# Arithmetic Dynamical Systems

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# Abstract

The main objects of study in this thesis are $\mathbb{Z}^d$–actions by automorphisms of compact abelian groups, which arise in a natural arithmetic setting. In particular, to a countable integral domain $D$ and units $\xi_1, \ldots, \xi_d \in D$ we associate a $\mathbb{Z}^d$–action by automorphisms of the compact abelian group $\widehat{D}$. This generalises the '$S$–integer dynamical systems' introduced by Chothi, Everest and Ward, where $d = 1$ and $D$ is a ring of $S$–integers in an $\mathbb{A}$–field. Familiar dynamical properties such as expansiveness and entropy are investigated in this setting, together with the emerging theory of expansive subdynamics introduced by Boyle and Lind. Homoclinic points are also examined. The main results are as follows.

1. Using results of Lind, Schmidt and Ward, an explicit entropy formula is given which applies whenever $D$ is an integrally closed domain (Theorems 3.3.4 and 3.3.8).

2. The well-known expansiveness criteria for toral automorphisms, involving the eigenvalues of associated integer matrices has been generalized by Schmidt, using complex affine varieties. This result is extended further using closed points of associated integral schemes, giving a valuation theoretic characterization of expansiveness (Theorem 4.2.4).

3. The results of recent joint work with Einsiedler, Lind and Ward concerning expansive subdynamics is considered using methods from algebraic geometry. This includes a novel classification of expansive subdynamics using valuations (Theorems 4.3.4 and 4.3.10).

4. Homoclinic points are investigated for both expansive and non-expansive systems. Using the work of Lind and Schmidt, the effect of certain algebraic properties, like integrality, is examined. There is a complete classification of homoclinic groups for $S$–integer dynamical systems (Theorem 5.2.1).

Generally speaking, proofs are only included for results which the author considers original. Wherever collaboration has been involved this is indicated clearly in the text. Some algebraic results which are needed are well known, but not readily available in the literature: for these proofs have been included.

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# Contents


**1 Introduction** **1**

1.1 Overview . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

1.2 Measure theory . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

1.3 Duality theory . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

1.4 Arithmetic dynamical systems . . . . . . . . . . . . . . . . . . . 6

1.5 Commutative algebra . . . . . . . . . . . . . . . . . . . . . . . . 7

1.6 Valuations, absolute values and Krull rings . . . . . . . . . . . . 11

1.7 Algebraic geometry . . . . . . . . . . . . . . . . . . . . . . . . . 18

1.8 Examples of arithmetic dynamical systems . . . . . . . . . . . . 22

**2 Ergodicity and mixing** **29**

2.1 Conditions for ergodicity and mixing . . . . . . . . . . . . . . . 29

2.2 Higher order mixing . . . . . . . . . . . . . . . . . . . . . . . . 32

**3 Entropy** **35**

3.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35

3.2 Completely positive entropy . . . . . . . . . . . . . . . . . . . . 38

3.3 Main results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38

**4 Expansiveness** **50**

4.1 Algebraic characterization . . . . . . . . . . . . . . . . . . . . . 50

4.2 Arithmetic characterization . . . . . . . . . . . . . . . . . . . . 51

4.3 Expansive subdynamics . . . . . . . . . . . . . . . . . . . . . . 59

**5 Homoclinic Points** **73**
5.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 73
5.2 The $\mathbb{A}$–field case . . . . . . . . . . . . . . . . . . . . . . . . . . . 75
5.3 The general case . . . . . . . . . . . . . . . . . . . . . . . . . . . 78
5.4 Homoclinic points and expansive subdynamics . . . . . . . . . . . . . . 87

# Chapter 1

# Introduction

## 1.1 Overview

Algebraic $\mathbb{Z}^d$–actions, that is actions of $\mathbb{Z}^d$ by continuous automorphisms of compact metrizable abelian groups, have revealed themselves to be intimately linked with commutative algebra. Indeed, to every such action, we may associate a countable module $M$ over $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ – the ring of Laurent polynomials in $d$ commuting variables. Using this association, dynamical properties can then be interpreted in terms of the algebraic properties of $M$. This framework for the study of such actions was introduced by Kitchens and Schmidt [13] and is developed in [17], [18], [28], [29], [31], and Schmidt's monograph [30], which provides an overview of the resulting theory.

The aim of this thesis is to investigate a subclass of such dynamical systems, namely *arithmetic dynamical systems*. An arithmetic dynamical system may be defined simply by stipulating that $M$, the module above, is an $R_d$–algebra with no zero divisors (that is, $M$ is an integral domain). An equivalent, more natural definition will be given in the next section. Although the approach taken will be mainly algebraic in nature, the underlying theme is entirely dynamical.

The algebraic manifestation of dynamical behaviour has at its core the duality theory of locally compact abelian groups, and an introduction to this may be found in [25]. For a more sophisticated treatment, using Fourier analysis, consult [27]. The most relevant results of this theory are summarized in the next section. Amongst

| | $\times\frac{1}{2}$ on $\mathbb{Z}[\frac{1}{2}]$ | $\times\frac{1}{2}$ on $\mathbb{Z}[\frac{1}{6}]$ | $\times\frac{1}{2}$ on $\mathbb{Q}$ |
|---|---|---|---|
| Mixing ? | Yes | Yes | Yes |
| Entropy | $\log 2$ | $\log 2$ | $\log 2$ |
| Expansive ? | Yes | No | No |
| Growth rate of Periodic points | $\log 2$ | $\log 2$ | 0 |
| Non-trivial homoclinic points ? | Yes | No | No |

Table 1: Three basic arithmetic dynamical systems.

these is the duality between discrete countable abelian groups and compact metrizable abelian groups, and this is the foundation for understanding dynamical properties in terms of their algebraic counterparts. In fact it is this duality which determines the correspondence between algebraic $\mathbb{Z}^d$–actions and countable $R_d$–modules. A more natural definition of an arithmetic dynamical system also arises from this dual point of view, and one of the aims of this thesis is to relate dynamical properties directly to the algebraic data which defines the system.

Although arithmetic dynamical systems arise as a natural subclass of algebraic ones, the motivation for their study lies in [4]. Examples such as those given in Table 1 exibit a suprising range of dynamical behaviour. Together, they illustrate another agenda of this work which, roughly speaking, is to 'fix' an action, 'perturb' the space and compare the resulting dynamical systems. The dynamical information in Table 1 of course refers to the actions and groups dual to those given. The results come mainly from [4], with the exception of the homoclinic point information which follows from Theorem 5.2.1 of Chapter 5. To make similar comparisons for more complex examples requires a good deal of the machinery contained in [30], and some commutative algebra. However, for arithmetic dynamical systems whose underlying ring is a Krull ring (see Section 1.6) in an $\mathbb{A}$–field (in the sense of Weil [39]), there is a convenient description of the dual group and the action (see Section 1.6 and Example 1.8.1). This makes more direct study of the dynamics possible. Unfortunately this description does not extend to other integral domains, or even Krull rings, and this makes certain results which we would like to generalize from [4] intractable. For example, topological questions about the distribution and growth rate of periodic points for arithmetic dynamical systems seem no easier to answer than in the more general algebraic case. On the other hand, algebraic interpretations of measurable properties, like ergodicity and mixing, are assembled relatively easily.

After some preliminaries, which make up the remainder of this chapter, ergodicity and mixing are discussed, in Chapter 2. In Chapter 3 attention is turned to entropy, and practical methods are given for its calculation through both general techniques and examples. Chapter 4 explores expansiveness and the extent to which some of the results from [4] extend to arbitrary arithmetic dynamical systems. Also considered here are some implications of recent joint work with Einsiedler, Lind and Ward [6] concerning expansive subdynamics. We conclude with a chapter on homoclinic behaviour including some further applications of [6].

## 1.2 Measure theory

A detailed introduction to measure theory may be found in [12]. Let $X$ be any set. A *$\sigma$–algebra* is a collection $\mathcal{B}$ of subsets of $X$ which satisfies the following.

1. $X \in \mathcal{B}$.

2. $B \in \mathcal{B} \Rightarrow X \setminus B \in \mathcal{B}$.

3. If $B_j \in \mathcal{B}$ for all $j \in \mathbb{N}$ then $\bigcup_{j=1}^{\infty} B_j \in \mathcal{B}$.

A *measure* is a map $\mu : \mathcal{B} \mapsto \mathbb{R}_+$ such that $\mu(\varnothing) = 0$ and $\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{i=1}^{\infty} \mu(B_j)$ whenever $B_1, B_2, B_3, \ldots$ is a sequence of pairwise disjoint subsets of $X$ which are members of $\mathcal{B}$. If $\mu(X) = 1$ then the measure $\mu$ is called a *probability measure* and the triple $(X, \mathcal{B}, \mu)$ is called a *probability space*.

If $X$ is a topological space then the $\sigma$–algebra generated by the open sets is called the *Borel $\sigma$–algebra*, usually denoted by $\mathcal{B}(X)$. If $X$ is a compact metrizable abelian group then there exists a unique translation invariant measure $\mu_X : \mathcal{B}(X) \mapsto [0, 1]$

called *Haar measure*. Translation invariance is the property that for all $B \in \mathcal{B}(X)$ and $x \in X$

$$\mu_X(x+B) = \mu_X(B).$$

Since in this thesis we will only be dealing with compact abelian groups, the presence of the unique Haar measure and the probability space $(X, \mathcal{B}(X), \mu_X)$ is implied by the context and rarely referred to directly.

Finally, a *measure preserving* $\mathbb{Z}^d$*–action* on the probability space $(X, \mathcal{B}, \mu)$ is an action $T : \mathbf{n} \mapsto T_{\mathbf{n}}$ of $\mathbb{Z}^d$ by invertible transformations $T_{\mathbf{n}} : X \mapsto X$, such that for all $\mathbf{n} \in \mathbb{Z}^d$ the transformation $T_{\mathbf{n}}$ is measurable (that is $T_{-\mathbf{n}}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$) and measure preserving (that is $\mu(T_{-\mathbf{n}}(B)) = \mu(B)$ for all $B \in \mathcal{B}$). Whenever it is useful, measure preserving $\mathbb{Z}^d$–actions will be distinguished from algebraic ones by using Roman letters to denote the former, and Greek letters to denote the latter.

If $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ is an algebraic $\mathbb{Z}^d$–action on the compact metrizable abelian group $X$, then for each $\mathbf{n} \in \mathbb{Z}^d$ there is a probability measure $\mu_{\mathbf{n}} : \mathcal{B}(X) \mapsto \mathbb{R}_+$ given by

$$\mu_{\mathbf{n}}(B) = \mu_X(\alpha_{-\mathbf{n}}(B)) \qquad B \in \mathcal{B}.$$

Now, for each $x \in X$ and $B \in B$

$$\begin{aligned} \mu_{\mathbf{n}}(\alpha_{\mathbf{n}}(x) + B) &= \mu_X(x + \alpha_{-\mathbf{n}}(B)) \\ &= \mu_X(\alpha_{-\mathbf{n}}(B)) \\ &= \mu_{\mathbf{n}}(B), \end{aligned}$$

so $\mu_{\mathbf{n}}$ is translation invariant, which means it is equivalent to Haar measure. It follows that $\alpha$ is a measure preserving $\mathbb{Z}^d$–action.

## 1.3 Duality theory

As is standard, all topological groups are assumed to be Hausdorff. Let $X$ be a locally compact abelian group. A continuous homomorphism $\gamma : X \mapsto \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, is called a *character* of $X$. The set of all continuous characters of $X$ forms a group which is referred to as the *dual group* of $X$, denoted by $\widehat{X}$. Let $r > 0$, $C$ a compact subset of $X$ and $U_r = \{z \in \mathbb{C} : |1 - z| < r\}$. Define

$$N(C, r) = \{\gamma \in \widehat{X} : \gamma(x) \in U_r \text{ for all } x \in C\}.$$

The family of sets $N(C, r)$ and their translates form a base for a topology on $\widehat{X}$ called the *compact open topology*. Once $\widehat{X}$ has been furnished with the compact open topology, $\widehat{X}$ is itself a locally compact abelian group. Moreover there is a topological isomorphism between $X$ and the dual group of $\widehat{X}$. This fact is known as the Pontryagin duality theorem. The isomorphism just described can be interpreted to mean that, for all $x \in X$ and $\gamma \in \widehat{X}$

$$\gamma(x) = x(\gamma).$$

This gives rise to the 'pairing' notation $\langle x, \gamma \rangle = \gamma(x) = x(\gamma)$. Proofs of these results, along with the following, may be found in [27] or [25].

1. Let $Y$ be a closed subgroup of $X$. The set

   $$Y^\perp = \{\gamma \in \widehat{X} : \langle x, \gamma \rangle = 1 \text{ for all } x \in Y\}$$

   is a closed subgroup of $\widehat{X}$, the *annihilator* of $Y$. Moreover, there are natural isomorphisms

   $$\widehat{Y} \cong \widehat{X}/Y^\perp \quad \text{and} \quad Y^\perp \cong \widehat{X/Y}.$$

2. Suppose $X$ is compact. Then $X$ is connected if and only if $\widehat{X}$ is torsion free.

3. The group $X$ is compact and metrizable if and only if $\widehat{X}$ is countable. Note also that any countable locally compact abelian group is discrete.

4. If $X$ and $Y$ are locally compact abelian groups and $\psi : X \mapsto Y$ is a continuous homomorphism then there is an induced dual map $\widehat{\psi} : \widehat{Y} \mapsto \widehat{X}$ defined by

   $$\langle \psi(x), \gamma \rangle = \langle x, \widehat{\psi}(\gamma) \rangle \qquad x \in X, \quad \gamma \in \widehat{Y}$$

   which is a continuous homomorphism. If $\psi$ is surjective then $\widehat{\psi}$ is injective. If $\psi$ is both an open mapping and injective then $\widehat{\psi}$ is surjective.

5. If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of compact metrizable groups then

$$\widehat{\prod_{\lambda\in\Lambda} X_\lambda} = \bigoplus_{\lambda\in\Lambda} \widehat{X}_\lambda$$

and if $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of countable discrete groups then

$$\widehat{\bigoplus_{\lambda\in\Lambda} X_\lambda} = \prod_{\lambda\in\Lambda} \widehat{X}_\lambda.$$

6. Suppose $\{X_j\}_{j\geq 1}$ is a family of countable discrete groups and that for each $j \geq 1$ there is an injective homomorphism $\psi_j : X_j \mapsto X_{j+1}$. Then the dual group of $\operatorname{inj\,lim}(X_j, \psi_j)$ is $\operatorname{proj\,lim}(\widehat{X}_j, \widehat{\psi}_j)$.

## 1.4 Arithmetic dynamical systems

Suppose that $D$ is an integral domain (always assumed to be countable) and $k$ is its field of fractions. If $D$ is endowed with the discrete topology then the group $X = \widehat{D}$ is compact and metrizable. Furthermore, given units $\xi_1, \ldots, \xi_d \in D$ we may define a $\mathbb{Z}^d$–action $\widehat{\alpha} : \mathbf{n} \mapsto \widehat{\alpha_\mathbf{n}}$ on $D$ by

$$\widehat{\alpha_\mathbf{n}}(a) = \xi^\mathbf{n} a \tag{1}$$

where $a \in D$, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\xi^\mathbf{n} = \xi_1^{n_1} \cdots \xi_d^{n_d}$. Dual to each automorphism $\widehat{\alpha_\mathbf{n}}$ there is a continuous automorphism $\alpha_\mathbf{n}$ of the compact abelian group $X$ and hence a corresponding $\mathbb{Z}^d$–action $\alpha : \mathbf{n} \mapsto \alpha_\mathbf{n}$ on $X$. The dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$, where $\xi = (\xi_1, \ldots, \xi_d) \in D^d$, is an *arithmetic dynamical system.*

As already mentioned the theory of algebraic $\mathbb{Z}^d$–actions is fundamental to the study of arithmetic dynamical systems. The main framework for this is as follows. Let $X$ be a compact metrizable abelian group, $d \geq 1$ and $\alpha : \mathbf{n} \mapsto \alpha_\mathbf{n}$ an action of $\mathbb{Z}^d$ by automorphisms of $X$. Let $R_d$ denote the ring of Laurent polynomials $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$. A typical element $f \in R_d$ has the form

$$f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^\mathbf{n}$$

where $c_f(\mathbf{n}) \in \mathbb{Z}$ and $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ for all $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. Note that $c_f(\mathbf{n}) = 0$ for all but a finite number of $\mathbf{n} \in \mathbb{Z}^d$. Under the action

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \widehat{\alpha_{\mathbf{n}}}(a) \qquad a \in \widehat{X},\ f \in R_d$$

the dual group $\widehat{X}$ becomes an $R_d$–module. By definition we have

$$\widehat{\alpha_{\mathbf{n}}}(a) = u^{\mathbf{n}} \cdot a$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \widehat{X}$. Conversely, if $M$ is a countable $R_d$–module then there is a natural action $\widehat{\alpha}^M : \mathbf{n} \mapsto \widehat{\alpha_{\mathbf{n}}}^M$ by $\mathbb{Z}^d$ on $M$ given by

$$\widehat{\alpha_{\mathbf{n}}}^M(a) = u^{\mathbf{n}} \cdot a$$

where $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$. Dual to this action there is an action of $\mathbb{Z}^d$ by automorphisms of the compact group $X = \widehat{M}$, this being

$$\alpha^M : \mathbf{n} \to \alpha_{\mathbf{n}}^M$$

where for each $\mathbf{n} \in \mathbb{Z}^d$, $\alpha_{\mathbf{n}}^M$ is the automorphism dual to $\widehat{\alpha_{\mathbf{n}}}^M$. In the next section we will consider how arithmetic dynamical systems exist as a subclass of algebraic ones.

## 1.5 Commutative algebra

Let $R$ be a commutative ring with 1. The set of all prime ideals of $R$ will be denoted by Spec $R$. This is a multifaceted object upon which, for example, a natural topology may be defined. However such features will not be considered at this stage. Given any ideal $\mathfrak{p} \in \text{Spec } R$ the supremum of the lengths, taken over all strictly decreasing chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$$

is the *height* of $\mathfrak{p}$. The *Krull dimension* of $R$ is the supremum of the heights of all prime ideals in $R$.

A *multiplicative* subset $S$ of $R$ is one which contains 1 and has the property that

$$a, b \in S \Rightarrow ab \in S.$$

For example, if $\mathfrak{p}$ is a prime ideal of $R$ then $R \setminus \mathfrak{p}$ is multiplicative. We can define an equivalence $\sim$ on $S \times R$ by

$$(s, a) \sim (s', a') \Leftrightarrow r(s'a - sa') = 0$$

for some $r \in S$. The notation $S^{-1}R$ will be used to denote the set of equivalence classes and $a/s$ for the equivalence class of $(s, a)$. When $S$ contains no zero divisors, the ring $R$ may be embedded in $S^{-1}R$ by identifying $a \in R$ with $a/1$. Upon defining

$$\frac{a}{s} + \frac{a'}{s'} = \frac{s'a + sa'}{ss'} \text{ and } \frac{a}{s}\frac{a'}{s'} = \frac{aa'}{ss'}$$

the set $S^{-1}R$ becomes a commutative ring with zero $0/1$ and multiplicative identity $1/1$. This ring is referred to as the *localization of $R$ with respect to $S$*. Note that if $0 \in S$ then $S^{-1}R = \{0/1\}$. If $S$ is the complement in $R$ of a prime ideal $\mathfrak{p} \subset R$ then instead of writing $S^{-1}R$ we write $R_{\mathfrak{p}}$ and call $R_{\mathfrak{p}}$ the *localization of $R$ at $\mathfrak{p}$*. Note that if $R$ is an integral domain then any localization of $R$ is a subring of the field of fractions of $R$. If $A$ is the localization of a commutative ring at a prime ideal then it is a local ring (that is, $A$ has a unique maximal ideal). For a local ring $A$ we often denote its maximal ideal by $\mathfrak{m}_A$.

Let $M$ be an $R_d$–module. The *annihilator* of an element $a \in M$ is the ideal $\mathrm{ann}(a)$ of $R_d$ given by

$$\mathrm{ann}(a) = \{f \in R_d : f \cdot a = 0\}.$$

This should not be confused with the definition arising in Section 1.3. An ideal $\mathfrak{p} \in \mathrm{Spec}\, R_d$ is said to be *associated* with $M$ if $\mathfrak{p} = \mathrm{ann}(a)$ for some $a \in M$. If $\mathfrak{p}$ is the only element in Spec $R_d$ associated with $M$ then we say that $M$ is *associated* with $\mathfrak{p}$. If $M$ is Noetherian then [15, Chapter 5, Section 6] shows that the set of prime ideals associated with $M$ is finite. When $M$ is an integral domain we can in fact say a little more.

**Lemma 1.5.1** *Let $D$ be an integral domain and $\xi_1, \ldots, \xi_d$ units of $D$. The substitution map $\theta_\xi : R_d \mapsto D$ defined by $\theta_\xi(f) = f(\xi_1, \ldots, \xi_d)$ is a homomorphism under which $D$ becomes an $R_d$–algebra and hence an $R_d$–module. Moreover, $D$ is associated with $\mathfrak{p}_\xi$, the kernel of $\theta_\xi$.*

*Proof.* It is readily verified that $\theta_\xi$ is a homomorphism and so $D$ becomes an $R_d$–algebra and hence an $R_d$–module in a natural way. That is, for each $f \in R_d$ and $a \in D$ we set $f \cdot a = \theta_\xi(f)a$. Let $0 \neq a \in D$ and suppose that $f \in R_d$ has $\theta_\xi(f)a = 0$. Since $D$ is an integral domain, it follows that $\theta_\xi(f) = 0$ and so $f \in \mathfrak{p}_\xi$. Hence $\mathfrak{p}_\xi$ is the annihilator of $a$ and, since $a$ was arbitrary, the only ideal in Spec $R_d$ associated with $D$ is $\mathfrak{p}_\xi$. $\square$

It is clear that every arithmetic dynamical system is algebraic. Furthermore, under the identification of $D$ as an $R_d$–module, the natural action of $\mathbb{Z}^d$ on $D$ coincides with the $\mathbb{Z}^d$–action given by (1). Conversely, suppose that $M$ is an $R_d$–algebra which is an integral domain. Then there is a ring homomorphism $\theta : R_d \mapsto M$ and $\theta(u_1), \ldots, \theta(u_d)$ are units of $M$. If

$$\xi = (\theta(u_1), \ldots, \theta(u_d)) \in M^d$$

then the pair $(X^M, \alpha^{(M,\xi)})$ defines an arithmetic dynamical system. It is important to note that not all $R_d$–modules which are integral domains give rise to arithmetic dynamical systems. For example, if $M = \mathbb{Z}[\sqrt{2}]$ and $\tau$ is the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})|\mathbb{Q})$, then $M$ becomes a $\mathbb{Z}[u^{\pm 1}]$–module by setting, for each $f = \sum_{n \in \mathbb{Z}} c_n u^n \in \mathbb{Z}[u^{\pm 1}]$ and $a \in M$,

$$fa = \sum_{n \in \mathbb{Z}} c_n \tau^n(a).$$

However this does not give $M$ a $\mathbb{Z}[u^{\pm 1}]$–algebra structure because

$$(u\sqrt{2})1 = -\sqrt{2} \neq \sqrt{2} = \sqrt{2}(u1).$$

We have seen that to any arithmetic dynamical system, a single element of Spec $R_d$ may be assigned and it therefore makes sense to call this *the associated prime* of the arithmetic dynamical system in question. The existence of this associated prime is extremely important and the first of many consequences of this is considered below. This result will prove to be useful in Chapter 3. Before stating it, the following definition is necessary. Let $M$ be a Noetherian $R_d$–module and suppose that there is a chain of submodules

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_s = M \tag{2}$$

such that, for all $i = 1, \ldots, s$, $N_i/N_{i-1} \cong R_d/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ which contains an associated prime of $M$. Then (2) is called a *prime filtration* of $M$. By [30, Corollary 6.2] the existence of such a chain is guaranteed for all Noetherian $R_d$–modules. However, if $M$ has only one associated prime then the following also holds.

**Lemma 1.5.2** *Let $\mathfrak{p} \in \operatorname{Spec} R_d$ and let $M$ be a Noetherian $R_d$–module associated with $\mathfrak{p}$. Then there exist integers $1 \le r \le s$ and submodules*

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_s = M \tag{3}$$

*such that for $i = 1, \ldots, s$ we have $N_i/N_{i-1} \cong R_d/\mathfrak{p}_i$ for prime ideals $\mathfrak{p} \subset \mathfrak{p}_i \subset R_d$. Hence (3) is a prime filtration of $M$. Furthermore, we may arrange that the filtration has $\mathfrak{p}_i = \mathfrak{p}$ for $i = 1, \ldots, r$ and $\mathfrak{p}_i \supsetneq \mathfrak{p}$ for $i = r+1, \ldots, s$.*

*Proof.* See [30, Proposition 6.1]. □

If $k$ is a field and $\xi_1, \ldots, \xi_d \in k^\times$, then there is a minimal subring of $k$ containing $\xi_1, \ldots, \xi_d$ as units, which will be denoted by $R_\xi$. If $\theta_\xi : R_d \mapsto k$ is the substitution map then it is easily seen that $R_\xi = \theta_\xi(R_d)$. Hence $R_\xi$ is also the minimal sub–$R_d$–algebra of $k$. We will frequently compare various arithmetic dynamical systems of the form $(X^D, \alpha^{(D,\xi)})$, where $D$ varies over subrings of $k$ containing $R_\xi$. Note that by Lemma 1.5.1 every member of the family of arithmetic dynamical systems

$$\{(X^D, \alpha^{(D,\xi)}) : D \text{ is a subring of } k \text{ containing } R_\xi\}$$

has the same associated prime.

Finally, it should be remarked that generators for the ideal $\mathfrak{p}_\xi$ are not always easy to find. For many examples however, the following observations are useful. Let $k$ be a field and $\xi_1, \ldots, \xi_d$ elements of $k$ such that $\xi_1, \ldots, \xi_{d-1}$ are algebraically independent over $F$, the prime subfield of $k$. Set $R = \mathbb{Z}$ or $R = \mathbb{F}_p$ according to the characteristic of $k$. Suppose that $\xi_d$ is integral over $A = R[\xi_1, \ldots, \xi_{d-1}]$ and let $f \in A[X]$ be a monic polynomial of minimal degree such that $f(\xi_d) = 0$. By [23, Theorem 9.2] since $A$ is an integrally closed domain (that is integrally closed in its own field of fractions) $f$ is actually the minimum polynomial for $\xi_d$ over $F(\xi_1, \ldots, \xi_{d-1})$. Using the division algorithm we establish that $A[\xi_d] \cong A[X]/(f(X))$. Now if $u_1, \ldots, u_d$ are independent variables over $R$, $f$ may be considered as a polynomial in these $d$ variables in a natural way. With $f$ considered as such we have

$$A[\xi_d] \cong R[u_1, \ldots, u_d]/(f).$$

Let $B = R[u_1, \ldots, u_d]/(f)$. Since $R_\xi$ is equal to the localization of $A[\xi_d]$ with respect to the multiplicative subgroup generated by $\{\xi_1, \ldots, \xi_d\}$, it follows that $R_\xi$ is isomorphic to the localization of $B$ with repect to the multiplicative subgroup $S$ of $B$ generated by $\{u_1, \ldots, u_d\}$. Here the $u_i$'s are of course identified with their images in $B$. By [23, Theorem 4.2]

$$S^{-1}B \cong R[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]/(f),$$

where $(f)$ is now taken to be the ideal in $R[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ generated by $f$. Hence if $k$ has characteristic zero then $R_\xi \cong R_d/(f)$. That is, $f$ generates the ideal $\mathfrak{p}_\xi$.

## 1.6 Valuations, absolute values and Krull rings

Valuations and valuation rings have proved useful in the study of algebraic number theory, function theory and algebraic geometry. They also have a role to play in the study of arithmetic dynamical systems. Let $G$ be a totally ordered abelian group whose ordering is preserved under the group operation. That is, for any $g_1, g_2, g_3 \in G$

$$g_1 \leq g_2 \Rightarrow g_1 + g_3 \leq g_2 + g_3.$$

A *valuation* on an integral domain $D$ is a map $v : D \to G \cup \{\infty\}$ such that for all $a, b \in D$

1. $v(ab) = v(a) + v(b)$.

2. $v(a + b) \geq \min\{v(a), v(b)\}$ (*ultrametric property*).

3. $v(a) = \infty \Leftrightarrow a = 0$.

Any such valuation has a unique extension to the field of fractions of $D$ given by

$$v(a/b) = v(a) - v(b)$$

where $a, b \in D$. Hence it is usual to consider valuations defined on fields. A valuation on a field $k$ is *discrete* if $G \cong \mathbb{Z}$ and a discrete valuation is *normalized* if its value group is set precisely to $\mathbb{Z}$. Two valuations $v, w$ on $k$ are *equivalent* if $v(a) > 0 \Rightarrow w(a) > 0$ for all $a \in k$. If $K|k$ is a field extension then a valuation $w$ on $K$ is said to *extend* a valuation $v$ on $k$ if $w|_k = v$. Corresponding to each equivalence class of valuations on $k$, there is an associated subring of $k$ given by

$$D_v = \{a \in k : v(a) \geq 0\}$$

where $v$ is any representative of the equivalence class. Such a ring is referred to as a *valuation ring*. Every valuation ring has a unique maximal ideal given by

$$\mathfrak{m}_v = \{a \in k : v(a) > 0\}.$$

For a more thorough description, see [5, Chapter 1]. An *absolute value* on $k$ is a real valued function $|\cdot| : k \mapsto \mathbb{R}_+$ such that, for all $a, b \in k$

1. $|a| \geq 0$, with equality if and only if $a = 0$.

2. $|a + b| \leq |a| + |b|$.

3. $|ab| = |a||b|$.

