where C' depends on S ($C' \to 1$ as $S/\delta \to 0$). Therefore,

$$|R_a^{\tilde{n}+1}(z_0) - R_a^{\tilde{n}+1}(w_0)| \geq C|(R_a^{\tilde{n}+1})'(\gamma(t))||z_0 - w_0|,$$

for all $t \in [0,1]$, where $C = C(S)$. In the neighbourhood $\mathcal{N}_\rho$ we use the linearisation described in Subsection 1.7;

$$\varphi_a \circ R_a(z) = g_a \circ \varphi_a(z),$$

where $g_a(z) = \lambda_a(z - p(a)) + p(a)$ and φ_a is conformal. By the continuity of φ_a, and Lemma 2.7, it is obvious that, for $z, w \in \mathcal{N}_\rho$,

$$|R_a^j(z) - R_a^j(w)| \geq C|(R_a^j)'(z')||z - w|,$$

whenever $R_a^k(z), R_a^k(w) \in \mathcal{N}_\rho$, for all $k \leq j$, and where $z' \in R^{\tilde{n}+1}(\gamma(t)) \in P_{\delta'} \setminus P_\delta$. Thus,

$$\begin{aligned}|R_a^j(z_0) - R_a^j(w_0)| &\geq C(\delta/\delta')C(S)C(\rho)|(R_a^j)'(\gamma(t))||z_0 - w_0| \\ &\geq (15/16)|(R_a^j)'(\gamma(t))||z_0 - w_0|,\end{aligned}$$

if the constants δ, a_0, ρ, S are chosen suitable. □

5. Initial distortion estimates

First, we estimate $|\xi_n'(a)|$ in the neighbourhood $\mathcal{N}_\rho$, where n is the very first iterates, (see Subsection 1.7 for the definition of $\mathcal{N}_\rho$). In the following there will be some conditions on $\mathcal{N}_\rho$, to get sufficiently good distortion estimates. However, ρ still does not depend on δ. We will study one critical orbit at a time. Let

$$x(a) = v(a) - p(a) = K_1 a^k + \mathcal{O}(a^{k+1}), \tag{5.1}$$

where $k \geq 0$, $v(a)$ is a critical value, i.e. $R(c(a), a) = v(a)$ and $p(a)$ is the repelling fixed point for R_a such that $R_0(v(0)) = p(0)$, (the case $k = 0$ means that $v(a) = p(a)$, which is a trivial case). Around $p(a)$ we use the linearisation by the conjugation $\varphi_a(z)$, described in Subsection 1.7. By the normalisation of $\varphi_a(z)$ we have $\varphi_a(v(a)) = v(a) + (v(a) - p(a))^2 + \ldots$. Let $\tilde{\xi}_n(a) = \varphi_a(\xi_n(a)) = g_a^n(\varphi_a(v(a)))$, be the curve in the new coordinates. Since $\varphi_a(p(a)) = p(a)$ we get

$$\tilde{\xi}_n(a) = \lambda_a^n (x(a) + \mathcal{O}(x(a)^2)) + p(a), \tag{5.2}$$

whenever $\xi_n(a) \subset \mathcal{N}_\rho$, where $\lambda_a = R'(p(a), a)$. Also, λ_a is analytic in a so

$$\lambda_a = \lambda_0 + c\lambda_0 a^l + \mathcal{O}(a^{l+1}), \tag{5.3}$$

for $a \leq a_0$. On the other hand, since φ_t is conformal there is an error $\varepsilon_n(a)$ with $|\varepsilon_n(a)| \leq \varepsilon$, where $\varepsilon = \varepsilon(\rho)0$, such that

$$\xi_n(a) = \lambda_a^n x(a) + p(a) + \varepsilon_n(a). \tag{5.4}$$

By the Chain Rule,

$$\xi_n'(a) = (\varphi_a^{-1})'(\tilde{\xi}_n(a))\tilde{\xi}_n'(a) + \partial_a \varphi_a^{-1}(\tilde{\xi}_n(a)).$$

If we can show that $|\tilde{\xi}_n'(a)|$ grows exponentially then $\xi_n'(a)$ and $\tilde{\xi}_n'(a)$ are comparable;

$$\left| \frac{\xi_n'(a)}{\tilde{\xi}_n'(a)} - (\varphi_a^{-1})'(\tilde{\xi}_n(a)) \right| \leq \frac{|\partial_a \varphi_a^{-1}(\tilde{\xi}_n(a))|}{|\tilde{\xi}_n'(a)|},$$

since $\partial_a \varphi_a^{-1}(z)$ is bounded from above. If we let $\mathcal{N}_\rho'$ be such that $|(\varphi_a^{-1})'(z) - (\varphi_0^{-1})'(p(0))| = |(\varphi_a^{-1})'(z) - 1| \leq 1/1000$ for all $z \in \mathcal{N}_\rho'$ and all $a \in [0, a_0]$, then we get

$$\left| \frac{\xi_n'(a)}{\tilde{\xi}_n'(a)} - 1 \right| \leq \frac{2|\partial_a \varphi_a^{-1}(\tilde{\xi}_n(a))|}{|\tilde{\xi}_n'(a)|}. \tag{5.5}$$

In particular, $|\xi_n'(a)|$ grows exponentially if $|\tilde{\xi}_n'(a)|$ does. By (5.2)

$$\tilde{\xi}_n'(a) = \left(n\frac{\lambda_a'}{\lambda_a} x(a)(1 + \mathcal{O}(x(a))) + x'(a)(1 + \mathcal{O}(x(a))) \right) e^{n \log \lambda_a} + p'(a). \tag{5.6}$$

Definition 5.1 (Large annular neighbourhoods). We define annular neighbourhoods

$$\begin{aligned}\Omega_1 &= \mathcal{N}_\rho \setminus \mathcal{N}_{\rho/(2\Lambda)},\\ \Omega_2 &= \mathcal{N}_{\rho/(2\Lambda)} \setminus \mathcal{N}_{\rho/(4\Lambda^2)},\\ \Omega_3 &= \mathcal{N}_{\rho/(4\Lambda^2)} \setminus \mathcal{N}_{\rho/(8\Lambda^3)},\\ \Omega &= \Omega_1 \cup \Omega_2 \cup \Omega_3,\end{aligned}$$

where Λ is the maximal multiplier $|R'(p(a), a)|$, over all repelling fixed points $p(a)$ and all $a \in [0, a_0]$.

This means that for any $a \in [0, a_0]$, if $\xi_n(a) \in \Omega_1$ then there are $k_1 \geq 1$ and $k_2 \geq 1$ such that $\xi_{n+k_1}(a) \in \Omega_2$ and $\xi_{n+k_2}(a) \in \Omega_3$ and $\xi_{n+j}(a) \in \Omega$ for all $0 \leq j \leq \max(k_1, k_2)$. That is, every critical orbit $\xi_n(a)$ has to pass over all Ω_i sooner or later. These annular neighbourhoods will be used frequently in Subsection 5.1.

Let k_0' and N be such that for all $a \in [k_0' a_0, a_0]$, we have that $\xi_N(a) \in \Omega$. By (5.4) we get

$$\frac{\rho}{16\Lambda^3} \leq |\lambda_a|^N |x(a)| \leq 2\rho \leq 1,$$

so

$$\frac{-\log|x(a)| + \log(\rho/(16\Lambda^3))}{\log|\lambda_a|} \leq N \leq \frac{-\log|x(a)|}{\log|\lambda_a|}. \tag{5.7}$$

Therefore, since $x'(a) \sim a^{k-1}$ we get

$$N\frac{\lambda_a'}{\lambda_a}x(a) + x'(a) \sim -k(\log a)a^k\frac{\lambda_a'}{\lambda_a} + K_1 k a^{k-1} \sim x'(a),$$

if a is small and since $\lambda_a \neq 0$. If $|\tilde{\xi}_N'(a)|$ is large, then (5.7) and (5.5) gives

$$\begin{aligned}|\xi_N'(a)| &\geq C|\tilde{\xi}_N'(a)| \geq C|x'(a)||\lambda_a|^N \geq Ce^{\log|x'(a)| + N\log|\lambda_a|}\\ &\geq Ce^{(k-1)\log a + N\log|\lambda_a|} \geq Ce^{N(1-\frac{k-1}{k})\log|\lambda_a|} \geq e^{\gamma' N},\end{aligned}$$

for some $\gamma' \leq (1/k)\log|\lambda_a|$. Since $x'(a)$ is dominant in (5.6), (5.5) implies

$$\left|\frac{\xi_N'(a)}{\lambda_a^N x'(a)} - 1\right| \leq 1/2000, \tag{5.8}$$

whenever $\xi_N(a) \in \Omega$, $a \leq a_0$ if a_0 is small enough.

Now, we present the following lemma. Recall that the index l stands for a specific critical "branch" $c_l(a)$ for $a \in [0, a_0]$ and $v_l = R_a(c_l(a))$.

Lemma 5.2. *Let N_l be an integer such that $\xi_{N_l,l}(\omega) \subset \Omega$, for some interval $\omega \subset [0, a_0]$, (and $\xi_k(\omega) \subset \mathcal{N}_\rho$ for all $k \leq N_l$). Then*

$$|\xi_{N_l,l}(a) - \xi_{N_l,l}(b)| \geq C_e|\lambda_0|^{N_l}|x_l'(a)||a-b|. \tag{5.9}$$

for all $a, b \in \omega$, or, equivalently,

$$|\xi_{N_l,l}(a) - \xi_{N_l,l}(b)| \geq C_e' e^{N_l\gamma_l'}|a-b| \tag{5.10}$$

for all $a, b \in \omega$, where $\gamma_l' = (1/k_l)\log|\lambda_0|$.

Moreover,

$$\left|\frac{(R_a^{N_l})'(v_l(a))}{(R_b^{N_l})'(v_l(b))} - 1\right| \leq 1/1000, \tag{5.11}$$

for all $a, b \in [0, a_0]$*, and*

$$|\xi_{N_l,l}(a) - \xi_{N_l,l}(b)| \geq C_e''|(R_t^{N_l})'(v_l(t))||x_l'(a)||a-b|,$$

for all $t \in \omega$*. The constants* C_e, C_e', C_e'' *only depend on the function* $x(a) = x_l(a)$ *defined in (5.1).*

Proof. Let $a > b$ and $a, b \in [0, a_0]$. In the following, we use that $|p(a) - p(b)| = \mathcal{O}(|a^k - b^k|)$ and that N is so large such that $\xi_N([a, b]) \subset \Omega$:

$$\begin{aligned}
|\xi_N(a) - \xi_N(b)| &\geq |\xi_N(a) - p(a) - (\xi_N(b) - p(b))| - |p(a) - p(b)| \\
&\geq \frac{7}{8}|\lambda_a^N x(a) - \lambda_b^N x(b)| - |p(a) - p(b)| \\
&\geq \frac{3}{4}|(\lambda_0(1 + ca^l))^N x(a) - (\lambda_0(1 + cb^l))^N x(b)| - |p(a) - p(b)| \\
&\geq \frac{3}{5}|\lambda_0|^N|(1 + Nca^l)K_1a^k - (1 + Ncb^l)K_1b^k| - |p(a) - p(b)| \\
&\geq \frac{3}{6}|\lambda_0|^N|K_1(a^k - b^k)(1 + Nc'a^l)| - |p(a) - p(b)| \qquad (5.12) \\
&\geq \frac{3}{7}|\lambda_0|^N|K_1(a - b)a^{k-1}(1 + Nc'(k + 1)a^l)|. \qquad (5.13)
\end{aligned}$$

Using the right inequality in (5.7), since $x(a) \sim a^k$, we have $N \leq -k \log a$, so $a^l \log a \to 0$ as $a \to 0$. Thus, by (5.13) we get

$$|\xi_N(a) - \xi_N(b)| \geq \frac{3kK_1}{8}|\lambda_0|^N|a - b|a^{k-1}$$

for all $a, b \in [0, a_0]$, if N is large. Also,

$$|\xi_N(a) - \xi_N(b)| \geq \frac{3}{8}|\lambda_0|^N|a - b||x'(a)|.$$

Moreover, by (5.7) we get

$$\begin{aligned}
|\xi_N(a) - \xi_N(b)| &\geq \frac{3kK_1}{8}e^{N\log|\lambda_0|}a^{k-1}|a - b| \\
&\geq \frac{3kK_1}{9}e^{-\log|x(a)|}a^{k-1}|a - b| \\
&\geq \frac{3kK_1}{10}e^{-(1-\frac{k-1}{k})\log|x(a)|}|a - b| \\
&\geq \frac{3kK_1}{11}e^{N\frac{\log|\lambda_0|}{k}}|a - b|
\end{aligned}$$

We have proved (5.10) and (5.9), and we need to verify (5.11). By the analyticity of λ_t we get that $|\lambda_t^N/\lambda_s^N - 1|$ is very small if $s, t \in [0, a_0]$ and a_0 is small enough. Also, by the linearisation described in Subsection 1.7, we get that

$$\left|\frac{(R_t^N)'(v(t))}{(R_t^N)'(p(t))} - 1\right| \leq \varepsilon \qquad (5.14)$$

where ε is sufficiently small if ρ is chosen appropriately. Now, (5.11) follows. The last inequality follows directly from (5.11). □

Now, we come to a very basic result in this theory of dynamical systems, which says that parameter and space derivatives are comparable as long as the space derivative grows exponentially. We here follow an analogue of the following important proposition in [4].

Define

$$Q_n = Q_{n,l}(a) = \frac{\partial R^n(v_l(a),a)}{\partial a} \Big/ \frac{\partial R^n(v_l(a),a)}{\partial z},$$

where $v_l(a)$ is a critical value, i.e. $v_l(a) = R(c_l(a),a)$ for some $c_l(a) \in \mathcal{C}(R_a)$.

Proposition 5.3. *Assume that* $a \in \mathcal{B}'_{m-1,l}$, *and that*

$$\left|\frac{\partial R^n(v_l(a),a)}{\partial z}\right| \ge e^{\gamma n}, \quad \text{for } n = 0,\dots,m,$$

$$\left|\frac{\partial R(z,a)}{\partial a}\right| \le B, \quad \text{for all } z \in \hat{\mathbb{C}},$$

where $\gamma \ge \underline{\gamma} - K\alpha$. *Then for* $n = N_l,\dots,m$

$$|Q_{n,l}(a) - Q_{N_l,l}(a)| \le |Q_{N_l,l}(a)|/1000,$$

where N_l *is an integer such that* $\xi_{N_l,l}(a) \in \Omega$, *and such that* $\xi_k(\omega) \subset \mathcal{N}_\rho$ *for all* $k \le N_l$.

Proof. As usual, set $N_l = N$, $\xi_{n,l}(a) = \xi_n(a)$ and $v_l(a) = v(a)$. First, we will use (5.8) and prove by induction, that

$$|\xi'_{N+k}(a)| \ge e^{\gamma'(N+k)},$$

where $\gamma' = \gamma/k - \varepsilon$, k is as in (5.1). Take $\varepsilon = (\underline{\gamma} - 1)/1000$, which is possible if N is sufficiently large. We get by equations (5.8) and (5.14) that

$$|\xi'_N(a)| \ge (1/2)|(R_a^N)'(v(a))||x'(a)| \ge (1/2)e^{\gamma N}|x'(a)| \ge e^{\gamma' N}.$$

So, assume that

$$|\xi'_{N+j}(a)| \ge e^{\gamma'(N+j)}, \text{ for all } j \le k.$$

We want to prove that

$$|\xi'_{N+j}(a)| \ge e^{\gamma'(N+j)}, \text{ for all } j \le k+1.$$

First note that the basic assumption on a and Lemma 2.2 gives

$$|R'(\xi_j(a),a)| \ge C_1^{-1} e^{-\alpha K j}. \tag{5.15}$$

By the Chain Rule we have the recursions (remember the notation $\xi_n(a) = R^{n-1}(v(a),a)$)

$$\frac{\partial R^{n+1}(v(a),a)}{\partial z} = \frac{\partial R(\xi_n(a),a)}{\partial z}\frac{\partial R^n(v(a),a)}{\partial z}, \tag{5.16}$$

$$\frac{\partial R^{n+1}(v(a),a)}{\partial a} = \frac{\partial R(\xi_n(a),a)}{\partial z}\frac{\partial R^n(v(a),a)}{\partial a} + \frac{\partial R(\xi_n(a),a)}{\partial a}. \tag{5.17}$$

Now, the recursion formulas (5.16) and (5.17), together with (5.15) and (5.8), gives

$$
\begin{aligned}
|\xi'_{N+k+1}(a)| &\geq |R'_a(\xi_{N+k}(a))||\xi'_{N+k}(a)|\left(1-\frac{|\partial_a R_a(\xi_{N+k}(a))|}{|R'_a(\xi_{N+k}(a))||\xi'_{N+k}(a)|}\right)\\
&\geq |(R_a^{k+1})'(\xi_N(a))||\xi'_N(a)|\prod_{j=0}^{k}\left(1-\frac{|\partial_a R_a(\xi_{N+j}(a))|}{|R'_a(\xi_{N+j}(a))||\xi'_{N+j}(a)|}\right)\\
&\geq e^{\gamma(k+1)}e^{\gamma' N}\prod_{j=0}^{k}(1-B'e^{K\alpha(N+j)}e^{-\gamma'(N+j)})\\
&\geq e^{(\gamma-\gamma')(k+1)}e^{\gamma'(N+k+1)}\prod_{j=0}^{k}(1-B'e^{(K\alpha-\gamma')(N+j)}) \geq e^{\gamma'(N+k+1)},
\end{aligned}
$$

if N is large enough, since $\gamma' \geq 2K\alpha$, (here $B' = BC_1$). The sum

$$
\sum_{j=0}^{\infty} B'e^{-(\gamma'-K\alpha)(N+j)} < \infty,
$$

and can be made arbitrarily small if N is large enough.

