Branched Coverings of Projective Space

by

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Vita

The author was born in New York in 1953. He received his B.A. from Harvard College in 1975.

To P.K.L. and the memory of P.F.L.

CONTENTS

Introduction

This thesis is concerned with the ramification of branched coverings, and with branched coverings of projective space. The following paragraphs give an account of our principal results.

Let $f : X \longrightarrow Y$ be a branched covering of irreducible varieties over an algebraically closed field k. We assume that X is normal, that Y is non-singular, and--in the present description--that f is generically étale. In order to measure the singularities of this mapping, we consider the local degree $e_f(x)$ of f at a point $x \in X$: if f is locally* e-to-one near x, then $e_f(x) =_{def} e$. One of the fundamental facts about the ramification of branched coverings is Zariski's theorem on the purity of the branch locus, which asserts that the ramification locus of f (i.e. $\{x \in X \mid e_f(x) > 1\}$) is either empty or has pure codimension one in X. Our first main result generalizes purity:

__THEOREM 1__. __Every irreducible component of__ $\{x \in X \mid e_f(x) > \ell\}$ __has codimension__ $\leq \ell$ __in__ X.

This was announced in [G-L]. The principal ingredient in the proof is a local theorem of Grothendieck's.

Theorem 1 is used to deduce the existence of higher order ramification points in various circumstances. For example, we give a new proof of the main result of [G-L]:

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*In the classical topology if $k = \mathbb{C}$, in the étale topology otherwise.

PROPOSITION. *If* $f : X \to \mathbb{P}^n$ *is a branched covering of degree* d, *then there exists at least one point* $x \in X$ *at which*

$$e_f(x) \geqslant \min(d, n+1).$$

The proposition asserts, for example, that if f has degree $\geqslant n + 1$, then $n + 1$ or more sheets of the covering must come together at some point of X. We recall that this result has a topological consequence:

COROLLARY. *If* X *is normal, and admits a branched covering* $f : X \to \mathbb{P}^n$ *of degree* $\leqslant n$, *then* X *is algebraically simply connected.*

The corollary is an indication of the fact that there are topological obstructions to expressing a variety as a branched covering of projective space with low degree. Our second main result shows that one has progressively stronger restrictions as the degree d becomes small compared to the dimension n:

THEOREM 2. *Let* X *be a non-singular complex projective variety of dimension* n, *and let* $f : X \to \mathbb{P}^n$ *be a branched covering of degree* d. *Then the induced maps*

$$f_* : \pi_i(X) \longrightarrow \pi_i(\mathbb{P}^n)$$

on homotopy groups are isomorphisms for $i \leqslant n + 1 - d$, *and surjective if* $i = n + 2 - d$.

It follows for instance that if $d \leqslant n$, then X is simply connected;

this result is due to Deligne [D2] and Fulton [F], who more generally extended the corollary stated above to the topological case. Theorem 2 was conjectured in [Lz], where the analogous result for complex cohomology was proved.

It is profitable to view Theorem 2 as an analogue for coverings of the well-known results of Barth and Larsen concerning the topology of small-codimensional subvarieties of projective space. Their theorem asserts that if X is a smooth complex projective variety of dimension n which admits an embedding $X \subseteq \mathbb{P}^{n+e}$ of codimension e, then $\pi_i(P^{n+e},X)=0$ for $i \leq n-e+1$. The connection between subvarieties and coverings is clarified by a basic construction upon which the proof of Theorem 2 relies. Canonically associated to a branched covering $f : X \to \mathbb{P}^n$ of degree d, there exists a vector bundle $E \to \mathbb{P}^n$ of rank d-1 having the property that f factors through an embedding of X in the total space of E. Moreover, the vector bundle E satisfies a strong positivity property: $E(-1)$ is generated by its global sections. This leads one to consider more generally a vector bundle $E \to \mathbb{P}^n$ of rank e, with $E(-1)$ generated by its global sections, and a smooth projective variety X of dimension n embedded in the total space of E:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & E \arrow[d] \\ & \mathbb{P}^n \end{tikzcd}$$

.

Using results of Goresky-MacPherson and Deligne, and a construction of Fulton's, we show that under these circumstances $\pi_i(E,X)=0$ for $i \leqslant n - e + 1$. This yields Theorem 2. And in fact, by taking E to be the direct sum of e copies of the hyperplane line bundle, one also recovers the Barth-Larsen theorem.

. . .