More generally, a *norm* on a commutative ring $R$ is a map $|\cdot| : R \mapsto \mathbb{R}_+$ which for all $a, b \in R$ satisfies (1), (2) and $|ab| \leq |a||b|$. If $|\cdot|$ only satisfies (2), $|ab| \leq |a||b|$ and $|a| \geq 0$ for all $a, b \in R$, then it is called a *partial norm*. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ on $k$ are equivalent if $|a|_1 > 1 \Rightarrow |a|_2 > 1$ for all $a \in k$. Any discrete valuation gives rise to an absolute value by setting

$$|\cdot|_v = c^{v(\cdot)} \tag{4}$$

where $0 < c < 1$. Note that if absolute values $|\cdot|_v$ and $|\cdot|_w$ arise from discrete valuations $v$ and $w$ in this way, equivalence of $v$ to $w$ implies and is implied by equivalence of $|\cdot|_v$ to $|\cdot|_w$. Given any absolute value $|\cdot|$ on $k$ there is a unique completion (up to equivalence of absolute values) of $k$ with respect to $|\cdot|$, into which $k$ can be embedded. For example, $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the usual archimedean absolute value. Other complete fields arising in a similar way, but with respect to different absolute values on $\mathbb{Q}$, all turn out to be fields of $p$–adic numbers, where $p$ is a rational prime. This is essentially Ostrowski's Theorem [26]. A concise and updated proof of this result, along with other details concerning the process of completion, may be found in [5] or [7].

Recall that an integrally closed domain is an integral domain which is integrally closed in its field of fractions. All valuation rings are integrally closed domains (see [23, Theorem 10.3]) and this property is of course enjoyed by all intersections of valuation rings in a given field. The following describes a particularly important class of such integrally closed domains.

Let $D$ be an integral domain and $k$ its field of fractions. Suppose that $\{D_\lambda\}_{\lambda \in \Lambda}$ is a family of discrete valuation rings with $D = \bigcap_{\lambda \in \Lambda} D_\lambda$, and with the property that for each $a \in k^\times$ there are at most a finite number of $\lambda \in \Lambda$ such that $v_\lambda(a) \neq 0$, where $v_\lambda$ is the normalized discrete valuation corresponding to $R_\lambda$ (this latter condition is sometimes called the *finite character property*). Then we call $D$ a *Krull ring*. By [23, Theorem 12.3], for each height 1 prime ideal $\mathfrak{p}$, the localization of $D$ at $\mathfrak{p}$ is a discrete valuation ring in $k$ and if $\mathcal{P}$ is the set of all height 1 prime ideals of $D$ then $\{D_\mathfrak{p}\}_{\mathfrak{p} \in \mathcal{P}}$ is a defining family for $D$. In fact [23, Theorem 12.3] shows that $\{D_\mathfrak{p}\}_{\mathfrak{p} \in \mathcal{P}}$ is the minimal defining family for $D$, in the sense that any other family of discrete valuation rings of $k$ defining $D$ contains $\{D_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}$ and if $\mathcal{P}' \subsetneq \mathcal{P}$ then

$$\bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}} \subsetneq \bigcap_{\mathfrak{p}\in\mathcal{P}'} D_{\mathfrak{p}}.$$

Hence, every Krull ring has a canonical defining family.

**Examples 1.6.1**

1. Let $k$ be an $\mathbb{A}$–field in the sense of Weil [39] (that is, a finite algebraic extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ – the field of rational functions over a finite field of characteristic $p$). Given any equivalence class of absolute values on $k$, by the above, there is a unique embedding of $k$ into a complete field (which itself corresponds to this equivalence class). Following Weil [39] such an embedding will be called a *place* of $k$. If $k$ has zero characteristic, the places which arise from distinct discrete valuations on $k$ are called *finite*, and the remaining places are termed *infinite*. If $k$ has characteristic $p$, all places correspond to discrete valuations. To describe the 'infinite' places of such a field, first consider the valuation $v_\infty$ on $\mathbb{F}_p(t)$ which is defined as follows. For any element $f/g \in \mathbb{F}_p(t)$ set $v_\infty(f/g) = \deg(g) - \deg(f)$. The equivalence class to which this valuation belongs gives rise to a place of $\mathbb{F}_p(t)$ which is referred to as the *infinite* place of $\mathbb{F}_p(t)$. The *infinite* places of $k$ are those which correspond to valuations on $k$ which extend $v_\infty$. All other places in the positive characteristic case are called *finite*. Irrespective of the characteristic of $k$, [39, Chapter 3] shows that the number of infinite places is always finite. Let $S$ be any collection of finite places of $k$ and let $D_\lambda$ denote the valuation ring in $k$ which corresponds to $\lambda \in S$. If $P$ is the set of all finite places of $k$ and $S$ is a proper subset of $P$, then $\{D_\lambda : \lambda \in P \setminus S\}$ is a defining family for a Krull ring $D_S$, referred to as the *ring of $S$–integers in $k$*. If $S = P$ then by convention we set $D_S = k$.

   An arithmetic dynamical system $(X^D, \alpha^{(D,\xi)})$ for which $D$ is a ring $S$–integers and $\xi$ is given by a single unit of $D$ is an *$S$–integer dynamical system*. Dynamical systems of this type were introduced in [4] and have received further treatment in [37] and [38].

2. Any Dedekind domain $D$ is a Krull ring (see [5, Theorem 2.4.5]). Each non-zero prime ideal of $D$ is of height 1. Hence, if $\mathcal{P}$ is the set of all non-zero prime ideals of $D$ then $\{D_{\mathfrak{p}} : \mathfrak{p} \in \mathcal{P}\}$ is the minimal defining family for $D$.

3. If $D$ is a unique factorization domain then $D$ is a Krull ring. Let $k$ be the field of fractions of $D$. Each height 1 prime ideal of $D$ is principal and generated by an irreducible $\pi \in D$. In addition $D_{(\pi)}$ is a discrete valuation ring in $k$. Moreover, if $\mathcal{P}$ denotes the set of all irreducibles of $D$ then $\{D_{(\pi)} : \pi \in \mathcal{P}\}$ is the minimal defining family for $D$.

Often it will be necessary to consider how Krull rings behave under certain ring extensions. The following results and many others concerning the fundamental properties of Krull rings may be found in [8, Chapter 1] or [23, Section 12].

**Proposition 1.6.2** *Let $D$ be a Krull ring with field of fractions $k$ and $K|k$ a finite algebraic extension. Then the integral closure of $D$ in $K$ is again a Krull ring.*

**Proposition 1.6.3** *Let $D$ be a Krull ring and $\{t_j : j \in J\}$ a family of algebraically independent indeterminates over $D$. Then the polynomial ring $D[t_j : j \in J]$ is a Krull ring.*

Given any Krull ring $D$, the canonical defining family for $D$ is obtained by localizing $D$ at each of its height 1 prime ideals. However there are often other defining families for $D$ which can be useful.

**Proposition 1.6.4** *Suppose that $D$ is a Krull ring with field of fractions $k$ and let $K|k$ be a finitely generated extension. If $D$ is integrally closed in $K$ then it may be expressed as the intersection of a family of discrete valuation rings of $K$ with the finite character property.*

*Proof.* Let $t_1, \ldots, t_r$ be a transcendence base for $K$ over $k$. The rings $A_1 = D[t_1, \ldots, t_r]$ and $A_2 = D[t_1^{-1}, \ldots, t_r^{-1}]$ are Krull rings by Proposition 1.6.3. If $B_1$ and $B_2$ denote the respective integral closures of $A_1$ and $A_2$ in $K$, then Proposition 1.6.2 shows that these too are Krull rings. Let $\{D_\lambda : \lambda \in \Lambda_1\}$ and $\{D_\lambda : \lambda \in \Lambda_2\}$ be their respective canonical defining families of discrete valuation rings of $K$. Suppose that $\eta \in B_1 \cap B_2$. Denote the common field of fractions of $A_1$ and $A_2$ by $L$. Since $A_1$ and $A_2$ are integrally closed in $L$, [23, Theorem 9.2] shows that the minimum polynomial for $\eta$ over $L$ has coefficients in $A_1 \cap A_2$. But $A_1 \cap A_2 = D$ and so this means $\eta \in D$. Thus $B_1 \cap B_2 = D$ which implies

$$D = \bigcap_{\lambda \in \Lambda_1 \cup \Lambda_2} D_\lambda.$$

Clearly also $\{D_\lambda : \lambda \in \Lambda_1 \cup \Lambda_2\}$ has the finite character property. □

**Theorem 1.6.5** *(Mori-Nagata integral closure theorem) If $D$ is a Noetherian integral domain with field of fractions $k$, then the integral closure of $D$ in $k$ is a Krull ring.*

*Proof.* See [8, Theorem 4.3]. □

Amongst the most well-studied of Krull rings are those lying in $\mathbb{A}$-fields, as described by Example 1.6.1(1). The dual groups of such rings have a particularly nice description, provided by the $S$–*adele ring*, where $S$ is the set of finite places of $k$ defining $D_S$ as in Example 1.6.1(1). For any place $\lambda$ of $k$, let $k_\lambda$ denote the completion of $k$ at $\lambda$ and $|\cdot|_\lambda$ the corresponding absolute value. For finite $\lambda$ there is a maximal compact subring of $k_\lambda$ given by

$$r_\lambda = \{y \in k_\lambda : |y|_\lambda \leq 1\}.$$

The $S$–adele ring $k^S$ is defined as follows

$$k^S = \{x = (x_\lambda) \in \prod_{\lambda \in T} k_\lambda : x_\lambda \in r_\lambda \text{ for all but a finite number of } \lambda \in S\}$$

where $T$ is the union of $S$ with the set of infinite places of $k$. Let $U$ be a finite subset of $T$ containing all infinite places of $k$, and define

$$k^S(U) = \prod_{\lambda \in U} k_\lambda \times \prod_{\lambda \in T \setminus U} r_\lambda.$$

The topology on $k^S$ is the smallest topology in which each set of the form $k^S(U)$ is open in $k^S$. Notice that $k^S(U)$ is locally compact with the usual product topology and we thus induce a locally compact ring structure on $k^S$. For a thorough description consult [34] or [39]. It is also worth noting that if $S$ is finite then $k^S$ is just a product of fields with the usual product topology. There is a natural way to embed the ring $D_S$ as a subring of $k^S$ by means of the homomorphism $\phi : D_S \mapsto k^S$ defined by $\phi(x) = (x, x, x, ...)$, *the canonical embedding*. This embedding is used to describe the dual group of $D_S$. In particular $\phi(D_S)$ is a discrete subgroup of $k^S$ (so that there is a local isomorphism between $k^S$ and $k^S/\phi(D_S))$, $\widehat{k^S} \cong k^S$ and $\phi(D_S)^\perp \cong \phi(D_S) \cong D_S$. By duality we have a natural isomorphism, $\widehat{D_S} \cong k^S/\phi(D_S)$.

We conclude this section by introducing two types of valuation which will be of use in Chapter 4. The first example highlights how certain 'unexpected' discrete valuations can arise on integral domains which are not Dedekind. It also illustrates how the finite character property can break down. The second gives an example of a non-discrete valuation.

**Example 1.6.6** Consider the field $\mathbb{Q}(t)$ and its subring $D = \mathbb{Z}[t]$, where $t$ is an indeterminate over $\mathbb{Q}$. The height 1 prime ideals of $D$ are given by the irreducibles of $D$ and the other non-trivial prime ideals have height 2 and are maximal. They are of the form $(p, f)$ where $p$ is a rational prime and $f$ is a polynomial whose reduction modulo $p$ is irreducible (see [32, Example 1.4.4]). Since $D$ is a unique factorization domain, it is easily seen that the localization of $D$ at each of its height 1 prime ideals is a discrete valuation ring of $\mathbb{Q}(t)$. In particular, the localization of $D$ at $p$, where $p$ is a rational prime, gives rise to a discrete valuation which extends $v_p$, the $p$-adic valuation on $\mathbb{Q}$. However, there are other discrete valuations on $\mathbb{Q}(t)$ which are extensions of $v_p$. Let $j \in \mathbb{N}$ and $T_j$ be an element of the complete field $\mathbb{Q}_p$ which is transcendental over $\mathbb{Q}$ and has $v_p(T_j) = j$ (by a simple cardinality argument there exist such $T_j$ for arbitrary $j \in \mathbb{N}$). Suppose that $f \in D$ is a monic polynomial whose reduction modulo $p$ is irreducible. The field $K = \mathbb{Q}(f)$ may be embedded in $\mathbb{Q}_p$ by sending $f$ to $T_j$. Restricting $v_p$ to the image of $K$ in $\mathbb{Q}_p$ gives a discrete valuation $v$ on $K$. Since $\mathbb{Q}(t)|K$ is a finite extension, by [5, Theorem 2.3.3] there exists an extension $w$ of $v$ to $\mathbb{Q}(t)$. Let $D_w$ denote the valuation ring in $\mathbb{Q}(t)$ which corresponds to $w$.

By construction $\mathbb{Z}[f]$ is contained in $D_w$ and because $t$ satisfies

$$f(X) - f(t) = 0,$$

it is integral over $\mathbb{Z}[f]$. Since valuation rings are integrally closed domains, it follows that $D$ is contained in $D_w$. If $\mathfrak{m}_w$ denotes the maximal ideal of $D_w$ then $\mathfrak{m}_w \cap \mathbb{Z}[t]$ contains both $p$ and $f$. Hence $\mathfrak{m}_w \cap \mathbb{Z}[t] = (p, f)$. Also, $w(f/p^i) = j - i$, and so $f/p^i \in D_w$ for $0 \leq i \leq j$ and $f/p^i \notin D_w$ for $i > j$. This shows that different choices of $j$ yield genuinely distinct valuation rings. Hence we have an infinite family of distinct valuations $\{w_j : j \in \mathbb{N}\}$ for which $w_j(p) \neq 0$. Thus, this family does not have the finite character property.

**Example 1.6.7** Suppose that $D_v$ is a discrete valuation ring and $k$ is its field of fractions. For any real number $\lambda \geq 0$ there is a valuation on the integral domain $D_v[t]$, where $t$ is an indeterminate over $k$, defined as follows. For $f = \sum_{n \in \mathbb{Z}_+} a_n t^n \in D_v[t]$, $a_n \in D_v$, define $v_\lambda : D_v[t] \mapsto \mathbb{R} \cup \{\infty\}$ by

$$v_\lambda(f) = \min_n \{v(a_n) + \lambda n\}.$$

This valuation extends in the usual way to $k(t)$. If $D_\lambda$ denotes the valuation ring in $k(t)$ corresponding to $v_\lambda$, then note that $k \cap D_\lambda = D_v$.

## 1.7 Algebraic geometry

This section is intended to give a brief introduction to the language of modern algebraic geometry, which will be used in Chapter 4. Detailed introductions can be found in [10] or [32]. Let $X$ be a topological space. A *sheaf* $\mathcal{O}$ of rings (always assumed to be commutative) on $X$ is a collection of rings and homomorphisms which satisfy the following.

1. For every open subset $U \subset X$ there is an associated ring $\mathcal{O}(U)$.

2. For every inclusion $V \subset U$ of open subsets of $X$ there is a ring homomorphism $\rho_{UV} : \mathcal{O}(U) \mapsto \mathcal{O}(V)$. These homomorphisms are called *restriction maps*.

3. $\mathcal{O}(\varnothing)$ is the trivial ring consisting of 1 element.

4. If $U$ is an open subset of $X$ then $\rho_{UU}$ is the identity map.

5. If $W \subset V \subset U$ are open subsets of $X$ then $\rho_{UW} = \rho_{VW} \cdot \rho_{UV}$.

6. If $U$ is an open set, $\{U_\lambda\}$ an open covering of $U$ and $s \in \mathcal{O}(U)$ is such that $\rho_{UU_\lambda}(s) = 0$ for all $\lambda$, then $s = 0$.

7. If $U$ is an open set, $\{U_\lambda\}$ an open covering of $U$ and if for each $\lambda$ there are elements $s_\lambda \in \mathcal{O}(U_\lambda)$ with the property that, for any $\lambda, \mu$

$$\rho_{U_\lambda(U_\lambda \cap U_\mu)}(s_\lambda) = \rho_{U_\mu(U_\lambda \cap U_\mu)}(s_\mu)$$

then there exists $s \in \mathcal{O}(U)$ such that $\rho_{UU_\lambda}(s) = s_\lambda$ for each $\lambda$.

If $\mathcal{O}_X$ is a sheaf of rings on $X$ and $x \in X$ is a point of $X$ then $\mathcal{O}_{X,x}$ denotes the *stalk* of $X$ at $x$, which is the direct limit of the rings $\mathcal{O}_X(U)$ as $U$ ranges over all open sets containing $x$, via the appropriate restriction maps. The pair $(X, \mathcal{O}_X)$ is called a *ringed space*. If for each $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring then $(X, \mathcal{O}_X)$ is a *locally ringed space*.

**Example 1.7.1** Let $k$ be a countable field and $A$ a subring. The *Zariski space* of valuation rings of $k$ having *centre* in $A$ is the set of all valuation rings of $k$ which contain $A$, denoted by $\mathrm{Zar}(k, A)$. If $A \subset B$ then clearly $\mathrm{Zar}(k, B) \subset \mathrm{Zar}(k, A)$. A topology may be introduced on $\mathrm{Zar}(k, A)$ by defining the open sets to be arbitrary unions of sets of the form

$$\mathrm{Zar}(k, A[a_1, \ldots, a_r])$$

where $a_1, \ldots, a_r \in k$. Note that a finite intersection of sets of this type is again of this type and hence this does indeed define a topology. To make $\mathrm{Zar}(k, A)$ into a locally ringed space, for any non-empty open set $U$ define $\mathcal{O}(U) = \bigcap_{R \in U} R$. If $V \subset U$ then $\bigcap_{R \in U} R$ is a subring of $\bigcap_{R \in V} R$ and hence the restriction maps are induced by inclusions. Also, for any $R \in \mathrm{Zar}(k, A)$ the stalk of $\mathrm{Zar}(k, A)$ at $R$ is simply $R$, which is a local ring.

A *morphism* of ringed spaces $\psi : (X, \mathcal{O}_X) \mapsto (Y, \mathcal{O}_Y)$ consists of a continuous map $\psi : X \mapsto Y$ and a collection of homomorphisms $\psi_U : \mathcal{O}_Y(U) \mapsto \mathcal{O}_X(\psi^{-1}U)$ for which the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\rho_{UV}"] \arrow[d, "\psi_U"'] & \mathcal{O}_Y(V) \arrow[d, "\psi_V"] \\
\mathcal{O}_X(\psi^{-1}U) \arrow[r, "\rho_{\psi^{-1}U\psi^{-1}V}"] & \mathcal{O}_X(\psi^{-1}V)
\end{tikzcd}$$

for all open subsets $V \subset U$ of $Y$. Often, when no confusion can arise, the notation $\psi : X \mapsto Y$ is also used for such a morphism. Given any $x \in X$ there is an induced homomorphism from $\mathcal{O}_{Y,\psi(x)}$ to $\mathcal{O}_{X,x}$ given by composing the direct limit of the maps

$$\psi_U : \mathcal{O}_Y(U) \mapsto \mathcal{O}_X(\psi^{-1}U)$$

(where $U$ ranges over all open neighbourhoods of $\psi(x)$) and the inclusion of $\operatorname{inj\,lim} \mathcal{O}(\psi^{-1}U)$ into $\mathcal{O}_{X,x}$. When dealing with locally ringed spaces, we say that $\psi$ is a *morphism of locally ringed spaces*, provided that the pre-image of the maximal ideal of $\mathcal{O}_{X,x}$ under this induced map, is equal to the maximal ideal of $\mathcal{O}_{Y,\psi(x)}$. The morphism $\psi$ is an *isomorphism* if the map between the topological spaces $X$ and $Y$ is a homeomorphism and each of the $\psi_U$ are isomorphisms.

Let $R$ be a commutative ring and $X = \text{Spec } R$. For any ideal $\mathfrak{a}$ of $R$, let $V(\mathfrak{a}) \subset X$ be the set of all prime ideals which contain $\mathfrak{a}$. The *Zariski topology* on $X$ is given by taking the closed sets to be all sets of the form $V(\mathfrak{a})$. Lemma 2.1 of [10, Chapter 2] shows that finite unions and arbitrary intersections of sets of the form $V(\mathfrak{a})$ are again of that form, verifying that this is a well defined topology. There is a natural structure sheaf $\mathcal{O}_X$ of rings on $X$ defined as follows. For $a \in R$, let $\mathfrak{a} = (a)$, $D(a) = X \setminus V(\mathfrak{a})$ and denote the localization of $R$ with respect to the set $\{a^j : j \in \mathbb{Z}_+\}$ by $R_a$. If $a, b \in R$ and $D(a) \subset D(b)$ then by [32, 2.2(2)], for some $j \in \mathbb{N}$ and $c \in R$

$$a^j = bc.$$

Hence, for every pair of elements $a, b \in R$ with $D(a) \subset D(b)$ there is a natural homomorphism $\rho_{D(b)D(a)} : R_b \mapsto R_a$ given by

$$\rho_{D(b)D(a)}\left(\frac{f}{b^i}\right) = \frac{fc^i}{a^{ij}}$$

For an open set $U \subset X$, define $\Lambda(U) = \{f \in R : D(f) \subset U\}$ and

$$\mathcal{O}(U) = \left\{(f_a)_{a\in\Lambda(U)} : f_a \in R_a \text{ and } \rho_{D(b)D(a)}(f_b) = f_a \text{ whenever } D(a) \subset D(b)\right\}.$$

If $V \subset U$ are open subsets of $X$ then $\Lambda(V) \subset \Lambda(U)$ and a map $\rho_{UV} : \mathcal{O}(U) \mapsto \mathcal{O}(V)$ may be defined by

$$\rho_{UV}\left((f_a)_{a\in\Lambda(U)}\right) = (f_a)_{a\in\Lambda(V)}.$$

Theorem 2.3.1 of [32] shows that the sets $\mathcal{O}(U)$ and maps $\rho_{UV}$ define a sheaf of rings $\mathcal{O}_X$ on $X$. This is referred to as the *structure sheaf* on $X$ and $(X, \mathcal{O}_X)$ is called the *spectrum* of $R$. We will be particularly interested in the case where $R$ is an integral domain with field of fractions $k$. Here, for non-trivial $U$, the ring $\mathcal{O}(U)$ may be identified with the subring of $k$ given by $\bigcap_{\mathfrak{p}\in U} R_{\mathfrak{p}}$.

An *affine scheme* is a locally ringed space which is isomorphic, as a locally ringed space, to the spectrum of some ring. A *scheme* is a locally ringed space where every point has a neighbourhood which, with the restricted topology and restricted sheaf, is an affine scheme. In general, if $(X, \mathcal{O}_X)$ is a scheme and $Y$ is an open subset of $X$ endowed with the subspace topology, then the pair $(Y, \mathcal{O}_X|_Y)$ is an *open subscheme* of $(X, \mathcal{O}_X)$. A morphism of schemes $\psi : Y \mapsto X$ is a *closed embedding* if every $x \in X$ has an affine neighbourhood $U$ such that $\psi^{-1}(U) \subset Y$ is an affine subscheme and $\psi_U$ is surjective. In such a case the scheme $(Y, \mathcal{O}_Y)$ is called a closed subscheme of $X$.

**Example 1.7.2** Let $R$ be a commutative ring and $\mathfrak{a}$ an ideal of $R$. Let $X = \text{Spec } R$ and $Y = \text{Spec } R/\mathfrak{a}$. The canonical map from $R$ to $R/\mathfrak{a}$ induces a morphism of schemes $\psi : Y \mapsto X$ which is a closed embedding. Under this morphism $Y$ is homeomorphic to the closed subset $V(\mathfrak{a})$ of $X$. Furthermore, for each $\mathfrak{p} \in Y$ the induced homomorphism from $\mathcal{O}_{X,\psi(\mathfrak{p})}$ to $\mathcal{O}_{Y,\mathfrak{p}}$ corresponds to the natural map from $R_{\psi(\mathfrak{p})}$ to $(R/\mathfrak{a})_{\mathfrak{p}}$, which is a surjection. This means each of the homomorphisms $\psi_U$ is surjective. Thus $Y$ is a closed subscheme of $X$.

**Example 1.7.3** The *affine d-space* over $\mathbb{C}$ is the set $A^d_{\mathbb{C}}$ of all $d$–tuples of elements of $\mathbb{C}$. Let $A$ be the polynomial ring $\mathbb{C}[t_1, \dots, t_d]$. For each ideal $\mathfrak{a} \subset A$ define

$$V_{\mathbb{C}}(\mathfrak{a}) = \{\mathbf{z} \in A^d_{\mathbb{C}} : f(\mathbf{z}) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

The *Zariski topology* on $A^d_{\mathbb{C}}$ is given by taking the closed sets to be all sets of the form $V_{\mathbb{C}}(\mathfrak{a})$ (see [10, Chapter 1]). An *affine complex variety* is any closed set of the form $V_{\mathbb{C}}(\mathfrak{p})$ where $\mathfrak{p} \in \operatorname{Spec} A$. Denote the set of all such (non-empty) varieties by $X$. For a closed subset $Y$ of $A^d_{\mathbb{C}}$ let $V(Y)$ be the set of varieties which are contained in $Y$. Taking the closed sets to be all sets of the form $V(Y)$ defines a topology on $X$. Moreover, [10, Proposition 2.6, Chapter 2] shows that $X$ is homeomorphic to Spec $A$ and the set of closed points of $X$ (corresponding to the maximal ideals of Spec $A$) is homeomorphic to $A^d_{\mathbb{C}}$. Hence the space $X$, together with the structure sheaf inherited from Spec $A$, is an affine scheme. Let

$$W = \bigcup_{j=1}^{d} V_{\mathbb{C}}(\mathfrak{a}_j)$$

where $\mathfrak{a}_j = (t_j)$, $j = 1, \dots, d$. Then $W$ is a closed subset of $A^d_{\mathbb{C}}$ and the complement of $V(W)$ in $X$ is an open subscheme of $X$ isomorphic to Spec $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. Closed subschemes of this spectrum will be of importance in Chapter 4.

## 1.8 Examples of arithmetic dynamical systems

**Example 1.8.1** Let $k$ be an $\mathbb{A}$–field, $S$ a set of finite places of $k$, $D_S$ the ring of $S$–integers in $k$ and $\xi_1, \dots, \xi_d$ units of $D_S$. The identification of $X = \widehat{D_S}$ and the quotient ring $k^S/\phi(D_S)$, as given in Section 1.6, allows a useful realization of the action dual to that given by (1). Let $T$ be the union of $S$ with the set of infinite places of $k$. Clearly for each $a \in D_S$ and $\mathbf{n} \in \mathbb{Z}^d$

$$\phi\widehat{\alpha_{\mathbf{n}}}(a) = \phi(\xi^{\mathbf{n}} a) = (\xi^{\mathbf{n}} a, \xi^{\mathbf{n}} a, \dots).$$

Let $\mathbf{n} \in \mathbb{Z}^d$ and $\tilde{\beta}_{\mathbf{n}} : k^S \mapsto k^S$ be defined by

$$\tilde{\beta}_{\mathbf{n}}(x) = (\xi^{\mathbf{n}} x_\lambda)_{\lambda \in T}$$

where $x = (x_\lambda)_{\lambda \in T} \in k^S$. Then

$$\tilde{\beta}_{\mathbf{n}}|_{\phi(D_S)} = \phi\widehat{\alpha_{\mathbf{n}}}.$$

Suppose that $\pi : k^S \mapsto k^S/\phi(D_S)$ is the quotient map and $\psi : \widehat{D_S} \mapsto k^S/\phi(D_S)$ is the canonical isomorphism described in Section 1.6. Upon defining $\beta_{\mathbf{n}} = \pi\tilde{\beta}_{\mathbf{n}}$, the self duality of $k^S$ means that

$$\begin{array}{ccc} \widehat{D_S} & \xrightarrow{\alpha_{\mathbf{n}}} & \widehat{D_S} \\ \psi\downarrow & & \downarrow\psi \\ k^S/\phi(D_S) & \xrightarrow{\beta_{\mathbf{n}}} & k^S/\phi(D_S) \end{array}$$

commutes. Many aspects of the dynamics of the original system can be analysed using the action $\tilde{\beta} : \mathbf{n} \to \tilde{\beta}_{\mathbf{n}}$ on $k^S$. This will be very useful in the sequel.

**Example 1.8.2** Let $D$ be the ring of integers in the field $\mathbb{Q}(\sqrt{17})$. By [5, Theorem 3.1.3], $D$ is a free $\mathbb{Z}$–module with basis $(1, \frac{1+\sqrt{17}}{2})$. The unit $\xi = 4 - \sqrt{17} \in D$ induces an automorphism $\widehat{\alpha}$ of $D$ (and hence an arithmetic dynamical system). Let $\alpha$ be the corresponding dual automorphism. Since there is a natural isomorphism $\psi : D \mapsto \mathbb{Z}^2$, the map

$$\beta = \widehat{\psi\widehat{\alpha}\psi^{-1}} = \widehat{\psi^{-1}}\alpha\widehat{\psi}$$

is an automorphism of $\mathbb{T}^2$. For a precise description of $\beta$ consider the following. Let $a + \frac{1+\sqrt{17}}{2}b \in D$, $a, b \in \mathbb{Z}$, and notice that

$$\widehat{\alpha}\left(a + \frac{1+\sqrt{17}}{2}b\right) = \xi\left(a + \frac{1+\sqrt{17}}{2}b\right) = 5a - 8b + \frac{1+\sqrt{17}}{2}(3b - 2a).$$

Thus $\beta$ may be calculated as follows. For each $(x, y) \in \mathbb{T}^2$ and $(a, b) \in \mathbb{Z}^2$ we have,

$$\begin{aligned} \langle \beta(x,y), (a,b) \rangle &= \langle (x,y), \psi\widehat{\alpha}\psi^{-1}(a,b) \rangle \\ &= \langle (x,y), (5a - 8b, 3b - 2a) \rangle \\ &= e^{2\pi i(5ax - 8bx + 3by - 2ay)} \\ &= \langle (5x - 2y, -8x + 3y), (a,b) \rangle. \end{aligned}$$

Hence $\beta$ is given by the matrix

$$\begin{pmatrix} 5 & -2 \\ -8 & 3 \end{pmatrix}.$$

Alternatively, this dynamical system may be studied using the techniques of the previous example. If $P$ is the set of all finite places of $\mathbb{Q}(\sqrt{17})$ then $D = \bigcap_{\lambda \in P} D_\lambda$. This is the ring of $S$–integers in $\mathbb{Q}(\sqrt{17})$ defined by $S = \varnothing$. The infinite places of $\mathbb{Q}(\sqrt{17})$ are given by $|\cdot|_\infty$ – the usual absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}(\sqrt{17})$ and $|\cdot|_{\tau\infty}$ – the absolute value defined by

$$|\cdot|_{\tau\infty} = |\tau(\cdot)|_\infty,$$

where $\tau$ is the non-trivial element in Gal $(\mathbb{Q}(\sqrt{17})|\mathbb{Q})$. If $k_\infty$ and $k_{\tau\infty}$ denote the respective complete fields then

$$\widehat{D} \cong (k_\infty \times k_{\tau\infty})/\phi(D)$$

and for each $n \in \mathbb{Z}$ the automorphism $\tilde{\beta}_n : k_\infty \times k_{\tau\infty} \mapsto k_\infty \times k_{\tau\infty}$, defined in Example 1.8.1, is given by

$$(x_1, x_2) \mapsto ((4 - \sqrt{17})^n x_1, (4 - \sqrt{17})^n x_2),$$

where $(x_1, x_2) \in k_\infty \times k_{\tau\infty}$.