By the definition of $Q_n(a)$, we have

$$
Q_{N+n}(a) = Q_N(a)\prod_{j=0}^{n}\left(1+\frac{\partial_a R_a(\xi_{N+j}(a))}{R'_a(\xi_{N+j}(a))\xi'_{N+j}(a)}\right).
$$

So,

$$
|Q_{N+n}(a) - Q_N(a)| \leq |Q_N(a)|/1000,
$$

if N is sufficiently large. □

Remark 5.4. The number $Q_N(a)$, for general $a \in [0, a_0]$ can be estimated by equation (5.8) and the fact that $\lambda_a^N/(R_a^N)'(v(a))$ is very close to 1 (equation (5.14)), in the following way;

$$
\left|\frac{Q_N(a)}{x'(a)} - 1\right| \leq \varepsilon_1,
$$

where ε_1 is small. If we want good argument distortion of an interval $\omega_0 = [k_0 a_0, a_0]$, i.e. the quotient $Q_N(a)/Q_N(a_0)$ is very close to 1 for all $a \in \omega_0$, then we must have

$$
\left|\frac{x'(a)}{x'(b)} - 1\right| \leq \varepsilon_2, \tag{5.18}
$$

for all $a \in \omega_0$, where ω_0 is a sufficiently small interval at the right end of $[0, a_0]$ and ε_2 is small enough. Equation (5.18) gives an estimate of the number k_0; it follows from (5.1) that it is enough to have

$$
|k_0^{k-1} - 1| \leq \varepsilon_3,
$$

for some suitable $\varepsilon_3 = \varepsilon_3(\varepsilon_2)$. In Subsection 5.1 we define a starting interval ω_0, at the very right end of $[0, a_0]$, so as to fulfil (5.18) for *all* critical points. More precisely, we have

Corollary 5.5. *If $\omega_0 = [k_0 a_0, a_0]$, where k_0 satisfies (5.18) for all critical points, then*

$$\left| \frac{Q_{N_l,l}(a)}{Q_{N_l,l}(a_0)} - 1 \right| \le 1/500, \tag{5.19}$$

for all $a \in \omega_0$ and all l, (if the ε_j:s are chosen suitable).

We also see that

$$\frac{1}{2}|x'(a)| \le |Q_N(a)| = \frac{|\xi_N'(a)|}{|(R^N)'(v(a),a)|} \le 2|x'(a)|,$$

if $a \le a_0$ for some sufficiently small $a_0 > 0$. In particular, if $x'(0) \neq 0$ then $Q_N(a) = K_2 + \mathcal{O}(a)$ in a neighbourhood of $a = 0$, so the equation (5.19) is valid in a whole interval $[0, a_0]$ instead of only a small ω_0 at the very right end of $[0, a_0]$. The parameter directions for which $x'(0) \neq 0$ are usually called *non-degenerate*.

5.1. **The first return. Definition of ω_0.** Here we shall define the interval $\omega_0 \subset [0, a_0]$. Also, we show that all critical orbits $\xi_{n,l}(\omega_0)$ will grow to size S, before leaving the repelling neighbourhood $\mathcal{N}_\rho$.

Assume that M_l is the first return time of $[0, a_0]$ into $J_{\Delta-2}$ under $\xi_{n,l}$, i.e. $M_l > 0$ is the least integer such that

$$\xi_{M_l,l}([0, a_0]) \cap J_{\Delta-2} \neq \emptyset.$$

Let $M = \min(M_l)$ and assume that $c_l(a')$ is a critical point which is mapped into $J_{\Delta-2}$, i.e. $\xi_{M,l}(a') \subset J_{\Delta-2}$, for some $a' \in [0, a_0]$. The curve $\xi_{M,l}([0, a_0])$ has to cross $J_{\Delta-1}$. If there is more than one index l for which $\xi_{M,l}(a) \subset J_{\Delta-2}$, for some $a \in [0, a_0]$, then we choose the least $a \in [0, a_0]$ such that $\xi_{M,l}(a) \subset J_{\Delta-2}$, for some l. Then it follows that $\xi_{M,l}([0, a]) \cap J_{\Delta-3} = \emptyset$ for all indices l. Now redefine a_0 such that $a_0 = a$ for this smallest a. We now study the orbits $\xi_{n,k}([0, a_0])$.

First, note that to every critical point $c_l(a_0)$ there is a number N_l such that $\xi_{N_l,l}(a_0) \in \Omega_2$, (see Definition 5.1. Also, compare with N_l in Lemma 5.2). Let us construct the interval $\omega_0 \subset [0, a_0]$:

SubLemma 1. *There is a $0 < k_0 < 1$, such that $\omega_0 = [k_0 a_0, a_0]$ has the following properties: Every curve $\xi_{N_l,l}(\omega_0)$, has length at least S. Moreover, for every interval $\omega \subset \omega_0$, such that $\xi_n(\omega) \subset \Omega$ (and $\xi_k(\omega) \subset \mathcal{N}_\rho$ for all $k \le n$), we have low argument distortion, (1.8) is fulfilled.*

Proof. As usual, let us drop the index l in the proof. By the definition of Ω_2 in Definition 5.1, there is an $N < M$ such that $\xi_N(a_0) \in \Omega_2$. Choose k_0' such that

$$\rho/(8\Lambda^3) \le |\xi_N(a_0) - \xi_N(k_0' a_0)| \le \rho, \tag{5.20}$$

and $\xi_N([k_0' a_0, a_0]) \subset \Omega$. Now, let $k_0' \le k_0 < 1$ be so that

$$\left| \frac{\xi_N'(a)}{\xi_N'(b)} - 1 \right| \le 1/200, \tag{5.21}$$

for all $a, b \in \omega_0 = [k_0 a_0, a_0]$. By (5.8) and (5.1), (5.21) is fulfilled if

$$|k_0^{k-1} - 1| \le \varepsilon_3, \tag{5.22}$$

for some sufficiently small ε_3. We choose the minimal k_0 such that (5.22) is fulfilled for every k corresponding to a critical point as in (5.1).

The length L of the curve $\xi_N(\omega_0)$ is

$$\begin{aligned} L &= |\xi_N'(a')||a_0 - a_0k_0| \geq (1/2)|\lambda_0|^N|x'(a')|a_0(1-k_0) \\ &\geq (1/4)|\lambda_0|^N kK_1(a_0k_0)^{k-1}a_0(1-k_0) = (1/4)|\lambda_0|^N kK_1a_0^k k_0^{k-1}(1-k_0). \end{aligned}$$

Reversing the inequality (5.12) we get

$$|\xi_N(a) - \xi_N(b)| \leq 2|\lambda_0|^N K_1(a^k - b^k), \tag{5.23}$$

if $\xi_N([a,b]) \subset \Omega$.

By (5.3) and (5.7), it follows that

$$\left|\frac{\lambda_a^N}{\lambda_b^N} - 1\right| \leq \varepsilon_4, \tag{5.24}$$

for all $a, b \in [0, a_0]$ if a_0 is small enough. Note that ε_3 and ε_4 are chosen such that $1/200$ in (5.21) is fulfilled. Putting $a = a_0$ and $b = k_0'a_0$ in (5.23), using (5.20) and (5.24), we get

$$\begin{aligned} \rho/(8\Lambda^3) \leq |\xi_N(a_0) - \xi_N(k_0'a_0)| &\leq \frac{8K_1}{3}|\lambda_0|^N|a_0^k - (k_0'a_0)^k| \\ &= \frac{8K_1}{3}|\lambda_0|^N a_0^k|1-(k_0')^k|, \end{aligned}$$

where Λ is the maximal multiplier over all repelling fixed points for R_a, $a \in [0, a_0]$, as in Definition 5.1. Thus,

$$\frac{L}{\rho/(8\Lambda^3)} \geq \frac{11k}{3}\frac{|\lambda_a|^N}{|\lambda_0|^N}\frac{k_0^{k-1}(1-k_0)}{(1-(k_0')^k)} \geq \frac{k_0^{k-1}(1-k_0)}{(1-(k_0')^k)}.$$

So the length L of the curve $\xi_N(\omega_0)$ is $L \geq S$, since $\rho \gg S = \delta/C_2$, if δ is small enough. Also, by (5.21), it is almost straight, i.e. it has bounded argument distortion.

By the definition of k_0, *all* critical orbits $\xi_{N_l,l}(\omega_0)$ of the interval $\omega_0 = (k_0a_0, a_0)$ grow to size at least S before they leave the neighbourhood $\mathcal{N}_\rho$, if δ is small enough compared to ρ.

It follows from the construction of ω_0 that if $\xi_{N_l,l}(\omega) \subset \Omega$ for some $\omega \subset \omega_0$, then $\xi_{N_l,l}(\omega)$ has small argument distortion for every l, i.e. (5.21) is satisfied. □

According to the partition rule, we cut the interval ω_0 so that $\omega \in \mathcal{P}_{N_l,l}$ means that $\xi_{N_l,l}(\omega)$ is of size at most S. Note that we do not delete any parameters from ω_0 until time M. However, the partition $\mathcal{P}_{N_l,l}$ may look different for different l.

Later we show that

$$\left|\frac{\xi_{M_l,l}'(a)}{\xi_{M_l,l}'(b)} - 1\right| \leq 1/100,$$

for all $a, b \in \omega \in \mathcal{P}_{M_l,l}, \omega \subset \omega_0$, where M_l are the first return times for the orbits of the sets $c_l(\omega_0)$ as above.

5.2. **The length of $c_i(\omega)$ is very small compared to the length of $\xi_n(\omega)$.** We now deal with the problem that the critical points may move as the parameter a moves, and therefore the sets $J_r(a)$ moves (see Definition 1.3). We will always assume that the parameter interval $[0, a_0]$ is chosen such that $a_0 \ll \delta^2$. Let ω be an interval in $[0, a_0]$. The idea is that the exponential growth of the derivative will imply that the part of the curve $\xi_n(\omega)$ which returns into U (call it $\xi_n(\omega')$) is almost straight and also that the corresponding parameter interval is exponentially small compared to the curve $\xi_n(\omega')$, i.e. $|\omega'| \ll l(\xi_n(\omega'))$, where $l(\xi_n(\omega'))$ is the length of $\xi_n(\omega')$. Let us assume that we have exponential growth of the z-derivative, i.e. $|(R_a^n)'(v(a))| \geq e^{\gamma n}$, for some $\gamma \geq \gamma_0$. The Mean Value Theorem, Remark 5.4 and Proposition 5.3 implies that

$$l(\xi_n(\omega')) = |\omega'||\xi_n'(a)| \geq (1/2)|\omega'||x'(a)|e^{\gamma n} \geq (1/2)|\omega'|e^{\gamma' n},$$

for some $a \in \omega'$, where $\gamma' \geq \gamma_0' = (1/k)\gamma_0$ (as in Lemma 5.2). Thus $|\omega'| \leq 2l(\xi_n(\omega'))e^{-\gamma' n}$. By the analyticity of $c_i(a)$, $|\omega'| \geq Cl(c_i(\omega'))$, where $c_i(\omega')$ is the set of critical points for the parameters ω' and $a \in \omega'$. Thus,

$$l(\xi_n(\omega')) \geq Ce^{\gamma' n} l(c_i(\omega')).$$

Now, the basic assumption (1.11) will imply

$$|\xi_n(a) - c_i(a)| \gg |\omega'|, \tag{5.25}$$

where $a \in \omega'$ and $\xi_n(\omega') \subset U_i$. Indeed, since $\xi_n(\omega') \subset U_i$ for some i we have $l(\xi_n(\omega')) \leq 3\delta$, if $\xi_n(\omega')$ is sufficiently straight, and therefore

$$|\omega'| \leq 6\delta e^{-\gamma' n} \leq e^{-\gamma' n} \ll e^{-\alpha n} \leq |\xi_n(a) - c_i(a)|,$$

since $\alpha \ll \gamma'$ and δ is small. Again, by the analyticity of $c_i(a)$, $|\omega'| \geq Cl(c_i(\omega'))$, and we get

$$|\xi_n(a) - c_i(a)| \gg l(c_i(\omega')), \tag{5.26}$$

if δ is sufficiently small. We have proved that the basic assumption (1.11) implies the condition (5.26) as long as we have exponential growth of the derivative ($\omega' \subset \mathcal{E}_n(\gamma)$, $\gamma \geq \gamma_0$) and control of the curvature.

Hence the parameter dependence for $J_r(a)$ is neglectable on a local level, i.e. if one looks only at $\xi_n(\omega')$ and compare it with the corresponding critical points $c_i(\omega')$.

5.3. **The relative sizes of $c_{ij}(\omega)$ and $c_{ik}(\omega)$.** We assume in this subsection that the interval ω_0 is defined as in Subsection 5.1, i.e. $\omega_0 = [k_0 a_0, a_0]$, where k_0 is very close to 1.

If $\xi_n(\omega')$ is almost straight (and $\omega' \subset \omega_0$), we want to show that the sets $c_{ij}(\omega')$ and $c_{ik}(\omega')$, for arbitrary i, j, k, are with a large amount of the relative sizes separated from each other:

$$\frac{|c_{ij}(a) - c_{ik}(a)|}{l(c_{ij}(\omega'))} \geq 100,$$

for $a \in \omega'$, $j \neq k$ and where $l(c_{ij}(\omega'))$ is the length of $c_{ij}(\omega')$.

Since the critical points $c_{ij}(a)$ are analytic functions of the parameter a, we have

$$c_{ij}(a) = c_i + k_{ij}a^r + \mathcal{O}(a^{r+1}), \tag{5.27}$$

for some $r \geq 1$, (the case $r = 0$ means that the critical points do not move under perturbation, and we can skip the definition of $\tilde{\delta}$ (see (1.8)), everything is then much easier). Let us, for simplicity, consider the largest critical star in the set $\mathcal{C}_i([0, a_0])$, i.e. the smallest r in (5.27). The other (smaller) critical stars can be treated in a similar way. Now, since $\tilde{\delta}$ is defined in terms of the diameter of the sets $\mathcal{C}_i([0, a_0])$, it follows that there is a constant C', only dependent on the unperturbed function $R(z)$ and the parameter direction, such that $\tilde{\delta} \leq C' a_0^r$, if a_0 is small enough. Assume that $\xi_n(\omega') \subset \tilde{U}$, for $\omega' \subset \omega_0$. If the basic assumption allows a return into $\tilde{U}$, then

$$e^{-\alpha n} \leq \tilde{\delta} \leq 2 k_0^{-r} C' a^r = C a^r,$$

for $a \in \omega_0 = [k_0 a_0, a_0]$, since k_0 is close to 1. By Proposition 5.3, Remark 5.4, and since the length of $\xi_n(\omega')$ is less than $2\tilde{\delta}$, we have

$$|x'(a)||\omega'|e^{\gamma n} \leq 2|\omega'||\xi_n'(a)| \leq 4\tilde{\delta},$$

for some $a \in \omega'$. By (5.27), $l(c_{ij}(\omega')) \sim |\omega'| a^{r-1}$. So, since $x'(a) \sim a^{k-1}$ as in Section 5,

$$l(c_{ij}(\omega')) \leq C e^{-\gamma n} \tilde{\delta} a^{r-1} a^{-(k-1)} \leq C e^{-\gamma n} a^{r-k}.$$

The distance $|c_{ij}(a) - c_{ik}(a)|$ is bounded from below by $C a^s$, for some integer $s \geq 0$ as shown in Subsection 1.3. Thus, if $s - r + k \geq 0$,

$$\begin{aligned} \frac{|c_{ij}(a) - c_{ik}(b)|}{l(c_{ij}(\omega'))} &\geq C a^s e^{\gamma n} a^{-r+k} \geq C e^{-n\alpha(s-r+k)/r} e^{\gamma n} \\ &= C e^{(\gamma - \alpha(s-r+k)/r)n} \geq 100, \end{aligned}$$

if n is large enough, and if $\gamma - \alpha(s - r + k)/r > 0$. If $s - r + k < 0$ then the estimate is trivial. Since every critical branch $c_{ij}(a)$ in the set $\mathcal{C}_i(a)$ gives rise to corresponding numbers s_{ij}, r_{ij}, k_{ij} we can apply this argument to all critical points. Let α be such that $\gamma - \alpha(s_{ij} - r_{ij} + k_{ij})/r > 0$ holds for all critical points.

6. Insignificant parameter dependence

We will in this section study the behaviour of a given orbit $\xi_{n,l}(\omega)$, between two return times. We count on the parameter dependence and show that under certain conditions, it is indeed insignificant. The last three lemmas of this section will be used inductively in the proof of the Main Distortion Lemma.

Definition 6.1 (Endpoint-stretch). Given a partition element $\omega \in \mathcal{P}_{n,l}$, we say that $\omega \in \mathcal{S}_{n,l}(\gamma')$ if

$$|\xi_{n,l}(a) - \xi_{n,l}(b)| \geq e^{\gamma' n}|a - b|. \tag{6.1}$$

for all $a, b \in \omega \in \mathcal{P}_{n,l}$.

The number γ' is highly dependent on *which* critical point that is iterated. In analogy with (5.1), let

$$x_l(a) = K_1 a^{k_l} + \mathcal{O}(a^{k_l+1}).$$

We call $\gamma'_{0,l} = (1/k_l)(\gamma_0 - \varepsilon)$, the "lower endpoint-stretch" exponent, where, eg. $\varepsilon = (\gamma_0 - 1)/1000$, and γ_0 is as in (1.7).

We begin with the the following basic fact which will be used frequently in the rest of this section.