This thesis is divided into two parts. Part I (§§ 1-3) is devoted to the ramification of branched coverings. In §1 we review some facts about the local degree. The proof of Theorem 2 occupies § 2, and applications are given in § 3. In Part II (§§ 4-6) we turn to low degree coverings of projective space. Section 4 contains an exposition of a construction of Deligne's. The proof of Theorem 2 appears in §§ 5 and 6.

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I am endebted to J. Hansen, J. Harris, A. Landman and R. MacPherson for numerous valuable conversations. The material in Part I grew out of a collaboration with T. Gaffney; the main result of Part I was conjectured jointly (at least in a special case), and its proof owes much to Gaffney's encouragement. I am fortunate also to be able to acknowledge the influence of P. Deligne. In comments on a preliminary version of [G-L] Deligne suggested the definition of the local degree which is used in Part I, and ideas he introduced in correspondence with Fulton play a central role in Part II. Above all, I wish to thank my advisor, W. Fulton. His assistance, enthusiasm, and patience have been invaluable.

PART I

We work in Part I over an algebraically closed field k. Varieties are reduced but possibly reducible, and we deal exclusively with closed points. All schemes that arise (e.g. as fibre products) may be assumed to have been given their reduced structures. By a branched covering, we mean a finite surjective morphism.

1. The Local Degree

We consider in this section a branched covering $f : X \longrightarrow Y$ of varieties. It is assumed that Y is normal, and that every irreducible component of X surjects onto Y. Our purpose is to recall some of the formal properties of the local degree, or ramification index, $e_f(x)$ of f at $x \in X$, which intuitively counts the number of sheets of the covering that come together at x.

The local degree is most easily defined when X and Y are complex varieties. Indeed, given $x \in X$, let $y=f(x)$. The inverse image of a sufficiently small connected neighborhood U(y) of y in the classical topology splits up into a disjoint union of connected neighborhoods of the preimages of y:

$$f^{-1} U(y) = \coprod_{f(x')=y} U(x'). \tag{1.1}$$

Consider now the restriction of f, $res(f) : U(x) \longrightarrow U(y)$. Since Y is normal, there exists an analytic subset $B \subsetneq U(y)$ such that every point in $U(y)-B$ has the same number--say e--of preimages in $U(x)$. In other words, the covering $f : X \longrightarrow Y$ is locally e-to-one near x. Naturally enough, one sets

$$e_f(x) = e.$$

For complex varieties, the various properties of the local degree which we shall use follow easily from this description. The reader may consult [M, Appendix to Chapter 6] for a more detailed account.

Suppose now that X and Y are varieties defined over an algebraically closed field k. One wants to imitate the geometric approach of the complex case, but the Zariski topology is too coarse to give the crucial splitting (1.1). Following a suggestion from P. Deligne, one circumvents this difficulty by passing to the étale topology. Then one can proceed in the same spirit as before. In the following paragraphs, we list for the reader's convenience the most frequently used properties of the resulting local degree. The actual definition and indications of proofs appear in [G-L, § 1 and the first paragraph of § 2], where Deligne's approach is worked out.

To begin with, we define the <u>geometric degree</u> $\deg_s f$ of the covering $f : X \longrightarrow Y$. If X is irreducible, $\deg_s f$ is the separable degree of the extension $k(X)/k(Y)$ of function fields. In general, we may define $\deg_s f$ as the sum of the geometric degrees of the coverings $X_i \to Y$ determined by the irreducible components X_i of X. A useful fact is that $\deg_s f = d$ if and only if almost every point of Y has precisely d pre-images in X.

The local degree $e_f(x)$ is a positive integer defined for each $x \in X$. It satisfies the basic additivity formula

$$(1.2) \qquad \sum_{f(x)=y} e_f(x) = \deg_s f.$$

(Note in particular that $\#\{f^{-1}(y)\} \leq \deg_s f$ for every $y \in Y$.) Furthermore, the function $x \mapsto e_f(x)$ is upper semicontinuous on X. If $S \subseteq X$ is an irreducible subvariety, we may accordingly define $e_f(S)$ as the generic value of $e_f(x)$ for $x \in S$.

As one would expect, if f is étale at x, then $e_f(x)=1$. On the other hand, when one wishes to exhibit points at which the local degree is large, the following lemma proves valuable. It corresponds to the fact over $\mathbf{C}$ that if $c_1, c_2 : [0,1] \to X$ are paths with $f \circ c_1 = f \circ c_2$, if $c_1(t) \neq c_2(t)$ for $t \neq 1$ but $c_1(1)=c_2(1)=x$, and if a and b are integers such that $e_f(c_1(t)) \geq a$, $e_f(c_2(t)) \geq b$ for $t \in [0,1)$, then $e_f(x) \geq a + b$.