**Example 1.8.3** Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be the arithmetic dynamical system generated by the data $D = \mathbb{Z}[\frac{1}{6}]$, $\xi_1 = 2$ and $\xi_2 = 3$. If $S$ is the set consisting of the 2–adic and 3–adic places of $\mathbb{Q}$ then $D$ is equal to the ring of $S$–integers in $\mathbb{Q}$. Hence $X$ is isomorphic to

$$Y = (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3) \Big/ \phi(\mathbb{Z}[\tfrac{1}{6}])$$

where $\phi$ is the canonical embedding. Furthermore, Example 1.8.1 shows that for each $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, $\alpha_{\mathbf{n}}$ may be identified with the automorphism of $Y$ which is given by

$$(x_1, x_2, x_3) + \phi(\mathbb{Z}[\tfrac{1}{6}]) \mapsto (2^{n_1} 3^{n_2} x_1, 2^{n_1} 3^{n_2} x_2, 2^{n_1} 3^{n_2} x_3) + \phi(\mathbb{Z}[\tfrac{1}{6}])$$

where $(x_1, x_2, x_3) + \phi(\mathbb{Z}[\frac{1}{6}]) \in Y$.

**Example 1.8.4** The well known example by Ledrappier [16] is given by the shift action of $\mathbb{Z}^2$ on the compact subgroup of $\mathbb{F}_2^{\mathbb{Z}^2}$ consisting of elements $(x_{m_1,m_2})$ which, for all $m_1, m_2 \in \mathbb{Z}$, satisfy the relation

$$x_{m_1,m_2} + x_{m_1+1,m_2} + x_{m_1,m_2+1} = 0.$$

The resulting dynamical system is algebraic and in [30] the corresponding dual $R_2$–module is identified as $M = R_2/(2, 1 + u_1 + u_2)$. Consider the subring $D = \mathbb{F}_2[t, \frac{1}{t^2+t}]$ of $\mathbb{F}_2(t)$. Set

$$\xi_1 = -t - 1 \text{ and } \xi_2 = t \tag{5}$$

to obtain an arithmetic dynamical system. It is readily verified that $R_\xi = D$ and $\mathfrak{p}_\xi = (2, 1 + u_1 + u_2)$. Hence there is an isomorphism $\psi : D \mapsto M$ and

$$\begin{tikzcd} D \arrow[r, "\widehat{\alpha_{\mathbf{n}}}^{(D,\xi)}"] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \\ M \arrow[r, "\widehat{\alpha_{\mathbf{n}}}^{M}"] & M \end{tikzcd}$$

commutes for all $\mathbf{n} \in \mathbb{Z}^d$. Thus Ledrappier's example can also be interpreted as an arithmetic dynamical system.

**Example 1.8.5** Let $D$ be a Krull ring for which $\mathcal{P}$, its set of height 1 prime ideals, is countably infinite, and let $k$ be its field of fractions. Write $\mathcal{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots\}$ and denote by $\Omega$ the probability space $\{0,1\}^{\mathbb{N}}$ equipped with the product measure $\mu_\rho = (\rho, 1-\rho)^{\mathbb{N}}$ for some $\rho \in [0,1]$. Let $\omega = (\omega_i) \in \Omega$ and define

$$D(\omega) = \{a \in k : v_{\mathfrak{p}_i}(a) \geq 0 \text{ for all } i \geq 1 \text{ such that } \omega_i = 0\}.$$

For each $\omega \neq (1,1,1,\dots)$, $D(\omega)$ is a Krull ring with defining family $\{D_{\mathfrak{p}_i} : w_i = 0\}$. If $\omega = (1,1,1,\dots)$ then clearly $D(\omega) = k$. If $\rho = 1$ then $\mu_\rho$–almost surely $\omega_i = 0$ for all $i \geq 1$, and hence $D(\omega) = D$. At the other extreme, if $\rho = 0$ then $\mu_\rho$–almost surely $D(\omega) = k$. If $0 < \rho < 1$ then $D(\omega)$ is a 'random ring' with $D \subset D(\omega) \subset k$.

Upon fixing units $\xi_1, \ldots, \xi_d \in D$ (which of course are also units of $D(\omega)$ for any $\omega \in \Omega$) there is an uncountable family of arithmetic dynamical systems given by

$$\{(X^{D(\omega)}, \alpha^{(D(\omega),\xi)}) : \omega \in \Omega\}.$$

In [36] such a family is considered with $D = \mathbb{Z}[\frac{1}{2}]$ and $\xi = 2$. The author shows that each of its members has a distinct dynamical zeta function and entropy $\log 2$. In Chapter 3 it will be shown that for any family of this kind, the entropy of each member is the same. Further work on such 'random' examples arising from Krull rings in $\mathbb{A}$–fields may be found in [37] and [38], for the case $d = 1$.

The following example illustrates several techniques which may be employed in a more general setting.

**Example 1.8.6** Let $t$ be an indeterminate, $D = \mathbb{Q}[t, \frac{1}{3t-t^2}]$, $\xi_1 = t$ and $\xi_2 = 3 - t$. Then $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ is an arithmetic dynamical system. It is readily verified that

$$D \cong \mathbb{Q}[\xi_1^{\pm 1}, \xi_2^{\pm 1}] \cong \mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]/\mathfrak{p}$$

where $u_1$ and $u_2$ are independent variables and $\mathfrak{p} \in \operatorname{Spec} \mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]$ is generated by the polynomial $3 - u_1 - u_2$. Since $\mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}] \cong \bigoplus_{\mathbb{Z}^2} \mathbb{Q}$, it follows that

$$\widehat{\mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]} \cong \widehat{\mathbb{Q}}^{\mathbb{Z}^2} \cong (\mathbb{Q}^S / \phi(\mathbb{Q}))^{\mathbb{Z}^2}$$

where $S$ is the set of all finite places of $\mathbb{Q}$ (given by the set of rational primes). Furthermore $\widehat{D} \cong \mathfrak{p}^\perp \subset \widehat{\mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]}$. For a realization of this compact group, notice that the image of $\mathfrak{p}$ in $\bigoplus_{\mathbb{Z}^2} \mathbb{Q}$ is precisely the set of elements of the form

$$b = (b_{m_1,m_2}) = (3a_{m_1,m_2} - a_{m_1-1,m_2} - a_{m_1,m_2-1}) \tag{6}$$

where $a = (a_{m_1,m_2}) \in \bigoplus_{\mathbb{Z}^2} \mathbb{Q}$. Now given $x = (x_{m_1,m_2}) \in \widehat{\mathbb{Q}}^{\mathbb{Z}^2}$,

$$
\begin{aligned}
\langle x, b\rangle &= \prod_{m_1,m_2} \langle x_{m_1,m_2}, b_{m_1,m_2}\rangle \\
&= \prod_{m_1,m_2} \langle x_{m_1,m_2}, 3a_{m_1,m_2} - a_{m_1-1,m_2} - a_{m_1,m_2-1}\rangle \\
&= \prod_{m_1,m_2} \langle 3x_{m_1,m_2}, a_{m_1,m_2}\rangle\langle -x_{m_1,m_2}, a_{m_1-1,m_2}\rangle\langle -x_{m_1,m_2}, a_{m_1,m_2-1}\rangle \\
&= \prod_{m_1,m_2} \langle 3x_{m_1,m_2}, a_{m_1,m_2}\rangle\langle -x_{m_1+1,m_2}, a_{m_1,m_2}\rangle\langle -x_{m_1,m_2+1}, a_{m_1,m_2}\rangle \\
&= \prod_{m_1,m_2} \langle 3x_{m_1,m_2} - x_{m_1+1,m_2} - x_{m_1,m_2+1}, a_{m_1,m_2}\rangle.
\end{aligned}
$$

If $\langle x, b\rangle = 1$ for all such $b$ then for all $(m_1, m_2) \in \mathbb{Z}^2$

$$3x_{m_1,m_2} - x_{m_1+1,m_2} - x_{m_1,m_2+1} = 0.$$

Therefore $\widehat{D}$ is isomorphic to

$$Y = \{(x_{m_1,m_2}) \in \widehat{\mathbb{Q}}^{\mathbb{Z}^2} : 3x_{m_1,m_2} - x_{m_1+1,m_2} - x_{m_1,m_2+1} = 0 \text{ for all } m_1, m_2 \in \mathbb{Z}\}.$$

Under the identification of $\mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]/\mathfrak{p}$ with $D$, for each $\mathbf{n} \in \mathbb{Z}^2$ multiplication by $\xi^{\mathbf{n}}$ on $D$ becomes multiplication by $u^{\mathbf{n}}$ mod $\mathfrak{p}$. Moreover, under the indentification of $\mathbb{Q}[u_1^{\pm 1}, u_2^{\pm 1}]$ with $\bigoplus_{\mathbb{Z}^2} \mathbb{Q}$, multiplication by $u^{\mathbf{n}}$ corresponds to the map which sends $(a_{\mathbf{m}}) \in \bigoplus_{\mathbb{Z}^2} \mathbb{Q}$ to $(a_{\mathbf{m}+\mathbf{n}})$. On $\widehat{\mathbb{Q}}^{\mathbb{Z}^2}$ the map dual to this is that which sends $(x_{\mathbf{m}}) \in \widehat{\mathbb{Q}}^{\mathbb{Z}^2}$ to $(x_{\mathbf{m}+\mathbf{n}})$. Consequently $\alpha$ may be identified with the restriction of the $\mathbb{Z}^2$–shift on $\widehat{\mathbb{Q}}^{\mathbb{Z}^2}$ to $Y$.

An alternative interpretation of this example may be obtained by using direct and inverse limits, as follows. Consider the arithmetic dynamical system $(X^{R_\xi}, \alpha^{(R_\xi,\xi)})$. Since $\mathfrak{p}_\xi = (3 - u_1 - u_2) \in \operatorname{Spec} R_2$, it follows that

$$R_\xi \cong R_2/(3 - u_1 - u_2).$$

Therefore $\alpha^{(R_\xi,\xi)}$ may be interpreted as the $\mathbb{Z}^2$–shift on the compact group

$$Y = \{(x_{m_1,m_2}) \in \mathbb{T}^{\mathbb{Z}^2} : 3x_{m_1,m_2} - x_{m_1+1,m_2} - x_{m_1,m_2+1} = 0 \text{ for all } m_1, m_2 \in \mathbb{Z}\}.$$

The dynamical system $(X^D, \alpha^{(D,\xi)})$ may be constructed using this as a building block. Notice that $D$ is isomorphic to $\operatorname{inj}\lim(R_\xi, \psi_j)$ where for each $j \geq 1$, $\psi_j$ is multiplication by $j+1$. Dually this gives $\widehat{D} \cong \operatorname{proj}\lim(Y, \widehat{\psi}_j)$, where for each $j \geq 1$, $\widehat{\psi}_j : Y \mapsto Y$ is given by

$$\widehat{\psi}_j((x_{m_1,m_2})) = ((j+1)x_{m_1,m_2}).$$

Thus $\widehat{D}$ may be identified with the subgroup of $\mathbb{T}^{\mathbb{Z}^2 \times \mathbb{N}}$ consisting of elements $(x_{m_1,m_2,j})$ which, for all $m_1, m_2 \in \mathbb{Z}$ and $j \in \mathbb{N}$, satisfy the relations

$$\begin{aligned} 3x_{m_1,m_2,j} - x_{m_1+1,m_2,j} - x_{m_1,m_2+1,j} &= 0, \\ x_{m_1,m_2,j} - (j+1)x_{m_1,m_2,j+1} &= 0. \end{aligned}$$

For each $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, $\alpha_{\mathbf{n}}^{(D,\xi)}$ corresponds to the automorphism of this group given by

$$(x_{m_1,m_2,j}) \mapsto (x_{m_1+n_1,m_2+n_2,j}).$$

This construction is perhaps a little more cumbersome than the first, but in certain circumstances, which will become apparent later, it will be most useful.

# Chapter 2

# Ergodicity and mixing

## 2.1 Conditions for ergodicity and mixing

Let $X$ be a compact abelian group, $\mathcal{B}(X)$ the $\sigma$–algebra of Borel sets of $X$ and $\mu_X$ the normalized Haar measure on $X$. Suppose that $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ is a $\mathbb{Z}^d$–action by continuous automorphisms of $X$. Recall that $\alpha$ is also a measure preserving $\mathbb{Z}^d$–action on the probability space $(X, \mathcal{B}(X), \mu_X)$. The action $\alpha$ is *ergodic* if and only if for any $B \in \mathcal{B}(X)$ with $\alpha_{\mathbf{n}}(B) = B$ for all $\mathbf{n} \in \mathbb{Z}^d$, $\mu_X(B) = 0$ or $\mu_X(B) = 1$. The automorphism $\alpha_{\mathbf{n}}$ is *ergodic* if and only if for any $B \in \mathcal{B}(X)$ with $\alpha_{\mathbf{n}}(B) = B$, $\mu_X(B) = 0$ or $\mu_X(B) = 1$. The action $\alpha$ is said to be *(strongly) mixing* if for any $B, C \in \mathcal{B}(X)$,

$$\lim_{\mathbf{n}\to\infty} \mu_X(B \cap \alpha_{\mathbf{n}}(C)) = \mu_X(B)\mu_X(C),$$

where $\mathbf{n} \to \infty$ means 'leaving finite sets'.

For arithmetic dynamical systems both ergodicity and mixing have straightforward algebraic interpretations deducible from the following result, which deals with the more general algebraic case.

**Proposition 2.1.1** *Let $X$ be a compact metrizable abelian group and suppose that $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ is a $\mathbb{Z}^d$–action by automorphisms of $X$. Let $M = \widehat{X}$ be the corresponding dual $R_d$–module.*

1. *For any* $\mathbf{n} \in \mathbb{Z}^d$ *the following are equivalent.*

   (a) $\alpha_{\mathbf{n}}^M$ *is ergodic.*

   (b) *No prime ideal* $\mathfrak{p}$ *associated with* $M$ *contains a polynomial of the form* $u^{l\mathbf{n}} - 1$ *with* $l \geq 1$.

2. *The following conditions are equivalent.*

   (a) $\alpha^M$ *is ergodic.*

   (b) *No prime ideal* $\mathfrak{p}$ *associated with* $M$ *contains a set of the form* $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ *with* $l \geq 1$.

3. *The following conditions are equivalent.*

   (a) $\alpha^M$ *is mixing.*

   (b) $\alpha_{\mathbf{n}}^M$ *is ergodic for all non-zero* $\mathbf{n} \in \mathbb{Z}^d$.

*Proof.* This is part of [30, Proposition 6.6]. □

**Corollary 2.1.2** *Let* $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ *be the arithmetic dynamical system arising from units* $\xi_1, \ldots, \xi_d$ *in the integral domain* $D$.

1. *For* $\mathbf{n} \in \mathbb{Z}^d$ *the automorphism* $\alpha_{\mathbf{n}}$ *is ergodic if and only if* $\xi^{l\mathbf{n}} \neq 1$ *for all* $l \geq 1$.

2. $\alpha$ *is ergodic if and only if given* $l \geq 1$ *there exists* $\mathbf{n} \in \mathbb{Z}^d$ *such that* $\xi^{l\mathbf{n}} \neq 1$.

3. $\alpha$ *is mixing if and only if* $\xi^{\mathbf{n}} \neq 1$ *for all non-zero* $\mathbf{n} \in \mathbb{Z}^d$.

*Proof.* 1. Let $D$ be considered as an $R_d$–module under the map $\theta_\xi : R_d \to D$. Lemma 1.5.1 shows that $D$ is associated to $\mathfrak{p}_\xi$. Now given $\mathbf{n} \in \mathbb{Z}^d$, condition 1(b) of Proposition 2.1.1 is equivalent to

$$\theta_\xi(u^{l\mathbf{n}} - 1) \neq 0 \Leftrightarrow \xi^{l\mathbf{n}} \neq 1 \text{ for all } l \geq 1.$$

2. Part 2(b) of Proposition 2.1.1 is the same as saying that given $l \geq 1$ there exists $\mathbf{n} \in \mathbb{Z}^d$ such that $\theta_\xi(u^{l\mathbf{n}} - 1) \neq 0$. Hence the result follows.

3. By part 3 of Proposition 2.1.1 $\alpha$ is mixing if and only if $\alpha_{\mathbf{n}}$ is ergodic for all non-zero $\mathbf{n} \in \mathbb{Z}^d$, which by the above implies and is implied by $\xi^{l\mathbf{n}} \neq 1$ for all $l \geq 1$ and non-zero $\mathbf{n} \in \mathbb{Z}^d$. Equivalently $\xi^{\mathbf{n}} \neq 1$ for all non-zero $\mathbf{n} \in \mathbb{Z}^d$. $\square$

Given any field $k$ and $\xi_1, \ldots, \xi_d \in k^\times$, the property that $\xi^{\mathbf{n}} \neq 1$ for all non-zero $\mathbf{n} \in \mathbb{Z}^d$ is determined only by the choice of $\xi_1, \ldots, \xi_d$. Hence, if the arithmetic dynamical system $(X^{R_\xi}, \alpha^{(R_\xi, \xi)})$ is mixing then so are all dynamical systems of the form $(X^D, \alpha^{(D,\xi)})$ where $D$ is any integral domain containing $\xi_1, \ldots, \xi_d$ as units. This observation should be compared with the results given in Table 1. A similar statement is true for ergodic properties.

If $\alpha$ is a $\mathbb{Z}$–action by automorphisms of a compact abelian group then it is well-known that ergodicity and mixing are equivalent. For an arithmetic dynamical system this is clear from Corollary 2.1.2 since conditions 1 and 3 are equivalent if $d = 1$.

**Example 2.1.3** Let $D = \mathbb{Z}[\frac{1}{6}]$, $\xi_1 = 2$, $\xi_2 = 8$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ the corresponding arithmetic dynamical system. Since $2^l 8^l = 2^{4l} \neq 1$ for all $l \geq 1$ it follows that $\alpha$ is ergodic. Moreover, for any $l \geq 1$, $n_1, n_2 \in \mathbb{Z}$,

$$2^{ln_1} 8^{ln_2} = 2^{l(n_1 + 3n_2)} = 1 \Leftrightarrow n_1 = -3n_2.$$

Hence $\alpha_{(n_1,n_2)}$ is ergodic if and only if $n_1 \neq -3n_2$, so $\alpha$ is not mixing.

**Example 2.1.4** In the above example, if instead of $\xi_2 = 8$ we set $\xi_2 = 3$, then $\xi_1^{n_1} \xi_2^{n_2} \neq 1$ for all non-zero $(n_1, n_2) \in \mathbb{Z}^2$ and hence the corresponding action is mixing. This dynamical system was introduced in Example 1.8.3 and despite its simplicity, it can be used to illustrate many important properties of arithmetic dynamical systems, typical in a more general setting. Here it appears as a special case of the following.

**Example 2.1.5** Let $k$ be a field and $\xi_1, \ldots, \xi_d \in k^\times$. Suppose that for each $1 \leq j \leq d$ there is a discrete valuation $v_j$ on $k$ such that $v_j(\xi_j) \neq 0$ and $v_j(\xi_i) = 0$ for all $i \neq j$. Then any arithmetic dynamical system of the form $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$, where $R_\xi \subset D \subset k$, is mixing. This is because if $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ is non-zero then for some $1 \leq j \leq d$

$$\begin{aligned} v_j(\xi^{\mathbf{n}}) &= \sum_{i=1}^{d} n_i v_j(\xi_i) \\ &= n_j v_j(\xi_j) \\ &\neq 0. \end{aligned}$$

If $\alpha$ were not mixing then this would contradict Corollary 2.1.2.

**Example 2.1.6** Ledrappier's example can also be handled by the above. Using the description provided by Example 1.8.4, this example is seen to correspond to the arithmetic dynamical system generated by the data $D = \mathbb{F}_2[t, \frac{1}{-t^2-t}]$, $\xi_1 = -t-1$ and $\xi_2 = t$, where $t$ is an indeterminate over $\mathbb{F}_2$. Set $v_1$ and $v_2$ to be the normalized discrete valuations corresponding to the valuation rings $\mathbb{F}_2[t]_{(t+1)}$ and $\mathbb{F}_2[t]_{(t)}$, repectively. Then $v_1(\xi_1) = 1$, $v_1(\xi_2) = 0$, $v_2(\xi_2) = 1$ and $v_2(\xi_1) = 0$. Therefore the above shows that Ledrappier's example is mixing.

## 2.2 Higher order mixing

Let $T : \mathbf{n} \mapsto T_{\mathbf{n}}$ be a measure preserving $\mathbb{Z}^d$–action on the probability space $(X, \mathcal{B}, \mu)$ and $r \geq 2$. The action $T$ is *mixing of order* $r$ (or $r$*–mixing*) if for all $B_1, \ldots, B_r \in \mathcal{B}$,

$$\lim_{\substack{\mathbf{n}_i,\mathbf{n}_j \in \mathbb{Z}^d \text{ and } \mathbf{n}_i - \mathbf{n}_j \to \infty \\ \text{for all } i,j=1,\ldots,r, i \neq j}} \mu\left(\bigcap_{i=1}^{r} T_{-\mathbf{n}_i}(B_i)\right) = \prod_{i=1}^{r} \mu(B_i).$$

A non-empty set $F \in \mathbb{Z}^d$ is *mixing* for $T$ if for all collections $\{B_{\mathbf{n}} : \mathbf{n} \in F\} \subset \mathcal{B}$, that is all collections of sets in $\mathcal{B}$ which can be indexed by $F$,

$$\lim_{i \to \infty} \mu\left(\bigcap_{\mathbf{n} \in F} T_{-i\mathbf{n}}(B_{\mathbf{n}})\right) = \prod_{\mathbf{n} \in F} \mu(B_{\mathbf{n}}).$$

Otherwise $F$ is said to be *non-mixing*.

If $T$ is $r$-mixing then every subset of $\mathbb{Z}^d$ of cardinality $r$ is mixing for $T$. Hence, if $T$ is mixing of all orders then every finite non-empty subset of $\mathbb{Z}^d$ is mixing for $T$. If $F$

is a finite subset of $\mathbb{Z}^d$ with cardinality $r \geq 2$ which is non-mixing for $T$ then $T$ is not mixing of order $r$. Finally, if $F$ is non-mixing for $T$ and $E \supset F$ then $E$ is non-mixing for $T$. Section 27 of [30] deals with many aspects of higher order mixing for algebraic $\mathbb{Z}^d$–actions and this will be useful for what follows. For example, from the general algebraic case it is easily deducible that for an arithmetic dynamical system whose underlying ring has characteristic zero, mixing is equivalent to mixing of all orders. In the positive characteristic case however, a more complicated picture of mixing arises, and although it will be demonstrated how one may identify particular non-mixing sets, a general technique for finding the exact order at which a given action fails to be mixing remains elusive.

Given any $\mathfrak{p} \in \operatorname{Spec} R_d$, the ring $R_d/\mathfrak{p}$ is an integral domain. Hence $R_d/\mathfrak{p}$ has a well defined characteristic which is either zero or some rational prime $p$. Denote this characteristic by $p(\mathfrak{p})$. Now suppose that $\mathfrak{p}$ is such that $\alpha^{R_d/\mathfrak{p}}$, the natural action on $\widehat{R_d/\mathfrak{p}}$ (see Section 1.4), is mixing. By [30, Theorem 27.3] if $p(\mathfrak{p}) = 0$ then $\alpha^{R_d/\mathfrak{p}}$ is mixing of all orders. If $p(\mathfrak{p}) \neq 0$ then $\alpha^{R_d/\mathfrak{p}}$ is mixing of all orders if and only if $\mathfrak{p}$ is generated by $p(\mathfrak{p}_\xi)$ (considered as an element of $R_d$). Furthermore, [30, Theorem 27.2] shows that for a general countable $R_d$–module $M$, the action $\alpha^M$ is $r$–mixing if and only if $\alpha^{R_d/\mathfrak{p}}$ is $r$–mixing for each prime ideal $\mathfrak{p}$ associated with $M$. The implications of some of these results are collected in the proposition below.

**Proposition 2.2.1** *Let $D$ be an integral domain, $\xi_1, \ldots, \xi_d$ units of $D$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ the corresponding arithmetic dynamical system.*

1. *For each $r \geq 2$ the action $\alpha$ is $r$–mixing if and only if $\alpha^{R_d/\mathfrak{p}_\xi}$ is $r$–mixing.*
2. *Suppose that $\alpha$ is mixing. If $D$ has zero characteristic then $\alpha$ is mixing of all orders. If $D$ has characteristic $p > 0$ then $\alpha$ is mixing of all orders if and only if $\mathfrak{p}_\xi = (p)$.*

Hence the only mixing arithmetic dynamical systems which fail to be mixing of all orders are those for which the underlying ring $D$ is of characteristic $p > 0$ and $\mathfrak{p}_\xi \neq (p)$ (equivalently, $D$ has positive characteristic and $\mathfrak{p}_\xi$ is non-principal).

If $M$ is a countable $R_d$–module it is also true (by [30, Theorem 27.2]) that a finite non-empty set $F \subset \mathbb{Z}^d$ is mixing for $\alpha^M$ if and only if it is mixing for $\alpha^{R_d/\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ associated with $M$. Therefore, for an arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ such a set is mixing for $\alpha^{(D,\xi)}$ if and only if it is mixing for $\alpha^{R_d/\mathfrak{p}_\xi}$. Let $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in R_d$, $p$ be a rational prime and define

$$\mathcal{S}_p(f) = \{\mathbf{n} \in \mathbb{Z}^d : c_f(\mathbf{n}) \not\equiv 0 \bmod p\}.$$

It turns out (see [30, Example 27.1(2)]) that for an action of the form $\alpha = \alpha^{R_d/\mathfrak{p}}$, where $(p) \subsetneq \mathfrak{p} \in \operatorname{Spec} R_d$, the set $\mathcal{S}_p(f)$ is non-mixing for $\alpha$ for all $f \in \mathfrak{p}$ with $f \not\equiv 0 \bmod p$. Thus for an arithmetic dynamical system whose underlying ring has characteristic $p$, if the associated prime $\mathfrak{p}_\xi$ strictly contains the prime ideal $(p)$ then for every $f \in \mathfrak{p}_\xi$ with $f \not\equiv 0 \bmod p$, the set $\mathcal{S}_p(f)$ is non-mixing.

**Example 2.2.2** Let $D = \mathbb{F}_3[t, \frac{1}{2+t}, \frac{1}{1+t+2t^2}, \frac{1}{1+t^3}]$. Set $\xi_1 = 2 + t$ and $\xi_2 = \frac{1+t+2t^2}{2+t}$ to obtain a mixing $\mathbb{Z}^2$–action $\alpha = \alpha^{(D,\xi)}$ on $\widehat{D}$. It is readily checked that the associated prime $\mathfrak{p}_\xi \in \operatorname{Spec} R_2$ contains the height 2 prime ideal $\mathfrak{p} = (3, u_2 + u_1 - u_1^{-1} + 1)$. Since $R_2$ has Krull dimension 3, and because $D$ is not a field, $\mathfrak{p}_\xi$ cannot be maximal. Therefore $\mathfrak{p}_\xi$ cannot be of height 3, which implies $\mathfrak{p}_\xi = \mathfrak{p}$. It follows that the set

$$\mathcal{S}_3(u_2 + u_1 - u_1^{-1} + 1) = \{(1,0), (0,1), (-1,0), (0,0)\}$$

is non-mixing for $\alpha$. Hence $\alpha$ is not $r$–mixing for all $r \geq 4$. However, the methods described above fail to determine whether $\alpha$ is mixing of order 3.

It should be noted that in general the family $\{\mathcal{S}_p(f) : f \in \mathfrak{p}_\xi\}$ is not necessarily sufficient to describe all sets which are non-mixing for $\alpha^{(D,\xi)}$. Further more involved procedures for identifying other non-mixing sets may be found in Section 28 of [30]. However the problem of finding all non-mixing sets for algebraic $\mathbb{Z}^d$–actions on compact, zero-dimensional abelian groups is currently an open one, even when the corresponding dual module has a single associated prime, as is the case for arithmetic dynamical systems.

# Chapter 3

# Entropy

In this section we take a close look at the entropy of arithmetic dynamical systems and give methods for its calculation with both general techniques (developed from [18]) and examples. It turns out that the entropy of an arithmetic dynamical system arising from units in an integral domain $D$, is closely related to the trancendence degree of the field of fractions of $D$ over its prime subfield. Also given is an explicit entropy formula for arithmetic dynamical systems whose underlying ring is an integrally closed domain. When this ring is not integrally closed, it is shown that in many cases a bound for the entropy still exists. Noteably, these results can be applied without manipulating complicated $R_d$–module filtrations. Finally, it is shown that if $\xi_1, \ldots, \xi_d \in k$ are fixed and every element of $k$ is algebraic over the field of fractions of $R_\xi$, then arithmetic dynamical systems arising from subrings of $k$ containing the integral closure of $R_\xi$ in $k$ all have the same entropy.

## 3.1 Preliminaries

There are several ways that entropy may be defined for algebraic dynamical systems and [30, Section 13] shows how these definitions coincide. Here the topological definition will be used. Let $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be a $\mathbb{Z}^d$–action by automorphisms of the compact metrizable abelian group $X$. Given an open cover $\mathcal{U}$ of $X$, set $N(\mathcal{U})$ equal to the least cardinality of a subcover of $\mathcal{U}$. It may be shown that

$$\log N(\mathcal{U} \vee \mathcal{V}) \leq \log N(\mathcal{U}) + \log N(\mathcal{V}) \tag{7}$$

for all open covers $\mathcal{U}$ and $\mathcal{V}$ of $X$. Here $\mathcal{U} \vee \mathcal{V}$ denotes the *join* of $\mathcal{U}$ and $\mathcal{V}$, which is the open cover defined by

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The property (7) will be referred to as *subadditivity*. For every rectangle $Q = \prod_{i=1}^{d}\{b_i, \ldots, b_i + l_i - 1\} \subset \mathbb{Z}^d$, set $\langle Q \rangle = \min_{i=1,\ldots,d} l_i$ and let $|Q|$ denote the cardinality of $Q$. Define the *entropy* of $\alpha$ with respect to the open cover $\mathcal{U}$ by

$$h(\alpha, \mathcal{U}) = \lim_{\langle Q \rangle \to \infty} \frac{1}{|Q|} \log N\left(\bigvee_{\mathbf{n} \in \mathbb{Z}^d} \alpha_{-\mathbf{n}}(\mathcal{U})\right).$$

This limit may be shown to exist using subadditivity. The *entropy* of the action $\alpha$ is

$$h(\alpha) = \sup_{\mathcal{U}} h(\alpha, \mathcal{U})$$

where $\mathcal{U}$ ranges over all open covers of $X$.