SubLemma 2. *Assume that* $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}'_{n,l}$ *for some* $\gamma \geq \gamma_0$ *and that* n *is a return time. Then* $\omega \in \mathcal{S}_{n,l}(\gamma'_l)$ *where* $\gamma'_l \geq \gamma'_{0,l} = (1/k_l)\gamma_0 - \varepsilon$.

Proof. Certainly, the condition $\omega \in \mathcal{P}_n$ means by definition that $\omega \in \mathcal{G}_n$, so the curve $\xi_n(\omega)$ is indeed almost straight. By Proposition 5.3 and Lemma 5.2 we get that the length L of $\xi_n(\omega)$ is

$$L = |\omega||\xi'_n(a)| \geq (1/2)|\omega||Q_N(a)||(R^n)'(v(a), a)| \geq (1/2)|x'(a)|e^{\gamma n}$$
$$= (1/2)(K_1 a^k + \ldots)e^{\gamma N} e^{\gamma(n-N)} = e^{\gamma' N} e^{\gamma(n-N)} \geq e^{\gamma' n},$$

where $(1/k)\gamma - \varepsilon \leq \gamma' \leq \gamma - \varepsilon$. □

The main ingredient is that the condition $\omega \in \mathcal{S}_{n,l}(\gamma'_l)$ enables us to neglect the parameter dependence for $j \leq (1+\sigma)n$ for some fraction σ of the time. We let α be sufficiently small and define σ so as to fulfil the following:

$$\frac{4K\alpha}{\underline{\gamma}'} \leq \sigma = \frac{\underline{\gamma}'}{4\max(\log B, \log B')}, \tag{6.2}$$

where $\underline{\gamma}' = (1/2)\min(\gamma'_{0,l}) = (1/2)\min((\gamma_0 - \varepsilon)/k_l)$ is half the lowest endpoint-stretch exponent over all critical points, and

$$B = \max|\partial_a R(z,a)| \quad \text{and} \quad B' = \max|\partial_a R'(z,a)|.$$

The left inequality in (6.2) ensures that given a return at time n, we may neglect the parameter dependence during the bound period. The crucial fact is that the expansion of the z-derivative will imply that the parameter dependence is always neglectable!

Recall that we drop the index l and therefore we shall often only write $\xi_n(\omega)$ instead of $\xi_{n,l}(\omega)$. Also, $\mathcal{S}_n(\gamma')$ and γ'_0 means $\mathcal{S}_{n,l}(\gamma'_l)$ and $\gamma'_{0,l}$ respectively, for some l.

Let $N = \min(N'_k)$, where

$$N'_k = \min_{a \in \omega_0, n \in \mathbb{N}} \{n : \xi_{n,k}(a) \in \Omega_2\}.$$

(See Section 5, Definition 5.1 for the definition of Ω_2). We choose a_0 so small such that

$$e^{-(3\underline{\gamma}'/4)N} \leq e^{-\Delta K} 10^{-10} = \bar{\varepsilon}, \quad \text{and} \quad e^{-N\alpha} \leq e^{-2\Delta K}. \tag{6.3}$$

This is possible if the perturbation is sufficiently small compared to $\delta = e^{-\Delta}$ (the number 10^{-10} has no real significance, but it must be small). We also assume that $C_\sigma \geq 1000\bar{\varepsilon}$.

Let us illustrate the meaning of the number $\bar{\varepsilon}$. Assume that $\omega \in \mathcal{P}_n \cap \mathcal{S}_n(\gamma')$. We have, in general,

$$|\xi_{n+j}(a) - \xi_{n+j}(b)| \geq ||R^j(\xi_n(a), a) - R^j(\xi_n(b), a)| \tag{6.4}$$
$$- |R^j(\xi_n(b), a) - R^j(\xi_n(b), b)||. \tag{6.5}$$

The idea is that the parameter dependence (the term (6.5)) is much smaller than the right hand side of (6.4). We estimate (6.5) by the trivial estimate $|\partial_a R(z,a)| \leq B$, for all $(z,a) \in \hat{\mathbb{C}} \times [0, a_0]$, and by the fact that $\omega \in \mathcal{S}_n(\gamma')$:

$$|R^j(\xi_n(b), a) - R^j(\xi_n(b), b)| \leq |a - b|B^j \leq |\xi_n(a) - \xi_n(b)|e^{-\gamma' n + j\log B}.$$

Now, assuming that $j \leq \sigma n$ and $\gamma' \geq \underline{\gamma}'$, we get

$$e^{-\gamma' n + j \log B} \leq e^{-n(\gamma' + \sigma \log B)} \leq e^{-(3\underline{\gamma}'/4)n} \leq e^{-(3\underline{\gamma}'/4)N} \leq \bar{\varepsilon}.$$

The following lemma deals with the behaviour of a curve $\xi_n(\omega)$, which has returned into U. We show that the curve $\xi_n(\omega)$ contracts with an amount equivalent to the derivative of R_a. Recall that J_r means the annular neighbourhood defined in Definition 1.3.

Lemma 6.2 (Critical step)**.** *Assume that $\xi_{n,l}(\omega) \subset J_r \subset U$ and $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$ where $\gamma \geq \gamma_0$. For $a, b \in \omega$, we have*

$$\begin{aligned} C_0^{-1} |R'(\xi_{n,l}(a), a)||\xi_{n,l}(a) - \xi_{n,l}(b)| &\leq |\xi_{n+1,l}(a) - \xi_{n+1,l}(b)| \\ &\leq C_0 |R'(\xi_{n,l}(a), a)||\xi_{n,l}(a) - \xi_{n,l}(b)|, \end{aligned} \tag{6.6}$$

if Δ is sufficiently large.

Proof. We want to use the fact that the parameter dependence is neglectable. So first, we show that

$$\begin{aligned} (C_0 - \varepsilon)^{-1} |R'(\xi_n(a), a)||\xi_n(a) - \xi_n(b)| &\leq |R(\xi_n(a), a) - R(\xi_n(b), a)| \\ &\leq (C_0 - \varepsilon)|R'(\xi_n(a), a)||\xi_n(a) - \xi_n(b)|, \end{aligned} \tag{6.7}$$

for some very small $\varepsilon > 0$. Let $\gamma(t) = \xi_n(a) + t(\xi_n(b) - \xi_n(a))$. The length of the curve $R_a(\gamma(t))$ for $0 \leq t \leq 1$ is

$$L = \int_0^1 |R'(\gamma(t), a)| dt = |\xi_n(a) - \xi_n(b)||R'(\gamma(t'), a)|,$$

for some $t' \in [0, 1]$. Note that

$$|\xi_n(a) - \xi_n(b)| \leq e^{-r}/r^2 \ll |\xi_n(a) - c(a)| \sim e^{-r}, \tag{6.8}$$

where $c(a)$ is the nearest critical point to $\xi_n(a)$. By (6.8), it is easy to verify that $R'_a(\gamma(t))$ changes very little, for $0 \leq t \leq 1$, if Δ is sufficiently large. Indeed, looking at the formula for R'_a in (2.2), we have $|\arg(R'_a(\gamma(t))/R'_a(\gamma(s)))| \leq 2N_i \arctan(1/r^2)$, where N_i is the number of critical points in the set $\mathcal{C}_i(a)$. Also, it is obvious that $|R'_a(\gamma(t))|/|R'_a(\gamma(s))| \leq 2(1 - 1/r^2)^{N_i}$, if Δ is large. So, if Δ is large enough, we have

$$\left| \frac{R'_a(\gamma(t))}{R'_a(\gamma(s))} - 1 \right| \leq 1/100,$$

for all $s, t \in [0, 1]$. The curvature $\kappa(t)$ of the curve $R_a(\gamma(t))$ is by Lemma 2.2 and Lemma 4.1 for the case $n = 1$,

$$\kappa(t) \leq \frac{|R''(\gamma(t), a)|}{|R'(\gamma(t), a)|^2} \leq C_1^3 e^{\tilde{d}_i r},$$

since the curvature of $\gamma(t)$ is zero.

If L is the length of $R_a(\gamma(t))$, then

$$\kappa(t) L \leq |\xi_n(a) - \xi_n(b)||R'(\gamma(t'), a)| C_1^3 e^{\tilde{d}_i r} \leq \frac{e^{-r}}{r^2} C_1 e^{-(\tilde{d}_i - 1)r} C_1^3 e^{\tilde{d}_i r} \leq \frac{C_1^4}{r^2},$$

which is very small if Δ is large. Thus the curve $R_a(\gamma(t))$ is almost straight, and therefore the length of the curve is almost the same as $|R(\xi_n(a), a) - R(\xi_n(b), a)|$, so (6.7) holds.

On the other hand, by Lemma 2.2 and the basic assumption $\mathcal{B}_n$ (the basic assumption implies that $r \le \alpha n + 1$),

$$|R'(\xi_n(a), a)| \ge C_1^{-1} e^{-(\tilde{d}_i - 1)r} \ge C_1^{-1} e^{-Kr} \ge C e^{-K\alpha n}.$$

By (6.1)

$$|R(\xi_n(b), a) - R(\xi_n(b), b)| \le B|a - b| \le B e^{-\gamma' n} |\xi_n(a) - \xi_n(b)|.$$

So, finally

$$\begin{aligned} |\xi_{n+1}(a) - \xi_{n+1}(b)| &\ge |R(\xi_n(a), a) - R(\xi_n(b), a)| - |R(\xi_n(b), a) - R(\xi_n(b), b)| \\ &\ge \big((C_0 - \varepsilon)^{-1} |R'(\xi_n(a), a)| - B e^{-\gamma' n}\big) |\xi_n(a) - \xi_n(b)| \\ &\ge C_0^{-1} |R'(\xi_n(a), a)| \xi_n(a) - \xi_n(b)| \end{aligned}$$

since $e^{-K\alpha n} \gg e^{-\gamma' n}$, if $\gamma' \ge \gamma_0' > K\alpha$, and n is large. The right inequality in (6.6) is proved in the same way. □

The following two lemmas deal with the bound period. First, we show that the bound period is bounded from above and below, and after that, we give an estimate of the derivative after the bound period. Before presenting these two lemmas, we give an estimate of the diameter of convex set $\mathcal{K}(\overline{\xi}_{n+1,l}(\omega), v_k(\omega))$, if $\xi_{n,l}(\omega) \subset J_r \subset U$ is a return, and if k is chosen such that HD-dist$(\xi_{n,l}(\omega), c_k(\omega))$ is minimal. Assume that $\omega \in \mathcal{P}_{n,l}$, and $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$. By Lemma 2.2, $|\xi_{n+1,l}(s) - v_i(s)| \le C_1 e^{-r\tilde{d}_i}$ for all $s \in \omega$.

In the case of an essential return in Lemma 6.2, by Lemma 2.2, we get

$$|\xi_{n+1,l}(a) - \xi_{n+1,l}(b)| \sim |R_a'(\xi_{n,l}(a))||\xi_{n,l}(a) - \xi_{n,l}(b)| \sim e^{-r\tilde{d}_i} \frac{e^{-r}}{r^2} = \frac{e^{-r\tilde{d}_i}}{r^2}.$$

Let us consider a host curve $\overline{\xi}_{n,l}(\omega) = \xi_{n,l}(\omega) \cup L_2$, as in Definition 1.11 instead, which has length e^{-r}/r^2. An easy argument very similar to that in the proof of Lemma 6.2 shows that any two points $z, w \in \overline{\xi}_{n,l}(\omega)$ has that

$$C^{-1} |R_a'(z')||z - w| \le |R_a(z) - R_a(w)| \le C |R_a'(z')||z - w|,$$

where C is close to 1. This, together with Lemma 2.2, implies that the endpoints of the curve $\overline{\xi}_{n+1,l}(\omega)$ is separated from each other with at least $e^{-\tilde{d}_i r}/r^2$. Altogether, this means that we have the following:

SubLemma 3. *Assume that $\xi_{n,l}(\omega) \subset J_r^i$, $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$. Assume that k is chosen such that HD-dist$(\xi_{n,l}(\omega), c_k(\omega))$ is minimal. Then*

$$C^{-1} \frac{e^{-r\tilde{d}_i}}{r^2} \le \text{ diam } (\mathcal{K}(\overline{\xi}_{n+1}(\omega), v_k(\omega))) \le C e^{-r\tilde{d}_i}, \tag{6.9}$$

where C does not depend on δ and diam(X) stands for the diameter of the set X.

Lemma 6.3. *Assume that $\xi_{n,l}(\omega) \subset J_r \subset U$ is a return and $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$, for some $\gamma \ge \gamma_0$. Then the bound period $p = p(\omega) \le (K\alpha/\gamma)n$ and*

$$\frac{(\tilde{d}_i - 1)r}{\Gamma} \le p \le \frac{\tilde{d}_i r}{\gamma}, \tag{6.10}$$

where $\Gamma = \sup_{z\in\hat{\mathbb{C}}} \log |R'(z)|$.

Proof. Let us consider the case of an essential return. If $\xi_n(\omega)$ is an inessential return, we use the host curve $\overline{\xi}_n(\omega)$. In that case the proof goes through in the same way. Assume that $\omega = [a, b]$. Set $z = \xi_n(a), w = \xi_n(b)$ and let $z_0 = \xi_{n+1}(a)$ and $w_0 = \xi_{n+1}(b)$. By Lemma 6.2 we have

$$|z_0 - w_0| \geq (1/C_0)|R'(z, a)||z - w|. \tag{6.11}$$

By the definition of bound period, Corollary 2.5 and (6.11),

$$\begin{aligned} e^{-\beta j} &\geq |R_a^j(z_0) - R_a^j(w_0)| \geq (1/C_0)|z_0 - w_0||(R^j)'(z_0, a)| \\ &\geq (1/C_0)|(R^j)'(z_0, a)||R'(z, a)||z - w|, \end{aligned}$$

for $j \leq p - 1$. By the definition of an essential return (or the host curve) we have $|z - w| \geq e^{-r}/(2r^2)$. Since $a \in \mathcal{E}_n(\gamma)$, Lemma 2.2 and Lemma 2.4 gives

$$e^{-\beta j} \geq (1/C_0C_1)|(R^j)'(z_0, a)|e^{-r(\tilde{d}_i-1)}e^{-r}/(2r^2) \geq C_0C_1e^{\gamma j}e^{-r\tilde{d}_i}/(2r^2).$$

Therefore,

$$e^{\gamma j}e^{-(\tilde{d}_i+(2/r)\log r)r} \leq Ce^{-\beta j}, \text{ for } j \leq \min(p, n).$$

The basic assumption implies $e^{-r+1} \geq |z - c_i(a)| \geq e^{-\alpha n}$, so $r \leq \alpha n + 1$. This gives

$$j \leq \frac{1}{\gamma}((\tilde{d}_i + (2/r)\log 2r)r - \beta j + \log C) \leq \frac{\tilde{d}_i r + \log C}{\gamma} \leq \frac{\tilde{d}_i(\alpha n + 1) + \log C}{\gamma} < n,$$

if n is sufficiently large, since $\gamma > \alpha\tilde{d}_i$. So, for sufficiently large $r \geq \Delta$

$$p \leq \frac{(\tilde{d}_i + (2/r)\log 2r)r}{\gamma + \beta} \leq \frac{\tilde{d}_i r}{\gamma},$$

and the right hand side of (6.10) is proved.

On the other hand, at time $j = p$ we have

$$|R^p(z_0, a) - R^p(w_0, b)| \geq e^{-\beta p}. \tag{6.12}$$

for some $z_0, w_0 \in \mathcal{K}(\overline{\xi}_{n+1}(\omega) \cup v_i(\omega))$ and some $a, b \in \omega$. By (1.2), Lemma 2.2 we get

$$\begin{aligned} |R^p(z_0, a) - R^p(w_0, b)| &\leq |R^p(z_0, a) - R^p(w_0, a)| + |R^p(w_0, a) - R^p(w_0, b)| \\ &\leq |(R_a^p)'(\gamma(t))||z_0 - w_0| + |a - b|B^p. \end{aligned}$$

By the right hand side of (6.10), which we just proved, and (6.2) we have $p \leq Kr/\gamma \leq K\alpha n/\gamma \leq \sigma n$. Since $\omega \in \mathcal{S}_n(\gamma')$, $\gamma' \geq \gamma_0'$, we get

$$|a - b|B^p \leq e^{-\gamma' n + p\log B} \leq e^{-n(\sigma \log B - \gamma')} \leq e^{-(3\underline{\gamma}'/4)n} \ll e^{-\beta n} \leq (1/4)e^{-\beta p}.$$

Since the diameter of the set $\mathcal{K}(\overline{\xi}_{n+1}(\omega), v_i(\omega))$ is at most $Ce^{-r\tilde{d}_i}$, we get $|z_0 - w_0| \leq Ce^{-r\tilde{d}_i}$. Let $\gamma(t) = z_0 + t(w_0 - z_0)$. We have

$$|R^p(z_0, a) - R^p(w_0, b)| \leq 2|(R_a^p)'(\gamma(t))||z_0 - w_0| \leq 4C_1e^{\Gamma p}e^{-r\tilde{d}_i} + (1/4)e^{-\beta p}, \tag{6.13}$$

for some $t \in [0,1]$. Finally, by (6.12) we have

$$p \geq \frac{(\tilde{d}_i - 1)r}{\Gamma},$$

since $\beta \ll \Gamma$. □

Definition 6.4 (Pseudo bound period). Assume that $\xi_n(\omega) \subset U' \setminus U$, i.e. a pseudo return. Then we define the *pseudo bound period* to be the largest integer p such that $\xi_{n+j}(\omega) \subset \mathcal{N}_\rho$, for all $0 < j \leq p$.

Lemma 6.5. *Assume that $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$, for some $\gamma \geq \gamma_0$. If $\xi_{n,l}(\omega) \subset U_i' \setminus U_i^2$ (i.e. $\xi_{n,l}(\omega)$ is a pseudo return or a shallow return), then*

$$|(R^p)'(\xi_{n,l}(a), a)| \geq (|\lambda_i| - \varepsilon)^{p/(\tilde{d}_i+1)},$$

for all $a \in \omega$, where p is its bound period, and where λ_i is the multiplier of the repelling fixed point $p_i(a)$ ($\varepsilon \leq (|\lambda_i| - 1)/1000$, provided δ is sufficiently small and $\delta \gg a_0$). For general returns, i.e. if $\xi_n(\omega) \subset U$, we have

$$|(R^p)'(\xi_{n,l}(a), a)| \geq e^{p\gamma/(\tilde{d}_i+1)}.$$

Proof. First, we assume that $\xi_n(a) \in U \setminus U^2$. By Lemma 6.3,

$$p \leq \frac{Kr}{\gamma} \leq \frac{K\Delta}{\gamma} = C'\Delta.$$

Now if $|\omega_0|$ (or, rather a_0) is sufficiently small compared to δ^2, we have

$$|\xi_j(a) - \xi_j(0)| \leq \delta^{4K},$$

for all $j \leq p \leq C'\Delta$ and $a \in \omega_0$. This means that

$$\begin{aligned} e^{-\beta j} + \delta^{4K} &\geq |\xi_{n+j}(a) - \xi_j(a)| + |\xi_j(0) - \xi_j(a)| \geq |\xi_{n+j}(a) - \xi_j(0)| \\ &= |\xi_{n+j}(a) - v_i|. \end{aligned} \tag{6.14}$$

Assume that $z = \xi_n(a) \in J_r^i$, i.e. $|z - c_i(a)| \sim e^{-r}$. By the definition of bound period we have

$$|R^p(z_0, a) - R^p(w_0, b)| \geq e^{-\beta p},$$

for some $z_0, w_0 \in \mathcal{K}(\overline{\xi}_{n+1}(\omega) \cup v_i(\omega))$ and some $a, b \in \omega$, as in Lemma 6.3. Let $\gamma(t) = z_0 + t(w_0 - z_0)$. We can use precisely the same calculations from equation (6.12) to equation (6.13) in Lemma 6.3. We have $|z_0 - w_0| \leq Ce^{-r\tilde{d}_i}$, again by (6.9). Therefore,

$$\begin{aligned} (3/4)e^{-\beta p} &\leq |R^p(z_0, a) - R^p(w_0, b)| \leq 2|(R_a^p)'(\gamma(t))||z_0 - w_0| \\ &\leq C|(R_a^p)'(\gamma(t))|e^{-r\tilde{d}_i}, \end{aligned}$$

for some $t \in [0,1]$. Now, using Lemma 2.4 we get

$$e^{-r\tilde{d}_i} D_{p-1} \geq Ce^{-\beta(p-1)} \geq Ce^{-\beta p},$$

with $D_k = |(R^k)'(R_a(z), a)|$. This implies

$$e^{-r(\tilde{d}_i - 1)} \geq Ce^{-\beta p(\tilde{d}_i - 1)/\tilde{d}_i} D_{p-1}^{(1-\tilde{d}_i)/\tilde{d}_i}.$$

By the Chain Rule and Lemma 2.2,

$$|(R^p)'(z,a)| = |R'(z,a)|D_{p-1} \geq C_1^{-1} e^{-r(\tilde{d}_i - 1)} D_{p-1}$$
$$\geq C e^{-\beta p(\tilde{d}_i-1)/\tilde{d}_i} D_{p-1}^{1/\tilde{d}_i}, \tag{6.15}$$

if p is sufficiently large, i.e. if Δ is sufficiently large (see Lemma 6.3). By Lemma 6.3 we have $e^{-\beta p} \leq e^{-\beta r/\Gamma} \leq e^{-\beta \Delta'/\Gamma} \leq \rho$ so $\xi_{n+p}(a) \in \mathcal{N}_\rho$, by (6.14). Thus the bound period for shallow returns, ends before leaving the neighbourhood $\mathcal{N}_\rho$ of the repelling fixed points. Therefore we can use the multiplier λ_i to get

$$|(R^p)'(z,a)| \geq (|\lambda_i| - \varepsilon)^{p/(\tilde{d}_i+1)}, \tag{6.16}$$

if Δ is sufficiently large, ($\varepsilon \leq (|\lambda_i| - 1)/1000$).

For pseudo returns, i.e. $z = \xi_n(a) \in U' \setminus U$, (so $\delta < |z - c_i| \leq \delta'$), we use the Taylor expansion around c_i;

$$R(z) = v_i + (z - c_i)^{d_i} + \ldots.$$

Assuming that $|z - c_i| \sim e^{-r}$ we get, for small perturbations, $|R_a(z) - v_i| \sim e^{-d_i r}$. By an argument similar to that of the beginning of the proof of Lemma 2.7, we get $p \leq C\Delta$. Thus, for small perturbations,

$$|R^p(z,a) - R^p(z,0)| \leq aB^p \leq aB^{C\Delta},$$

which can be made arbitrary small if $a \leq a_0$ is sufficiently small compared to δ. Thus, by Lemma 2.7, with $\lambda_i = R'(p_i(0), 0)$, we get

$$|R_a^p(R_a(z)) - v_i| \leq 2|R_0^p(R_a(z)) - v_i| \leq 2(|\lambda_i| + \varepsilon)^p e^{-d_i r} \leq 2\rho \leq 1,$$

which implies $p \leq d_i r / \log(|\lambda_i| + \varepsilon) \leq d_i r/\gamma$. Thus, we get the same upper bound on p as in Lemma 6.3 and also, we can use the same estimate as in (6.16).

For general returns, if $a \in \mathcal{E}_n(\gamma)$ then (6.15) gives

$$|(R^p)'(z,a)| \geq e^{\gamma p/(\tilde{d}_i+1)}.$$

□

In the following lemmas in this section, for every partition element ω, we roughly prove that

$$|\xi_{n+j}(a) - \xi_{n+j}(b)| \sim |(R^j)'(\xi_n(a),a)||\xi_n(a) - \xi_n(b)|,$$

for all $a, b \in \omega$ under the basic assumption and if the derivative has expanded exponentially up to time n. These "inductive" lemmas will be used repeatedly in the proof of the Main Distortion Lemma, which in turn, will give control of the geometry, by Lemma 7.1. Let us start with the bound period:

Lemma 6.6 (Bound period)**.** *Assume that $\xi_{n,l}(\omega) \subset U$ and $\omega \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\gamma, l) \cap \mathcal{B}_{n,l}$ for some $\gamma \geq \gamma_0$. Then for all $a, b \in \omega$*

$$C_0'^{-1}|(R^j)'(\xi_{n,l}(a),a)||\xi_{n,l}(a) - \xi_{n,l}(b)| \leq |\xi_{n+j,l}(a) - \xi_{n+j,l}(b)| \leq C_0'|(R^j)'(\xi_{n,l}(a),a)||\xi_{n,l}(a) - \xi_{n,l}(b)|, \tag{6.17}$$

for $j \leq p$.

Proof. The condition $\omega \in \mathcal{P}_n$ implies by definition that (1.8) is fulfilled. By Sublemma 2 we have $\omega \in \mathcal{S}_n(\gamma')$ for some $\gamma' \geq \gamma'_0$. We have

$$|\xi_{n+j}(a) - \xi_{n+j}(b)| \geq |R^j(\xi_n(a), a) - R^j(\xi_n(b), a)| \\ - |R^j(\xi_n(b), a) - R^j(\xi_n(b), b)|.$$

By Corollary 2.5,

$$|R^j(\xi_{n+1}(a), a) - R^j(\xi_{n+1}(b), a)| \geq C_0^{-1}|(R^j)'(\xi_{n+1}(a), a)||\xi_{n+1}(a) - \xi_{n+1}(b)|.$$

Lemma 6.2, Lemma 2.2 and the basic assumption implies

$$\begin{aligned} |\xi_{n+1}(a) - \xi_{n+1}(b)| &\geq C_0^{-1}|R'(\xi_n(a), a)||\xi_n(a) - \xi_n(b)| \\ &\geq Ce^{-(K-1)\alpha n}|\xi_n(a) - \xi_n(b)| \geq e^{-K\alpha n}e^{\gamma' n}|a - b| \\ &= e^{(\gamma' - K\alpha)n}|a - b|. \end{aligned}$$

So, $\omega \in \mathcal{S}_{n+1}(\gamma'_1)$, where $\gamma'_1 = \gamma' - K\alpha$. This implies

$$|R^j(\xi_{n+1}(b), a) - R^j(\xi_{n+1}(b), b)| \leq B^j e^{-\gamma'_1(n+1)}|\xi_{n+1}(a) - \xi_{n+1}(b)|.$$

Therefore, we get

$$|\xi_{n+j}(a) - \xi_{n+j}(b)| \geq \left[C_0^{-1}|(R^{j-1})'(\xi_{n+1}(a), a)| - e^{(j-1)\log B - \gamma'(n+1)}\right] \\ \cdot |\xi_{n+1}(a) - \xi_{n+1}(b)|. \tag{6.18}$$

The second term in (6.18) is less than $\bar{\varepsilon}$ when $j \leq \sigma(n+1)$, where σ is as in (6.2). By Lemma 6.3,

$$p \leq \frac{\tilde{d}_i r}{\gamma} \leq \frac{K\alpha n}{\gamma} \leq \frac{K\alpha n}{\gamma_0} \leq \sigma(n+1),$$

by (6.2). Since, by Lemma 2.4, $|(R^{j-1})'(\xi_{n+1}(a), a)| \geq C_0^{-1}e^{\gamma(j-1)}$, which is much greater than $\bar{\varepsilon}$, we get

$$|\xi_{n+j}(a) - \xi_{n+j}(b)| \geq C_0'^{-1}|(R^j)'(\xi_n(a), a)||\xi_n(a) - \xi_n(b)|.$$

The other inequality is proved analogously. □

The following lemma show that points $\xi_n(a)$ and $\xi_n(b)$ repel each other during the free period. During the free period, we have a uniform expansion from Lemma 3.8. So, assume that $R_a^j(z) \notin U$ for $0 \leq j \leq n_0$. For sufficiently large time periods n_0 we have

$$|(R_a^{n_0})'(z)| \geq C\lambda^{n_0} \inf_{0 \leq j \leq n_0} |R'_a(R_a^j(z))| \geq C\lambda^{n_0}e^{-\Delta K} \geq e^{(\log\lambda - \varepsilon/2)n_0},$$

where $\varepsilon = (\gamma_0 - 1)/1000$. The condition $\mathcal{S}_n(\gamma')$ enables us to go forward σn iterates with insignificant parameter dependence. Therefore, we may choose the perturbation a_0 sufficiently small so as to fulfil $\sigma N \geq n_0$, (see (6.3). This expansion is much larger than the parameter dependence $\bar{\varepsilon}$ in (6.3), and we will get the endpoint stretch condition satisfied at time $(1+\sigma)n$, ($a \in \mathcal{S}_{(1+\sigma)n}(\gamma')$). We can go on in the same manner and until a return occurs.

Lemma 6.7 (Free period)**.** *Assume that $\omega \in \mathcal{P}_{\nu_j,l}$, $\omega \subset \mathcal{E}_{\nu_j}(\gamma, l) \cap \mathcal{B}_{\nu_j,l}$, for $\gamma \geq \gamma_0$ and $|\xi_{k,l}(a) - \xi_{k,l}(b)| \leq S$ for all $a, b \in \omega$ and all k such that $\nu_j + p_j \leq k \leq m \leq \nu_{j+1}$. Take any i with $\nu_j + p_j \leq i \leq m$. If $\xi_{k,l}(\omega) \cap U = \emptyset$, for all $\nu_j + p_j \leq k \leq m$ then*

$$|\xi_{m,l}(a) - \xi_{m,l}(b)| \geq (7/8)^t |(R^{m-i})'(\xi_{i,l}(a), a)||\xi_{i,l}(a) - \xi_{i,l}(b)|$$

for all $a, b \in \omega$, where $t \leq \log(m-i)/\log(1+\sigma)$ and σ is as in (6.2). If $\xi_{k,l}(\omega) \cap U = \emptyset$, for $\nu_j + p_j \leq k \leq m \leq \nu_{j+1}$ and $\xi_{m,l}(\omega) \subset P_{\delta'}^c$, then

$$|\xi_{m,l}(a) - \xi_{m,l}(b)| \geq C_\sigma e^{(99/100)(\log \lambda)(m-i)} |\xi_{i,l}(a) - \xi_{i,l}(b)|,$$

for all $a, b \in \omega$.

Proof. **Part I. The case $i = \nu_j + p_j$.** We first show that $\omega \in \mathcal{S}_{\nu_j + p_j}(\gamma_1')$, for some slightly smaller $\gamma_1' > 0$ instead of $\gamma' = (1/k)\gamma - \varepsilon$. By Sublemma 2, Lemma 6.5 and Lemma 6.6 we get

$$\begin{aligned} |\xi_{\nu_j+p_j}(a) - \xi_{\nu_j+p_j}(b)| &\geq C_0'^{-1} |(R_a^{p_j})'(\xi_{\nu_j}(a))||\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \\ &\geq C_0'^{-1} e^{\gamma p_j/(K+1)} |\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \\ &\geq C_0'^{-1} e^{\gamma p_j/(K+1)} e^{\gamma' \nu_j} |a - b| \geq e^{\gamma_1'(\nu_j + p_j)} |a - b|, \end{aligned}$$

for some $\gamma_1' \leq \gamma'$. Indeed, since $p \leq K\alpha\nu_j/\gamma$, we have, eg. $\gamma_1' \geq \gamma' - K\alpha \geq \gamma_0' - K\alpha \geq \underline{\gamma}'$. We want to use the fact that the parameter dependence is neglectable as long as $n - i \leq \sigma i$ as in Lemma 6.6, where σ is as in (6.2). Indeed,

$$|R_a^{n-i}(\xi_i(b), a) - R_a^{n-i}(\xi_i(b), b)| \leq |a - b| B^{n-i} \leq \overline{\varepsilon}.$$

For $m \leq \nu_{j+1}$, Lemma 4.3 implies

$$|R_a^{m-i}(\xi_i(a)) - R_a^{m-i}(\xi_i(b))| \geq (15/16)|(R_a^{m-i})'(\xi_i(a))||\xi_i(a) - \xi_i(b)|.$$

There are two cases. First, assume that $m - i \leq \sigma i$. We estimate $|(R_a^{m-i})'(\xi_i(a))|$ by Lemma 3.8;

$$|(R_a^{m-i})'(\xi_i(a))| \geq C\lambda^{m-i} \inf_{i \leq j \leq m} |R_a'(\xi_j(a))|.$$

So, since $\xi_j(a) \cap U = \emptyset$ for all $i \leq j \leq m$,

$$|R_a'(\xi_j(a))| \geq e^{-\Delta K} \geq 10^{10}\overline{\varepsilon},$$

we get

$$\begin{aligned} |\xi_m(a) - \xi_m(b)| &\geq |R^{m-i}(\xi_i(a), a) - R^{m-i}(\xi_i(b), a)| \\ &\quad - |R^{m-i}(\xi_i(b), a) - R^{m-i}(\xi_i(b), b)| \\ &\geq ((15/16)|(R_a^{m-i})'(\xi_i(a))| - e^{(m-i)\log B - \gamma_1' i})|\xi_i(a) - \xi_i(b)| \\ &\geq (7/8)|(R_a^{m-i})'(\xi_i(a))||\xi_i(a) - \xi_i(b)|. \end{aligned}$$

If $\xi_m(\omega) \subset P_{\delta'}^c$ then we estimate $|(R_a^{m-i})'(\xi_i(a))|$ with the Outside Expansion Lemma;

$$|(R_a^{m-i})'(\xi_i(a))| \geq C_\sigma \lambda^{m-i} \geq C_\sigma \geq 1000\overline{\varepsilon}.$$

Thus,

$$|\xi_m(a) - \xi_m(b)| \geq (7/8)C_\sigma \lambda^{m-i} |\xi_i(a) - \xi_i(b)|.$$

On the other hand, if $m-i>\sigma i$ then let n be such that $n-i=\sigma i$. Now we estimate $|(R_a^{n-i})'(\xi_i(a))|$ with Lemma 3.8;

$$\begin{aligned}|(R_a^{n-i})'(\xi_i(a))| &\geq C\lambda^{n-i}\inf_{i\leq j\leq n}|R_a'(\xi_j(a))|\\ &\geq Ce^{(n-i)\log\lambda-\Delta K}\geq e^{(n-i)(\log\lambda-\varepsilon/2)},\end{aligned}$$

if $n-i=\sigma i$ is sufficiently large, i.e. if $i\geq N$ for sufficiently large N, ($\varepsilon=(\gamma_0-1)/1000$). Since σ is chosen such that the parameter dependence $e^{(n-i)\log B-\gamma_1' i}\leq\varepsilon$, we get by the initial endpoint-stretch condition (5.10)

$$\begin{aligned}(6.19)\quad |\xi_n(a)-\xi_n(b)| &\geq |R^{n-i}(\xi_n(a),a)-R^{n-i}(\xi_n(b),a)|\\ &\quad-|R^{n-i}(\xi_n(b),a)-R^{n-i}(\xi_n(b),b)|\\ &\geq((15/16)|(R_a^{n-i})'(\xi_i(a))|-e^{(n-i)\log B-\gamma_1' i})|\xi_i(a)-\xi_i(b)|\\ &\geq(7/8)|(R_a^{n-i})'(\xi_i(a))||\xi_i(a)-\xi_i(b)|\\ &\geq(7/8)e^{(n-i)(\log\lambda-\varepsilon/2)}e^{\gamma_1' i}|a-b|\\ &\geq e^{(n-i)(\log\lambda-\varepsilon)+i\gamma_1'}|a-b|\geq e^{n\gamma_1'}|a-b|,\end{aligned}$$

since $\gamma_1'\leq\log\lambda-\varepsilon$ (see after Definition 6.1). So the endpoint-stretch condition (6.1) holds also at time n, i.e. $\omega\in\mathcal{S}_n(\gamma_1')$. Repeating this argument gives a sequence of times n_k, with $n_0=i$, and exponents $\gamma_k'\geq\gamma_1'$ such that

$$|\xi_{n_t}(a)-\xi_{n_t}(b)|\geq\prod_{k=0}^{t-1}((15/16)|(R_a^{j_k})'(\xi_{n_k}(a))|-e^{j_k\log B-\gamma_k' n_k})|\xi_{n_0}(a)-\xi_{n_0}(b)|,$$

where $j_k=n_{k+1}-n_k=\sigma n_k$,. Thus, (recall $n_0=i$),

$$\begin{aligned}|\xi_{n_t}(a)-\xi_{n_t}(b)| &\geq(7/8)^t\prod_{k=0}^{t-1}|(R_a^{j_k})'(\xi_{n_k}(a))||\xi_{n_0}(a)-\xi_{n_0}(b)|\\ &=(7/8)^t|(R_a^{n_l-n_0})'(\xi_{n_0}(a))||\xi_{n_0}(a)-\xi_{n_0}(b)|.\end{aligned}$$

In the case $\xi_m(\omega)\subset P_{\delta'}^c$, by the Outside Expansion Lemma we get

$$\begin{aligned}|\xi_m(a)-\xi_m(b)| &\geq(7/8)^t|(R^{m-i})'(\xi_i(a),a)||\xi_i(a)-\xi_i(b)|\\ &\geq(7/8)^tC_\sigma\lambda^{m-i}|\xi_{n_0}(a)-\xi_{n_0}(b)|\\ &\geq C_\sigma e^{((99/100)\log\lambda)(m-i)}|\xi_i(a)-\xi_i(b)|.\end{aligned}$$

Part II. The case $\nu_j+p_j\leq i\leq\nu_{j+1}$. We only have to show that $\omega\in\mathcal{S}_i(\gamma_1'')$, for some $\gamma_1''\geq\gamma_0'-2K\alpha\geq\underline{\gamma}'$.