In principle the entropy of any algebraic $\mathbb{Z}^d$–action may be calculated using the methods exposed by Lind, Schmidt and Ward in [18]. In particular, for a Noetherian $R_d$–module $M$, the authors give a formula in terms of the associated primes of $M$ and their multiplicities. Central to this is a quantity introduced by Mahler [20, 21] in a completely non-dynamical context, the *Mahler measure*. For a polynomial $f \in R_d$, it is defined by $\mathbb{M}(f) = 0$ when $f = 0$ and

$$\mathbb{M}(f) = \exp\left(\int_{\mathbb{S}^d} \log|f(\mathbf{z})| d\mathbf{z}\right)$$

otherwise. Here $d\mathbf{z}$ denotes integration with respect to the normalized Haar measure on $\mathbb{S}^d$ and $f$ is interpreted naturally as a function on $\mathbb{S}^d$. When $M$ is Noetherian, recall that there are only finitely many prime ideals associated with $M$. Let those which are principal be generated by polynomials $f_1, \ldots, f_r \in R_d$. Given any prime filtration of $M$ the multiplicity of each prime ideal $(f_i)$, $1 \leq i \leq r$, depends only on $M$

(see [18, Proposition 6.9]). Let these corresponding multiplicities be $c_1, \dots, c_r$. The Lind, Schmidt, Ward entropy formula is

$$h(\alpha^M) = \sum_{i=1}^{r} c_i |\log \mathbb{M}(f_i)|. \tag{8}$$

However, if $M$ is not Noetherian, although in theory the entropy of the corresponding dynamical system may still be calculated [30, Proposition 18.6], this may involve the Mahler measure of infinitely many polynomials. Fortunately, for an arithmetic dynamical system, this problem is easily overcome by Lemma 1.5.1.

As already indicated the results of Lind, Schmidt and Ward are implicit in what follows. The following are taken from [30, Chapter 5] (see also [18]).

**Proposition 3.1.1**

1. *Let* $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R_d$ *and suppose that* $\mathfrak{p} \subsetneq \mathfrak{q}$. *If* $h(\alpha^{R_d/\mathfrak{p}}) < \infty$ *then* $h(\alpha^{R_d/\mathfrak{q}}) = 0$.

2. *Let* $\mathfrak{p} \in \operatorname{Spec} R_d$. *Then*

$$h(\alpha^{R_d/\mathfrak{p}}) = \begin{cases} |\log \mathbb{M}(f)| & \textit{if } \mathfrak{p} = (f) \textit{ is principal.} \\ 0 & \textit{if } \mathfrak{p} \textit{ is non-principal.} \end{cases}$$

3. *Let* $X$ *be a compact metrizable abelian group and let* $\alpha$ *be a* $\mathbb{Z}^d$*–action by automorphisms of* $X$.

    (a) *If* $Y$ *is an* $\alpha$*–invariant subgroup of* $X$ *then*

    $$h(\alpha) = h(\alpha^{X/Y}) + h(\alpha^Y) \qquad \textit{(the addition formula).}$$

    (b) *If* $\{Y_j\}_{j\geq 1}$ *is a decreasing chain of closed* $\alpha$*–invariant subgroups of* $X$ *such that* $\bigcap_{j\geq 1} Y_j = \{0\}$ *then*

    $$h(\alpha) = \sup_{j\geq 1} h(\alpha^{X/Y_j}).$$

Note that the entropy formula (8) follows from parts 2 and 3(a) above.

## 3.2 Completely positive entropy

Completely positive entropy is a property of certain measure preserving $\mathbb{Z}^d$–actions on probability spaces, usually defined using the measure theoretic definition of entropy (see [30, Section 13]). Since every algebraic $\mathbb{Z}^d$–action may also be regarded as a measure preserving $\mathbb{Z}^d$–action on a probability space, it is not surprising that there is an algebraic formulation of completely positive entropy. Because the measure theoretic definition of entropy has not been introduced, it seems sensible to use the algebraic formulation here. However, it must be remarked that equivalence of the following to the usual definition of completely positive entropy is by no means trivial. A proof of this equivalence may be found in [30, Theorem 20.8]. Let $\alpha$ be an action of $\mathbb{Z}^d$ by automorphisms of the compact abelian group $X$, and $M$ the corresponding dual $R_d$-module. Then $\alpha$ has *completely positive entropy* if and only if the action $\alpha^{R_d/\mathfrak{p}}$ has positive entropy for all $\mathfrak{p} \in \operatorname{Spec} R_d$ associated with $M$. As an immediate consequence, we have the following.

**Proposition 3.2.1** *For an arithmetic dynamical system the properties of positive entropy and completely positive entropy are equivalent.*

*Proof.* Follows from Lemma 1.5.1. $\square$

## 3.3 Main results

We begin with another easy consequence of Lemma 1.5.1.

**Proposition 3.3.1** *Let $D$ be an integral domain, $\xi_1, \ldots, \xi_d$ units of $D$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ the corresponding arithmetic dynamical system. If $\mathfrak{p}_\xi = \{0\}$ then $h(\alpha) = \infty$.*

*Proof.* Both $R_\xi$ and $D/R_\xi$ may be regarded as $R_d$–modules with respective natural $\mathbb{Z}^d$–actions $\widehat{\alpha}^{R_\xi}$ and $\widehat{\alpha}^{D/R_\xi}$. Note that $\widehat{R_\xi} \cong \widehat{D}/R_\xi^\perp$ and $\widehat{D/R_\xi} \cong R_\xi^\perp$, so that by the addition formula

$$h(\alpha) = h(\alpha^{R_\xi}) + h(\alpha^{D/R_\xi}).$$

But $h(\alpha^{R_\xi}) = h(\alpha^{R_d/\mathfrak{p}_\xi})$. The result follows by part 2 of Proposition 3.1.1. □

**Proposition 3.3.2** *Let* $(X,\alpha) = (X^D, \alpha^{(D,\xi)})$ *be an arithmetic dynamical system. Then* $\alpha$ *has positive entropy (equivalently, completely positive entropy) if and only if it is mixing and its associated prime is principal. Furthermore, if* $\alpha$ *is mixing and* $\mathfrak{p}_\xi = (f)$ *for some* $0 \neq f \in R_d$ *then*

$$h(\alpha) = \begin{cases} c|\log \mathbb{M}(f)| & \textit{if there is a Noetherian submodule of } D \textit{ for which} \\ & \mathfrak{p}_\xi \textit{ occurs in a prime filtration with maximal} \\ & \textit{multiplicity } c \textit{ say,} \\ \infty & \textit{otherwise.} \end{cases}$$

*Proof.* Let $D$ be considered as an $R_d$–module via the map $\theta_\xi$. By [30, Theorem 19.5] and Proposition 2.1.1, $h(\alpha^{R_\xi}) > 0$ if and only if $\alpha^{R_\xi}$ is mixing and $\mathfrak{p}_\xi$ is principal. Since $D$ is countable it is possible to select a chain

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

of Noetherian submodules of $D$ such that $\bigcup_{j\geq 1} N_j = D$. If $\mathfrak{p}_\xi = \{0\}$ then the previous result shows that $h(\alpha) = \infty$. Hence assume that $\mathfrak{p}_\xi \neq \{0\}$. By the entropy formula (8), for each $j \geq 1$ there exists an integer $c(j) \geq 1$ such that

$$h(\alpha^{N_j}) = c(j)h(\alpha^{R_\xi}).$$

Moreover by part 3(b) of Proposition 3.1.1,

$$h(\alpha) = \sup_{j\geq 1} h(\alpha^{N_j}) = \sup_{j\geq 1} c(j)h(\alpha^{R_\xi}). \tag{9}$$

The first part of the proposition now follows. Suppose that $\alpha$ is mixing and $\mathfrak{p}_\xi = (f)$ for some non-zero $f \in R_d$. If there is a Noetherian submodule $N$ of $D$ for which $\mathfrak{p}_\xi$ occurs in a prime filtration with maximal multiplicity $c$ say, then without loss of generality we may set $N_1 = N$. In such a case (9) implies that

$$h(\alpha) = c\ h(\alpha^{R_\xi}) = c|\log \mathbb{M}(f)|.$$

If there is no such submodule then the desired result again follows by equation (9). $\square$

The constant $c$ appearing above will be referred to as the *multiplicity constant* for the dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$. Note that $c$ depends on both $D$ and the choice of $\xi$. Later it will be shown how the existence of this constant can, in many cases, be determined simply by considering field extensions, instead of module filtrations.

**Example 3.3.3** Suppose that $D$ is an integral domain of positive characteristic $p$, $\xi_1, \ldots, \xi_d$ are units of $D$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ is the corresponding arithmetic dynamical system. If $\alpha$ is not mixing or $\mathfrak{p}_\xi$ is non-principal then $h(\alpha) = 0$. Clearly $\theta_\xi(f) = 0$ for all $f \in R_d$ with $p|f$. Therefore $(p) \subset \mathfrak{p}_\xi$. Hence if $\mathfrak{p}_\xi$ is principal, $(p) = \mathfrak{p}_\xi$. Since $\mathbb{M}(p) = p$, it follows that $h(\alpha) = \infty$ or $h(\alpha) = c \log p$ for some integer $c \geq 0$.

Let $D$ be an integral domain and $k$ its field of fractions. Suppose that $k$ has finite transcendence degree over its prime subfield. If the characteristic of $k$ is non-zero then this degree will be denoted by $t_D$. If the characteristic of $k$ is zero then we set $t_D = 1 + \text{tr.deg.}(k|\mathbb{Q})$. If $k$ has infinite transcendence degree over its prime subfield then we simply write $t_D = \infty$.

**Theorem 3.3.4** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be a mixing arithmetic dynamical system arising from units $\xi_1, \ldots, \xi_d$ in the integral domain $D$. If $d > t_D$ then $h(\alpha) = 0$. If $d < t_D$ then $h(\alpha) = \infty$ when $\mathfrak{p}_\xi$ is principal and $h(\alpha) = 0$ when $\mathfrak{p}_\xi$ is non-principal.*

*Proof.* A proof is given for the case $\text{char}(D) = 0$. The proof for the non-zero characteristic case follows a very similar line of argument. First suppose $d > t_D$. If at least one of $\xi_1, \ldots, \xi_d$ is transcendental over $\mathbb{Q}$ then (after some relabelling) we may assume $\xi_1, \ldots, \xi_e$ are algebraically independent over $\mathbb{Q}$, where $1 \leq e \leq d-2$ and $e$ is maximal. Both of the sets $\{\xi_1, \ldots, \xi_e, \xi_{e+1}\}$, $\{\xi_1, \ldots, \xi_e, \xi_{e+2}\}$ are algebraically dependent over $\mathbb{Q}$. Hence there exist non-zero polynomials $f \in \mathbb{Z}[u_1^{\pm 1}, \ldots, u_e^{\pm 1}, u_{e+1}^{\pm 1}]$ and $g \in \mathbb{Z}[u_1^{\pm 1}, \ldots, u_e^{\pm 1}, u_{e+2}^{\pm 1}]$ such that $\theta_\xi(f) = \theta_\xi(g) = 0$. Because of the algebraic independence of $\xi_1, \ldots, \xi_e$ we may obtain a factor of $f$, $f'$ say, which lies in $\mathfrak{p}_\xi$ but has no factor in $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_e^{\pm 1}]$. Similarly $g$ has a factor $g'$ say, in $\mathfrak{p}_\xi$ which has no factor in $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_e^{\pm 1}]$. Now $f'$ and $g'$ are coprime which shows that $\mathfrak{p}_\xi$ cannot be principal. If none of $\xi_1, \ldots, \xi_d$ are transcendental over $\mathbb{Q}$ then again $\mathfrak{p}_\xi$ must be non-principal. Thus in both cases by Proposition 3.3.2 it follows that $h(\alpha) = 0$. Now suppose that $d < t_D$. If $\mathfrak{p}_\xi$ is non-principal then the result follows immediately, so assume that $\mathfrak{p}_\xi$ is principal. If no subset of $\{\xi_1, \ldots, \xi_d\}$ is algebraically independent over $\mathbb{Q}$ then $\mathbb{Q}(\xi_1, \ldots, \xi_d)$ is an algebraic extension and hence there must be some element of $D$ which is transcendental over $\mathbb{Q}(\xi_1, \ldots, \xi_d)$. Now suppose that at least one of $\xi_1, \ldots, \xi_d$ is transcendental over $\mathbb{Q}$. With appropriate relabelling we may assume that $\xi_1, \ldots, \xi_e$ are algebraically independent over $\mathbb{Q}$, where $1 \leq e \leq d$ and $e$ is maximal. If $e = d$ then it is clear that $\mathfrak{p}_\xi = \{0\}$ and thus $h(\alpha) = \infty$. If this is not the case then $e < d < t_D$. This means again that there must be some element of $D$ which is transcendental over $\mathbb{Q}(\xi_1, \ldots, \xi_d)$. Let $\eta$ be such an element. For each integer $i \geq 1$ set

$$N_i = \sum_{j=0}^{i} R_d \cdot \eta^j.$$

Then for $i \geq 1$ we have

$$N_i / N_{i-1} \cong R_d \cdot \eta^i \Bigg/ \left( \sum_{j=0}^{i-1} R_d \cdot \eta^j \right) \cap (R_d \cdot \eta^i)$$

$$\cong R_d \cdot \eta^i / \{0\} \cong R_d / \mathfrak{p}_\xi.$$

Hence for all $i \geq 1$ it follows that $N_i$ is a submodule of $D$ for which $\mathfrak{p}_\xi$ arises via a prime filtration with multiplicity $i$. Since $\alpha$ is mixing and $\mathfrak{p}_\xi$ is principal, Proposition 3.3.2 shows that $h(\alpha) = \infty$. $\square$

If $d = t_D$ then the range of possible entropies is more interesting, as the following examples show.

**Example 3.3.5** ($d = t_D = 2$ and zero entropy) Let $t$ be an indeterminate over $\mathbb{Z}$, $D = \mathbb{Z}[t, \frac{1}{6}]$, $\xi_1 = 2$ and $\xi_2 = 3$. This gives a mixing dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ . Here the associated prime $\mathfrak{p}_\xi$ is generated by the polynomials $u_1 - 2$

and $u_2 - 3$ which means it is non-principal. Proposition 3.3.2 implies that $h(\alpha) = 0$. Note that for this dynamical system, such a result is to be expected because $(X, \alpha)$ is simply an infinite direct product of copies of a zero entropy system.

**Example 3.3.6** ($d = t_D = 3$ and finite, non-zero entropy) Let $t_1$ and $t_2$ be two algebraically independent indeterminates over $\mathbb{Z}$ and let

$$D = \mathbb{Z}[t_1, t_2, \tfrac{1}{1-t_1^2}, \tfrac{1}{t_2^2+3}, \tfrac{1}{t_1^2-t_2^2-5}].$$

If we take $\xi_1 = 1 - t_1^2$, $\xi_2 = t_2^2 + 3$ and $\xi_3 = t_1^2 - t_2^2 - 5$ then we obtain a mixing arithmetic dynamical system. Here

$$R_\xi = \mathbb{Z}[t_1^2, t_2^2, \tfrac{1}{1-t_1^2}, \tfrac{1}{t_2^2+3}, \tfrac{1}{t_1^2-t_2^2-5}].$$

Hence $D$ may be expressed explicitly as a Noetherian $R_3$–module in the following way

$$D = R_3 \cdot 1 + R_3 \cdot t_1 + R_3 \cdot t_2 + R_3 \cdot t_1 t_2.$$

Now let $a_0 = 0$, $a_1 = 1$, $a_2 = t_1$, $a_3 = t_2$, $a_4 = t_1 t_2$ and for $0 \leq i \leq 4$ define

$$N_i = \sum_{j=0}^{i} R_3 \cdot a_j$$

then for $1 \leq i \leq 4$ we have

$$\begin{aligned} N_i/N_{i-1} &\cong R_3 \cdot a_i/(N_{i-1} \cap R_3 \cdot a_i) \\ &\cong R_3 \cdot a_i \\ &\cong R_3/\mathfrak{p}_\xi. \end{aligned}$$

Thus $N_0 \subset N_1 \subset \cdots \subset N_4$ is a prime filtration of $D$. The multiplicity of $\mathfrak{p}_\xi$ in this filtration is clearly 4. Moreover, the prime ideal $\mathfrak{p}_\xi$ is principal and is given by

$$\mathfrak{p}_\xi = (u_3 + u_2 + u_1 + 1).$$

Therefore by Proposition 3.3.2

$$h(\alpha) = 4|\log \mathbb{M}(u_3 + u_2 + u_1 + 1)|.$$

In fact the Mahler measure of the polynomial $u_3 + u_2 + u_1 + 1$ has been calculated by Smyth in [33] (see also [2]):

$$\mathbb{M}(u_3 + u_2 + u_1 + 1) = \exp\left(\frac{7\zeta(3)}{2\pi^2}\right)$$

where $\zeta$ is the Reimann zeta function. Thus, for this dynamical system we have

$$h(\alpha) = \frac{14\zeta(3)}{\pi^2}.$$

**Example 3.3.7** ($d = t_D = 2$ and infinite entropy) Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ and $t$ be an indeterminate over $\mathbb{Q}$. Let $D = \overline{\mathbb{Q}}[t, \frac{1}{t^2+t}]$, $\xi_1 = t$ and $\xi_2 = -t-1$ to obtain a mixing $\mathbb{Z}^2$–action. Note that $\mathfrak{p}_\xi$ is principal and is generated by the polynomial $u_2 + u_1 + 1 \in R_2$. Let $p_1, p_2, \ldots$ be a sequence of distinct rational primes, and for each $i \geq 1$ define

$$N_i = \sum_{j=1}^{i} R_2 \cdot \sqrt{p_j}.$$

Then for $i > 1$,

$$\begin{aligned} N_i/N_{i-1} &\cong R_2 \cdot \sqrt{p_i} \\ &\cong R_2/\mathfrak{p}_\xi. \end{aligned}$$

Clearly $N_1 \cong R_2/\mathfrak{p}_\xi$ and so for all $i \geq 1$, $N_i$ is a submodule of $D$ for which $\mathfrak{p}_\xi$ arises in a prime filtration with multiplicity $i$. Proposition 3.3.2 gives $h(\alpha) = \infty$.

The final theorem in this chapter limits the range of possible entropies for many arithmetic dynamical systems. Although Proposition 3.3.2 gives a formula for the entropy of any such system, the multiplicity constant appearing there may be difficult to calculate. However, the following helps in this respect, not only by establishing its existence without having to handle complicated module filtrations, but in many cases realizing it explicitly as the degree of an appropriate field extension.

**Theorem 3.3.8** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be a mixing arithmetic dynamical system generated by units $\xi_1, \ldots, \xi_d$ in the integral domain $D$. Suppose that $d = t_D$ and*

$\mathfrak{p}_\xi = (f)$ *is principal. Let* $k$ *be the field of fractions of* $D$ *and* $F_\xi$ *the field of fractions of* $R_\xi$*. Then*

$$h(\alpha) \leq \deg(k|F_\xi)|\log \mathbb{M}(f)|. \tag{10}$$

*Moreover, if* $D$ *is an integrally closed domain then equality holds.*

To prove the theorem we need the following lemmas.

**Lemma 3.3.9** *Let* $k$ *be a field,* $\xi_1, \ldots, \xi_d \in k^\times$ *and* $k$ *be considered as an* $R_d$*–module via the map* $\theta_\xi$*. Suppose that* $N$ *is a submodule of* $k$ *with the property that for any* $a \in k$ *there exists* $0 \neq c \in R_\xi$ *such that* $ca \in N$*. If* $\mathfrak{p}_\xi$ *is non-zero then* $h(\alpha^{k/N}) = 0$*.*

*Proof.* For a non-trivial case assume that $\alpha$ is mixing and $N \neq k$. Since $k/N$ is countable we may select a chain

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

of Noetherian submodules such that $\bigcup_{j\geq 1} N_j = k/N$. Let $j \in \mathbb{N}$ be fixed and suppose that $\pi : k \mapsto k/N$ is the natural map. Let $b \in N_j$ and $a \in k$ be such that $\pi(a) = b$. By assumption there exists $0 \neq c \in R_\xi$ such that $ca \in N$. That is, there is some $f \in R_d$ such that $\theta_\xi(f)a \in N$. Hence

$$\pi(\theta_\xi(f)a) = 0_{k/N} \Rightarrow (\pi\theta_\xi(f))b = 0_{k/N}.$$

Therefore $f \in \mathrm{ann}(b)$. Furthermore $\theta_\xi(f) = c \neq 0$ and this means $\mathrm{ann}(b) \supsetneq \mathfrak{p}_\xi$. Therefore the associated primes of $N_j$ all lie strictly above $\mathfrak{p}_\xi$. Using the assumption $\mathfrak{p}_\xi \neq \{0\}$ it follows that $h(\alpha^{R_d/\mathfrak{p}_\xi}) < \infty$ and Proposition 3.1.1 part 1 yields $h(\alpha^{R_d/\mathfrak{p}}) = 0$ for each $\mathfrak{p} \in \mathrm{Spec}\, R_d$ associated with $N_j$. The Lind, Schmidt, Ward entropy formula then gives $h(\alpha^{N_j}) = 0$. Thus, by part 3(b) of Proposition 3.1.1 we have $h(\alpha^{k/N}) = 0$. $\square$

**Lemma 3.3.10** *Let* $D$ *be an integral domain and* $L$ *its field of fractions. If* $a$ *is algebraic over* $L$ *then there exists* $c \neq 0$ *in* $D$ *such that* $ca$ *is integral over* $D$*.*

*Proof.* See [15, Chapter 9, Section 1, Proposition 1]. $\square$

*Proof of Theorem 3.3.8.* Since $d = t_D$ and $\mathfrak{p}_\xi$ is principal, using the proof of Theorem 3.3.4, we see that the extension $k|F_\xi$ is algebraic. If $k|F_\xi$ is not finite or $\mathfrak{p}_\xi = \{0\}$ then (10) holds trivially. For the moment supppose that $\deg(k|F_\xi) = n$ for some $n \in \mathbb{N}$. Let $a_1, \ldots, a_n$ be a basis for $k$ over $F_\xi$. Every element of $k$ may be written in the form

$$\frac{b_1}{c_1}a_1 + \cdots + \frac{b_n}{c_n}a_n$$

where $b_1, \ldots, b_n, c_1, \ldots, c_n \in R_\xi$. Alternatively we may write this as

$$\frac{b_1'a_1 + \cdots + b_n'a_n}{c} \tag{11}$$

where $b_1', \ldots, b_n', c \in R_\xi$. Consider the $R_d$–module

$$N = R_d \cdot a_1 + \cdots + R_d \cdot a_n.$$

Since $N$ is a free module associated with $\mathfrak{p}_\xi$, the multiplicity of $\mathfrak{p}_\xi$ in any prime filtration of $N$ must be $n$. Therefore

$$\begin{aligned} h(\alpha^N) &= nh(\alpha^{R_d/\mathfrak{p}_\xi}) \\ &= n|\log \mathbb{M}(f)|. \end{aligned}$$

Now consider the $R_d$–module $k/N$. From (11) it is clear that for any $a \in k \setminus N$ there exists $0 \neq c \in R_\xi$ such that $ac \in N$. By Lemma 3.3.9 it follows that $h(\alpha^{k/N}) = 0$. Therefore by the addition formula

$$\begin{aligned} h(\alpha^k) &= h(\alpha^{k/N}) + h(\alpha^N) \\ &= n|\log \mathbb{M}(f)|. \end{aligned} \tag{12}$$

Also, the addition formula shows that $h(\alpha^D) \leq h(\alpha^k)$ and so the required inequality (10) follows.

Now suppose that $D$ is an integrally closed domain. If $k|F_\xi$ is finite, then (12) gives an expression for $h(\alpha^k)$. If $k|F_\xi$ is not finite then, since $k$ is countable, there exists a chain

$$F_\xi = k_0 \subset k_1 \subset k_2 \subset \cdots$$

of subfields of $k$ such that $\bigcup_{j=1}^{\infty} k_j = k$, $\deg(k_j|F_\xi) < \infty$ for all $j \in \mathbb{N}$ and

$$\deg(k_1|F_\xi), \deg(k_2|F_\xi), \deg(k_3|F_\xi), \ldots$$

is a strictly increasing sequence of positive integers. By repeatedly applying the above arguments to each of the fields $k_j$ and using part 3(b) of Proposition 3.1.1 it follows that $h(\alpha^k) = \infty$.

Since $D$ is an integrally closed domain it must contain the integral closure of $R_\xi$ in $k$. Denote this by $\overline{R}_\xi$. Application of the addition formula gives

$$h(\alpha^{\overline{R}_\xi}) = h(\alpha^k) - h(\alpha^{k/\overline{R}_\xi}).$$

Using Lemma 3.3.10 and Lemma 3.3.9 we find that $h(\alpha^{k/\overline{R}_\xi}) = 0$ and thus

$$h(\alpha^{\overline{R}_\xi}) = h(\alpha^k).$$

However

$$h(\alpha^{\overline{R}_\xi}) \leq h(\alpha^D) \leq h(\alpha^k)$$

and so this forces $h(\alpha^D) = h(\alpha^k)$. □

The assumptions that the associated prime $\mathfrak{p}_\xi$ is principal and that the action is mixing are not particularly restrictive since if either is not the case then Proposition 3.3.2 shows that the entropy of the dynamical system is zero. It should be remarked that not only does this theorem determine the existence of the multiplicity constant, but also gives a maximum possible value for it, namely $\deg(k|F_\xi)$. Also, if $k$ is finitely generated over its prime subfield then, with the assumption that $d = t_D$, the extension $k|F_\xi$ is necessarily finite. Hence for any arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$, where the field of fractions of $D$ is finitely generated over its prime subfield, using the results of this section we are able to give finitely many possible values for $h(\alpha)$. Moreover, when $D$ is an integrally closed domain we can now give a precise value for $h(\alpha)$. Table 2 provides a summary for this case. Note that the fields $k$ and $F_\xi$ appearing in this table are as defined in Theorem 3.3.8. The entry in this table which is missing is an entropy value for the case $\mathfrak{p}_\xi = \{0\}$ and

| | $h(\alpha)$ | | |
|---|---|---|---|
| | $\alpha$ not mixing or $\mathfrak{p}_\xi$ non-principal | $\alpha$ mixing and $\mathfrak{p}_\xi = \{0\}$ | $\alpha$ mixing and $\mathfrak{p}_\xi = (f)$ non-zero |
| $d < t_D$ | 0 | $\infty$ | $\infty$ |
| $d = t_D$ | 0 | $\infty$ | $\deg(k\|F_\xi)\|\log \mathbb{M}(f)\|$ |
| $d > t_D$ | 0 | – | 0 |

Table 2: Summary of entropy values in the integrally closed case.

$d > t_D$. This is because it is impossible for the homomorphism $\theta_\xi : R_d \mapsto D$ to have a trivial kernel when $d > t_D$, $\xi_1, \ldots, \xi_d$ necessarily being algebraically dependent.

In light of Theorem 3.3.8, consider again Example 3.3.6. Here $F_\xi = \mathbb{Q}(t_1^2, t_2^2)$ and $k = F_\xi(t_1, t_2)$. The minimal polynomial for $t_1$ over $F_\xi$ is $X^2 + \xi_1 - 1$ and the minimal polynomial for $t_2$ over $F_\xi(t_1)$ is $X^2 - \xi_2 + 3$. Hence by the tower law for field extensions, deg $(k|F_\xi) = 4$. Since $D$ is a unique factorization domain it is an integrally closed domain and so Theorem 3.3.8 may be applied to give

$$h(\alpha^{(D,\xi)}) = 4|\log \mathbb{M}(u_3 + u_2 + u_1 + 1)|.$$

In [4] the authors give an entropy formula which applies to arithmetic dynamical systems arising from a single unit in a ring of $S$–integers in an $\mathbb{A}$–field $k$. One consequence of this formula is that if $S'$ is a set of places of $k$ containing $S$, then the arithmetic dynamical system which arises from the ring of $S'$–integers and the same unit, has the same entropy as the original one (a result which has its origins in [19]). The following corollary to Theorem 3.3.8 shows that an analogue of this result exists in a much more general setting.

**Corollary 3.3.11** *Let $k$ be a field, $\xi_1, \ldots, \xi_d \in k^\times$ and $F_\xi$ the field of fractions of $R_\xi$. If $k|F_\xi$ is an algebraic extension then for any subring $D$ of $k$ containing $\overline{R}_\xi$ – the integral closure of $R_\xi$ in $k$,*

$$h(\alpha^{(D,\xi)}) = h(\alpha^{(\overline{R}_\xi,\xi)}).$$

Let $k$ be a field, $\xi_1, \ldots, \xi_d \in k^\times$, $F_\xi$ the field of fractions of $R_\xi$ and suppose that $k|F_\xi$ is a finite algebraic extension. By the Mori-Nagata integral closure theorem, the integral closure of $R_\xi$ in $F_\xi$ is a Krull ring. Hence by Proposition 1.6.2 the integral closure of $R_\xi$ in $k$ is also a Krull ring. Denote this ring by $D$. If $\mathcal{P}$ is the set of height 1 prime ideals of $D$ and $\mathcal{S} \subset \mathcal{P}$ then $D_{\mathcal{S}} = \bigcap_{\mathfrak{p} \in \mathcal{P} \setminus \mathcal{S}} D_{\mathfrak{p}}$ is an integrally closed subring of $k$ containing $D$. Hence Corollary 3.3.11 shows that

$$h(\alpha^{(D_{\mathcal{S}},\xi)}) = h(\alpha^{(D,\xi)}).$$

Furthermore if $\mathcal{S}' \subset \mathcal{P}$ contains $\mathcal{S}$ it is now easily seen that

$$h(\alpha^{(D_{\mathcal{S}'},\xi)}) = h(\alpha^{(D_{\mathcal{S}},\xi)}).$$

This observation generalizes the analogous one for $S$–integer dynamical systems, discussed previously.