We have just proved that

$$|\xi_i(a)-\xi_i(b)|\geq(7/8)^t|(R_a^{i-\nu_j-p_j})'(\xi_{\nu_j+p_j}(a))||\xi_{\nu_j+p_j}(a)-\xi_{\nu_j+p_j}(b)|,$$

where $t\leq\log(i-p_j-\nu_j)/\log(1+\sigma)$. To estimate $|(R_a^{i-\nu_j-p_j})'(\xi_{\nu_j+p_j}(a))|$ we use Lemma 3.8;

$$(6.20)\qquad |(R_a^{i-\nu_j-p_j})'(\xi_{\nu_j+p_j}(a))|\geq C\lambda^{i-\nu_j-p_j}\inf_{\nu_j+p_j\leq j\leq i-\nu_j-p_j}|R_a'(\xi_j(a))|.$$

We have that $e^{-K\Delta} \geq e^{-N\alpha}$ as in (6.3), and therefore

$$|R'_a(\xi_j(a))| \geq e^{-K\Delta} \geq e^{-j(K-1)\alpha} \geq e^{-j(K-1)\alpha},$$

since $j \geq N$. Thus,

$$\begin{aligned} |\xi_i(a) - \xi_i(b)| &\geq (7/8)^t |(R_a^{i-\nu_j-p_j})'(\xi_{\nu_j+p_j}(a))||\xi_{\nu_j+p_j}(a) - \xi_{\nu_j+p_j}(b)| \\ &\geq Ce^{(i-\nu_j-p_j)(\log\lambda)-i(K-1)\alpha} e^{\gamma_1'(\nu_j+p_j)}|a-b| \\ &\geq e^{(\gamma_1'-K\alpha)i}|a-b| = e^{\gamma_1'' i}|a-b|, \end{aligned}$$

where $\gamma_1'' = \gamma_1' - K\alpha \geq \gamma_0' - 2K\alpha \geq \underline{\gamma}'$. Hence the lemma is proved also for arbitrary $i \leq \nu_{j+1}$. □

Finally, using the three Lemmas 6.2, 6.6 and 6.7, let us prove that a curve $\xi_\nu(\omega)$ has indeed expanded between two consecutive returns.

Lemma 6.8. *Assume that $\xi_{\nu_j,l}(\omega) \subset J_r^i$, $\omega \in \mathcal{P}_{\nu_j,l}$, $\omega \subset \mathcal{E}_{\nu_j}(\gamma, l) \cap \mathcal{B}_{\nu_j,l}$, where $\gamma \geq \gamma_0$ and $|\xi_{k,l}(a) - \xi_{k,l}(b)| \leq S$ for all k such that $\nu_j + p_j \leq k \leq \nu_{j+1}$. For the next free return $\xi_{\nu_{j+1},l}(\omega)$ where $a, b \in \omega$ we have*

$$\begin{aligned} |\xi_{\nu_{j+1},l}(a) - \xi_{\nu_{j+1},l}(b)| &\geq |\xi_{\nu_j,l}(a) - \xi_{\nu_j,l}(b)| e^{r(1-2\tilde{d}_i\beta/\gamma)} r^2 \\ &\geq 2|\xi_{\nu_j,l}(a) - \xi_{\nu_j,l}(b)|. \end{aligned}$$

Moreover, we have $\omega \in \mathcal{S}_{n,l}(\gamma')$, $\omega \subset A_n(\gamma, l)$, for all n such that $\nu_j \leq n \leq \nu_{j+1}$, for some $\gamma' \geq \underline{\gamma}'$ and $\gamma \geq \underline{\gamma}$.

Proof. By Sublemma 2, $\omega \in \mathcal{S}_{\nu_j}(\gamma')$, for some $\gamma' \geq \gamma_0'$. An interval $\xi_\nu(\omega)$ in an essential return position has length $\sim e^{-r}/r^2$. Put

$$\begin{aligned} l_{\nu_j} &= |\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| = k\frac{e^{-r}}{r^2}, \\ l_{\nu_{j+1}} &= |\xi_{\nu_{j+1}}(a) - \xi_{\nu_{j+1}}(b)|, \end{aligned}$$

for some $0 \leq k \leq 1$, and $a, b \in \omega$, (the case $k \approx 1$ means that we are considering an essential return).

By Lemma 6.6 and Lemma 2.2

$$\begin{aligned} |\xi_{\nu_j+p_j}(a) - \xi_{\nu_j+p_j}(b)| &\geq kC\frac{e^{-r}}{r^2}|(R_a^{p_j})'(\xi_{\nu_j}(a))| \\ &\geq kC\frac{e^{-r\tilde{d}_i}}{r^2}|(R_a^{p_j-1})'(\xi_{\nu_j+1}(a))|. \end{aligned}$$

By the definition of the bound period,

$$|R^{p_j-1}(z_0, a) - R^{p_j-1}(w_0, b)| \geq e^{-\beta p_j},$$

for some $z_0, w_0 \in \mathcal{K}(\overline{\xi}_{\nu_j+1}(\omega) \cup v_i(\omega))$ and some $a, b \in \omega$. By Corollary 2.5

$$|R^{p_j-1}(z_0, a) - R^{p_j-1}(w_0, b)| \leq |(R_a^{p_j-1})'(z_0)||z_0 - w_0| + |a-b|B^{p_j-1}.$$

Since $\omega \in \mathcal{S}_{\nu_j}(\gamma')$, $\gamma' \geq \gamma_0'$, by Sublemma 2, we get (with $p = p_j$)

$$|a-b|B^{p-1} \leq e^{-\gamma' n + p\log B} \leq e^{-n(\sigma \log B - \gamma')} \leq e^{-(3\underline{\gamma}'/4)n} \ll e^{-\beta n} \leq (1/4)e^{-\beta p}.$$

Therefore, by Lemma 2.4

$$(3/4)e^{-\beta p} \leq |(R_a^{p-1})'(z_0)||z_0 - w_0| \leq |(R_a^{p-1})'(\xi_{\nu_j+1}(a))||z_0 - w_0|.$$

Now, since the diameter of the convex set $\mathcal{K}(\overline{\xi}_{\nu_j+1}(\omega) \cup v_i(\omega))$ is by bounded from above by $Ce^{-r\tilde{d}_i}$ (see (6.9)), we get $|z_0 - w_0| \le Ce^{-r\tilde{d}_i}$. Thus,

$$\begin{aligned}|\xi_{\nu_j+p_j}(a) - \xi_{\nu_j+p_j}(b)| &\ge kC\frac{e^{-r\tilde{d}_i}}{r^2}|(R_a^{p_j-1})'(\xi_{\nu_j+1}(a))| \\ &\ge kC(3/4r^2)e^{-\beta p_j} \ge ke^{-2\beta p_j}.\end{aligned}$$

By Lemma 6.7 and Lemma 6.3, (with q_j as the free period)

$$\begin{aligned}|\xi_{\nu_{j+1}}(a) - \xi_{\nu_{j+1}}(b)| &\ge kC\frac{e^{-r}}{r^2}|(R_a^{p_j})'(\xi_{\nu_j}(a))|C_\sigma e^{(99/100)(\log\lambda)q_j} \\ &\ge kC\frac{e^{-\beta p_j}}{r^2} \ge kC\frac{e^{-\tilde{d}_i\beta r/\gamma}}{r^2} \ge ke^{-2\tilde{d}_i\beta r/\gamma}.\end{aligned}$$

Therefore,

$$l_{\nu'} \ge l_\nu e^{r(1-2\tilde{d}_i\beta/\gamma)}r^2 \ge 2l_\nu.$$

To prove the last statement of the lemma, we use Lemma 6.2, Lemma 6.6 and Lemma 6.7 to conclude that in any situation where $\nu_j \le n \le \nu_{j+1}$ we have

$$|\xi_n(a) - \xi_n(b)| \ge C(7/8)^t|(R_a^{n-\nu_j})'(\xi_{\nu_j}(a))||\xi_{\nu_j}(a) - \xi_{\nu_j}(b)|,$$

where $t \le \log(n - \nu_j + p_j)/\log(1+\sigma)$ if $n \ge \nu_j + p_j$ and $t = 0$ if $n < \nu_j + p_j$. Since $\omega \in \mathcal{P}_{\nu_j}$ and $\omega \subset \mathcal{E}_{\nu_j}(\gamma) \cap \mathcal{B}_{\nu_j}$ we have, by Sublemma 2 that $\mathcal{S}_{\nu_j}(\gamma')$ for some $\gamma' \ge \gamma_0'$. The basic assumption on ω and the condition $\omega \in \mathcal{E}_{\nu_j}(\gamma)$, implies $\omega \subset \mathcal{E}_n(\gamma - K\alpha)$. Indeed, by Lemma 3.8,

$$\begin{aligned}|(R_a^{\nu_j})'(v(a))||(R_a^{n-\nu_j})'(\nu_j(a))| &\ge e^{\gamma\nu_j}C\lambda^{n-\nu_j}\inf_{\nu_j\le j\le n}|R_a'(\xi_j(a))| \\ &\ge e^{\gamma\nu_j}C\lambda^{n-\nu_j}e^{-(K-1)\alpha j} \ge e^{(\gamma-K\alpha)n},\end{aligned}$$

if n is large enough. Note that $\gamma_0 - K\alpha \ge \underline{\gamma}$ according to (1.7). Therefore,

$$\begin{aligned}|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| &\ge C(7/8)^t|(R_a^{n-\nu_j})'(\xi_{\nu_j}(a))||(R_a^{\nu_j})'(v(a))|Q_N(a)|\omega|| \\ &\ge C(7/8)^t|(R_a^n)'(v(a))||x'(a)||a-b| \\ &\ge C(7/8)^te^{(\gamma-K\alpha)n}|x'(a)||a-b| \ge e^{\gamma_1 n}|a-b|,\end{aligned}$$

where $\gamma_1 \ge \gamma - 2K\alpha \ge \gamma_0 - 2K\alpha$, if n is sufficiently large. Therefore,

$$|\xi_n(a) - \xi_n(b)| \ge e^{\gamma' n}|a-b|$$

where $\gamma' \ge \gamma_0' - 2K\alpha \ge \underline{\gamma}'$. □

7. Strong distortion estimates and geometry

This section is devoted to two main distortion estimates. We show strong distortion estimates, i.e. not only for the absolute value, but also for the argument. Recall Proposition 5.3, which shows that parameter derivatives and z-derivatives are comparable. The Main Distortion Lemma presented soon, gives a strong distortion estimate on the derivatives of R^n for different parameters, i.e. the quotient $(R^n)'(v(a),a)/(R^n)'(v(b),b)$ is very close to 1 for all $a, b \in \omega \in \mathcal{P}_n$. This implies immediately Lemma 7.1, which shows that the tangent slope of the curve $\xi_n(a)$

varies very little. This introduces the notion of *argument distortion*, i.e. low distortion of the argument of $(R^n)'(v(a),a)$ which will imply low distortion of $\xi_n'(t)$ for $t \in (a,b) = \omega \in \mathcal{P}_n$.

Lemma 7.1. *Assume that ω is an interval in ω_0, $\omega \subset A_n(\gamma,l) \cap \mathcal{B}'_{n-1,l}$ for some $\gamma \geq \underline{\gamma}$. Moreover, assume that*

$$\left| \frac{(R^k)'(v_l(a),a)}{(R^k)'(v_l(b),b)} - 1 \right| \leq 1/200, \tag{7.1}$$

for all $a,b \in \omega$ and $N_l \leq k \leq n$. Then

$$\left| \frac{\xi_{n,l}'(a)}{\xi_{n,l}'(b)} - 1 \right| \leq 1/100, \tag{7.2}$$

for all $a,b \in \omega$.

Proof. By Proposition 5.3 and Corollary 5.5, we have

$$(Q_{N_l,l}(a_0) + \varepsilon(a))(R^n)'(v_l(a),a) = \xi_{n,l}'(a),$$

where $|\varepsilon(a)| \leq |Q_{N_l,l}(a_0)|/500$. Equation (7.1) implies

$$\left| \frac{\xi_{n,l}'(a)}{\xi_{n,l}'(b)} \frac{(Q_{N_l,l}(a_0) + \varepsilon(a))}{(Q_{N_l,l}(a_0) + \varepsilon(b))} - 1 \right| \leq 1/500,$$

and hence (7.2) follows. □

Equation (7.2) gives precisely the control of geometry we need. The curvature $\kappa_n(a)$ of the curve $\xi_n(a)$ is bounded by

$$\kappa_n(a) \leq \frac{|\xi_n''(a)|}{|\xi_n'(a)|^2}.$$

If (7.2) is satisfied the curvature may in principle be locally large, but in average, the curvature must be low. We have the following;

$$1/100 \geq |\log \xi_n'(a) - \log \xi_n'(b)| = \left| \int_0^1 \frac{\xi_n''(\gamma(t))}{\xi_n'(\gamma(t))^2} \xi_n'(\gamma(t))\gamma'(t)dt \right|, \tag{7.3}$$

for all $a,b \in \omega$, for any partition element $\omega \in \mathcal{P}_n$.

We are ready to prove the following important lemma.

Lemma 7.2 (Main Distortion Lemma)**.** *Assume that $\omega \in \mathcal{P}_{\nu_s,l}, \omega \subset \mathcal{E}_{\nu_s}(\gamma,l) \cap \mathcal{B}_{\nu_s,l}$ for $\gamma \geq \gamma_0$. Moreover, assume that $|\xi_{k,l}(a) - \xi_{k,l}(b)| \leq S$ for all $a,b \in \omega$ and all $\nu_s + p_s \leq k \leq n$, where $\nu_s \leq n \leq \nu_{s+1}$. Then*

$$\left| \frac{(R^k)'(v_l(a),a)}{(R^k)'(v_l(b),b)} - 1 \right| \leq 1/200, \tag{7.4}$$

for $N_l \leq k \leq n$.

Proof. By Lemma 2.1 is suffices to show that

$$\sum_k \frac{|R'(\xi_k(a),a) - R'(\xi_k(b),b)|}{|R'(\xi_k(b),b)|} \leq \log(1 + 1/400). \tag{7.5}$$

If (7.5) holds, then the lemma will follow from equation (5.11) and the Chain Rule.

Let us first see how the terms look like near critical points, i.e when $\xi_m(a), \xi_m(b) \in U_i'$. Assume that $\xi_m(\omega) \subset J_r$, $(r \geq \Delta')$. We have $|\xi_m(a) - c(a)| \sim |\xi_m(b) - c(b)| \sim e^{-r}$. By Lemma 6.8 it follows that

$$\begin{aligned} |R'(\xi_m(b), a) - R'(\xi_m(b), b)| &\leq |a-b||\partial_a R'(\xi_m(b), a')| \\ &\leq B' e^{-\gamma' m} |\xi_m(a) - \xi_m(b)|, \end{aligned} \tag{7.6}$$

where $B' = \max |\partial_a R'(z, a)|$. Also, since $K\alpha \ll \gamma'$,

$$e^{-r(\tilde{d}_i - 2)} \geq e^{-\alpha m(\tilde{d}_i - 2)} \geq e^{-(K-2)\alpha m} \gg e^{-\gamma' m}.$$

Therefore,

$$\begin{aligned} |R'(\xi_m(a), a) - & R'(\xi_m(b), b)| \\ &\leq |R'(\xi_m(a), a) - R'(\xi_m(b), a)| + |R'(\xi_m(b), a) - R'(\xi_m(b), b)| \\ &\leq 2|R''(z, a)||\xi_m(a) - \xi_m(b)| + |a-b||\partial_a R'(\xi_m(b), a')| \\ &\leq 4C_1 e^{-r(\tilde{d}_i - 2)} |\xi_m(a) - \xi_m(b)|, \end{aligned}$$

for some z on the line segment joining $\xi_m(a)$ and $\xi_m(b)$. The denominator in (7.5) is also estimated from below using Lemma 2.2 again by

$$|R'(\xi_m(b), b)| \geq C_1^{-1} e^{-(\tilde{d}_i - 1)r}.$$

For the rest of this proof, set $\tilde{d}_i = d$. In particular, near critical points, i.e. for $\xi_m(a), \xi_m(b) \in U'$ we have the estimate

$$\frac{|R'(\xi_m(a), a) - R'(\xi_m(b), b)|}{|R'(\xi_m(b), b)|} \leq C \frac{|\xi_m(a) - \xi_m(b)|}{e^{-r}}.$$

We have $|\xi_k(a) - c(b)| \geq \delta'$ when $\xi_k(b) \notin U'$. Now, by Lemma 6.8 we have that $\omega \subset \mathcal{S}_n(\gamma')$ for some $\gamma' \geq \underline{\gamma}$. This implies that, in general, the terms in (7.5) can be estimated by

$$\frac{|R'(\xi_k(a), a) - R'(\xi_k(b), b)|}{|R'(\xi_k(b), b)|} \leq C \frac{|\xi_k(a) - \xi_k(b)|}{|\xi_k(a) - c(b)|},$$

where C does not depend on δ. So (7.5) holds if

$$\sum_k \frac{|\xi_k(a) - \xi_k(b)|}{|\xi_k(a) - c(b)|} \leq \varepsilon, \tag{7.7}$$

for some sufficiently small ε only depending on δ'.

Now we can estimate the contribution from the bound orbits. Note first that $|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \leq e^{-r}/r^2$ and that $|(R^i)'(v(a), a)| \sim |(R^i)'(z, a)|$ for $|z - v(a)|$ small if $i \leq p(a)$, by Lemma 2.4. By assumption we have $\omega \in \mathcal{P}_{\nu_j}$, and $\omega \subset \mathcal{E}_n(\gamma)$ where $\gamma \geq \gamma_0$, for every return ν_j, for $j \leq s$. From the definition of the bound period, Lemma 2.2 and Lemma 6.6 we have

$$\begin{aligned} |\xi_i(a) - \xi_i(b)| &\leq C_0 C_1 |(R^{i-\nu_j})'(\xi_{\nu_j}(b), b)||\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \\ &\leq C_0 C_1 \frac{|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| e^{-\beta(i-\nu_j)}}{|\xi_{\nu_j}(b) - c(b)|}, \end{aligned}$$

for the nearest $c(b) \in \mathcal{C}(b)$. Moreover, if $N_0 > 0$ is the smallest integer such that $\xi_{\nu_j+N_0}(\omega) \cap U' \neq \emptyset$, then for $i - \nu_j \geq N_0$

$$\begin{aligned}|\xi_i(b) - c(b)| &\geq ||\xi_i(b) - \xi_{i-\nu_j}(b)| - |\xi_{i-\nu_j}(b) - c(b)|| \\ &\geq e^{-\alpha(i-\nu_j)} - e^{-\beta(i-\nu_j)} \geq \frac{1}{2}e^{-\alpha(i-\nu_j)}.\end{aligned}$$

If $i - \nu_j \leq N_0$ then $|\xi_i(b) - c(b)| \geq \delta'$. So in any case

$$\begin{aligned}\sum_{i=\nu_j}^{\nu_j+p_j-1} \frac{|R'(\xi_i(a),a) - R'(\xi_i(b),b)|}{|R'(\xi_i(b),b)|} &\leq C \sum_{i=\nu_j}^{\nu_j+p_j-1} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b) - c(b)|} \\ &\leq C \sum_{i=\nu_j}^{\nu_j+p_j-1} \frac{|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)|e^{-\beta(i-\nu_j)}}{|\xi_{\nu_j}(b) - c(b)|e^{-\alpha(i-\nu_j)}} \\ &\leq C \frac{|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)|}{e^{-r_j}},\end{aligned}$$

where $C = C(\delta')$. By Lemma 6.8,

$$2|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \leq |\xi_{\nu_{j+1}}(a) - \xi_{\nu_{j+1}}(b)|,$$

if δ is sufficiently small, summing over all returns ν_j we get

$$\begin{aligned}\sum_j \sum_{i=\nu_j}^{\nu_j+p_j-1} \frac{|R'(\xi_i(a),a) - R'(\xi_i(b),b)|}{|R'(\xi_i(b),b)|} &\leq C \sum_j \frac{|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)|}{e^{-r_j}} \\ &\leq C \sum_{r\geq\Delta} \max_{j\in(r)} \frac{|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)|}{e^{-r_j}} \\ &\leq C \sum_{r\geq\Delta} \frac{1}{r^2} \leq \varepsilon_b,\end{aligned}$$

where (r) is the j:s which have that $\xi_{\nu_j}(\omega) \cap J_r \neq \emptyset$. Note that the constant ε_p can be made arbitrarily small if δ is small. So the contribution to the sum (7.5) from the bound periods only depends on δ.

To estimate the contribution from the free periods, assume that ν_j are the return times for the parameters a and b for $\nu_j \leq n = \nu_l$ (the return times for a and b are equal; $\nu_j(a) = \nu_j(b)$ since $a, b \in \mathcal{P}_n$). According to Lemma 6.7 we can estimate the sum of $|\xi_i(a) - \xi_i(b)|$ for $\nu_j + p_j \leq i \leq \nu_{j+1} - 1$ until a return occurs by the last term times a constant:

$$\sum_{i=\nu_j+p_j}^{\nu_{j+1}-1} |\xi_i(a) - \xi_i(b)| \leq C|\xi_{\nu_{j+1}-1}(a) - \xi_{\nu_{j+1}-1}(b)|.$$

We have $|\xi_i(a) - c_i(a)| \geq \delta$ for all critical points c_i and for $i = \nu_j + p_j, \dots, \nu_{j+1} - 1$. Again, by Lemma 6.8, $2|\xi_{\nu_j}(a) - \xi_{\nu_j}(b)| \leq |\xi_{\nu_{j+1}}(a) - \xi_{\nu_{j+1}}(b)|$, if δ is sufficiently small, summing over all returns ν_j we get

$$\sum_{j=0}^{s} \sum_{i=\nu_j+p_j}^{\nu_{j+1}-1} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b) - c(b)|} \leq C \frac{|\xi_{\nu_l-1}(a) - \xi_{\nu_l-1}(b)|}{\delta} \leq \varepsilon_f,$$

where C does not depend on δ. Here ε_f is sufficiently small, if C_2 is large enough, since $|\xi_j(a) - \xi_j(b)| \leq S = \delta/C_2$, for all $j \leq \nu_l$.

The lemma is proved for $\nu_j \leq n \leq \nu_{j+1} + p_{j+1}$. We have to deal with the case when $\nu_j + p_j < n < \nu_{j+1}$, for some j. For $\nu_j + p_j < i < n$, Lemma 6.7 gives

$$|\xi_n(a) - \xi_n(b)| \geq C_\sigma e^{(99/100)(\log\lambda)(n-i)}|\xi_i(a) - \xi_i(b)|$$

as long as $|\xi_n(a) - \xi_n(b)| \leq S$ and if $\xi_n([a,b]) \subset P^c_{\delta'}$. Assume that ν_j was the last return time. The contribution of the iterates after $\nu_j + p_j$ is thus

$$\sum_{i=\nu_j+p_j}^{n} \frac{|R'(\xi_i(a),a) - R'(\xi_i(b),b)|}{|R'(\xi_i(b),b)|} \leq \sum_{i=\nu_j+p_j}^{n} \frac{C|\xi_i(a) - \xi_i(b)|}{\delta}$$
$$\leq C\frac{|\xi_n(a) - \xi_n(b)|}{\delta} \leq C\frac{S}{\delta} \leq \varepsilon_s,$$

where again ε_s is sufficiently small if C_2 is chosen appropriately.

Finally, we have to show that if $\xi_n(\omega) \subset P_{\delta'}$, then we still have strong distortion estimates. However, in this case, the distortion cannot be estimated by the last term in (7.7), since a contraction for a pseudo return depends on δ. Assume that $\tilde{n} \geq \nu_j + p_j$ is the last pseudo return time, so $\xi_{\tilde{n}}(\omega) \subset U' \setminus U$. By Lemma 6.7 we have

$$|\xi_n(a) - \xi_n(b)| \geq (7/8)^t |(R_a^{n-\tilde{n}-1})'(\xi_{\tilde{n}+1}(a))||\xi_{\tilde{n}+1}(a) - \xi_{\tilde{n}+1}(b)|$$
$$\geq (7/8)^t (\mu - \varepsilon)^{n-\tilde{n}-1}|\xi_{\tilde{n}+1}(a) - \xi_{\tilde{n}+1}(b)|$$

where $t \leq \log(n - \tilde{n} - 1)/\log(1+\sigma)$ and μ is the minimal multiplier $|R'(p(a),a)|$ over all $a \in [0,a_0]$ and $\varepsilon \leq (\mu-1)/1000$. It follows immediately that

$$\sum_{j=\tilde{n}}^{n} \frac{|R'(\xi_j(a)) - R'(\xi_j(b))|}{|R'(\xi_j(b),b)|} = \frac{|\xi_{\tilde{n}}(a) - \xi_{\tilde{n}}(b)|}{|\xi_{\tilde{n}}(b) - c(b)|} + \sum_{j=\tilde{n}+1}^{n} \frac{|R'(\xi_j(a)) - R'(\xi_j(b))|}{|R'(\xi_j(b),b)|}$$
$$\leq S/\delta + C\frac{|\xi_n(a) - \xi_n(b)|}{\mu} \leq C\frac{S}{\delta} \leq \varepsilon_{pseudo},$$

which is small if C_2 is chosen appropriately.

Choosing C_2 such that $\varepsilon_b + \varepsilon_f + \varepsilon_s + \varepsilon_{pseudo}$ is sufficiently small then (7.5) holds, hence (7.4) holds, and the proof is finished. □

We end the geometry part of this thesis with the following important proposition:

Proposition 7.3. *Assume that $\omega \in \mathcal{P}_{n,l}$ and $\omega \subset \mathcal{E}_n(\gamma,l) \cap \mathcal{B}_{n,l}$ for some $\gamma \geq \gamma_0$, then $\omega \in \mathcal{G}_{k,l}$ for all $N_l \leq k \leq n$. Moreover, if $\omega \in \mathcal{P}_{\nu_{s+1}-1,l}$, $\omega \subset \mathcal{E}_{\nu_s}(\gamma,l) \cap \mathcal{B}_{\nu_s,l}$, then $\omega \in \mathcal{G}_{\nu_{s+1},l}$.*

Proof. The first statement follows directly from the Main Distortion Lemma and Lemma 7.1. If $\omega \subset \mathcal{B}_{\nu_s,l}$ then $\omega \subset \mathcal{B}'_{n,l}$ for all $n < \nu_{s+1}$, since $e^{-\alpha\nu_s} \leq \delta$, by (6.3). Also, Lemma 6.8 implies that $\omega \subset A_{\nu_{s+1}}(\gamma,l)$ for some $\gamma \geq \underline{\gamma}$. Again, Lemma 7.1 and the Main Distortion Lemma implies $\omega \in \mathcal{G}_{\nu_{s+1}}$. □

So we have indeed control of the geometry whenever a partition element interval ω has that $|(R_a^n)'(v_l(a))|$ grows exponentially for all $a \in \omega$. Also, the second statement ensures that given a "good" partition element $\omega \subset \mathcal{E}_n(\gamma,l)$ for some $\gamma \geq \gamma_0$, where n is a return time, we can ensure that the curve is straight at the next return, so we can indeed make partitions. This is a crucial fact about the geometry in the whole thesis and we show in the following sections that the measure of the set of parameters belonging to $\mathcal{E}_n(\overline{\gamma}, l)$ has positive measure, where $\overline{\gamma} = (1-\tau)\log\lambda$.

8. Large deviations

We prove that the set $\cap_n \mathcal{F}_n$ has positive measure following [3], where the large deviation method is developed, and [13], where an exposition can be found. Having control of the geometry, the use of the large deviation argument will be very similar to [4]. The main idea is that we start from a partition element $\omega \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$ at time n, so that we may use the binding information of all other critical points up to time $2n$. In the time interval $(n, 2n)$ we first delete those parameters not satisfying the basic assumption (Proposition 8.2). Next, we delete those parameters not satisfying the free assumption, and give an estimate of the measure of the set deleted (Proposition 8.11). For the remaining parameters the exponent $\overline{\gamma}$ is restored (Proposition 9.1).

Let us first estimate the measure of those parameters not satisfying the basic assumption:

Lemma 8.1. *Assume that $\xi_{\nu,l}(\omega) \subset J_r$ is an essential return and $\omega \in \mathcal{P}_{\nu,l}$, $\omega \subset \mathcal{E}_\nu(\gamma, l) \cap \mathcal{B}_{\nu,l}$, for some $\gamma \geq \gamma_0$. Let ν' be the next return time. Let $\omega' \subset \omega$ be the parameters not satisfying $\mathcal{B}_{\nu'}$. Then*

$$\frac{|\omega'|}{|\omega|} \leq e^{-(\alpha/2)\nu}.$$

Proof. In view of Proposition 7.3, the tangent slope of $\xi_n(t)$, for $\nu \leq n \leq \nu'$, is under sufficient control until the next return time ν'. If l_n is the length of the curve $\xi_n(\omega)$, by Lemma 6.8

$$l_{\nu'}(\omega) \geq e^{-2K\beta r/\gamma} \geq e^{-2K\beta\alpha\nu/\gamma},$$

if the length of $\xi_n(\omega)$ never exceeded S between the return times ν and ν'. By strong argument distortion, the fraction which fails to fulfil the basic assumption at each return, is equivalent to

$$\begin{aligned}\frac{m(\{a \in \omega : |\xi_{\nu'}(a) - c(a)| \geq e^{-\alpha n}\})}{m(\omega)} &\leq C\frac{e^{-\alpha\nu'}}{e^{-2K\beta\alpha\nu/\gamma}} \\ &\leq Ce^{-\alpha\nu(1-(2K\beta/\gamma))} \leq e^{-(\alpha/2)\nu}.\end{aligned}$$

If the length of $\xi_n(\omega)$ has reached size S for some $\nu \leq n \leq \nu'$, it means that ω is partitioned into smaller elements ω_j so that $\cup\omega_j = \omega$. On each element ω_j, we have that the fraction which fails to fulfil the basic assumption is less than $e^{-\alpha\nu'}/S \leq Ce^{-\alpha\nu}$. So in any case, (8.1) holds. □

In fact, by the uniform distortion of the a-derivative, repeating the Lemma 8.1 for every return we will get that the set satisfying $\mathcal{B}_n$ for all n can be estimated by

$$m\Big(\bigcap_n \mathcal{B}_n\Big) \geq |\omega_0| \prod_n (1 - e^{-(\alpha/2)n}) > 0.$$

Proposition 8.2. *Assume that* $\overline{\omega} \in \mathcal{P}_{n,l}$, $\overline{\omega} \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$. *Put* $\omega = \{a \in \overline{\omega} : a \subset \mathcal{B}'_{2n,l}\}$. *Then* $\omega \subset \mathcal{E}_{2n}(\gamma_0, l)$ *and the Lebesgue measure of* $m(\omega)$ *of the set* ω *satisfies* $m(\omega) \geq m(\overline{\omega})(1 - Ce^{-(\alpha/2)n})$.

Proof. We shall inductively construct the set ω, deleting those parameters from $\overline{\omega}$, not satisfying the basic assumption at every return. Lemma 8.1 implies that the measure of the set deleted between two consecutive return times ν and ν' is less than $e^{-(\alpha/2)\nu}$, if the former return was essential. For inessential returns in $(n, 2n)$, the basic assumption is automatically fulfilled. To see this, we may consider the last essential return $\xi_\nu(\omega)$ into, say J_r. By Lemma 6.8, the interval $\xi_{\nu'}(\omega)$ has length at least twice the length of $\xi_\nu(\omega)$. Thus, since the return ν' is inessential it means that $\xi_{\nu'}(\omega) \subset J_{r'}$, where $r' \leq r$, which directly implies that $\omega \subset \mathcal{B}'_{\nu'}$.