**Example 3.3.12** Let $t$ be an indeterminate over $\mathbb{Q}$ and let $k = \mathbb{Q}(t)$. Set $\xi_1 = t + 2$ and $\xi_2 = -3t - 5$. Notice that $\xi^{\mathbf{n}} \neq 1$ for all non-zero $\mathbf{n} \in \mathbb{Z}^d$ and so all arithmetic dynamical systems arising from subrings of $k$ containing $R_\xi$ are mixing. Since $D = R_\xi = \mathbb{Z}[t, \frac{1}{t+2}, \frac{1}{-3t-5}]$ is a unique factorization domain, it is integrally closed in its own field of fractions, which is of course $k$. Let $\mathcal{P}$ be the set of height 1 prime ideals of $D$ (recall that these are given by the irreducibles in $D$, because $D$ is a unique factorization domain). Define

$$\mathcal{S} = \{(p) : p \text{ is a rational prime}\} \cup \{(t^3 - 1)\}.$$

Then $D_{\mathcal{S}} = \mathbb{Q}[t, \frac{1}{t+2}, \frac{1}{-3t-5}, \frac{1}{t^3-1}]$. By the above $h(\alpha^{(D_{\mathcal{S}},\xi)}) = h(\alpha^{(D,\xi)})$. But

$$\begin{aligned} D &\cong R_d/\mathfrak{p}_\xi \\ &= R_d/(u_2 + 3u_1 - 1). \end{aligned}$$

Using [33, Theorem 1] we calculate $\mathbb{M}(u_2 + 3u_1 - 1) = 3$. Thus

$$h(\alpha^{(D_{\mathcal{S}},\xi)}) = \log 3.$$

An important condition of Corollary 3.3.11 is that the extension $k|F_\xi$ is algebraic. The example below illustrates why.

**Example 3.3.13** Let $t$ be an indeterminate over $\mathbb{Q}$ and let $k = \mathbb{Q}(t)$. If $\xi = 5$ then $R_\xi = \mathbb{Z}[\frac{1}{5}]$ and so $F_\xi$, the field of fractions of $R_\xi$, is simply $\mathbb{Q}$. Also every element of $k \setminus F_\xi$ is trancendental over $F_\xi$ and hence $\overline{R}_\xi$, the integral closure of $R_\xi$ in $k$, is the same as the integral closure of $R_\xi$ in $F_\xi$ which is clearly $R_\xi$. Since $\mathfrak{p}_\xi = (u_1 - 5)$ it follows that

$$\begin{aligned} h(\alpha^{(\overline{R}_\xi,\xi)}) &= h(\alpha^{(R_\xi,\xi)}) \\ &= h(\alpha^{R_1/(u_1-5)}) \\ &= |\log \mathbb{M}(u_1 - 5)| \\ &= 5. \end{aligned}$$

However the ring $D = \mathbb{Z}[\frac{1}{5}, t] \subset k$ contains $\overline{R}_\xi$ but has $t_D = 2$. Therefore by Theorem 3.3.4, because $d = 1 < 2$ and $\mathfrak{p}_\xi$ is principal,

$$h(\alpha^{(D,\xi)}) = \infty.$$

# Chapter 4

# Expansiveness

## 4.1 Algebraic characterization

The dynamical property of expansiveness for $\mathbb{Z}$–actions by homeomorphisms of compact metric spaces has been the subject of extensive study. For example, it is well known that there are no expansive homeomorphisms of the torus $\mathbb{T}$ [35, Section 5.6]. Recently, there has been interest in expansive actions by groups other than $\mathbb{Z}$. For example, Morris [24, Theorem 2.9] has shown that a $\mathbb{Z}^d$–action by homeomorphisms of $\mathbb{T}$ cannot be expansive. Studies of countable group actions by automorphisms of compact metrizable groups have yielded some interesting algebraic consequences of expansiveness. Although the definition of expansiveness is usually given in terms of a metric, it is actually a topological property [35, Corollary 5.22.1]. Let $X$ be a compact metrizable group, $G$ a countable group and $\alpha : G \mapsto \mathrm{Aut}(X)$ a $G$–action by automorphisms of $X$. Then $\alpha$ is *expansive* if and only if there is a neighbourhood $N$ of the identity $1_X$ in $X$ such that

$$\bigcap_{g \in G} \alpha_g(N) = \{1_X\}$$

where $\alpha_g$ is the image of $g$ in $\mathrm{Aut}(X)$.

The property of expansiveness, for dynamical systems $(X, \alpha)$ as defined above, receives analysis in [13]. Here the authors highlight an important connection between expansiveness and the *descending chain condition*. This is the condition that any descending chain

$$X \supset X_1 \supset X_2 \supset \cdots$$

of closed $\alpha$–invariant subgroups of $X$ eventually becomes stationary. When the countable group $G$ is abelian and finitely generated [13, Theorem 5.2] shows that the descending chain condition is implied by expansiveness, and if $X$ is zero dimensional then the converse also holds.

For an algebraic $\mathbb{Z}^d$–action, conditions for expansiveness can be expressed in terms of the dual module and its associated primes. This result originally appeared in [29] (see also [30, Proposition 5.4 and Theorem 6.5]). Before stating it, the following definition is required. For any ideal $\mathfrak{a}$ of $R_d$ the *complex variety* of $\mathfrak{a}$ is

$$V_{\mathbb{C}}(\mathfrak{a}) = \{\mathbf{z} \in (\mathbb{C}^{\times})^d : f(\mathbf{z}) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

**Theorem 4.1.1** *Let $(X, \alpha)$ be an algebraic $\mathbb{Z}^d$–action and $M$ the corresponding dual $R_d$–module. Then $\alpha$ is expansive if and only if $M$ is Noetherian, with associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ say, and for any $(z_1, \ldots, z_d) \in \bigcup_{j=1}^{r} V_{\mathbb{C}}(\mathfrak{p}_j)$, $|z_i| \neq 1$ for some $i \in \{1, \ldots, d\}$.*

## 4.2 Arithmetic characterization

To contextualize the arithmetic characterization of expansiveness in the general case, it is useful to consider first the situation for dynamical systems of the form $(X^D, \alpha^{(D,\xi)})$ where $D$ is a ring of $S$–integers in an $\mathbb{A}$–field. Recall that in this situation, the adelic description of the dual group of $D$ is available, as given in Section 1.6. Also there is a convenient description of the action, given by Example 1.8.1.

**Proposition 4.2.1** *Let $D_S$ be a ring of $S$–integers in an $\mathbb{A}$–field $k$. Let $T$ denote the union of the finite places $S$ and the infinite places of $k$. If $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ is the dynamical system arising from units $\xi_1, \ldots, \xi_d$ in $D_S$ then $\alpha$ is expansive if and only if for any $\lambda \in T$, $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$.*

*Proof.* Recall that $X \cong k^S/\phi(D_S)$ and that there is a local isometry between $X$ and $k^S$. Hence it is enough to check expansiveness of the action $\tilde{\beta} : \mathbf{n} \mapsto \tilde{\beta}_{\mathbf{n}}$, as defined in Example 1.8.1. First suppose that for any $\lambda \in T$ there is some $i \in \{1, \ldots, d\}$ such that $|\xi_i|_\lambda \neq 1$. Note that this means $T$ is necessarily finite and so $k^S$ is a finite product of fields. Hence if $U_\lambda = \{x_\lambda \in k_\lambda : |x_\lambda|_\lambda < 1\}$ then $U = \prod_{\lambda \in T} U_\lambda$ is a neighbourhood of zero in $k^S$. Let $x = (x_\lambda)$ be any non-zero element of $U$, then for some $\lambda \in T$ we have $x_\lambda \neq 0$. But for such a $\lambda$ there is some $i \in \{1, \ldots, d\}$ such that $|\xi_i|_\lambda \neq 1$. Therefore there exists $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that

$$\begin{aligned} |\xi^{\mathbf{n}} x_\lambda|_\lambda &= |\xi_1|_\lambda^{n_1} \cdots |\xi_d|_\lambda^{n_d} |x_\lambda|_\lambda \\ &> 1. \end{aligned}$$

That is $\xi^{\mathbf{n}} x_\lambda \notin U_\lambda$. Hence $\tilde{\beta}_{\mathbf{n}}(x) \notin U$. This shows that $U$ is an expansive neighbourhood of zero. Now suppose that there is some $\lambda \in T$ with $|\xi_i|_\lambda = 1$ for all $i = 1, \ldots, d$ and let $U$ be any neighbourhood of zero in $k^S$. Then there exists a non-zero $x = (x_\lambda) \in U$ with $x_\lambda = 0$ for those $\lambda$ with $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$ and $x_\lambda \neq 0$ otherwise. Hence for all $\lambda \in T$ and $\mathbf{n} \in \mathbb{Z}^d$, $|\xi^{\mathbf{n}} x_\lambda|_\lambda = |x_\lambda|_\lambda$. This means $\tilde{\beta}_{\mathbf{n}}(x) \in U$ for all $\mathbf{n} \in \mathbb{Z}^d$. Thus $\tilde{\beta}$ is non-expansive. $\square$

When comparing Proposition 4.2.1 and Theorem 4.1.1, there are obvious similarities. In fact it can be shown that in the situation above, the points of the complex variety $V_{\mathbb{C}}(\mathfrak{p}_\xi)$ correspond to the infinite places of the $\mathbb{A}$–field in question, restricted to its minimal subfield containing $\xi_1, \ldots, \xi_d$. The Noetherian condition of Theorem 4.1.1 is taken care of by the finite places appearing in Proposition 4.2.1 and the details of this will become apparent when the general case is considered. However there is a problem that occurs in the general case which is not highlighted by Proposition 4.2.1. Because the set of all discrete valuations of an $\mathbb{A}$–field $k$ satisfy the finite character property, given $\xi_1, \ldots, \xi_d \in k$, there can only be finitely many $\lambda$ for which $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$. Hence, if for any $\lambda \in S$ there exists $i \in \{1, \ldots, d\}$ such that $|\xi_i|_\lambda \neq 1$, this means that $S$ must be a finite set. When this is the case it can be shown that $D_S$ is a finitely generated ring (that is, finitely generated over $\mathbb{F}_p$ or $\mathbb{Z}$, depending on the characteristic). The importance of this is manifest in the following.

**Lemma 4.2.2** *Let $A$ be a Noetherian ring and $B$ an $A$–algebra. Then $B$ is a Noetherian $A$–module if and only if every element of $B$ is integral over $A$ and $B$ is finitely generated as an algebra over $A$.*

*Proof.* See [15, Chapter 9, Section 1]. □

Hence in the general case, if we wish to find a set of absolute values which play the part of the set $S$ above, to get a true analogue of Proposition 4.2.1 this set must detect when the underlying ring $D$ is finitely generated. Unfortunately, it does not seem that there is such a set for an arbitrary integral domain $D$, even if non-discrete valuations are also considered. Therefore, attention is now restricted to finitely generated integral domains. Theorem 4.1.1 and the lemma above show that this restriction is not unreasonable, since no choice of $\xi$ will generate an expansive action if $D$ is not finitely generated.

Suppose that $D$ is a finitely generated integral domain with field of fractions $k$. Let $\{D_\lambda : \lambda \in \mathcal{S}_D\}$ be the family of discrete valuation rings of $k$ which do not contain $D$, and for each $\lambda \in \mathcal{S}_D$ let $|\cdot|_\lambda$ be an absolute value representing the equivalence class corresponding $D_\lambda$ (as given by (4)). Thus, when $D$ is a ring of $S$–integers, $\mathcal{S}_D$ corresponds to $S$.

**Proposition 4.2.3** *Let $D$ be a finitely generated integral domain and $\xi_1, \ldots, \xi_d$ units of $D$. Then $D$ is a Noetherian $R_d$–module under the map $\theta_\xi : R_d \mapsto D$ if and only if for any $\lambda \in \mathcal{S}_D$, $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$.*

*Proof.* By Lemma 4.2.2, $D$ is a Noetherian $R_d$–module under the map $\theta_\xi$ if and only if the extension $R_\xi \subset D$ is integral, that is if and only if $R_\xi$ and $D$ have the same integral closure in $k$, the field of fractions of $D$. By [23, Theorem 10.4] the integral closure of a subring of $k$ is the intersection of all valuation rings of $k$ which contain that ring. By applying the Mori-Nagata integral closure theorem, it follows that the integral closure of $D$ in $k$ is simply the intersection of all discrete valuation rings of $k$ which contain $D$. Denote this ring by $A$. Since $R_\xi$ is a Noetherian ring the same theorem may be applied to show that the integral closure of $R_\xi$ in $F_\xi$ (the field of fractions of $R_\xi$) is a Krull ring. Denote this ring by $B$. If $L$ is the algebraic closure of $F_\xi$ in $k$ then since $k$ is finitely generated over $F_\xi$, it follows that $L|F_\xi$ is a finite extension. Hence by Proposition 1.6.2 the integral closure of $B$ in $L$ is a Krull ring, $C$ say. Note that $C$ is also the integral closure of $R_\xi$ in $k$ and so $C$ is the intersection of all valuation rings of $k$ which contain $R_\xi$. However, by Proposition 1.6.4, $C$ may be expressed as the intersection of discrete valuation rings of $k$ which contain $C$ (which of course also contain $R_\xi$). Therefore it follows that $C$ is the intersection of all discrete valuation rings of $k$ which contain $R_\xi$. Let $\Lambda_A$ be the family of all discrete valuation rings of $k$ which contain $A$ and $\Lambda_C$ the family of all discrete valuation rings of $k$ which contain $C$. Since $\Lambda_C$ is necessarily contained in $\Lambda_A$, it follows that $A = C$ if and only if $\Lambda_A$ is contained in $\Lambda_C$. Furthermore, because $\Lambda_A$ is also equal to the family of discrete valuation rings of $k$ which contain $D$, it follows that $A = C$ if and only if $R_\xi$ is not contained in $D_\lambda$ for all $\lambda \in \mathcal{S}_D$. If $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$ then it is clear that $R_\xi$ is not contained in $D_\lambda$. Conversely, if $R_\xi$ is not contained in $D_\lambda$ then there is some $a \in R_\xi$ such that $|a|_\lambda > 1$. However, $a$ may be written in the form $a = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}}\xi^{\mathbf{n}}$, where $a_{\mathbf{n}} \in \mathbb{Z}$ or $a_{\mathbf{n}} \in \mathbb{F}_p$ depending on the characteristic of $k$, $a_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\xi^{\mathbf{n}} = \xi_1^{n_1} \cdots \xi_d^{n_d}$. Therefore

$$|a|_\lambda \leq \max_{\mathbf{n}\in\mathbb{Z}^d} \{|a_{\mathbf{n}}|_\lambda |\xi^{\mathbf{n}}|_\lambda\}.$$

If the characteristic is positive then $|a_{\mathbf{n}}|_\lambda = 1$ and if the characteristic is zero then $|a_{\mathbf{n}}|_\lambda \leq 1$. In either case, for some $\mathbf{n} \in \mathbb{Z}^d$ we must have $|\xi^{\mathbf{n}}|_\lambda \geq |a|_\lambda > 1$. This can only happen if $|\xi_i| \neq 1$ for some $i \in \{1, \ldots, d\}$. $\square$

Let $k$ be a field and $\xi_1, \ldots, \xi_d \in k^\times$. Suppose that $\theta_\xi : R_d \mapsto k$ is the substitution map and $\mathfrak{p}_\xi$ is the associated prime. For any $\mathbf{z} \in V_{\mathbb{C}}(\mathfrak{p}_\xi)$ there is a corresponding real valued function $|\cdot| : R_\xi \mapsto \mathbb{R}_+$, defined as follows. For each $a \in R_\xi$ choose some $f_a \in \theta_\xi^{-1}(a)$. Set

$$|a| = |f_a(\mathbf{z})|.$$

This is well defined because if we choose $g_a \in \theta_\xi^{-1}(a)$ with $f_a \neq g_a$, then since $f_a = g_a + h$ for some $h \in \mathfrak{p}_\xi$,

$$\begin{aligned} g_a(\mathbf{z}) &= f_a(\mathbf{z}) + h(\mathbf{z}) \\ &= f_a(\mathbf{z}). \end{aligned}$$

Note that in general $|\cdot|$ is not an absolute value because potentially $|a| = 0$ for some non-zero $a \in R_\xi$. However, $|\cdot|$ still has the status of a partial norm. If the family of all such partial norms is $\{|\cdot|_\lambda : \lambda \in \mathcal{S}_V\}$ then for an arithmetic dynamical system, Theorem 4.1.1 can be restated in the following way.

**Theorem 4.2.4** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be an arithmetic dynamical system arising from units $\xi_1, \ldots, \xi_d$ in the finitely generated integral domain $D$. Then $\alpha$ is expansive if and only if for any $\lambda \in \mathcal{S}_V \cup \mathcal{S}_D$, $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$.*

*Proof.* Use Theorem 4.1.1 and Proposition 4.2.3. □

In the general case, as well as emulating the role of the infinite and finite places for $S$–integer dynamical systems, the sets $\mathcal{S}_V$ and $\mathcal{S}_D$ have a rich structure. Let $A = \mathbb{C}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ and $X = \text{Spec } A$. By definition, the points of $\mathcal{S}_V$ parameterize the variety $V_{\mathbb{C}}(\mathfrak{p}_\xi)$ and this in turn corresponds to the set of closed points of $X$ contained in $V(\mathfrak{a}_\xi)$, where $\mathfrak{a}_\xi$ is the ideal generated by $\mathfrak{p}_\xi$ in $A$. Hence $\mathcal{S}_V$ may be thought of as the set of closed points of the affine scheme, Spec $A/\mathfrak{a}_\xi$. For any $\mathbf{z} = (z_1, \ldots, z_d) \in (\mathbb{C}^\times)^d$ let $\mathfrak{m}_\mathbf{z} \in X$ be the maximal ideal of $A$ generated by $u_1 - z_1, \ldots, u_d - z_d$. Then the condition that

$$\lambda \in \mathcal{S}_V \Rightarrow |\xi_i|_\lambda \neq 1 \text{ for some } i \in \{1, \ldots, d\}$$

is equivalent to

$$\mathfrak{m}_\mathbf{z} \notin V(\mathfrak{a}_\xi) \text{ for all } \mathbf{z} \in \mathbb{S}^d.$$

Let $Y$ be equal to the set $\mathcal{S}_D$ together with an additional element $\eta$, called the *generic point* of $Y$. Define a subset of $Y$ to be open if it contains $\eta$. It is readily checked that this defines a topology on $Y$. Let $k$ denote the field of fractions of $D$ and set $D_\eta = k$. For each open set $U \subset Y$ let

$$\mathcal{O}_Y(U) = \bigcap_{\lambda \in U} D_\lambda.$$

For open sets $V \subset U$ there is an inclusion $\mathcal{O}_Y(U) \subset \mathcal{O}_Y(V)$ and hence a natural injective homomorphism $\rho_{UV} : \mathcal{O}_Y(U) \mapsto \mathcal{O}_Y(V)$. It is readily checked that this defines a sheaf of rings $\mathcal{O}_Y$ on $Y$, and that $(Y, \mathcal{O}_Y)$ is a locally ringed space. Moreover $(Y, \mathcal{O}_Y)$ is a scheme because each $\lambda \in Y$ has a neighbourhood isomorphic to Spec $D_\lambda$. The set of closed points of this scheme clearly corresponds to $\mathcal{S}_D$ and the condition that

$$\lambda \in \mathcal{S}_D \Rightarrow |\xi_i|_\lambda \neq 1 \text{ for some } i \in \{1, \ldots, d\}$$

is equivalent to

$$R_\xi \not\subset \mathcal{O}_{Y,\lambda} \text{ for all } \lambda \neq \eta.$$

We now apply Theorem 4.2.4 and the above discussion to some arithmetic dynamical systems.

**Example 4.2.5** Let $k$ be an $\mathbb{A}$–field and $\xi_1, \ldots, \xi_d \in k^\times$. If for every infinite place $\lambda$ of $k$, $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$, then there is precisely one ring of $S$–integers $D_S \subset k$, for which the arithmetic dynamical system $(X, \alpha) = (X^{D_S}, \alpha^{(D_S, \xi)})$ is expansive. To see this, note that by assumption $\xi_1, \ldots, \xi_d$ must be units of $D_S$. Hence $S$ must contain the set of finite places $\lambda$ of $k$ for which $|\xi_i|_\lambda \neq 1$ for some $i \in \{1, \ldots, d\}$. Furthermore, this inclusion must be strict otherwise Proposition 4.2.1 would be violated.

**Example 4.2.6** Let $D = \mathbb{Z}[t^{\pm 1}, \frac{1}{at+b}]$, where $t$ is an indeterminate and $a, b$ are non-zero integers. Set $\xi_1 = t$ and $\xi_2 = at + b$. Here $D$ is simply the intersection of all discrete valuation rings in $\mathbb{Q}(t)$ for which $\xi_1$ and $\xi_2$ are both units. Hence $\mathcal{S}_D$ is exactly the set of $\lambda$ for which either $|\xi_1|_\lambda \neq 1$ or $|\xi_2|_\lambda \neq 1$. Suppose that $\lambda \in \mathcal{S}_V$ corresponds to the point in $(\mathbb{C}^\times)^2$ with first co-ordinate $x+y\sqrt{-1}$, where $x, y$ are real. If $|\xi_1|_\lambda = 1$ then $|t|_\lambda = 1$ which implies

$$x^2 + y^2 = 1. \tag{13}$$

If $|\xi_2|_\lambda = 1$ then $|at+b|_\lambda = 1$ and therefore $(ax+b)^2 + a^2y^2 = 1$. Hence if (13) also holds then

$$y^2 = 2a^2 + 2b^2 + 2a^2b^2 - a^4 - b^4 - 1$$

for some real $y$. This means that the arithmetic dynamical system $(X^D, \alpha^{(D,\xi)})$ is non-expansive for those $a, b$ which satisfy

$$2a^2 + 2b^2 + 2a^2b^2 - a^4 - b^4 - 1 \geq 0. \tag{14}$$

Since the quartic equation

$$X^4 - 2(b^2+1)X^2 + b^4 - 2b^2 + 1 = 0$$

has roots $X = \pm(b+1), X = \pm(b-1)$, it follows that for a given $b \in \mathbb{Z}$ the choices of $a$ for which (14) holds are

$$a = b-1, b, b+1, -b-1, -b, -b+1.$$

In particular, once $b$ is chosen, there are only finitely many choices of $a$ which, with the defining data $D = \mathbb{Z}[t^{\pm 1}, \frac{1}{at+b}]$, $\xi_1 = t$ and $\xi_2 = at+b$, will generate a non-expansive arithmetic dynamical system.

**Example 4.2.7** Let $D = \mathbb{Z}[t, \frac{1}{15}, \frac{1}{t^2+t+1}]$, where $t$ is an indeterminate. Set $\xi_1 = 5$ and $\xi_2 = \frac{t^2+t+1}{3}$. We claim that $(X^D, \alpha^{(D,\xi)})$ is non-expansive. First note that $|\xi_1|_\lambda = 5$ for all $\lambda \in \mathcal{S}_V$ and so we need to look at $\mathcal{S}_D$ to detect non-expansiveness. By Example 1.6.6 there is a non-archimedean absolute value $|\cdot|_\lambda$ on $\mathbb{Q}(t)$, the field of fractions of $D$, which extends the 3–adic absolute value on $\mathbb{Q}$ and has $|\xi_2|_\lambda = 1$. Since $|\cdot|_\lambda$ extends the 3–adic absolute value, we also have $|\xi_1|_\lambda = 1$ and $D \not\subset D_\lambda$, that is $\lambda \in \mathcal{S}_D$. Hence by Theorem 4.2.4 this arithmetic dynamical system cannot be expansive.

**Example 4.2.8** Let $D = \mathbb{Z}[t^{\pm 1}, \frac{1}{2t-1}, \frac{1}{t-3}]$. If $\xi_1 = t^2$ and $\xi_2 = 2t^2 - 7t + 3$ then the arithmetic dynamical system $(X^D, \alpha^{(D,\xi)})$ is expansive. First consider $\mathcal{S}_D$. Let $Y$ be the scheme obtained by adjoining a generic point to $\mathcal{S}_D$ and

$$\begin{aligned} Y_1 &= \{\lambda \in Y : \mathbb{Z}[t] \subset \mathcal{O}_{Y,\lambda}\} \\ Y_2 &= \{\lambda \in Y : \mathbb{Z}[\tfrac{1}{t}] \subset \mathcal{O}_{Y,\lambda}\}. \end{aligned}$$

Note that $Y_1$ and $Y_2$ are open subschemes of $Y$ and $Y = Y_1 \cup Y_2$. For each $\lambda \in Y_1$, let $\mathfrak{m}_\lambda$ denote the maximal ideal of $\mathcal{O}_{Y_1,\lambda}$. Similarly for $\lambda \in Y_2$. If $X_1 = \text{Spec } \mathbb{Z}[t]$ and $X_2 = \text{Spec } \mathbb{Z}[\frac{1}{t}]$, then there are morphisms $\psi_1 : Y_1 \mapsto X_1$ and $\psi_2 : Y_2 \mapsto X_2$ given by $\lambda \mapsto \mathfrak{m}_\lambda \cap \mathbb{Z}[t]$ and $\lambda \mapsto \mathfrak{m}_\lambda \cap \mathbb{Z}[\frac{1}{t}]$, respectively. For any open set $U$ in $X_1$ notice that $\mathcal{O}_{X_1}(U)$ is a subring of $\mathcal{O}_{Y_1}(\psi_1^{-1}U)$ and hence there is a natural injective homomorphism from $\mathcal{O}_{X_1}(U)$ into $\mathcal{O}_{Y_1}(\psi_1^{-1}U)$. Similarly for $X_2$. The values of $|\xi_1|_\lambda, \ldots, |\xi_d|_\lambda$, $\lambda \in \mathcal{S}_D$ may be established by considering the images of $Y_1$ and $Y_2$ in $X_1$ and $X_2$, respectively. By looking at the generators of $D$ we establish that $\psi_1(Y_1)$ is contained in

$$P_1 = \{(t), (2t-1), (t-3), (2, t-3), (3, 2t-1), (p, t-3), (p, 2t-1)\}$$

and $\psi_2(Y_2)$ is contained in

$$P_2 = \{(\tfrac{1}{t}), (2 - \tfrac{1}{t}), (1 - \tfrac{3}{t}), (2, 1 - \tfrac{3}{t}), (3, 2 - \tfrac{1}{t}), (p, 1 - \tfrac{3}{t}), (p, 2 - \tfrac{1}{t})\}.$$

In both cases $p$ is any rational prime, not equal to 2 or 3. By considering each of these ideals in turn, it can be seen that for any $\lambda \in \mathcal{S}_D$, either $|\xi_1|_\lambda \neq 1$ or $|\xi_2|_\lambda \neq 1$. For example, consider $\mathfrak{p} = (2, 1 - \frac{3}{t}) \in P_2$ and notice that $\xi_2$ can be written in the form $a/b$, where $a \in \mathfrak{p}$ and $b \in \mathbb{Z}[\frac{1}{t}] \setminus \mathfrak{p}$. This means that if $D_\lambda$ is any discrete valuation ring for which $\mathfrak{m}_\lambda \cap \mathbb{Z}[\frac{1}{t}] = \mathfrak{p}$, then

$$|\xi_2|_\lambda = |a/b|_\lambda = |a|_\lambda |b|_\lambda^{-1} = |a|_\lambda > 1.$$

A similar approach can be used with all other elements of $P_1$ and $P_2$.

Now consider $\mathcal{S}_V$. Suppose that $\lambda \in \mathcal{S}_V$ corresponds to the point in $(\mathbb{C}^\times)^2$ with first co-ordinate $x + y\sqrt{-1}$, where $x, y$ are real, and that both $|\xi_1|_\lambda = 1$ and $|\xi_2|_\lambda = 1$.

This implies that

$$x^2 + y^2 = 1 \tag{15}$$

and

$$(2x^2 - 2y^2 - 7x + 3)^2 + y^2(4x - 7)^2 = 1. \tag{16}$$

Combining (15) and (16) yields

$$24x^2 - 70x + 49 = 0$$

which has two real solutions strictly greater than 1. Thus $x^2 > 1$ and so (15) has no real solution for $y$, which is a contradiction. Therefore there can be no $\lambda \in \mathcal{S}_V$ with both $|\xi_1|_\lambda = 1$ and $|\xi_2|_\lambda = 1$.

## 4.3 Expansive subdynamics

Suppose that $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ is an expansive algebraic $\mathbb{Z}^d$–action on the compact abelian group $X$. We aim to investigate the 'expansive subdynamics' of $\alpha$. To this end, the approach introduced by Boyle and Lind [3] will be followed. This involves considering subsets of $\mathbb{R}^d$ and their intersections with $\mathbb{Z}^d$. For ease of notation the intersection of a given subset $F$ of $\mathbb{R}^d$ with $\mathbb{Z}^d$ will be written $F_{\mathbb{Z}}$. For any real number $r > 0$ define

$$F^r = \{\mathbf{v} \in \mathbb{R}^d : \inf_{\mathbf{w} \in F} \|\mathbf{v} - \mathbf{w}\| \leq r\}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. Geometrically, $F^r$ can be thought of as the thickening of $F$ by $r$ (see Figure 1). A subset $F$ of $\mathbb{R}^d$ is called *expansive for* $\alpha$ if there is some $r > 0$ and a neighbourhood $N$ of $0_X$ such that

$$\bigcap_{\mathbf{n} \in F^r_{\mathbb{Z}}} \alpha_{\mathbf{n}}(N) = \{0_X\}.$$

If $F$ fails to meet this condition then $F$ is called *non-expansive for* $\alpha$. Particular attention will be paid to linear subspaces of $\mathbb{R}^d$. Let $\mathsf{G}_l = \mathsf{G}_{d,l}$ be the Grassmann

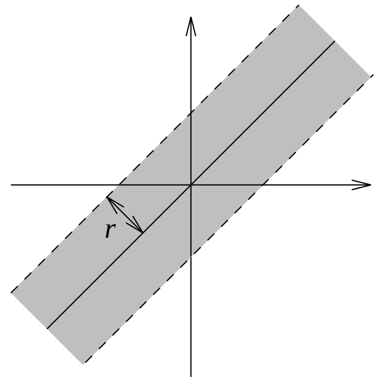


Figure 1: A thickened line in $\mathbb{R}^2$.

manifold of $l$–dimensional subspaces of $\mathbb{R}^d$. The elements of $\mathsf{G}_l$ will be called $l$–planes. The topology of $\mathsf{G}_l$ is given by stating that two $l$–planes are close if their intersections with the unit sphere are close in the Hausdorff metric. Following Boyle and Lind, let

$$\begin{aligned}\mathsf{E}_l(\alpha) &= \{F \in \mathsf{G}_l : F \text{ is expansive for } \alpha\} \\ \mathsf{N}_l(\alpha) &= \{F \in \mathsf{G}_l : F \text{ is non-expansive for } \alpha\}.\end{aligned}$$

**Example 4.3.1** Consider the arithmetic dynamical system generated by the data $D = \mathbb{Z}[\frac{1}{6}]$, $\xi_1 = 2$ and $\xi_2 = 3$. Proposition 4.2.1 shows that $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ is expansive. Recall that there is a local isomorphism between $X$ and the product of fields

$$Y = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$$

and that $\alpha$ can be studied via the lifted action given by multiplication by $2^{n_1}3^{n_2}$ componentwise on $Y$. Since $\alpha$ is an expansive $\mathbb{Z}^2$–action, $\mathsf{N}_l(\alpha)$ is non-trivial only for $l = 1$. Let $F \subset \mathbb{R}^2$ be a thickened line passing through the origin. By using the adelic description of $X = \widehat{D}$, it becomes apparent that $\alpha$ is non-expansive along $F$ if and only if $\{|2^{n_1}3^{n_2}|_\lambda : (n_1, n_2) \in F_{\mathbb{Z}}\}$ is bounded, where $|\cdot|_\lambda$ is either the 2–adic, 3–adic or archimedean absolute value on $\mathbb{Q}$. For the 2–adic place this can only happen if $F$

is such that $n_1$ is bounded (that is, $F$ is a thickening of the $x$–axis). For the 3–adic place this can only happen if $F$ is such that $n_2$ is bounded (that is, $F$ is a thickening of the $y$–axis). For the infinite place, this can only happen if there is a real number $r$ such that for all $(n_1, n_2) \in F$, $|2^{n_1}3^{n_2}| < r$, that is

$$n_1 \log 2 + n_2 \log 3 < \log r.$$

Here $|\cdot|$ denotes the usual archimedean absolute value on $\mathbb{R}$. Hence the line $y = (-\log 2/\log 3)x$ is also non-expansive. Thus $\mathsf{N}_1(\alpha)$ comprises the three lines $x = 0$, $y = 0$ and $y = (-\log 2/\log 3)x$.

A fundamental result of [3] is that for an expansive $\mathbb{Z}^d$–action $\alpha$, $\mathsf{N}_l(\alpha)$ consists precisely of those $l$–planes which are subspaces of some $(d-1)$–plane in $\mathsf{N}_{d-1}(\alpha)$. Hence the sets $\mathsf{N}_{d-1}(\alpha)$ and $\mathsf{E}_{d-1}(\alpha)$ completely describe the expansive subdynamics of $\alpha$. The remainder of this section is therefore concerned with methods for determining the expansiveness (or otherwise) of $(d-1)$–planes.

For any $(d-1)$–plane $F$ there are two corresponding half-spaces $H_1, H_2$ for which $F$ is the boundary. More specifically, $H_1 \cup H_2 = \mathbb{R}^d$ and $H_1 \cap H_2 = F$. The following lemma reveals how these half-spaces are useful for determining whether or not $F$ is expansive for $\alpha$.

**Lemma 4.3.2** *Let $\alpha$ be an algebraic $\mathbb{Z}^d$–action and $F \in \mathsf{G}_{d-1}$. Then $F \in \mathsf{N}_{d-1}$ if and only if there is a half-space $H$ which is non-expansive for $\alpha$ and has boundary $F$.*

*Proof.* See [6, Lemma 2.9]. □

Thus expansiveness along a $(d-1)$–plane can be determined by considering the half-spaces for which it is the boundary. Let $\mathsf{S}_{d-1}$ denote the unit $(d-1)$–sphere $\{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| = 1\}$ and $H_{\mathbf{v}}$ the half-space with outward normal $\mathbf{v} \in \mathsf{S}_{d-1}$, that is $H_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{R}^d : \mathbf{w} \cdot \mathbf{v} \leq 0\}$. The set of all half-spaces in $\mathbb{R}^d$ is parameterized by $\mathsf{S}_{d-1}$ via the correspondence $\mathbf{v} \leftrightarrow H_{\mathbf{v}}$. For an algebraic $\mathbb{Z}^d$–action this perspective provides a more immediate algebraic characterization of expansive subdynamics than that given by using $(d-1)$–planes.

**Theorem 4.3.3** *Suppose that $(X, \alpha)$ is an expansive algebraic $\mathbb{Z}^d$–action and $M$ is the corresponding dual module. Let $\mathbf{v} \in \mathsf{S}_{d-1}$, then $H = H_{\mathbf{v}}$ is expansive for $\alpha$ if and only if for every associated prime $\mathfrak{p}$ of $M$*

1. *$R_d/\mathfrak{p}$ is a Noetherian module over the ring $A_H = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in H_{\mathbb{Z}}]$.*

2. *$[0, \infty)\mathbf{v} \cap \mathcal{V}(\mathfrak{p}) = \varnothing$*

*where $\mathcal{V}(\mathfrak{p}) = \{(\log |z_1|, \ldots, \log |z_d|) : (z_1, \ldots, z_d) \in V_{\mathbb{C}}(\mathfrak{p})\}$ and $[0, \infty)\mathbf{v}$ denotes the ray $\{r\mathbf{v} : r \geq 0\} \subset \mathbb{R}^d$ through $\mathbf{v}$.*

*Proof.* See [6, Theorem 4.9] $\square$

Before the implications of the above for arithmetic dynamical systems are considered, recall the definition of the locally ringed space introduced in Example 1.7.1, the Zariski space $\mathrm{Zar}(k, A)$ of all valuation rings of a field $k$ having centre in a subring $A$. The use of Zariski spaces allows similarities between conditions 1 and 2 of Theorem 4.3.3 to become apparent. The Zariski space $\mathrm{Zar}(k)$ of all valuation rings of $k$ is $\mathrm{Zar}(k, \mathbb{Z})$ or $\mathrm{Zar}(k, \mathbb{F}_p)$, depending on the characteristic of $k$. For a subring $A$ of a field $k$ denote the complement of $\mathrm{Zar}(k, A)$ in $\mathrm{Zar}(k)$ by $\mathcal{V}(k, A)$.

Suppose that $k$ is a field, $\xi_1, \ldots, \xi_d \in k^{\times}$ and $\theta_{\xi} : R_d \mapsto k$ is the substitution map. We claim that $R_d/\mathfrak{p}_{\xi}$ is a Noetherian module over the ring $A_H = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in H_{\mathbb{Z}}]$ if and only if

$$\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_{\xi}) = \varnothing$$

where $R_H = \theta_{\xi}(A_H)$. If $R_d/\mathfrak{p}_{\xi} \cong R_{\xi}$ is a Noetherian $A_H$–module, then clearly it is a finitely generated $A_H$–module. By [23, Theorem 9.1], this means that every element of $R_{\xi}$ is integral over $R_H$. Conversely, if $R_H \subset R_{\xi}$ is an integral extension, because $R_{\xi}$ is a finitely generated ring (and hence a finitely generated $A_H$–algebra) [15, Chapter 9, Section 1, Proposition 2] shows that $R_{\xi}$ is finitely generated as an $A_H$–module. Lemma 4.3 of [6] implies that $R_{\xi}$ must then be a Noetherian $A_H$–module. Thus, $R_d/\mathfrak{p}_{\xi}$ is a Noetherian $A_H$–module if and only if the extension $R_H \subset R_{\xi}$ is integral. The latter condition is equivalent to saying that $R_H$ and $R_{\xi}$ have the same integral closure in $k$. Since the integral closure of a subring of a field is the intersection of all valuation rings of that field which contain it (see for example [14, Corollary 7.3.3.2] or [23, Theorem 10.4]), it follows that $R_H \subset R_\xi$ is integral if and only if $\mathrm{Zar}(k, R_H) = \mathrm{Zar}(k, R_\xi)$. That is, $\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_\xi) = \varnothing$. Thus Theorem 4.3.3 can be restated as follows.

**Theorem 4.3.4** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be an expansive arithmetic dynamical system. For any $\mathbf{v} \in \mathsf{S}_{d-1}$, the corresponding half-space $H = H_{\mathbf{v}}$ is expansive for $\alpha$ if and only if*

1. $\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_\xi) = \varnothing$.

2. $[0, \infty)\mathbf{v} \cap \mathcal{V}(\mathfrak{p}_\xi) = \varnothing$

*where $R_H$ is the subring of $D$ generated by $\{\xi^{\mathbf{n}} : \mathbf{n} \in H_{\mathbb{Z}}\}$ and $k$ is any field containing $\xi_1, \ldots, \xi_d$.*

With the aid of Theorem 4.3.4 the expansive subdynamics of a range of illustrative examples is now investigated. In all these examples, the changing roles of $\mathcal{V}(k, R_\xi)$ and $\mathcal{V}(\mathfrak{p}_\xi)$ become clear. To begin with, the opening example of this section is again considered.

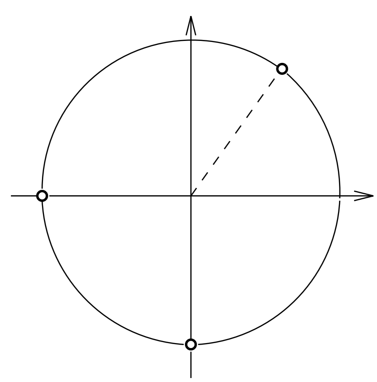

Figure 2: Non-expansive set for Example 4.3.5.

**Example 4.3.5** Let $D = \mathbb{Z}[\frac{1}{6}]$, $\xi_1 = 2$ and $\xi_2 = 3$. It has already been shown that $(X, \alpha) = (X_D, \alpha^{(D,\xi)})$ is an expansive arithmetic dynamical system. We wish to find the points of $\mathsf{S}_1$ which describe the non-expansive directions for $(X, \alpha)$. Clearly $\mathfrak{p}_\xi = (u_1 - 2, u_2 - 3)$ and so $V_{\mathbb{C}}(\mathfrak{p}_\xi) = \{(2, 3)\}$. Therefore, $\mathcal{V}(\mathfrak{p}_\xi) = \{(\log 2, \log 3)\}$ and the intersection of the ray passing from the origin through this point is shown in Figure 2. The Zariski space of all valuation rings in $\mathbb{Q}$ is parameterized by Spec $\mathbb{Q}$ and it is easy to see that

$$\mathcal{V}(\mathbb{Q}, \mathbb{Z}[\tfrac{1}{6}]) = \{\mathbb{Z}_{(2)}, \mathbb{Z}_{(3)}\}.$$

Furthermore, for a half-space $H = H_{\mathbf{v}}$, the subspace $\mathrm{Zar}(\mathbb{Q}, R_H)$ of $\mathrm{Zar}(\mathbb{Q})$ is precisely the complement of the above unless, $\mathbf{v} = (-1, 0)$ in which case $\mathrm{Zar}(\mathbb{Q}, R_H)$ contains $\mathbb{Z}_{(2)}$, or $\mathbf{v} = (0, -1)$ in which case $\mathrm{Zar}(\mathbb{Q}, R_H)$ contains $\mathbb{Z}_{(3)}$. In either case, condition (1) of Theorem 4.3.4 is violated. Figure 2 gives a complete picture of the expansive subdynamics of this example, as parameterized by $\mathsf{S}_1$.

**Example 4.3.6** Let $D$ be the ring of integers in the $\mathbb{A}$–field $k = \mathbb{Q}(\sqrt{3})$. Theorem 3.1.3 of [5] shows that $D = \mathbb{Z}[\sqrt{3}]$. The localization of $D$ at $(1 + \sqrt{3}) \in$ Spec $D$ is a discrete valuation ring $D_\lambda$ giving rise to a finite place $\lambda$ of $k$. Let $S = \{\lambda\}$. If $D_S$ is the ring of $S$–integers in $k$, $\xi_1 = 2 + \sqrt{3}$ and $\xi_2 = 2$ then Proposition 4.2.1 may be used to show that $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ is an expansive arithmetic dynamical system. We wish to examine the expansive subdynamics of $(X, \alpha)$. First note that $\mathcal{V}(k, R_\xi) = \{D_\lambda\}$ and for any $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$

$$\begin{aligned} v_\lambda(\xi^{\mathbf{n}}) &= v_\lambda(2 + \sqrt{3})n_1 + v_\lambda(2)n_2 \\ &= 2n_2 \end{aligned}$$

where $v_\lambda$ is the normalized valuation corresponding to $\lambda$. Hence, for a half-space $H$ the ring $R_H$ contains an element with $\lambda$–adic absolute value less than 1, except when $H$ has outward normal $(0, -1)$. That is $D_\lambda \notin \mathrm{Zar}(k, R_H)$ provided $H$ is not the half-space with outward normal $(0, -1)$. There are two other non-expansive directions which are derived from

$$\mathcal{V}(\mathfrak{p}_\xi) = \{(\log(2 + \sqrt{3}), \log 2), (\log(2 - \sqrt{3}), \log 2)\}.$$

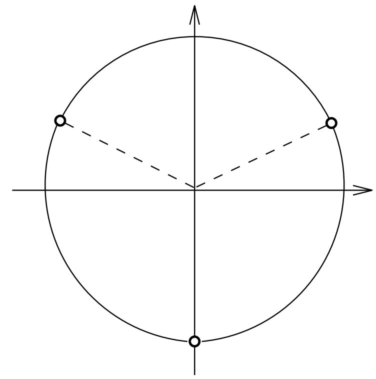

Figure 3: Non-expansive set for Example 4.3.6.

Figure 3 shows where the rays in $\mathbb{R}^2$ passing through these points intersect $\mathsf{S}_1$.

**Example 4.3.7** Let $D$ be the ring of integers in $k = \mathbb{Q}(\sqrt{2}+\sqrt{5})$, $\xi_1 = 1+\sqrt{2}$, $\xi_2 = 2+\sqrt{5}$, $\xi_3 = 3+\sqrt{2}\sqrt{5}$ and $(X,\alpha) = (X^D, \alpha^{(D,\xi)})$. Again Proposition 4.2.1 may be used to show that this arithmetic dynamical system is expansive. Since $R_\xi$ is an integral extension of $\mathbb{Z}$ it follows that $\mathcal{V}(k, R_\xi) = \varnothing$. Therefore it is only necessary to consider $\mathcal{V}(\mathfrak{p}_\xi)$, which consists of the points

$$\{(\log|\tau(\xi_1)|, \log|\tau(\xi_2)|, \log|\tau(\xi_3)|) : \tau \in \mathrm{Gal}(k|\mathbb{Q})\}.$$

Figure 4(a) shows where the rays in $\mathbb{R}^3$ passing through these points intersect $\mathsf{S}_2$. Figure 4(b) shows the corresponding non-expansive planes in relative position in $\mathbb{R}^3$.

**Example 4.3.8** Let $D = \mathbb{F}_2[t_1, t_2, \frac{1}{t_1}, \frac{1}{t_1+t_2}, \frac{1}{t_2+1}]$ where $t_1$ and $t_2$ are algebraically independent indeterminates over $\mathbb{F}_2$, $\xi_1 = t_1$, $\xi_2 = t_1+t_2$, $\xi_3 = t_2+1$ and $(X,\alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ the resulting arithmetic dynamical system. The associated prime $\mathfrak{p}_\xi$ contains 2 and the polynomial $1+u_1+u_2+u_3$, and these elements of $R_3$ generate a prime ideal $\mathfrak{p} \in \operatorname{Spec} R_3$ of height 2. Since $R_3/\mathfrak{p}_\xi \cong D$ is not a field, this means that $\mathfrak{p}_\xi$ cannot be maximal. Moreover $R_3$ has Krull dimension 3 and so $\mathfrak{p}_\xi$ has height 2. Therefore $\mathfrak{p}_\xi = \mathfrak{p}$. Hence, as an $R_3$–module, $D \cong R_3/(2, 1+u_1+u_2+u_3)$.

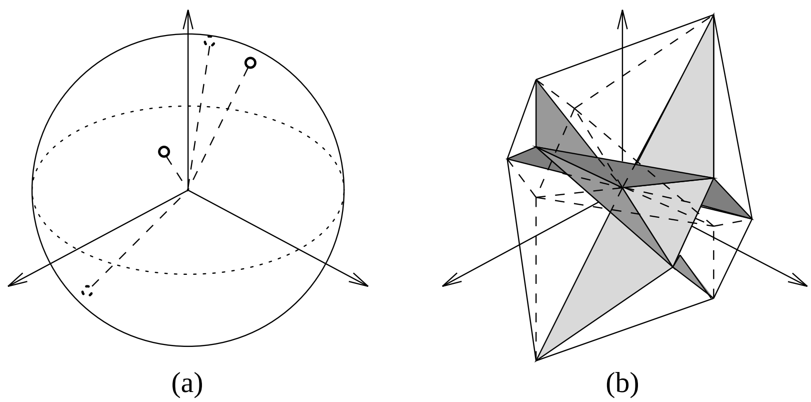


Figure 4: Non-expansive set for Example 4.3.7.

The algebraic dynamical system corresponding to this module has been studied in [3], where the authors use alternative methods to investigate expansive subdynamics. However, Theorem 4.3.4 can also be applied, using the isomorphism described above. Since $2 \in \mathfrak{p}_\xi$, the complex variety $V_{\mathbb{C}}(\mathfrak{p}_\xi)$ is empty (and because we are dealing with a cyclic module this shows that $\alpha$ is expansive). Hence, it is only necessary to consider $\mathcal{V}(k, R_\xi)$. We will deal with the octant $\{\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^d : n_1, n_2, n_3 \geq 0\}$ (the other 7 octants can be treated similarly). Suppose that $T_1, T_2$ are algebraically independent indeterminates over $\mathbb{F}_2$. Example 1.6.7 shows that for any $\lambda > 0$ there is a real valued valuation $v_\lambda$ on $L = \mathbb{F}_2(T_1, T_2)$ which has $v_\lambda(T_1) = 1$ and $v_\lambda(T_2) = \lambda$. If $k = \mathbb{F}_2(t_1, t_2)$ then there is an isomorphism $\theta : k \mapsto L$ given by sending $\frac{1}{t_1}$ to $T_1$ and $\frac{1}{t_2+1}$ to $T_2$. This induces a valuation $w_\lambda$ on $k$ given by $w_\lambda(\cdot) = v_\lambda(\theta(\cdot))$ which has $w_\lambda(t_1) = -1$, $w_\lambda(t_2 + 1) = -\lambda$ and

$$\begin{aligned} w_\lambda(t_1 + t_2) &= v_\lambda\left(\frac{T_1 + T_2 + T_1T_2}{T_1T_2}\right) \\ &= \begin{cases} -1 & \text{if } \lambda < 1 \\ -\lambda & \text{if } \lambda \geq 1 \end{cases} \end{aligned}$$

Note that for each $\lambda$, the valuation ring $D_\lambda$ corresponding to $w_\lambda$ lies in $\mathcal{V}(k, R_\xi)$. Now for all $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^d$

$$w_\lambda(t_1^{n_1}(t_1 + t_2)^{n_2}(t_2 + 1)^{n_3}) = w_\lambda(t_1)n_1 + w_\lambda(t_1 + t_2)n_2 + w_\lambda(t_2 + 1)n_3 \qquad (17)$$

and so if $\lambda < 1$ and $H$ is the half-space with outward normal $(1, 1, \lambda)$, then $w_\lambda(a) \geq 0$ for all $a \in R_H$. If $\lambda \geq 1$ and $H$ is the half-space with outward normal $(1, \lambda, \lambda)$, then (17) again shows that $w_\lambda(a) \geq 0$ for all $a \in R_H$. Hence in either case, $D_\lambda \in \mathcal{V}(k, R_H)$. If the isomorphism $\theta : k \mapsto L$ is replaced by $\theta' : k \mapsto L$, given by sending $\frac{1}{t_2}$ to $T_1$ and $\frac{1}{t_1}$ to $T_2$, then using completely analogous methods to above, for every $0 < \lambda < 1$ we obtain a valuation $w_\lambda$ on $k$ with $w_\lambda(t_1) = -\lambda$, $w_\lambda(t_2 + 1) = -\lambda$ and $w_\lambda(t_1 + t_2) = -1$. This means that for a half-space $H$ with outward normal $(\lambda, 1, \lambda)$, again the valuation ring corresponding to $w_\lambda$ lies in $\mathrm{Zar}(k, R_H)$. Thus in all cases just described $\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_\xi) \neq \varnothing$ and so by Theorem 4.3.4 the corresponding half-spaces $R_H$ are non-expansive. Figure 5(a) shows the curves produced in $\mathsf{S}_2$ by

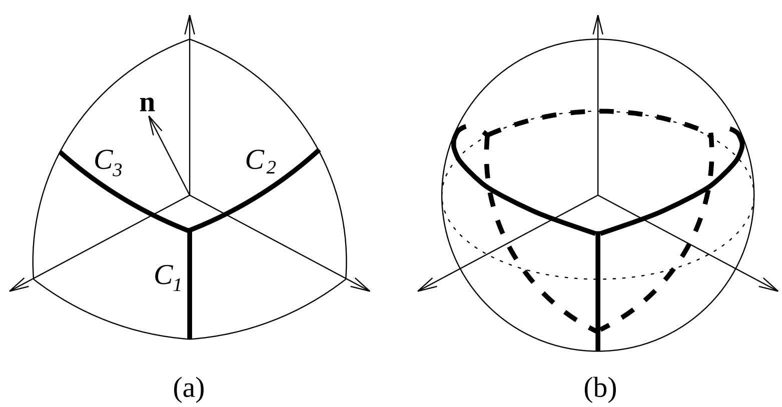


Figure 5: Non-expansive set for Example 4.3.8.

varying $\lambda$, each point of which represents a non-expansive half-space. The curves $C_1$ and $C_2$ arise from the isomorphism $\theta$ with the choices $\lambda < 1$ and $\lambda \geq 1$ respectively. The curve $C_3$ arises from the isomorphism $\theta'$ with $\lambda < 1$. Note that selecting $\lambda \geq 1$ with the isomorphism $\theta'$ duplicates the curve $C_1$.

To see that the regions bounded by, but not including these curves represent expansive directions, consider the following. Suppose that $\mathbf{n} = (n_1, n_2, n_3) \in \mathsf{S}_2$ satisfies $n_1, n_2, n_3 \geq 0$, $n_3 > n_1$ and $n_3 > n_2$, so that $\mathbf{n}$ lies inside the region shown in Figure 5(a) which is bounded by the curves $C_2$ and $C_3$. Then we may choose positive integers $l, m$, with $l$ even, so that

$$1 < \frac{l}{m} < \min\left\{\frac{n_3}{n_2}, \frac{n_3}{n_1}\right\}.$$

If $H$ is the half-space with outward normal $\mathbf{n}$ then $(t_1 + t_2)^l/(t_2 + 1)^m \in R_H$ and $t_1^l/(t_2 + 1)^m \in R_H$. Clearly also $\frac{1}{t_2+1} \in R_H$, giving $(t_1 + t_2)^l/(t_2 + 1)^{l-1} \in R_H$ and $t_1^l/(t_2 + 1)^{l-1} \in R_H$. Hence

$$1 + \frac{(t_1 + t_2)^l}{(t_2 + 1)^{l-1}} + \frac{t_1^l}{(t_2 + 1)^{l-1}} + \frac{1}{(t_2 + 1)^{l-1}} = t_2 \in R_H.$$

Since $\frac{1}{t_1} \in R_H$ and $\frac{1}{t_1+t_2} \in R_H$, this shows that $R_H = R_\xi$ and so $\mathrm{Zar}(k, R_H) = \mathrm{Zar}(k, R_\xi)$. The other 2 regions in Figure 5(a) can be dealt with similarly. Figure 5(b) shows the complete non-expansive set for this example, taking into account the other octants.

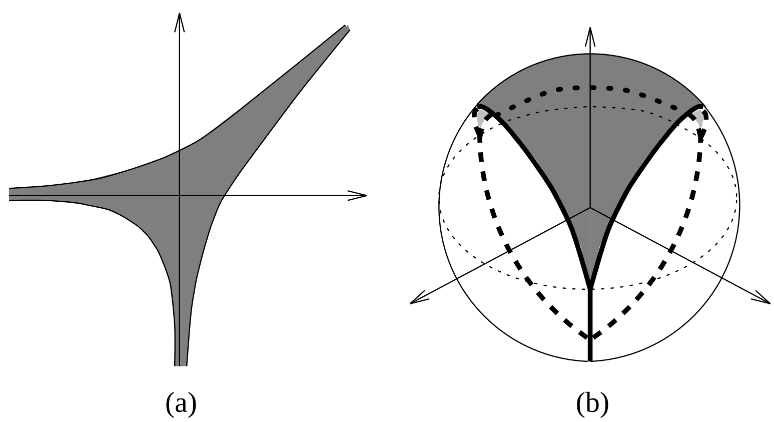


Figure 6: Non-expansive set for Example 4.3.9.

**Example 4.3.9** Let $t$ be an indeterminate, $k = \mathbb{Q}(t)$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ the arithmetic dynamical system generated by the data $D = \mathbb{Z}[t^{\pm 1}, \frac{1}{2}, \frac{1}{t+1}]$, $\xi_1 = t$, $\xi_2 = -t - 1$ and $\xi_3 = 2$. The generators $\xi_1, \xi_2, \xi_3$ parameterize the complex variety $V_{\mathbb{C}}(\mathfrak{p}_\xi)$ and hence the logarithmic image in $\mathbb{R}^3$ is

$$\mathcal{V}(\mathfrak{p}_\xi) = \{(\log|z|, \log|z + 1|, \log 2) : z \in \mathbb{C}^\times\}.$$

Since $D = R_\xi$, this is sufficient to show that $\alpha$ is expansive. The set $\mathcal{V}(\mathfrak{p}_\xi)$ is shown in Figure 6(a) and Figure 6(b) shows where the rays in $\mathbb{R}^3$ passing through $\mathcal{V}(\mathfrak{p}_\xi)$ intersect the upper hemisphere of $\mathsf{S}^2$, each point representing a non-expansive half space.

To illustrate how to calculate the half-spaces $H$ which are non-expansive due to the non-trivial intersection of $\mathrm{Zar}(k, R_H)$ and $\mathcal{V}(k, R_\xi)$, consider for example the octant $\{\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^d : n_1, n_2 \geq 0, n_3 < 0\}$ (as in the previous example, the other 7 octants can be treated similarly). Example 1.6.7 shows that for each $\lambda \geq 0$ there is a real valued valuation $v_\lambda$ on $k$ which has $v_\lambda(t) = -\lambda$, $v_\lambda(-t-1) = -\lambda$ and $v_\lambda(2) = 1$. Let $D_\lambda$ denote the valuation ring of $k$ which corresponds to $v_\lambda$. If $H$ is the half-space with outward normal $(\lambda, \lambda, -1)$ then for all $\mathbf{n} \in H_{\mathbb{Z}}$

$$\begin{aligned} v_\lambda(t^{n_1}(-t-1)^{n_2}2^{n_3}) &= -\lambda n_1 - \lambda n_2 + n_3 \\ &\geq 0. \end{aligned}$$

Hence $D_\lambda \in \mathrm{Zar}(k, R_\xi)$. Clearly also $D_\lambda \in \mathcal{V}(k, R_\xi)$ which shows that $H$ is non-expansive for $\alpha$. By varying $\lambda$ we obtain a quarter meridian on $\mathsf{S}_2$ running from the south pole to the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, each point of which represents a non-expansive half-space. To see that these are the only non-expansive directions in this octant, consider the following.

Suppose $H$ is the half-space with outward normal $\mathbf{n} = (n_1, n_2, n_3)$, where $\mathbf{n}$ satisfies $n_1 \geq 0, n_2 \geq 0$ and $n_3 < 0$. We wish to show that if $n_1 \neq n_2$ then $H$ is expansive for $\alpha$. If $n_2 > n_1$ then there exist positive integers $l, m$ with $l \geq m$ such that $(t+1)^l/t^m \in R_H$. Since $R_H$ also contains $\frac{1}{t}$ this means

$$\frac{(t+1)^l}{t^{l-1}} - \sum_{j=1}^{l} \binom{l}{j} \frac{1}{t^{j-1}} = t \in R_H.$$

It now follows that $\frac{1}{2} \in R_H$. Also, $\frac{1}{t+1} \in R_H$ and so $R_H = R_\xi$. Thus $\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_\xi) = \varnothing$. If $n_1 > n_2$ then there exist positive integers $l, m$ with $l \geq m$ such that $t^l/(t+1)^m \in R_H$. Since $R_H$ also contains $\frac{1}{t+1}$ and $\frac{1}{t}$, we have

$$-1 + \sum_{j=0}^{l} \binom{l}{j} \frac{t^{l-j}}{(t+1)^{l-1}} = t \in R_H.$$

Again this means $\frac{1}{2} \in R_H$, which implies $R_H = R_\xi$ and so $\mathrm{Zar}(k, R_H) \cap \mathcal{V}(k, R_\xi) = \varnothing$.

Upon considering the other 7 octants, two other quarter meridians representing non-expansive half-spaces arise. These run from the south pole to the points $(-1, 0, 0)$ and $(0, -1, 0)$. The complete set of non-expansive half-spaces for this example, as parameterized by $\mathsf{S}_2$, are shown in Figure 6(b).

It has been shown, for an arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ how the set

$$\mathsf{N}(\alpha) = \{\mathbf{v} \in \mathsf{S}_{d-1} : H_\mathbf{v} \text{ is non-expansive for } \alpha\}$$

can be calculated using the logarithmic image of the complex variety $\mathcal{V}(\mathfrak{p}_\xi)$ and the locally ringed space $\mathcal{V}(k, R_\xi)$, where $k$ is any field containing $D$. A closer look at the examples above reveals that in many cases it is only necessary to consider elements of $\mathcal{V}(k, R_\xi)$ which correspond to real-valued valuations. This is in fact always the case as we shall now see.