By the fact that $\omega \subset \mathcal{E}_n(\overline{\gamma}, i, *) \cap \mathcal{B}_{n,i,*}$, we can use the binding information of all other critical points for all returns in the time interval $(n, 2n)$. Also, since $p \leq K\alpha n/\gamma \ll 1$, the same exponent $\overline{\gamma}$ can be used during the bound periods. Indeed, if p is the bound period following $\nu \in (n, 2n)$ and q the free period, we get by Lemma 6.5 and the Outside Expansion Lemma

$$|(R_a^{\nu'-\nu})'(\xi_\nu(a))| \geq e^{p\overline{\gamma}/(K+1)} C_\sigma \lambda^q \geq e^{p\overline{\gamma}(2K)}.$$

whenever $a \in \omega_1$ for some partition element $\omega_1 \subset \overline{\omega}$ such that $\omega_1 \subset \mathcal{B}'_{\nu,i} \cap \mathcal{E}_\nu(\gamma, i)$, for some $\gamma \geq \gamma_0$. We have consecutive free returns $\nu_j(a)$, where $j \in (n, 2n)$, and assume for simplicity that $n = \nu_0$. Inductively, using Lemma 6.5, removing the parameters which do not satisfy the basic assumption, we get

$$|(R_a^{\nu_j})'(v(a))| = \prod_{k=0}^{j-1} |(R_a^{\nu_{k+1}-\nu_k})'(\nu_k(a))||(R_a^n)'(v(a))| \geq e^{\overline{\gamma}n} = e^{\gamma_j \nu_j},$$

where $\gamma_j \geq \overline{\gamma}(\nu_j/n) \geq \gamma_0$. In general,

$$|(R_a^k)'(v(a), a)| \geq e^{(\gamma_j - K\alpha)k},$$

for all $k \leq \nu_j$, by the basic assumption. So, indeed, at every return we have, for any partition element ω_j, that $\omega_j \subset \mathcal{B}'_{\nu_j} \cap A_{\nu_j}(\gamma, i)$, for some $\gamma \geq \gamma_0$. By the assumption on $\overline{\omega}$, this implies that $\omega_j \subset \mathcal{B}'_{\nu_j} \cap \mathcal{E}_{\nu_j}(\gamma, i)$. According to Proposition 7.3, the geometry is under good control, and we can go on to the next return until time $2n$. It follows that $\omega \subset \mathcal{E}_{2n}(\gamma_0, i)$. □

Remark 8.3. Proposition 8.2 immediately implies that the partition elements in the set ω satisfies $\omega \in \mathcal{G}_{2n,l}$, i.e. we have good geometry control.

Thus, we can delete parameters not satisfying the basic assumption in the time interval $(n, 2n)$ and still have control of the geometry. We will now use the large deviation argument to show that the set of parameters $\omega \cap \mathcal{E}_{2n}(\overline{\gamma}, l)$ is only a slightly smaller set.

We define the notion of *escape*, which briefly means that a curve segment $\xi_n(\omega)$ has reached the length $S = \delta/C_2$. The number C_2 does not depend on δ, so we may choose δ such that $S/e^\Gamma \geq \delta/\Delta^2 \gg \delta^2$, where $e^\Gamma = \max|R'(z, a)|$.

Definition 8.4 (Escape). We say that the curve $\xi_{n,l}(\omega)$ has escaped, or equivalently, ω is in escape position if $\xi_{n,l}(\omega)$ has length at least $S/e^\Gamma = \delta/(e^\Gamma C_2)$.

The notion of escape is already introduced in [4], and the above definition is an analogue of that in [4]. The following lemma shows that the escape time is bounded from above in proportion to the "depth" of the return.

Lemma 8.5. *Assume $\xi_{\nu,l}(\omega)$ is an essential return and $\omega \in \mathcal{P}_{\nu,l}, \omega \subset \mathcal{E}_{\nu'}(\gamma, l) \cap \mathcal{B}_{\nu',l}$, for $\gamma \geq \gamma_0$, where ν' is the next essential return time after ν or the time when $\xi_{\nu',l}(\omega)$ has escaped, whichever comes first. Let q be the time spent on the first free period and the following inessential returns until the next essential return or until escape occurs. Then*

$$q \leq \frac{3Kr}{\gamma} \leq hr,$$

where $h = \frac{4K}{\gamma_0}$.

Proof. The length of $\xi_n(\omega)$ must by definition not exceed S/e^{Γ}, because then it escapes. Assume first that no returns takes place at all after time ν. Since $\omega \subset \mathcal{E}_{\nu'}(\gamma) \cap \mathcal{B}_{\nu'}$ for $\gamma \geq \gamma_0$, Proposition 7.3 implies that the geometry is under full control during the time period (ν, ν') considered. By Lemma 6.6, Lemma 6.7 and Lemma 3.8 implies

(8.1)

$$S \geq \frac{e^{-r}}{2r^2}|(R_a^{p_0})'(\xi_\nu(a))|(7/8)^t|(R_a^{q_0})'(\xi_{\nu+p_0}(a))| \geq Ce^{-2\beta p_0}e^{q_0(\log\lambda)}e^{-\Delta(K-1)},$$

where $p_0 + q_0 = \mu$ and $t \leq \log q_0 / \log(1+\sigma)$. Thus (8.1) implies

$$\log C - 2\beta p_0 + (99/100)q_0 \log\lambda \leq (K-1)\Delta.$$

Since $p_0 \leq \tilde{d}_i r/\gamma \leq Kr/\gamma$ we get

$$q_0 \leq \frac{4K\beta r}{\gamma \log\lambda} + \frac{(K-1)\Delta}{\log\lambda} \leq \frac{2Kr}{\gamma},$$

since $\beta < 4\log\lambda$, $\gamma \leq \log\lambda$ and $r \geq \Delta$. Lemma 6.8, together with the strong argument distortion, shows that the length l_{m_j} of an inessential return $\xi_{m_j}(\omega)$ enlarges by a factor at least $e^{(1-2K\beta/\gamma)r_j}$. Thus the length L after all inessential returns satisfies

$$e^{-2\beta Kr/\gamma}\prod_j e^{r_j(1-2K\beta/\gamma)+q_j\log\lambda} \leq L \leq S$$

where q_j are the corresponding free periods. This implies

$$\sum_j r_j(1-2K\beta/\gamma) + q_j\log\lambda \leq \frac{2K\beta r}{\gamma}.$$

Assume that the number of inessential returns is s. The total time spent on inessential returns and its bound and free periods becomes

$$\begin{aligned} q &= \sum_{j=0}^{s} p_j + q_j = p_0 + q_0 + \sum_{j=1}^{s} (p_j + q_j) \leq \frac{Kr}{\gamma} + \frac{2Kr}{\gamma} + \frac{K}{\gamma} \sum_{j=1}^{s} (r_j + \gamma q_j) \\ &\leq \frac{3Kr}{\gamma} + \frac{K}{\gamma(1 - 2K\beta/\gamma)} \sum_{j=1}^{s} r_j (1 - 2K\beta/\gamma) + q_j \log \lambda \\ &\leq \frac{3Kr}{\gamma} + \frac{4K^2 \beta r}{\gamma^2} \leq \frac{4Kr}{\gamma}. \end{aligned}$$

We used that $\beta \leq \gamma/(4K)$, which follows from (1.7), and $\gamma \leq \log \lambda$. □

If $\xi_n(\omega)$ is an escape situation then it will return with a length at least $\sim S$. By Proposition 7.3 and since $\delta^2 \ll S$, when the curve $\xi_n(\omega)$ has returned into U^2, it has very low argument distortion, i.e. it is almost straight, compared to δ^2.

Now, let us return to the set $\overline{\omega} \in \mathcal{P}_{n,l}$, $\omega \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$ and put $\omega_1 = \{a \in \overline{\omega} : a \in \mathcal{B}'_{2n,l}\}$. The measure of parameters not satisfying the basic assumption which are deleted every return, is exponentially small in terms of the return time (Lemma 8.1). Therefore, the portion of a partition element $\omega \subset \omega_1$ which has consecutive essential returns to $J_{r_{i-1}}$ and J_{r_i} in the time interval $(n, 2n)$, is

$$C \frac{e^{-r_i}}{e^{-2\tilde{d}_{i-1}\beta r_{i-1}/\gamma}}, \tag{8.2}$$

by Lemma 6.8, if no escape has taken place between the two returns. This is the core of the geometry; that we can estimate the portion as in (8.2).

More generally, the subset $\omega_s \subset \omega$ that has a specific history, i.e. specific essential returns to $J_{r_1}, J_{r_2}, \ldots, J_{r_s}$, starting at J_{r_0} can be written

$$\frac{m(\omega_s)}{m(\omega)} := \varphi_s \leq C^s \prod_{i=1}^{s} \frac{e^{-r_i}}{e^{-2\tilde{d}_{i-1}\beta r_{i-1}/\gamma}},$$

where $m(E)$ is the Lebesgue measure of E.

Assume $\xi_n(\omega) \cap U \neq \emptyset$. Then a part P_1 of $\xi_n(\omega)$ lies inside U and the other part P_2 lies outside U. We now make the following convention. If P_2 is smaller than $\delta/(2\Delta^2)$ then we just adjoin P_2 to P_1. Otherwise cut P_2 off and iterate it further. It will grow to size S, and hence escape, rapidly during the pseudo bound period (see Definition 6.4) because

$$\left(\frac{\delta}{\Delta^2}\right)^{\tilde{d}_i\beta/\gamma} = e^{-\frac{\tilde{d}_i \Delta\beta}{\gamma}} e^{-\frac{2\tilde{d}_i\beta\log\Delta}{\gamma}} = e^{-\Delta(\frac{\tilde{d}_i\beta}{\gamma} + \frac{2\log\Delta}{\Delta})} \gg S = e^{-\Delta}/C_2,$$

if Δ is sufficiently large.

Fix $r_1 + r_2 + \ldots + r_s = R$ for the moment. Let us calculate the number of possible combinations of the sequence $r_1, r_2, \ldots, r_s$ for s and R fixed, i.e. the number of possibilities of dividing the number R into s parts $r_i \geq 0$. This number is the same as the number of possibilities of putting $s-1$ balls in $R+s-1$ boxes, with at most one ball in each. If we, for a start, neglect the fact that every curve returned into J_r

is divided into r^2 smaller sets for each essential return, the number of combinations is thus

$$\binom{R+s-1}{s-1}.$$

Using Stirling's formula for large R and that $R \geq s\Delta \geq \Delta$ we get

$$\frac{1}{\sqrt{2\pi}} \frac{(R+s-1)^{R+s-1} e^{-R-s+1}}{R^R e^{-R} (s-1)^{s-1} e^{-s}} \sqrt{\frac{R+s-1}{R(s-1)}} \leq C \frac{R^{R+\frac{R}{\Delta}} \left(1+\frac{1}{\Delta}\right)^{(1+\frac{1}{\Delta})R}}{R^R \left(\frac{R}{\Delta}\right)^{R/\Delta}}$$
$$\leq \left(C^{1/R} \Delta^{1/\Delta} (1+\frac{1}{\Delta})^{1+\frac{1}{\Delta}} \right)^R \leq 2(1+\eta(\Delta))^R,$$

if Δ is large enough, where $\eta(\Delta) \to 0$ as $\Delta \to \infty$. Since the critical set is finite the total number of combinations of the returns $\nu_i(a)$ is estimated by

$$C(1+\eta(\Delta))^R \prod_{i=1}^{s} r_i^2 \leq C e^{\frac{1}{16}R} (1+\eta(\Delta))^R.$$

Definition 8.6. Let $\omega_{n,l}(a) \in \mathcal{P}_{n,l}$ be the set of parameters laying in the same partition element as a, for the first n iterates.

Definition 8.7 (Escape time). If $\xi_{\nu,l}(\omega_{\nu,l}(a))$ is an essential return into U^2 then define $E_l(a,\nu) = \{\inf k > 0 : \omega_{\nu+k,l}(a)$ is in escape position and $k \geq p(a)\}$. $E_l(a,\nu)$ is called the *escape time*.

According to Lemma 8.5, if the number of essential returns is s before escape takes place, then the escape time is

$$E(a,\nu) = E_l(a,\nu) \leq \sum_{i=0}^{s} hr_i = hR + hr_0$$

if ν is a return time into J_{r_0}.

For $R = \sum_{i=1}^{s} r_i$ let $A_{s,R}$ be the set of parameters $a \in \omega$, which have a specific starting return $\nu(a)$ into J_r and after that have s essential returns without escaping until the $(s+1)$:st return. By the above calculations we have at most $(1+\eta(\Delta))^R$ possibilities, i.e. the interval ω is at most divided into $(1+\eta(\Delta))^R$ intervals. Let $\hat{\omega}_s$ be the largest one. We can rewrite the fraction φ_s as

$$\varphi_s \leq C^s e^{-\frac{7}{8}\sum_{i=1}^{s} r_i + 2\tilde{d}_{i-1}\beta r_0/\gamma} = C^s e^{-\frac{7}{8}R + 2\tilde{d}_0 \beta r_0/\gamma}. \tag{8.3}$$

The fraction $|\omega_s|$ of $|\omega|$ not escaping after s essential returns starting at r_0 with $R = \sum_{i=1}^{s} r_i$ is thus estimated by

$$|A_{s,R}| \leq |\hat{\omega}_s| (1+\eta(\Delta))^R e^{R/16}. \tag{8.4}$$

We want to make an estimate of the escape time for a given return ν_0 into J_{r_0}. Assume that $E(a,\nu_0) = t$. Then $t \leq hR + hr_0$, so $R \geq \frac{t}{h} - r_0$, where $h = 4K/\gamma_0$. Summing over all possible combinations of sequences r_i satisfying $R \geq \frac{t}{h} - r_0$ we get from (8.3) and (8.4)

$$\begin{aligned} m(\{a \in \omega : E(a,\nu_0) = t\}) &= \sum_{\substack{R \geq \frac{t}{h} - r_0, \\ s \leq R/\Delta}} |A_{s,R}| \leq \sum_{\substack{R \geq \frac{t}{h} - r_0, \\ s \leq R/\Delta}} |\hat{\omega}_s| (1+\eta(\Delta))^R e^{R/16} \\ &\leq |\omega| \sum_{R=\frac{t}{h}-r_0}^{\infty} \sum_{s=1}^{R/\Delta} (1+\eta(\Delta))^R e^{R/16} C^s e^{-\frac{7}{8}R + 2\tilde{d}_0 \beta r_0/\gamma} \\ &\leq |\omega| \sum_{R=\frac{t}{h}-r_0}^{\infty} \frac{C^{R/\Delta} - 1}{C-1} e^{R(-\frac{3}{4}+\eta(\Delta)) + 2\tilde{d}_0\beta r_0/\gamma} \\ &\leq C e^{(\frac{t}{h} - r_0)(\eta(\Delta) - \frac{3}{4}) + 2\tilde{d}_0 \beta r_0/\gamma} \leq C e^{-\frac{3t}{4h} + (\frac{3}{4} + \frac{2\tilde{d}_0\beta}{\gamma}) r_0}. \end{aligned}$$

Since $2\tilde{d}_0\beta/\gamma$ is small we get with $t \geq 2hr_0$ an estimate for large escape times;

$$m(\{a \in \omega : E(a,\nu_0) = t\}) \leq C e^{-\frac{t}{3h}}. \tag{8.5}$$

We have $2hr_0 = 6Kr_0/\gamma \leq 6Kr_0/\gamma_0 \leq 6K\alpha\nu_0/\gamma_0$. Since $\iota = 6K\alpha/\gamma_0$ is small we get that so the estimate (8.5) holds for $t \in (\nu_0(1+\iota), 2n)$, assuming that $n \leq \nu_0 \leq 2n$.

Assume $\nu_i(a) \in U^2$ are deep returns for $i = 1, \ldots, s$. Their corresponding escape times are $E(a,\nu_i)$. We want to estimate the sum of all escape times in time intervals of the type $(n, 2n)$. Define

$$T_n(a) = T_{n,l}(a) = \sum_{i=0}^{s(a)-1} E_l(a, \nu_i(a)),$$

where $s(a) = s$ is the largest integer such that $n \leq \nu_0(a) < \nu_1(a) < \ldots < \nu_{s-1}(a) \leq 2n$, where $\nu_k(a)$ are essential deep returns after escape has occurred. We assume that all s escape periods have ended before time $2n$, so by definition $E(\nu_{s-1}, a) \leq n - \nu_{s-1}$.

Remark 8.8 (Blind escapes). If there is a deep return $\nu_s(a) \leq n$, which does not escape until after $2n$, then we delete those parameters if $E(a,\nu_s) > \iota n$. According to (8.5), those parameters correspond to an exponentially small fraction of the interval $\omega_{\nu_s}(a)$. If $E(a,\nu_s) \leq \iota n$, ($\iota = 6K\alpha/\gamma_0$) then this escape time will normally not be counted in the sum $T_n(a)$. However, we simply disregard from these "blind escapes". It will make very little difference at the end, by the fact that we may choose $\tau \geq 100\iota$, see (1.7).

We will estimate

$$\frac{1}{\omega} \int_\omega e^{\theta T_n(a)} da,$$

for some suitable θ. Choose $\theta = 1/(6h)$, (remember $h = 4K/\gamma_0$). Then $\theta > \tau$, (see (1.7)). The shallow returns into $U \setminus U^2$ are treated in the usual manner but we do not speak of escape times from such situations. We do this since shallow returns do not deteriorate the expansion of the derivative (see Lemma 6.5). If $\omega_n(a)$ is in escape position then the escape time stops and the *free escape* orbit begins until it returns to U^2 again. Since the escape time is larger than the bound period by definition, the sum of all free escape orbits in the time interval $(n, 2n)$ is at least $n - T_n(a)$. If we let M be the first return time for a given critical orbit $\xi_{M,i}(\omega)$, let $m \in \mathbb{N}$ be such that $2^m M = n$. Then

$$F_{2n,l}(a) \geq \sum_{k=0}^{m} 2^k M - T_{2^k M,l}(a). \tag{8.6}$$

Note that, by definition $T_{n,l}(a)$ is constant on each component of $\mathcal{P}_{n,l}$.