Suppose that $G$ is a finitely generated abelian group of $\mathbb{Z}$–rank $d$. A homomorphism from $G$ into the additive group of real numbers is called a *character of $G$*. It is easy to see that the group of all such characters is isomorphic to $\mathbb{R}^d$. Moreover, under the equivalence relation

$$\chi_1 \sim \chi_2 \Leftrightarrow \chi_1 = r \cdot \chi_2$$

for some $r > 0$, the set of equivalence classes $[\chi]$ corresponding to non-trivial characters, can be identified with $\mathsf{S}_{d-1}$. This is the *character sphere* of $G$. Suppose that $R$ is a commutative ring and $M$ is a finitely generated $RG$–module. Following Bieri and Groves [1], define the submonoid $G_\chi = \{g \in G : \chi(g) \geq 0\} \subset G$ and

$$\Sigma_M = \{[\chi] : M \text{ is not finitely generated over } RG_\chi\}.$$

Bieri and Groves give a formula for $\Sigma_M$ in terms of valuations with value group a subgroup of $\mathbb{R}$. This formula will be considered in a moment. First, for the case of a module $M$ associated with an ideal $\mathfrak{p} \in \mathrm{Spec}\, R_d$, let us see how $\Sigma_M$ corresponds to the set $\mathsf{N}(\alpha)$ when $\alpha = \alpha^M$ is an expansive algebraic $\mathbb{Z}^d$–action. Let $H \subset \mathbb{R}^d$ be a half-space. Lemma 4.3 of [6] shows that condition 1 of Theorem 4.3.3 is satisfied if and only if $M$ is a finitely generated $A_H$–module, where $A_H$ is as defined in Theorem 4.3.3. Because of the natural correspondence between the set of half-spaces of $\mathbb{R}^d$ and $\mathsf{S}_{d-1}$, there is a correspondence between half-spaces and elements of the character sphere for $G = \langle u_1, \ldots, u_d \rangle$, the unit group of $R_d$. Hence, for each half-space $H$ the ring $A_H$ may be identified with $\mathbb{Z}G_\chi$, where $H = H_{\mathbf{v}}$, $[\chi] = [\chi]_{-\mathbf{v}}$ and $\mathbf{v} \in \mathsf{S}_{d-1}$, because

$$\begin{aligned} \mathbf{n} \in H_{\mathbb{Z}} &\Leftrightarrow \mathbf{v} \cdot \mathbf{n} \leq 0 \\ &\Leftrightarrow \chi(u^{\mathbf{n}}) \geq 0 \\ &\Leftrightarrow u^{\mathbf{n}} \in G_\chi. \end{aligned}$$

Thus, $-\Sigma_M$ represents the set of half-spaces which do not satify condition 1 of Theorem 4.3.3.

Let $D$ be an integral domain, $\xi, \ldots, \xi_d$ units of $D$ and $\theta_\xi : R_d \mapsto D$ the substitution map. Denote the subgroup of the unit group of $D$ generated by $\xi_1, \ldots, \xi_d$ by $\Xi$. Let $v$ be a valuation on $\theta_\xi(\mathbb{Z}) \subset D$ and $w : D \mapsto \mathbb{R} \cup \{\infty\}$ a valuation with $w|_{\theta_\xi(\mathbb{Z})} = v$. Again, following Bieri and Groves, let $\Delta_D^v(\Xi)$ denote the set of all characters $\chi = w|_\Xi$ induced by all such valuations $w$. If $D$ is a Noetherian $R_d$–module under the map $\theta_\xi$ then [1, 8.1] shows that

$$\Sigma_D = \bigcup_v [\Delta_D^v(\Xi)]$$

where $v$ runs through all valuations on $\theta_\xi(\mathbb{Z})$ (including the trivial one) and $[\Delta_D^v(\Xi)]$ denotes the set of equivalence classes obtained from the non-trivial elements of $\Delta_D^v(\Xi)$. Corollary 8.4 of [1] shows that the set $\mathrm{dis}\Sigma_D \subset \Sigma_D$ consisting of equivalence classes $[\chi] \in \Sigma_D$ which contain a non-trivial character $\chi$ with $\chi(\Xi) \subset \mathbb{Q}$, is dense in $\Sigma_D$. Moreover, by [1, Corollary 6.2] every such $\chi$ is induced by a discrete valuation on $D$. Since the character sphere of $\Xi$ is obtained from non-trivial characters, every element of $\mathrm{dis}\Sigma_D$ is induced by at least one element of $\mathcal{S}_D$. Denote the equivalence class of characters induced by $\lambda \in \mathcal{S}_D$ by $[\chi_\lambda]$. A fundamental result of [1] is that $\Sigma_D$ is a closed subset of the character sphere, so $\Sigma_D$ is in fact the closure of the set

$$\{[\chi_\lambda] : \lambda \in \mathcal{S}_D\}.$$

Each element of $\mathcal{S}_V$ also induces a character of $\Xi$ as follows. For each $\lambda \in \mathcal{S}_V$ define $\chi_\lambda : \Xi \mapsto \mathbb{R}$ by

$$\chi_\lambda(a) = -\log |a|_\lambda.$$

We now have the following formulation of Theorem 4.3.3.

**Theorem 4.3.10** *Let* $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ *be an expansive arithmetic dynamical system. Then* $\mathsf{N}(\alpha)$ *is the closure of*

$$-\{[\chi_\lambda] : \lambda \in \mathcal{S}_V \cup \mathcal{S}_D\}.$$

*Proof.* Let $N_1$ denote the subset of $\mathsf{S}_{d-1}$ consisting of elements $\mathbf{v}$ for which $H_\mathbf{v}$ violates condition 1 of Theorem 4.3.3 and $N_2$ denote the subset of $\mathsf{S}_{d-1}$ obtained by normalizing the elements of $\mathcal{V}(\mathfrak{p}_\xi)$ to unit length. Then $\mathsf{N}(\alpha) = N_1 \cup N_2$. Clearly, $N_2 = -\{[\chi_\lambda] : \lambda \in \mathcal{S}_V\}$ and the above discussion shows that $-\{[\chi_\lambda] : \lambda \in \mathcal{S}_D\}$ is dense in $N_1$. Thus $-\{[\chi_\lambda] : \lambda \in \mathcal{S}_V \cup \mathcal{S}_D\}$ is dense in $N_1 \cup N_2$. Furthermore, [3, Lemma 3.4] implies $\mathsf{N}(\alpha)$ is a closed subset of $\mathsf{S}_{d-1}$ and hence the result folows. $\square$

# Chapter 5

# Homoclinic Points

## 5.1 Preliminaries

Let $X$ be a compact abelian group and $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ an algebraic $\mathbb{Z}^d$–action by automorphisms of $X$. A point $x \in X$ is called *homoclinic* (to zero) if $\alpha_{\mathbf{n}}(x) \to 0$ as $\|\mathbf{n}\| \to \infty$. The set of all such points in $X$ is denoted by $\Delta_\alpha(X)$. It is routine to check that $\Delta_\alpha(X)$ is a subgroup of $X$. In this chapter we investigate the relationships between homoclinic groups and other dynamical properties for arithmetic dynamical systems. For such a system, the influence of algebraic characteristics of its defining data on the homoclinic group will also be considered. Immediately from the definition, it is possible to deduce the following two results. The first rules out the possibility of non-trivial homoclinic groups for non-mixing systems and the second is mainly structural in nature. This will be useful in Section 5.3, although of interest in its own right.

**Lemma 5.1.1** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be an arithmetic dynamical system. If $\alpha$ is not mixing then $\Delta_\alpha(X) = \{0\}$.*

*Proof.* According to Proposition 2.1.1, if $\alpha$ is not mixing then there is a non-zero $\mathbf{m} \in \mathbb{Z}^d$ such that $\xi^{\mathbf{m}} = 1$. Therefore for all $j \in \mathbb{Z}$, $\alpha_{j\mathbf{m}}$ is the identity map. Hence, if $J = \{j\mathbf{m} : j \in \mathbb{Z}\}$ then for all non-zero $x \in X$, $\alpha_{\mathbf{n}}(x) \not\to 0$ as $\|\mathbf{n}\| \to \infty$, $\mathbf{n} \in J$.

Hence the only homoclinic point is 0. $\square$

**Proposition 5.1.2** *Let $k$ be a field, $\xi_1, \ldots, \xi_d \in k^\times$, $A, B$ subrings of $k$ which contain $\xi_1, \ldots, \xi_d$ as units, $(X, \alpha) = (X^A, \alpha^{(A,\xi)})$ and $(Y, \beta) = (X^B, \beta^{(B,\xi)})$. If $A \subset B$ then*

$$\Delta_\alpha(X) = \{0\} \Rightarrow \Delta_\beta(Y) = \{0\}.$$

*Proof.* There is a surjective homomorphism $\psi : C \mapsto B$ from a direct sum $C$ of copies of $A$ onto $B$. Moreover, if $\widehat{\gamma}$ denotes the $\mathbb{Z}^d$–action given by the componentwise action of $\widehat{\alpha}$ on $C$, then $\psi\widehat{\gamma} = \widehat{\beta}\psi$. Dually, this means $Y$ can be regarded as a $\gamma$–invariant subgroup $W$ of $X^C = \widehat{C}$, where $\gamma$ is the algebraic $\mathbb{Z}^d$–action dual to $\widehat{\gamma}$. Furthermore under this identification, $\beta$ becomes the restriction of $\gamma$ to $W$. Therefore,

$$\begin{aligned} \Delta_\beta(Y) &\cong \Delta_\gamma(W) \\ &= \Delta_\gamma(X^C) \cap W. \end{aligned}$$

Since $\Delta_\alpha(X) = \{0\}$, it follows that $\Delta_\gamma(X^C) = \{0\}$ and so $\Delta_\beta(Y) = \{0\}$. $\square$

For a basic example of an arithmetic dynamical system with non-trivial homoclinic group, consider the following.

**Example 5.1.3** Let $D = \mathbb{Z}[\frac{1}{2}]$, $\xi = 2$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ the corresponding arithmetic dynamical system. The action $\alpha$ may be realized using the adelic description of $X$ and Example 1.8.1. Following this approach, the group $X$ is identified with $(\mathbb{R} \times \mathbb{Q}_2)/\phi(\mathbb{Z}[\frac{1}{2}])$, where $\phi$ is the canonical embedding. For every $n \in \mathbb{Z}$ the automorphism $\alpha_n$ is identified with the map given by

$$(x, y) + \phi(\mathbb{Z}[\tfrac{1}{2}]) \mapsto (2^n x, 2^n y) + \phi(\mathbb{Z}[\tfrac{1}{2}]).$$

For each $a \in \mathbb{Z}[\frac{1}{2}]$ the point $x_a = (0, a) + \phi(\mathbb{Z}[\frac{1}{2}])$ is homoclinic since $(0, 2^n a) + \phi(\mathbb{Z}[\frac{1}{2}]) \to 0$ as $\|n\| \to \infty$ (here it is important to notice that $(0, 2^n a)$ is congruent to $(2^n a, 0)$ modulo $\phi(\mathbb{Z}[\frac{1}{2}])$). In fact, it will be shown in the next section that all homoclinic points for this example are of the form $x_a$.

Lind and Schmidt [17] have dealt with numerous facets of homoclinic groups for expansive algebraic $\mathbb{Z}^d$–actions, and some consequences of their results will be considered in Section 5.3. Also in that section, we introduce a class of arithmetic dynamical systems which extends the class of expansive ones, and homoclinic groups are investigated for this wider class. Building on the approach used by Lind and Schmidt, it is shown that homoclinic groups are not entirely unpredictable for a large class of non-expansive algebraic $\mathbb{Z}^d$–actions.

An approach common to both [17] and the analysis which follows is the investigation of the relationship between homoclinic points and entropy. In a different setting, King [11] has given an example of an ergodic transformation with zero entropy and non-trivial homoclinic points. This relates to a problem originally posed by Gordin which arose in [9]. For the following class of arithmetic dynamical systems this is not possible.

## 5.2 The $\mathbb{A}$–field case

When $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ is an arithmetic dynamical system arising from units in a ring of $S$–integers in an $\mathbb{A}$–field $k$, recall the description of $X$ and $\alpha$ given in Section 1.6 and Example 1.8.1. For the rest of this section, $X$ will be identified with the group $k^S/\phi(D_S)$ and $\alpha$ with the componentwise action described in Example 1.8.1. This is particularly useful for calculating $\Delta_\alpha(X)$. In [22] the authors treat a special case of the following, in a slightly different context. The approach used there (in particular that used in the proof of Theorem 1.2) for toral automorphisms can easily be extended to cover a larger class of algebraic dynamical systems, namely those of the form $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$.

Let $\alpha : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be an algebraic $\mathbb{Z}^d$–action on the compact group $X$ and $F \subset \mathbb{Z}^d$. Suppose $x \in X$ is such that $\alpha_{\mathbf{n}}(x) \to 0$ as $\|\mathbf{n}\| \to \infty$ and $\mathbf{n} \in F$. Such a point will be referred to as *homoclinic for* $\alpha|_F$. The set of all these points, denoted by $\Delta_{\alpha|F}(X)$, is again easily seen to be a subgroup of $X$. Also, if $F$ is infinite then $\Delta_\alpha(X) \subset \Delta_{\alpha|F}(X)$ and this fact is often useful for showing that $\Delta_\alpha(X)$ is trivial. Another fundamental result is that if $E$ and $F$ are both infinite subsets of $\mathbb{Z}^d$ then

$$\Delta_{\alpha|E\cup F}(X) \subset \Delta_{\alpha|E}(X) \cap \Delta_{\alpha|F}(X).$$

**Theorem 5.2.1** *Suppose that $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ is an arithmetic dynamical system arising from units in a ring of $S$–integers. Then $\Delta_\alpha(X)$ is non-trivial if and only if $\alpha$ is an expansive $\mathbb{Z}$–action. Moreover, if this is the case then $\Delta_\alpha(X) \cong D_S$.*

**Corollary 5.2.2** *For the above class of arithmetic dynamical systems, $\Delta_\alpha(X)$ is non-trivial if and only if $\alpha$ is expansive and has positive entropy.*

*Proof.* Apply Theorem 5.2.1 and Theorem 3.3.4. □

To prove Theorem 5.2.1, the problem is considered in two stages. The idea is to first deal with $\mathbb{Z}$–actions and then extend to actions of $\mathbb{Z}^d$.

**Lemma 5.2.3** *Suppose that $(X, \alpha) = (X^{D_S}, \alpha^{(D_S,\xi)})$ is an arithmetic dynamical system arising from a single unit in a ring of $S$–integers. Then $\Delta_\alpha(X)$ is non-trivial if and only if $\alpha$ is expansive. Moreover, if this is the case then $\Delta_\alpha(X) \cong D_S$.*

*Proof.* Let $k$ be the field of fractions of $D_S$ and suppose that $T$ is the union of the set $S$ together with the infinite places of $k$. Let $Y^+ \subset k^S$ be the set of points $y = (y_\lambda) \in k^S$ which have $y_\lambda = 0$ for all $\lambda$ satisfying $|\xi|_\lambda \geq 1$. We claim that $\Delta_{\alpha|\mathbb{Z}_+}(X)$ consists entirely of points of the form $y + \phi(D_S)$ where $y \in Y^+$. Clearly, any point of this form is homoclinic for $\alpha|_{\mathbb{Z}_+}$ and so it remains to show that if $x \in X$ is homoclinic for $\alpha|_{\mathbb{Z}_+}$ then $x = y + \phi(D_S)$ for some $y \in Y^+$. Suppose that $x \in X$ satisfies $\alpha_n(x) \to 0$ as $n \to \infty$. Because of the local isomorphism between $X$ and $k^S$, for a sufficiently large $m > 0$ it follows that $\alpha_m(x)$ must be of the form $y + \phi(D_S)$ for some $y \in Y^+$. Furthermore, since $\alpha_m$ is an automorphism, by applying $\alpha_{-m}$ it follows that $x$ is also of the required form. Let $Y^- \subset k^S$ denote the set of points $y = (y_\lambda) \in k^S$ which have $y_\lambda = 0$ for all $\lambda$ satisfying $|\xi|_\lambda \leq 1$. Using completely analogous methods to above, it can be shown that $\Delta_{\alpha|\mathbb{Z}_-}(X)$ consists of all points of the form $y + \phi(D_S)$, where $y \in Y^-$. Hence the group $W = \Delta_{\alpha|\mathbb{Z}_+}(X) \cap \Delta_{\alpha|\mathbb{Z}_-}(X)$ consists of points of the form

$y + \phi(D_S)$, where $y \in Y^+$ and satisfies $y \equiv y' \bmod \phi(D_S)$ for some $y' \in Y^-$. Thus, for some $a \in D_S$, $y = (y_\lambda)$ has

$$y_\lambda = a \text{ for each } \lambda \in T \text{ satisfying } |\xi|_\lambda \leq 1. \tag{18}$$

Recall that $\alpha$ is expansive if and only if $|\xi|_\lambda \neq 1$ for all $\lambda \in T$. If $\alpha$ is non-expansive then there exists $\lambda \in T$ such that $|\xi|_\lambda = 1$. By the definition of $Y^+$, for this $\lambda$, $y_\lambda = 0$ and hence (18) implies that $a = 0$. Therefore $y_\lambda = 0$ for all $\lambda \in T$. Thus $W = \{0\}$ and hence $\Delta_\alpha(X) = \{0\}$. If $\alpha$ is expansive then (18) implies that each point of $W$ is of the form $y_a = (y_\lambda^{(a)}) + \phi(D_S)$, where

$$y_\lambda^{(a)} = \begin{cases} 0 & \text{for those } \lambda \text{ with } |\xi|_\lambda < 1 \\ a & \text{for those } \lambda \text{ with } |\xi|_\lambda > 1 \end{cases}$$

and $a \in D_S$. Since $\Delta_\alpha(X) \subset W$, upon checking that each point of the above form is homoclinic, it follows that $\Delta_\alpha(X) = W$.

It is readily verified that the map $\psi : D_S \mapsto W$ given by $\psi(a) = y_a$ is a surjective homomorphism. If $y_a = y_b$ then there is some $c \in D_S$ such that $y_\lambda^{(a)} = y_\lambda^{(b)} + c$ for all $\lambda \in T$. By the product formula for the global field $k$, there is necessarily some $\lambda \in T$ with $|\xi|_\lambda < 1$. Therefore, for this $\lambda$, $y_\lambda^{(a)} = y_\lambda^{(b)} = 0$. But then $c = 0$ and so $a = b$. Thus $\psi$ is an isomorphism. $\square$

*Proof of Theorem 5.2.1.* If $\alpha$ is a $\mathbb{Z}$–action then Lemma 5.2.3 shows that $\Delta_\alpha(X)$ is non-trivial if and only if $\alpha$ is expansive. Also, if this is the case then the same lemma shows that $\Delta_\alpha(X) \cong D_S$. Hence it remains to show that if $\alpha$ is a $\mathbb{Z}^d$–action with $d > 1$, then $\Delta_\alpha(X) = \{0\}$.

First suppose that there exists $\mu \in T$ such that $|\xi_i|_\mu = 1$ for some $i \in \{1, \ldots, d\}$. After some relabelling, we may assume that $i = 1$. Let $F = \{(m, 0, \ldots, 0) : m \in \mathbb{Z}\}$. Since $\alpha|_F$ is equivalent to a non-expansive $\mathbb{Z}$–action, Lemma 5.2.3 may be applied to show that $\Delta_{\alpha|F}(X) = \{0\}$, giving $\Delta_\alpha(X) = \{0\}$. Now let $\mu \in T$ and suppose that there are at least two of $\xi_1, \ldots, \xi_d$ which have $\mu$–adic absolute value not equal to 1. After some relabelling, we can assume that both $|\xi_1| \neq 1$ and $|\xi_2| \neq 1$. For each $m \in \mathbb{Z}$ set

$$l(m) = \lceil -m \log |\xi_1|_\mu / \log |\xi_2|_\mu \rceil.$$

Define the following subsets of $\mathbb{Z}^d$

$$E = \{(m, l(m), 0, \ldots, 0) \in \mathbb{Z}^d : m \in \mathbb{Z}_+\}$$

and

$$F = \{(m, l(m), 0, \ldots, 0) \in \mathbb{Z}^d : m \in \mathbb{Z}_-\}.$$

Notice that when $\mathbf{n} \in E$, $|\xi^{\mathbf{n}}|_\mu \to 1$ as $\|\mathbf{n}\| \to \infty$. Similarly for $\mathbf{n} \in F$. Using completely analogous methods to the proof of Lemma 5.2.3 it follows that

$$\Delta_{\alpha|E}(X) \subset \{(x_\lambda) + \phi(D_S) : x_\lambda = 0 \text{ when } \lambda = \mu \text{ or } |\xi_1|_\lambda > 1\}$$

and

$$\Delta_{\alpha|F}(X) \subset \{(x_\lambda) + \phi(D_S) : x_\lambda = 0 \text{ when } \lambda = \mu \text{ or } |\xi_1|_\lambda < 1\}.$$

Therefore $\Delta_{\alpha|E}(X) \cap \Delta_{\alpha|F}(X) = \{0\}$. This implies that $\Delta_{\alpha|E\cup F}(X) = \{0\}$ and hence $\Delta_\alpha(X) = \{0\}$. $\square$

## 5.3 The general case

In this section, homoclinic groups for more general arithmetic dynamical systems are considered. In what follows, for an arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ generated by units $\xi_1, \ldots, \xi_d$ in the integral domain $D$, the condition that

$$\lambda \in \mathcal{S}_V \Rightarrow \text{ there exists } i \in \{1, \ldots, d\} \text{ such that } |\xi_i|_\lambda \neq 1$$

will be of particular significance. If $(X, \alpha)$ satisfies this condition then it will be said to have an *expansive variety*. It is easy to see that arithmetic dynamical systems with an underlying ring of positive characteristic have expansive varieties, because for these systems $\mathcal{S}_V = \varnothing$. Also, using Theorem 4.2.4, it can be seen that every expansive arithmetic dynamical system has an expansive variety, and an arithmetic dynamical system with an expansive variety is expansive whenever $D$ is finitely generated and

$$\lambda \in \mathcal{S}_D \Rightarrow \text{ there exists } i \in \{1, \ldots, d\} \text{ such that } |\xi_i|_\lambda \neq 1. \tag{19}$$

Using Fourier analysis and some results from [17] it is possible to determine whether or not many arithmetic dynamical systems with expansive varieties enjoy non-trivial homoclinic groups. Before considering this problem further, a straightforward consequence of the results of [17] is given.

**Theorem 5.3.1** *Let $(X, \alpha)$ be an expansive arithmetic dynamical system. Then $\Delta_\alpha(X) \neq \{0\}$ if and only if $\alpha$ has positive entropy. Moreover, if this is the case then $\Delta_\alpha(X)$ is dense in $X$.*

*Proof.* This follows from [17, Theorems 4.1 and 4.2] and the fact that positive entropy and completely positive entropy are equivalent conditions for arithmetic dynamical systems. □

**Corollary 5.3.2** *If the arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ has an expansive variety and zero entropy then $\Delta_\alpha(X) = \{0\}$.*

*Proof.* The arithmetic dynamical system $(X^{R_\xi}, \alpha^{(R_\xi,\xi)})$ is expansive and has zero entropy. Therefore, the above shows that it must also have trivial homoclinic group. Applying Proposition 5.1.2 with $A = R_\xi$ and $B = D$ gives the result. □

When an arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ has an expansive variety and positive entropy, it is not necessarily true that $\Delta_\alpha(X)$ is non-trivial (although of course if (19) is satisfied then Theorem 5.3.1 shows that $\Delta_\alpha(X)$ is dense in $X$). For example, if $D = \mathbb{Z}[\frac{1}{6}]$ and $\xi = 2$ then $\mathcal{S}_V$ gives a single norm $|\cdot|$ which corresponds to the archimedean absolute value on $\mathbb{Q}$. Clearly $|\xi|_\lambda \neq 1$ and so $(X, \alpha)$ has an expansive variety. Also, Theorem 3.3.8 shows that $h(\alpha) = \log 2$. However, this dynamical system has trivial homoclinic group by Lemma 5.2.3. If $D$ is replaced by $\mathbb{Z}[\frac{1}{2}]$ then the same lemma shows that the corresponding dynamical system has non-trivial homoclinic group. Roughly speaking, despite retaining the same generator for the action, it seems that in passing from $\mathbb{Z}[\frac{1}{2}]$ to $\mathbb{Z}[\frac{1}{6}]$ all homoclinic points are lost. The key issue here is the nature of the extension $R_\xi \subset D$, and the effect of this on the homoclinic group will be explored in what follows. For example, if

$(X,\alpha) = (X^D, \alpha^{(D,\xi)})$, $D$ is a subring of the field of fractions of $R_\xi$ and the extension $R_\xi \subset D$ is not integral, then it will be shown that $\Delta_\alpha(X) = \{0\}$, irrespective of entropy. To begin with, some illustrative examples are considered, including a more detailed treatment of the example just described, to prepare for the general approach adopted later.

**Example 5.3.3** Let $(X,\alpha) = (X^D, \alpha^{(D,\xi)})$ be the arithmetic dynamical system generated by the data $D = \mathbb{Z}[\frac{1}{6}]$ and $\xi = 2$. Then $R_\xi = \mathbb{Z}[\frac{1}{2}]$ and $Y = \Delta_{\alpha^{(R_\xi,\xi)}}(X^{R_\xi})$ is non-trivial. One description of $Y$ is given by Example 5.1.3. Alternatively, $Y$ can be described as follows. The action of $\alpha^{(R_\xi,\xi)}$ on $X^{R_\xi}$ may be identified with the $\mathbb{Z}$–shift on the compact subgroup $W$ of $\mathbb{T}^{\mathbb{Z}}$ consisting of those $(x_m) \in \mathbb{T}^{\mathbb{Z}}$ which satisfy

$$x_{m+1} - 2x_m = 0$$

for all $m \in \mathbb{Z}$. Under this identification, $Y$ is the subgroup of $W$ generated by elements of the form

$$(x_m) = \left(\dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 0, 0, 0, \dots\right)$$

where the zeros can start at any coordinate of $(x_m)$. Note that each element of $Y$ can only have coordinate entries whose denominator is a power of 2. The ring $\mathbb{Z}[\frac{1}{6}]$ is isomorphic to $\mathbb{Z}[\frac{1}{2}, t]/(3t-1)$, where $t$ is an indeterminate, and it follows that $X$ can be identified with the compact subgroup $W'$ of $\mathbb{T}^{\mathbb{Z}\times\mathbb{Z}_+}$ consisting of elements $(x_{m_1,m_2})$ which, for all $m_1, m_2 \in \mathbb{Z}$, satisfy the relations

$$\begin{aligned} x_{m_1+1,m_2} - 2x_{m_1,m_2} &= 0, \\ 3x_{m_1,m_2+1} - x_{m_1,m_2} &= 0. \end{aligned} \tag{20}$$

Under this identification, for each $n \in \mathbb{Z}$ the automorphism $\alpha_n$ is given by

$$(x_{m_1,m_2}) \mapsto (x_{m_1+n,m_2})$$

where $(x_{m_1,m_2}) \in W'$. For $(x_{m_1,m_2})$ to be homoclinic with respect to the resulting $\mathbb{Z}$–action, for each $m_2 \in \mathbb{Z}_+$

$$(\dots, x_{-2,m_2}, x_{-1,m_2}, x_{0,m_2}, x_{1,m_2}, x_{2,m_2}, \dots) \tag{21}$$

should look like an element of $Y$. In particular, each coordinate of (21) must be rational and have denominator a power of 2. Suppose that $(x_{m_1,m_2})$ is a non-zero homoclinic point. Since $(x_{m_1,m_2}) \neq 0$, there exists $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{Z}_+$ such that $x_{m_1,m_2} \neq 0$. Equation (20) implies that for all $j \in \mathbb{Z}_+$

$$3^j x_{m_1,m_2+j} \equiv x_{m_1,m_2} \bmod 1.$$

Therefore, for sufficiently large $j$, $x_{m_1,m_2+j}$ does not have denominator a power of 2, which gives a contradiction. Thus $(x_{m_1,m_2}) = 0$, showing that $\Delta_\alpha(X) = \{0\}$.

**Example 5.3.4** Let $D = \mathbb{Z}[\frac{1}{2}, t, \frac{1}{t^2+3t+1}]$, where $t$ is an indeterminate, and set $\xi = 2$. Then $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ has an expansive variety because $\mathcal{S}_V$ consists of a single element $\lambda$ corresponding to the archimedean absolute value on $\mathbb{Q}$, and $|\xi|_\lambda = 2$. Note also that this dynamical system is non-expansive, since $\mathcal{S}_D$ contains, for example, an element $\mu$ corresponding to the valuation ring $D_\mu = \mathbb{Z}[\frac{1}{2}, t]_{(t^2+3t+1)}$ and $|\xi|_\mu = 1$. Let $t_1, t_2$ be algebraically independent indeterminates over $\mathbb{Z}$ and $A = \mathbb{Z}[\frac{1}{2}, t_1^{\pm 1}, t_2]$. The substitution map from $A$ onto $D$ given by sending $t_1$ to $t^2 + 3t + 1$ and $t_2$ to $t$ has kernel $\mathfrak{p} = (t_2^2 + 3t_2 - t_1 + 1)$ and hence $D \cong A/\mathfrak{p}$. Therefore, upon setting $W = (\mathbb{R} \times \mathbb{Q}_2)/\phi(\mathbb{Z}[\frac{1}{2}])$, it follows that $X^D$ may be identified with the compact subgroup $Y$ of $W^{\mathbb{Z}\times\mathbb{Z}_+}$ consisting of those $(x_{m_1,m_2}) \in W^{\mathbb{Z}\times\mathbb{Z}_+}$ which for all $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{Z}_+$ satisfy

$$x_{m_1,m_2+2} = x_{m_1+1,m_2} - 3x_{m_1,m_2+1} - x_{m_1,m_2}. \tag{22}$$

The action $\alpha$ may be realized on $Y$ as the coordinatewise application of the action described in Example 5.1.3. Hence the homoclinic points of $(X, \alpha)$, realized as a subgroup of $Y$, must in each of their coordinates look like the homoclinic points described Example 5.1.3. Let the subgroup of $W$ which consists of such points be denoted by $V$. An element of $\Delta_\alpha(X)$ corresponds to $(x_{m_1,m_2}) \in Y$ as follows. To construct $(x_{m_1,m_2})$, we freely choose the coordinates

$$\ldots, x_{-1,0}, x_{0,0}, x_{1,0}, \ldots$$

and

$$\ldots, x_{-1,1}, x_{0,1}, x_{1,1}, \ldots$$

from $V$ and calculate the remaining coordinates uniquely using (22). Note that because $V$ is a subgroup of $W$, the relation (22) guarantees that for every $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{Z}_+$, $x_{m_1,m_2} \in V$. Thus

$$\Delta_\alpha(X) \cong (V^2)^{\mathbb{Z}}.$$

In contrast to the $\mathbb{A}$–field case, this gives an example of an uncountable homoclinic group.

Let $\ell^2(\mathbb{Z}^d, \mathbb{R})$ denote the Hilbert space of square-summable real-valued functions on $\mathbb{Z}^d$. The *convolution* of $\Psi = (\Psi_{\mathbf{m}}), \Phi = (\Phi_{\mathbf{m}}) \in \ell^2(\mathbb{Z}^d, \mathbb{R})$ is defined by

$$\Phi * \Psi|_{\mathbf{m}} = \sum_{\mathbf{l} \in \mathbb{Z}^d} \Psi_{\mathbf{l}} \Phi_{\mathbf{m}-\mathbf{l}}.$$

For each $h = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_h(\mathbf{m}) u^{\mathbf{m}} \in R_d$, define $\tilde{h} = (\tilde{h}_{\mathbf{m}}) \in \ell^2(\mathbb{Z}^d, \mathbb{R})$ by $\tilde{h}_{\mathbf{m}} = c_h(-\mathbf{m})$. The following is taken from [17, Lemma 4.5].

**Lemma 5.3.5** *Let $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \in R_d$ and $(X, \alpha) = (X^M, \alpha^M)$ the algebraic dynamical system corresponding to the $R_d$–module $M = R_d/(f)$. If $(X, \alpha)$ is expansive and mixing then, upon identifying $X$ with the compact group*

$$\left\{ (x_{\mathbf{m}}) \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) x_{\mathbf{n}+\mathbf{m}} = 0 \textit{ for all } \mathbf{n} \in \mathbb{Z}^d \right\}$$

*and defining the map $\pi : \ell^2(\mathbb{Z}^d, \mathbb{R}) \mapsto \mathbb{T}^{\mathbb{Z}^d}$ by coordinate reduction* mod 1, *there exists $\Psi \in \ell^2(\mathbb{Z}^d, \mathbb{R})$ such that $\pi(\Psi)$ is a non-trivial homoclinic point of $(X, \alpha)$. Moreover, under this identification, there is an isomorphism $\theta : R_d/(f) \mapsto \Delta_\alpha(X)$ given by*

$$\theta(h + (f)) = \pi(\tilde{h} * \Psi)$$

**Theorem 5.3.6** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be an arithmetic dynamical system with an expansive variety and suppose that $D$ contains an element of the form $a/b$, where $b \in R_\xi$ and $a$ is integral over $R_\xi$, satisfying*

$$a_r a^r + a_{r-1} a^{r-1} + \cdots + a_1 a + a_0 = 0 \tag{23}$$

*for some $a_0, \ldots, a_r \in R_\xi$, $a_r = 1$. If there is a discrete valuation $v$ on $R_\xi$ with $v(R_\xi) \geq 0$ and $v(b) > v(a_0)$ then $\Delta_\alpha(X) = \{0\}$.*

*Proof.* Firstly, if $\mathfrak{p}_\xi$ is non-principal then Proposition 3.3.2 shows that $(X, \alpha)$ has zero entropy and Corollary 5.3.2 implies $\Delta_\alpha(X) = \{0\}$. Therefore, it can be assumed that $\mathfrak{p}_\xi = (f)$, for some $f \in R_d$. Let $B = R_d[t_1, t_2]$, where $t_1, t_2$ are algebraically independent indeterminates over $R_d$. There exists a polynomial

$$g(t_1) = a'_r t_1^r + a'_{r-1} t_1^{r-1} + \cdots + a'_0 \in B$$

such that for $i = 0, \ldots, r$, $a'_i \in R_d$ satisfies $\theta_\xi(a'_i) = a_i$. Hence the polynomial in $R_\xi[t_1, t_2]$, obtained by applying $\theta_\xi$ to $a'_0, \ldots, a'_r$, has $a$ as a zero. If $b' \in R_d$ is an element of $\theta_\xi^{-1}(b)$, then polynomial in $R_\xi[t_1, t_2]$ obtained from

$$h(t_1, t_2) = b' t_2 - t_1 \in B$$

by applying $\theta_\xi$ to $b'$, has $(a, a/b)$ as a zero. Thus, the substitution map from $B$ onto $C = R_\xi[a, a/b]$ obtained by applying $\theta_\xi$ to elements of $R_d$, sending $t_1$ to $a$ and $t_2$ to $a/b$, contains $g$ and $h$ in its kernel. Hence there is a surjective homomorphism from the ring $R = B/(f, g, h)$ to $C$. By duality, this means that $\widehat{C}$ may be regarded as a subgroup $Y$ of $X^R = \widehat{R}$. Moreover, $R$ can be viewed as an $R_d$–module in a natural way and if $(X^R, \alpha^R)$ is the resulting algebraic dynamical system, then $Y$ is an $\alpha^R$–invariant subgroup of $X^R$ and the arithmetic dynamical system $(X^C, \alpha^{(C,\xi)})$ can be identified with $(Y, \alpha^R)$. Our aim will be to show that $\Delta_{\alpha^R}(X^R) = \{0\}$, implying that $\Delta_{\alpha^R}(Y) = \{0\}$. Then applying Proposition 5.1.2 will show that $\Delta_\alpha(X) = \{0\}$.

Write

$$\begin{aligned} f &= \sum_{\mathbf{m}} c_f(\mathbf{m}) u^{\mathbf{m}}, \\ b' &= \sum_{\mathbf{m}} c_{b'}(\mathbf{m}) u^{\mathbf{m}}, \\ g &= \sum_{\mathbf{m}, l} c_g(\mathbf{m}, l) u^{\mathbf{m}} t_1^l, \end{aligned}$$

where $c_f(\mathbf{m}), c_{b'}(\mathbf{m}), c_g(\mathbf{m}, l) \in \mathbb{Z}$, $l \in \{0, \ldots, r\}$ and $\mathbf{m}$ ranges over $\mathbb{Z}^d$. Then the compact group $X^R$ can be identified with the subgroup of $\mathbb{T}^{\mathbb{Z}^d \times \mathbb{Z}_+^2}$ consisting of elements $(x_{\mathbf{m},i,j})$ which satisfy the relations

$$\begin{aligned} \sum_{\mathbf{m}} c_f(\mathbf{m}) x_{\mathbf{m}+\mathbf{n},i,j} &= 0 & (24) \\ \sum_{\mathbf{m},l} c_g(\mathbf{m}, l) x_{\mathbf{m}+\mathbf{n},i+l,j} &= 0 & (25) \\ \sum_{\mathbf{m}} c_{b'}(\mathbf{m}) x_{\mathbf{m}+\mathbf{n},i,j+1} &= x_{\mathbf{m}+\mathbf{n},i,j} & (26) \end{aligned}$$

for all $\mathbf{n} \in \mathbb{Z}^d$ and $i, j \in \mathbb{Z}_+$. Furthermore, under this identification, for each $\mathbf{n} \in \mathbb{Z}^d$ the automorphism $\alpha^R_{\mathbf{n}}$ corresponds to the map

$$(x_{\mathbf{m},i,j}) \mapsto (x_{\mathbf{m}+\mathbf{n},i,j}). \tag{27}$$

Let $W$ be the subgroup of $\mathbb{T}^{\mathbb{Z}^d \times \mathbb{Z}_+^2}$ consisting of elements $(x_{\mathbf{m},i,j})$ which satisfy relation (24). It is easy to see that (27) induces an algebraic $\mathbb{Z}^d$–action $\beta$ on $W$ and under the identification described above, $X^R$ can be regarded as a subgroup of $W$ and $\alpha^R$ as the appropriate restriction of $\beta$. Also, since $(W, \beta)$ may be viewed as a direct product of copies of $(X^{R_d/(f)}, \alpha^{R_d/(f)})$, it is possible to use Lemma 5.3.5 to calculate $\Delta_\beta(W)$. This gives an isomorphism $\psi : R_\xi^{\mathbb{Z}^2} \mapsto \Delta_\beta(W)$, described as follows. Let $(e_{i,j}) \in R_\xi^{\mathbb{Z}^2}$ and for each $i, j \in \mathbb{Z}$, let $e'_{i,j} \in R_d$ be such that $\theta_\xi(e'_{i,j}) = e_{i,j}$. Set

$$\psi\left((e_{i,j})_{i,j\in\mathbb{Z}}\right) = \left(\pi(\widetilde{e'}_{i,j} * \Psi)\right)_{i,j\in\mathbb{Z}}$$

where $\pi$ and $\Psi$ are as defined in Lemma 5.3.5. Note that this is well defined because of the nature of the isomorphism described in Lemma 5.3.5.

Suppose that $\psi$ sends $(e_{i,j}) = (\theta_\xi(e'_{i,j})) \in R_\xi^{\mathbb{Z}^2}$ to an element of $W$ which satisfies (25) and (26). Then upon writing, for each $i, j \in \mathbb{Z}$

$$e'_{i,j} = \sum_{\mathbf{m}} c_{e'}(\mathbf{m}, i, j) u^{\mathbf{m}},$$

relation (25) implies, for all $\mathbf{n} \in \mathbb{Z}^d$

$$\begin{aligned} \sum_{\mathbf{m}_1,\mathbf{m}_2,l} c_g(\mathbf{m}_1, i) c_{e'}(-\mathbf{m}_2, i+l, j) \Psi_{\mathbf{m}_1+\mathbf{n}-\mathbf{m}_2} &\equiv 0 \bmod 1 \\ \Rightarrow \sum_{\mathbf{m}_1,\mathbf{m}_2,l} c_g(-\mathbf{m}_1, i) c_{e'}(-\mathbf{m}_2, i+l, j) \Psi_{\mathbf{n}-(\mathbf{m}_1+\mathbf{m}_2)} &\equiv 0 \bmod 1. \end{aligned}$$

Therefore, for all $i, j \in \mathbb{Z}_+$

$$\pi\left(\widetilde{\sum_l a'_l e'_{i+l,j}} * \Psi\right) = 0.$$

So, applying $\psi^{-1}$ gives

$$\sum_l a_l e_{i+l,j} = 0. \tag{28}$$

Since (26) also holds, using similar methods to above, for all $i, j \in \mathbb{Z}$ we have

$$e_{i+1,j} = b e_{i,j+1} \tag{29}$$

Combining (28) and (29) gives

$$a_0 e_{i,j} = \sum_{l=1}^{r} a_l b^l e_{i,j+l}. \tag{30}$$

Hence

$$v(e_{i,j}) \geq \min_{1 \leq l \leq r}\{l v(b) + v(a_l) + v(e_{i,j+l})\} - v(a_0). \tag{31}$$

Suppose that $e_{i,j} \neq 0$ for some $i, j \in \mathbb{Z}_+$. By (30), this means $e_{i,j+l} \neq 0$ for some $1 \leq l \leq r$. Since $v(b) > v(a_0)$, by (31) it follows that

$$\begin{aligned} v(e_{i,j}) &> \min_{1 \leq l \leq r}\{v(e_{i,j+l})\} \\ &= v(e_{i,j_1}) \end{aligned}$$

for some $j < j_1 \leq j + l$. This implies $e_{i,j_1} \neq 0$, because $e_{i,j} \neq 0$. By repeating this argument with $j_1$ in place of $j$, we obtain $j_1 < j_2 \leq j_1 + l$ such that

$$v(e_{i,j_1}) > v(e_{i,j_2})$$

and $e_{i,j_2} \neq 0$. Continuing in this way yields a sequence $j_1, j_2, j_3, \ldots$ such that

$$v(e_{i,j}) > v(e_{i,j_1}) > v(e_{i,j_2}) > \ldots$$

and since $v$ is a discrete valuation with $v(R_\xi) \geq 0$, we must have $v(e_{i,j}) = \infty$. So $e_{i,j} = 0$, which gives a contradiction. It follows that $e_{i,j} = 0$ for all $i, j \in \mathbb{Z}_+$, and hence $\Delta_{\alpha^R}(X^R) = \{0\}$. □

**Corollary 5.3.7** *Let $k$ be a field and $\xi_1, \ldots, \xi_d \in k^\times$. If $D \supset R_\xi$ is a subring of the field of fractions of $R_\xi$ and $R_\xi \subset D$ is not integral, then given that the arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ has an expansive variety, $\Delta_\alpha(X) = \{0\}$.*

*Proof.* If $R_\xi \subset D$ is not integral then there is an element $a/b \in D$, where $a, b \in R_\xi$, which is not integral over $R_\xi$. Moreover, by the Mori-Nagata integral closure theorem, there is a discrete valuation $v$ on $R_\xi$ which shows this. That is, $v(R_\xi) \geq 0$ and $v(a/b) < 0$. Hence, $v(a) < v(b)$ and the result follows by Theorem 5.3.6. $\square$

If the condition that the arithmetic dynamical system has an expansive variety is dropped in Corollary 5.3.7, the following example illustrates that a trivial homoclinic group cannot necessarily be expected.

**Example 5.3.8** Let $k = \mathbb{Q}(t, \eta)$, where $t$ is an indeterminate over $\mathbb{Q}$ and

$$\eta = \sqrt{(t^2 - 5t + 1)(t^2 - t + 1)}.$$

Set $\xi_1 = t$, $\xi_2 = \frac{1}{2}t^{-1}\eta - \frac{1}{2}(t - 3 + t^{-1})$, $D = \mathbb{Q}[t^{\pm 1}, \eta] \subset k$ and let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be the corresponding arithmetic dynamical system. Note $\xi_2$ is a unit of $D$ because $\xi_2^{-1} = -\frac{1}{2}t^{-1}\eta - \frac{1}{2}(t - 3 + t^{-1})$. Also, it is clear that the field of fractions of $R_\xi$ is $k$. Since

$$Y^2 + (1 + t^2 - 3t)t^{-1}Y + 1 \in \mathbb{Q}(t)[Y]$$

is the minimal polynomial for $\xi_2$ and $\xi_2^{-1}$ over $\mathbb{Q}(t)$, and because $\mathbb{Z}[t^{\pm 1}]$ is an integrally closed domain, it follows that $R_\xi \cong R_2/\mathfrak{p}_\xi$, where the associated prime $\mathfrak{p}_\xi$ is generated by $u_2^2 + (1 + u_1^2 - 3u_1)u_1^{-1}u_2 + 1 \in R_2$. Multiplying this polynomial by $-u_2^{-1}$ gives

$$\mathfrak{p}_\xi = (3 - u_1 - u_1^{-1} - u_2 - u_2^{-1}).$$

Now $\mathcal{S}_V$ contains an element $\lambda$ corresponding to $(1, \frac{1}{2} + \frac{1}{2}\sqrt{3}\sqrt{-1}) \in (\mathbb{C}^\times)^2$ and $|\xi_1|_\lambda = |\xi_2|_\lambda = 1$. Hence $(X, \alpha)$ does not have an expansive variety. Thus $(X, \alpha)$ satisfies the requisites of Corollary 5.3.7, with the exception of the variety condition. Furthermore, despite the fact that the extension $R_\xi \subset D$ is clearly not integral, it will be shown that $\Delta_\alpha(X)$ is non-trivial.

Notice that $\eta \in R_\xi$ and $D \cong \operatorname{inj}\lim(R_\xi, \psi_j)$, where for $j \geq 1$, $\psi_j$ is multiplication by $j+1$. By duality, $X \cong \operatorname{proj}\lim(X^{R_\xi}, \widehat{\psi}_j)$. Using similar methods to Example 1.8.6, $X$ can be identified with the compact subgroup of $\mathbb{T}^{\mathbb{Z}^2 \times \mathbb{N}}$ consisting of elements $(x_{\mathbf{m},j}) = (x_{m_1,m_2,j})$, which for all $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $j \in \mathbb{N}$ satisfy

$$\begin{aligned} 3x_{m_1,m_2,j} - x_{m_1+1,m_2,j} - x_{m_1-1,m_2,j} - x_{m_1,m_2+1,j} - x_{m_1,m_2-1,j} &= 0, & (32) \\ x_{m_1,m_2,j} - (j+1)x_{m_1,m_2,j+1} &= 0. & (33) \end{aligned}$$

Also, for each $\mathbf{n} \in \mathbb{Z}^2$, the automorphism $\alpha_{\mathbf{n}}$ can be identified with the map

$$(x_{\mathbf{m},j}) \mapsto (x_{\mathbf{m}+\mathbf{n},j}). \tag{34}$$

Under the above identification, $X$ may be considered as a subgroup $V$ of the compact group $W \subset \mathbb{T}^{\mathbb{Z}^2 \times \mathbb{N}}$ consisting of elements $(x_{\mathbf{m},j}) \in \mathbb{T}^{\mathbb{Z}^2 \times \mathbb{N}}$ which satisfy (32). Moreover, (34) induces an algebraic $\mathbb{Z}^d$–action $\beta$ on $W$, $V$ is a $\beta$–invariant subgroup of $W$ and $(V, \beta|_V)$ has a natural identification with $(X, \alpha)$. Therefore

$$\begin{aligned} \Delta_\alpha(X) &\cong \Delta_\beta(V) \\ &= V \cap \Delta_\beta(W). \end{aligned}$$

The algebraic dynamical system $(W, \beta)$ may be regarded as a direct product of copies of $(X^{R_2/\mathfrak{p}_\xi}, \alpha^{R_2/\mathfrak{p}_\xi})$, a system which has been studied in [17, Example 7.3]. Here the authors show that there exists $\Psi \in \ell^2(\mathbb{Z}^2, \mathbb{R})$ with the property that, for any $r \in \mathbb{R}$, $\pi(r\Psi)$ is a non-trivial homoclinic point of $(X^{R_2/\mathfrak{p}_\xi}, \alpha^{R_2/\mathfrak{p}_\xi})$. If $\Phi = r\Psi$ for some $r \in \mathbb{R}$ and for each $j \in \mathbb{N}$ we set

$$\Phi_j = \frac{1}{j!}\Phi$$

then $x_\Phi = (\pi(\Phi_1), \pi(\Phi_2), \pi(\Phi_3), \dots)$ satisfies both (32) and (33) and $x_\Phi \in V \cap \Delta_\beta(W)$. Thus, $\Delta_\alpha(X)$ is non-trivial. In fact, by varying the choice of $r$, it can be seen that $\Delta_\alpha(X)$ is uncountable.

## 5.4 Homoclinic points and expansive subdynamics

Let $\alpha$ be an algebraic $\mathbb{Z}^d$–action. In the study of expansive subdynamics, a concept which is fundamental is that many dynamical properties are constant (or vary nicely) within connected components of $\mathsf{E}_l(\alpha)$, the set of expansive $l$–spaces in the Grassmann manifold $\mathsf{G}_l$, $1 \leq l \leq d$. Also, these properties may undergo marked changes between one maximal connected component of $\mathsf{E}_l(\alpha)$ and another. Such maximal connected components are called *expansive components*. Boyle and Lind discuss these ideas in [3, Section 5], formalising the notion of a dynamical property being defined 'along' an element of $\mathsf{E}_l(\alpha)$, which we now proceed to do for homoclinic points.

Let $(X, \alpha)$ be an algebraic dynamical system. A point $x \in X$ will be called *homoclinic along* a subset $F$ of $\mathbb{R}^d$ if there exists $r \geq 0$ such that

$$\alpha_{\mathbf{n}}(x) \to 0 \text{ as } \|\mathbf{n}\| \to \infty, \mathbf{n} \in F^r_{\mathbb{Z}}. \tag{35}$$

Let $x \in X$ and suppose that for some $r > 0$, $x$ satisfies (35). Then $x$ satisfies (35) for all $r \geq 0$. To see this, it is enough to notice that given $s, r > 0$, all points of $F^s_{\mathbb{Z}}$ lie within a bounded distance of $F^r_{\mathbb{Z}}$. The set of all $x \in X$ satisfying (35), for some $r > 0$ will be denoted by $\Delta^F_\alpha(X)$. This is again easily seen to be a subgroup of $X$.

**Theorem 5.4.1** *Let $\alpha$ be an expansive algebraic $\mathbb{Z}^d$–action on a compact abelian group $X$. If $\mathcal{C}$ is a connected component of $\mathsf{E}_l(\alpha)$ then for any $V, W \in \mathcal{C}$*

$$\Delta^V_\alpha(X) = \Delta^W_\alpha(X)$$

*Proof.* See [6, Theorem 9.6] □

Hence every $l$–plane in a connected component of $\mathsf{E}_l(\alpha)$ has the same homoclinic group. The idea of common homoclinic points existing in this way for commuting toral automorphisms was introduced in [22]. The theorem above generalizes this notion to arbitrary algebraic $\mathbb{Z}^d$–actions. In light of Theorem 5.4.1, it makes sense to refer to the group of common homoclinic points of a connected component $\mathcal{C}$ of $\mathsf{E}_l(\alpha)$. This group will be denoted by $\Delta^{\mathcal{C}}_\alpha(X)$.

Let $\alpha$ be an algebraic $\mathbb{Z}^d$–action and $1 \leq l \leq d$. An $l$–plane $V \in \mathsf{G}_l$ is called *rational* if $V \cap \mathbb{Q}^d$ spans $V$. For such a plane there is a correponding restriction of $\alpha$, $\alpha|_{V_{\mathbb{Z}}} = \alpha|_{V \cap \mathbb{Z}^d}$, which is an algebraic $\mathbb{Z}^l$–action. Therefore, there is a straightforward notion of entropy along $V$, that is the *entropy along* $V$ is just $h(\alpha|_{V_{\mathbb{Z}}})$.

**Theorem 5.4.2** *Let $\alpha$ be an expansive algebraic $\mathbb{Z}^d$–action on a compact abelian group $X$ and $1 \leq l \leq d$. If $h(\alpha|_{V_{\mathbb{Z}}}) = 0$ for some rational $V \in \mathsf{E}_l(\alpha)$, then $h(\alpha|_{V_{\mathbb{Z}}}) = 0$ for all rational $V \in \mathsf{E}_l(\alpha)$.*

*Proof.* This is part of [3, Theorem 6.3]. □

Therefore, the property of having positive or zero entropy is constant along every rational expansive $l$–plane. Combining Theorems 5.4.1 and 5.4.2 gives the following.

**Corollary 5.4.3** *If $\Delta_\alpha^{\mathcal{C}}(X)$ is non-trivial for some expansive component $\mathcal{C}$ of $\mathsf{E}_l(\alpha)$ then $\Delta_\alpha^{\mathcal{C}}(X)$ is non-trivial for every expansive component $\mathcal{C}$ of $\mathsf{E}_l(\alpha)$.*

*Proof.* By [3, Lemma 3.4] $\mathcal{C}$ is an open subset of $\mathsf{G}_l$. Hence the set of rational $l$–planes is dense in $\mathcal{C}$. By definition, there is a rational $l$–plane $V$ in $\mathcal{C}$ for which $\Delta_\alpha^V(X) \neq \{0\}$. Since $V$ is rational, this means $\Delta_{\alpha|V_{\mathbb{Z}}}(X) \neq \{0\}$. Therefore by [17, Theorem 4.1] it follows that $h(\alpha|_{V_{\mathbb{Z}}}) > 0$. Hence if $\mathcal{B}$ is another expansive component, then there is a rational $l$–plane $W$ in $\mathcal{B}$ with $h(\alpha|_{W_{\mathbb{Z}}}) > 0$, by Theorem 5.4.2. Again applying [17, Theorem 4.1] gives $\Delta_{\alpha|W_{\mathbb{Z}}}(X) \neq \{0\}$, which implies $\Delta_\alpha^W(X) \neq \{0\}$. Thus $\Delta_\alpha^{\mathcal{C}}(X) \neq \{0\}$. □

The *entropy rank* of an expansive algebraic $\mathbb{Z}^d$–action $\alpha$ is defined to be the largest $1 \leq l \leq d$ such that there is a rational $l$–plane $V$ with $h(\alpha|_{V_{\mathbb{Z}}}) > 0$. By convention, if there is no such $l$–plane, the entropy rank is set at zero. Proposition 7.3 of [6] shows that the entropy rank of an algebraic $\mathbb{Z}^d$–action arising from an $R_d$–module of the form $R_d/\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} R_d$, is equal to the Krull dimension of $R_d/\mathfrak{p}$. Since an expansive arithmetic dynamical system $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ has an underlying ring $D$ which is necessarily integral over $R_\xi \cong R_d/\mathfrak{p}_\xi$, by [23, Section 9], the Krull dimension of $D$ and $R_d/\mathfrak{p}_\xi$ is the same. Therefore, the entropy rank of $\alpha$ is equal to the Krull dimension of $D$.

**Corollary 5.4.4** *Let $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$ be an expansive arithmetic dynamical system. Then for any expansive component $\mathcal{C}$ of $\mathsf{E}_l(\alpha)$, $1 \leq l \leq d$*

$$\Delta_\alpha^{\mathcal{C}}(X) \text{ is } \begin{cases} \textit{trivial} & \textit{if } l > \operatorname{kdim}(D), \\ \textit{dense in } X & \textit{otherwise.} \end{cases}$$

*Here* $\mathrm{kdim}(D)$ *is the Krull dimension of* $D$.

*Proof.* If $l > \mathrm{kdim}(D)$ then each rational $l$–plane $V \in \mathsf{E}_l(\alpha)$ has $h(\alpha|_{V_{\mathbb{Z}}}) = 0$. Hence by Theorem 5.3.1, $\Delta_\alpha^V(X) = \{0\}$. It follows that all expansive components $\mathcal{C}$ of $\mathsf{E}_l(\alpha)$ have $\Delta_\alpha^{\mathcal{C}}(X) = \{0\}$. If $l \leq \mathrm{kdim}(D)$ then by Theorem 5.4.2, for each rational $l$–plane $V \in \mathsf{E}_l(\alpha)$, $h(\alpha|_{V_{\mathbb{Z}}}) > 0$. Therefore, Theorem 5.3.1 implies that $\Delta_\alpha^V(X)$ is dense in $X$. Thus $\Delta_\alpha^{\mathcal{C}}(X)$ is dense in $X$ for each expansive component $\mathcal{C}$. $\square$

**Corollary 5.4.5** *If in the above,* $D$ *is a ring of* $S$*–integers then*

$$\Delta_\alpha^{\mathcal{C}}(X) \text{ is } \begin{cases} \textit{isomorphic to } D & \textit{if } l = 1, \\ \textit{trivial} & \textit{otherwise.} \end{cases}$$

*Proof.* This follows from the fact that rings of $S$–integers have Krull dimension 1, using Lemma 5.2.3 in place of Theorem 5.3.1. $\square$

It is important to note that despite the isomorphism described in Corollary 5.4.5, as subgroups of $X$, the homoclinic groups $\Delta_\alpha^{\mathcal{C}}(X)$ can intersect trivially. This demonstrates the idea of a 'phase transition' when passing from one expansive component to another, as described in [3, Section 5]. The following example illustrates this further.

**Example 5.4.6** Let $D$ be the ring of integers in the $\mathbb{A}$–field $k = \mathbb{Q}(\sqrt{2} + \sqrt{5})$, $\xi_1 = 1 + \sqrt{2}$, $\xi_2 = 2 + \sqrt{5}$, $\xi_3 = 3 + \sqrt{2}\sqrt{5}$ and $(X, \alpha) = (X^D, \alpha^{(D,\xi)})$. Proposition 4.2.1 may be used to show that this arithmetic dynamical system is expansive. Note that by Theorem 3.3.4 this dynamical system has zero entropy. Hence Corollary 5.2.2 gives $\Delta_\alpha(X) = \{0\}$. However, there are non-trivial homoclinic groups isomorphic to $D$ for each expansive component of $E_1(\alpha)$. The expansive components of $E_1(\alpha)$ can be determined from Figure 4(b), by pairing off the 'cones' between the non-expansive planes shown there. The intersection of the four non-expansive planes creates 14 open cones in $\mathbb{R}^3$ and these pair off to give 7 expansive components of $\mathsf{E}_1(\alpha)$. By choosing a line in each of these components, Theorem 5.4.1 and Lemma 5.2.3 then allow the calculation of $\Delta_\alpha^{\mathcal{C}}(X)$ for each expansive component $\mathcal{C}$. The results of doing so are

| Line in $\mathcal{C}$ | $\Delta_\alpha^{\mathcal{C}}(X)$ |
|---|---|
| $r(2,-2,1)$ | $\{(0,0,a,0)+\phi(D) : a \in D\}$ |
| $r(0,-0,1)$ | $\{(0,0,a,a)+\phi(D) : a \in D\}$ |
| $r(-2,2,1)$ | $\{(0,0,0,a)+\phi(D) : a \in D\}$ |
| $r(1,0,0)$ | $\{(0,a,0,a)+\phi(D) : a \in D\}$ |
| $r(-2,-2,1)$ | $\{(a,0,a,a)+\phi(D) : a \in D\}$ |
| $r(2,2,1)$ | $\{(0,a,a,a)+\phi(D) : a \in D\}$ |
| $r(0,1,0)$ | $\{(0,a,a,0)+\phi(D) : a \in D\}$ |

Table 3: Homoclinic groups for Example 5.4.6.

summarized in Table 3. Here the homoclinic groups are given as subgroups of the compact group

$$\left(\prod_\lambda k_\lambda\right) \Big/ \phi(D)$$

where $\lambda$ runs over the four infinite places of $k$, which correspond to the absolute values given by

$$|\cdot|_\lambda = |\gamma_\lambda(\cdot)|_\infty,$$

$|\cdot|_\infty$ being the restriction of the usual archimedean absolute value on $\mathbb{R}$ to $k$ and $\gamma_\lambda \in \mathrm{Gal}(k|\mathbb{Q})$.

In fact, a simple combinatorial argument shows that for an expansive arithmetic dynamical system, arising from units in a ring of $S$–integers in an $\mathbb{A}$–field $k$, the number of distinct homoclinic groups arising in this way is always bounded. This is evident immediately from the explicit form of the homoclinic groups possible in the proof of Lemma 5.2.3. In particular, if $T$ is the union of $S$ with the set of infinite places of $k$, then the number of distinct non-trivial homoclinic groups corresponding to expansive components of $\mathsf{E}_1(\alpha)$ cannot exceed

$$2^{|T|-1} - 1$$

where $|T|$ is the cardinality of $T$. Note that this is necessarily finite by Proposition 4.2.1 and the fact that the places of an $\mathbb{A}$–field satisfy the finite character property. Also, the example above shows that this bound may be achieved.

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