Lemma 8.9. *Let $\overline{\omega} \in \mathcal{P}_{n,l}$, $\overline{\omega} \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$. Put $\omega' = \{a \in \overline{\omega} : a \in \mathcal{B}'_{2n,l}\}$. Moreover, assume that $\omega \in \mathcal{P}_{\nu,l}$ for some $\omega \subset \omega'$, and some $n \leq \nu \leq 2n$, where $\xi_{\nu,l}(\omega)$ is an essential deep return into J_r^i. Then*

$$\int_{\{a\in\omega: 2hr \leq E_l(a,\nu) \leq n-\nu\}} e^{\theta E_l(a,\nu)} da \leq C e^{-r/3} |\omega|,$$

$$\int_{\{a\in\omega: E_l(a,\nu) \leq 2hr\}} e^{\theta E_l(a,\nu)} da \leq C e^{r/3} |\omega|.$$

Proof. By (8.5)

$$\begin{aligned}\int_{\{a\in\omega: 2hr \leq E(a,\nu) \leq n-\nu\}} e^{\theta E(a,\nu)} da &\leq \sum_{t \geq 2hr} C e^{-\frac{t}{3h}} e^{\theta t} |\omega| \leq C' e^{-t(\frac{1}{3h} - \theta)} |\omega| \\ &= C' e^{-t\theta} |\omega| \leq C' e^{-r/3} |\omega|,\end{aligned}$$

since $\theta = 1/6h$. The last integral follows directly. □

Lemma 8.10. *Let $\overline{\omega} \in \mathcal{P}_{n,l}$, $\overline{\omega} \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$. Put $\omega = \{a \in \overline{\omega} : a \in \mathcal{B}'_{2n,l}\}$. Then*

$$\int_{\omega} e^{\theta T_{n,l}(a)} da \leq e^{\tau^2 n} |\omega|.$$

Proof. Let $s = s(a)$ be the largest integer such that $\nu_s(a) < 2n$ and let ω^0 be the subset of ω such that every $a \in \omega^0$ has escaped precisely s times, for some fixed s. For every parameter $a \in \omega^0$, $\omega^s \subset \ldots \subset \omega^1 \subset \omega^0$ is a nested sequence of parameters following a for s consecutive deep essential returns after escape situations. So $\xi_{\nu_k}(\omega_{\nu_k}(a))$ are essential deep returns after escape situations.

Since $T_n(a) = \sum_{i=0}^{s} E(a, \nu_i)$ and $E(a, \nu_i)$ is constant on ω^{i-1} but not on ω^i we get

$$\int_{\omega^s} e^{\theta T_n(a)} da = e^{\theta \sum_{i=0}^{s-2} E(a,\nu_i)} \int_{\omega^{s-1}} e^{\theta E(a,\nu_{s-1})} da.$$

The set ω^{s-1} is a union of sets $\omega^{s-1,r}$ where $\xi_{\nu_{s-1}}(\omega^{s-1,r}) \subset J_r$, i.e

$$\omega^{s-1} = \bigcup_{r=2\Delta}^{\infty} \omega^{s-1,r}.$$

In the following equations we use that $\xi_{\nu_{s-1}}(\omega_{\nu_{s-1}}(a))$ has length $\mathcal{O}(\delta) \sim S$ (since it has escaped before the deep free return). So by Lemma 8.9

$$\begin{aligned}
\int_{\omega^{s-1}} e^{\theta E(a,\nu_{s-1})} da &\leq |\omega^{s-1}| + \sum_{r=2\Delta}^{\infty} \int_{\omega^{s-1,r}} e^{\theta E(a,\nu_{s-1})} da \\
&= |\omega^{s-1}| + \sum_{r=2\Delta}^{\infty} \Bigg[\int_{\{a\in\omega^{s-1,r}:2hr\leq E(a,\nu_{s-1})\leq n-\nu_{s-1}\}} e^{\theta E(a,\nu_{s-1})} da \\
&+ \int_{\{a\in\omega^{s-1,r}:E(a,\nu_{s-1})\leq 2hr\}} e^{\theta E(a,\nu_{s-1})} da \Bigg] \\
&\leq |\omega^{s-1}| + C \sum_{r=2\Delta}^{\infty} (e^{-r/3} + e^{r/3}) |\omega^{s-1,r}| \\
&\leq |\omega^{s-1}| + C \sum_{r=2\Delta}^{\infty} (e^{-r/3} + e^{r/3}) \frac{e^{-r}}{\delta} |\omega^{s-1}| \\
&= |\omega^{s-1}|(1 + Ce^{-\Delta/3)}) = |\omega^{s-1}|(1+\eta(\Delta)),
\end{aligned}$$

where $\eta(\Delta) \to 0$ as $\Delta \to \infty$.

To calculate the integral over ω^{s-2} instead we note first that the set ω_{s-2} is also a union of $\omega^{s-2,r}$, i.e.

$$\omega^{s-2} = \bigcup_{r=2\Delta}^{\infty} \omega^{s-2,r}.$$

For each r the set $\omega_{s-2,r}$ is again a union of sets of the type ω^{s-1} on which $E(a,\nu_{s-2})$ is constant. Hence,

$$\begin{aligned}
\int_{\omega^{s-2,r}} e^{\theta(E(a,\nu_{s-1})+E(a,\nu_{s-2}))} da &= \sum_{\omega^{s-1}\subset\omega^{s-2,r}} e^{\theta E(a,\nu_{s-2})} \int_{\omega^{s-2,r}\cap\omega^{s-1}} e^{\theta E(a,\nu_{s-1})} da \\
&\leq \sum_{\omega^{s-1}\subset\omega^{s-2,r}} e^{\theta E(a,\nu_{s-2})} (1+\eta(\Delta)) |\omega^{s-1}| \\
&= (1+\eta(\Delta)) \int_{\omega^{s-2,r}} e^{\theta E(a,\nu_{s-2})} da.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\omega^{s-2}} e^{\theta(E(a,\nu_{s-1})+E(a,\nu_{s-2}))} da &= \sum_{r=2\Delta}^{\infty} \int_{\omega^{s-2,r}} e^{\theta(E(a,\nu_{s-1})+E(a,\nu_{s-2}))} da \\
&= (1+\eta(\Delta)) \sum_{r=2\Delta}^{\infty} \int_{\omega^{s-2,r}} e^{\theta E(a,\nu_{s-2})} da \\
&= (1+\eta(\Delta)) \int_{\omega^{s-2}} e^{\theta E(a,\nu_{s-2})} da \\
&\leq (1+\eta(\Delta))^2 |\omega^{s-2}|.
\end{aligned}$$

Repeating this argument s times we get finally

$$\int_{\omega^0} e^{\theta T_n(a)} da \leq (1+\eta(\Delta))^s |\omega^0| \leq e^{\tau^2 n} |\omega^0|,$$

for some suitable $\tau > 0$ with $\theta > \tau$ if Δ is sufficiently large. We used also that $s < n$. Since this is valid for all such sets ω^0, the lemma follows. □

We can now show that the set of parameters in $\cap_n \mathcal{F}_n$, has positive Lebesgue measure.

Proposition 8.11. *Assume that* $\overline{\omega} \in \mathcal{P}_{n,l}$, $\overline{\omega} \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$. *Put* $\omega = \{a \in \overline{\omega} : a \in \mathcal{B}'_{2n,l}\}$. *Then*

$$m(\{a \in \omega : T_{n,l}(a) > \tau n\}) \leq e^{-\tau(\theta-\tau)n} |\omega|.$$

Proof. We have

$$\begin{aligned} e^{\theta \tau n} m(\{a \in \omega : T_n(a) > \tau n\}) &\leq \int_{\{a \in \omega : T_n(a) \geq \tau n\}} e^{\theta T_n(a)} da \\ &\leq \int_{\omega} e^{\theta T_n(a)} da \leq e^{\tau^2 n} |\omega|, \end{aligned}$$

by Lemma 8.10. So

$$m(\{a \in \omega : T_n(a) > \tau n\}) \leq e^{-\theta \tau n} e^{\tau^2 n} |\omega| = e^{-\tau(\theta-\tau)n} |\omega|.$$

□

9. Conclusion and proof of Theorem B

The following proposition follows an analogue in [13]. Roughly one can say that once the basic and free assumption holds, then the original exponent $\overline{\gamma}$ is restored.

Proposition 9.1. *Assume that* $\omega \in \mathcal{P}_{2n,l}$ *and* $\omega \subset \mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$. *If, in addition* $\omega \subset \mathcal{B}'_{2n,l} \cap \mathcal{F}_{2n,l}$, *then* $\omega \subset \mathcal{E}_{2n}(\overline{\gamma}, l) \cap \mathcal{B}_{2n,l}$, *i.e.*

$$|(R_a^k)'(v_l(a))| \geq e^{(\overline{\gamma} - K\alpha)k},$$

for all $k \leq 2n$.

Proof. Fix any $a \in \omega$. First we estimate $|(R_a^p)'(z)|$ for $z = \xi_j(a) \in J_r$ for some $j \leq 2n$. Since $p \leq 2K\alpha n/\gamma \leq \alpha_0 n < n$, Lemma 6.5 gives

$$|(R_a^p)'(z)| > e^{\gamma p/(K+1)}.$$

For simplicity put $m = 2n$. Now, let p_i be the bound periods, μ_i as the free deep periods and ν_i as the free deep return times, where $\nu_0 = 0$ and $\nu_1 = M$, the first return time. The Outside Expansion Lemma, Lemma 6.5 and the Chain Rule gives

$$\begin{aligned} |(R_a^m)'(v(a))| &= |(R_a^{m-\nu_s})'(\xi_{\nu_s}(a))| \prod_{i=0}^{s-1} |(R_a^{\nu_{i+1}-\nu_i})'(\xi_{\nu_i}(a))| \\ &\geq |(R_a^{m-\nu_s})'(\xi_{\nu_s}(a))| C^s \prod_{i=0}^{s-1} e^{p_i \gamma/(K+1)} e^{\mu_i \log \lambda}. \end{aligned}$$

If $m-\nu_s > p_s$ then $|(R_a^{m-\nu_s})'(\xi_{\nu_s}(a))| \geq e^{p_s\gamma/(K+1)}|(R_a^{m-\nu_s-p_s})'(z)|$. The basic assumption implies $|R_a'(\xi_m(a))| \geq C_1^{-1}|\xi_m(a)-c(a)|^{d-1} \geq e^{-(K-1)\alpha m}$. Lemma 3.8 implies

$$|(R_a^{m-\nu_s})'(\xi_{\nu_s}(a))| \geq Ce^{p_s\gamma/(K+1)}e^{(m-\nu_s-p_s)\log\lambda}e^{-(K-1)\alpha m}.$$

If $m-\nu_s \leq p_s$ instead then

$$|(R_a^{m-\nu_s})'(\xi_{\nu_s}(a))| \geq e^{-(K-1)\alpha m}.$$

So in any case

$$\begin{aligned}|(R_a^m)'(v(a))| &\geq C^s e^{\gamma\sum_i p_i/(K+1)}e^{\log\lambda\sum_i \mu_i}e^{-(K-1)\alpha m}\\ &= C^s e^{\gamma(m-F_m(a))/(K+1)}e^{F_m(a)\log\lambda}e^{-(K-1)\alpha m}.\end{aligned}$$

We have $p_i \geq C'\Delta$ (see Lemma 6.3) so $m-F_m(a)\geq sC'\Delta$ and therefore

$$C^s e^{\gamma(m-F_m(a))/(K+1)} \geq e^{s\log C+\gamma s\Delta/(K+1)} \geq 1$$

if Δ is large enough. Thus, if $a\in\mathcal{F}_{2n}$

$$\begin{aligned}|(R_a^m)'(v(a))| &\geq e^{F_m(a)\log\lambda}e^{-(K-1)\alpha m}\\ &\geq e^{m((1-\tau)\log\lambda-(K-1)\alpha)} \geq e^{\gamma m},\end{aligned}$$

where $\gamma=(1-\tau)\log\lambda - K\alpha$. □

Proof of Theorem B. We will use induction over time intervals of the type $(n,2n)$.

To start the induction, choose the first return as in Subsection 5.1, which gives a starting interval ω_0, where $|\omega_0|\ll\delta$, and such that tangent slope distortion is very low. Therefore, $\omega_0\subset\mathcal{E}_M(\gamma,l)$ for all l and $\gamma\geq\overline{\gamma}=(1-\tau)\log\lambda$, where $\xi_{M,l}(\omega)\subset J_{\Delta-1}$ for some partition element $\omega\in\mathcal{P}_{M,l}$. Also, λ here is the minimum of the expansion in Lemma 3.6 and all μ_i^{1/d_i}, where $\mu_i=\sup_{a\in[0,a_0]}|R'(p_i(a),a)|$, and $p_i(a)$ are the repelling fixed points. Thus we can use binding information until time M/α_0 is reached for all critical orbits $\xi_{n,l}(a)$, $a\in\omega_0$.

We now give an inductive method of how to handle finitely many critical points. We use the ideas described in [2]. Assume that we have constructed sets $\Omega_k=\mathcal{E}_n(\overline{\gamma},k)\cap\mathcal{B}_{n,k}$ of Lebesgue measure $|\omega_0|(1-Ce^{-n})$, for all k and for some n, such that $m(\cap_k\Omega_k)\geq 1/2$. Before continuing the orbit of $c_l(a)$, we have to delete parameters in Ω_l, so that we can use binding information of the other critical points up to time $2n$. Consider the set

$$E_{\alpha_0 n,k}=A_{\alpha_0 n}(\overline{\gamma},k)\setminus A_{2\alpha_0 n}(\overline{\gamma},k)$$

for $k\neq l$ which is a union of finitely many intervals which are deleted during the time interval $(\alpha_0 n, 2\alpha_0 n)$. We shall show that intervals in the sets $E_{\alpha_0 n,k}$ are very much larger than the partition elements in $\mathcal{P}_{n,l}$. Indeed, if $l_{\alpha_0 k}$ is the length of the curve $\xi_{\alpha_0 j,k}(\omega_1)$, (the orbit of some critical point $c_k(a)$) and $n\leq j\leq 2n$, where $\omega_1\in\mathcal{P}_{\alpha_0 j,k},\omega_1\subset A_{\alpha_0 n}(\overline{\gamma},k)$, $k\neq l$. Then by Proposition 5.3

$$l_{\alpha_0 j}(\omega_1)\sim|\omega_1||(R_a^{\alpha_0 j})'(v_k(a))|\leq|\omega_1|e^{\Gamma\alpha_0 j}$$

Assume now that this ω_1 is deleted, i.e. $\omega_1 \subset E_{\alpha_0 n,k}$. We delete only parameter intervals which covers some whole partition element fully. This means that

$$l_{\alpha_0 j} \geq \frac{e^{-r}}{2r^2} \geq e^{-2\alpha\alpha_0 n},$$

if $\omega_1 \in \mathcal{B}_{\alpha_0 n}$. Thus, the size of the partition elements of $E_{\alpha_0 n,k}$ can be estimated by

$$|\omega_1| \geq Ce^{-2\alpha\alpha_0 n - \Gamma\alpha_0 n} = Ce^{-\alpha_0 n(2\alpha+\Gamma)}.$$

On the other hand, to estimate the size of the partition elements $\mathcal{P}_{n,l}$ of $\mathcal{E}_n(\overline{\gamma}, l)$, we use Proposition 5.3 to get

$$S \geq l_n(\omega_2) \sim |\omega_2||(R_a^n)'(v_l(a))| \geq |\omega_2|e^{\overline{\gamma}n},$$

for some $\omega_2 \subset \mathcal{E}_n(\overline{\gamma}, l), \omega_2 \in \mathcal{P}_{n,l}$. So

$$\frac{|\omega_1|}{|\omega_2|} \geq Ce^{n(\overline{\gamma} - \alpha_0(2\alpha+\Gamma))} \gg 1,$$

since $\overline{\gamma} > \alpha_0(2\alpha + \Gamma)$, (follows from (1.7)). Thus intervals in the sets $E_{\alpha_0 n,k}$ are very much larger than the partition elements $\mathcal{P}_{n,l}$. Therefore the partition $\mathcal{P}_{n,l}$ will not be damaged, we will only delete whole elements which intersect $E_{\alpha_0 n,k}$ for some $k \neq l$. In particular, if $E_{\alpha_0 n,k}$ only partly covers some partition element in $\mathcal{P}_{n,l}$ then delete the whole partition element. Thus we delete a little more than necessary, but this fraction is very little by the fact that $|\omega_1| \gg |\omega_2|$. This ensures that the new set

$$\mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*}$$

consists of whole partition elements in $\mathcal{P}_{n,l}$ for which we can use binding information of all critical points up to time $2n$. With $\Omega = \cap\Omega_k$, note that

$$\big((\mathcal{E}_n(\overline{\gamma}, l) \cap \mathcal{B}_{n,l}) \setminus (\mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*})\big) \cap \Omega = \emptyset.$$

Now, in view of (8.6), by Proposition 8.2 and Proposition 8.11 we get

$$\begin{aligned} m\Big((\mathcal{F}_{2n,l} \cap \mathcal{B}_{2n,l}) \cap (\mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*})\Big) \\ \geq m(\mathcal{E}_n(\overline{\gamma}, l, *) \cap \mathcal{B}_{n,l,*})(1 - e^{-\tau(\theta-\tau)n})(1 - Ce^{-(\alpha/2)n}). \end{aligned}$$

Proposition 9.1 gives

$$m(\mathcal{E}_{2n}(\overline{\gamma}, l)) \geq m(\mathcal{E}_n(\overline{\gamma}, l, *))(1 - e^{-C_4 n}) > 0,$$

for some $C_4 > 0$, for every l. Thus, if $\tilde{N}$ is the number of critical points, the measure of the set deleted from Ω is at most

$$\tilde{N}e^{-C_4 n}.$$

Applying the above procedure to every time interval $(n, 2n)$ we get

$$m\Big(\bigcap_{l=1}^{\tilde{N}} \bigcap_{n=0}^{\infty} A_n(\overline{\gamma}, l)\Big) \geq |\omega_0| \prod_n (1 - \tilde{N}e^{-C_4 n}) > 0.$$

The proof of Theorem B is finished. $\square$

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