(1.3) Let T be an integral variety, and let $c_1, c_2 : T \to X$ be distinct morphisms with $f \circ c_1 = f \circ c_2$. Suppose that $c_1(t)=c_2(t)=x$ for some $t \in T$. Then

$$e_f(x) \geqslant e_f(\overline{c_1(T)}) + e_f(\overline{c_2(T)}).$$

The most important technical feature of Deligne's definition is that one has the ability to localize much as in the complex setting. This allows one to reduce many questions to the case in which $e_f(x)=\deg_s f$. Specifically, suppose given a point $x \in X$, and let $y=f(x)$. Then there exist connected étale neighborhoods $p : (W,w) \longrightarrow (X,x)$ * of x, and $q : (V,v) \longrightarrow (Y,y)$ of y, plus a finite map $g : W \longrightarrow V$ sending w to v, which satisfy the following properties:

(1.4a) The diagram

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes.

(1.4b) Every irreducible component of W surjects onto V.

(1.4c) $\deg_s g = e_f(x)$

(1.4d) $e_g(w') = e_f(p(w'))$ for all $w' \in W$.

Note that (b) is automatic: since p is étale and X is pure

*I.e. $p : W \longrightarrow X$ is an étale morphism, and $w \in W$ is a point mapping to x.

pure dimensional, so too is W, which implies (b). Observe also from (c) and (d) that $e_g(w) = \deg_s g$, i.e. w is the only point lying over v=g(w). As in (1.1), W arises as a connected component of $X \times_Y V$.

Finally, we indicate two additional properties which were not stated in [G-L].

(1.5) Suppose in addition to the previous assumptions that $f : X \longrightarrow Y$ is generically étale. Then $e_f(x) = 1$ if and only if f is étale at x.

Indeed, suppose that $e_f(x) = 1$. If x is the only point lying over y=f(x), then the assertion is easy. For in this case $e_f(x)=\deg_s f=1$, which implies first of all that X is irreducible. Being reduced, it is therefore integral. But since f is generically étale, of geometric degree one, it must be birational. Fibally, as Y is normal, it follows that f is an isomorphism. In general one uses (1.4) to reduce to the case just treated.

(1.6) Suppose that X is irreducible. Let $\pi : X^* \longrightarrow X$ be the normalization of X, and denote by f* the composition $X^* \longrightarrow X \longrightarrow Y$. Then

$$e_f(x) = \sum_{\pi(x^*)=x} e_{f^*}(x^*).$$

We leave (1.6) as an exercise for the reader.

2. A Generalization of Purity of the Branch Locus

Let $f : X \longrightarrow Y$ be a branched covering of n-dimensional integral varieties. As long as Y is normal, so that we can speak about the local degree, we define the ℓth ramification locus of f to be the set

$$(2.1) \qquad R_\ell(f) = \{x \in X \mid e_f(x) > \ell\}.$$

Thus $R_\ell(f) \subsetneq R_{\ell-1}(f)$, and $R_0(f)=X$. The main result of Part I concerns the dimensions of these loci:

THEOREM 2.2. *Assume that* X *is normal, and that* Y *is non-singular. Consider a chain*

$$T_\ell \subsetneq T_{\ell-1} \subsetneq \cdots \subsetneq T_1 \subsetneq T_0 = X$$

of irreducible components T_i *of* $R_i(f)$. *Then* T_i *has codimension at most one in* T_{i-1}. *In particular, every irreducible component of* $R_\ell(f)$ *has codimension* $\leq \ell$ *in* X.

This generalizes Zariski's theorem on the purity of the branch locus [Z]. In fact, if $f : X \longrightarrow Y$ is generically étale, then by (1.5)

$$R_1(f) = \{x \in X \mid f \text{ not étale at } x\}.$$

Hence the case $\ell = 1$ of Theorem 2.2 is equivalent to Zariski's assertion that the ramification locus of a generically étale covering $f : X \longrightarrow Y$ as above is either empty or has pure codimension one in X. We refer the reader to [A-M] for a brief survey of some other generalizations of purity. Note by contrast that for $\ell > 1$, the locus $R_\ell(f)$ may have codimension less than ℓ. For example, if $\ell + 1$ is prime to char(k), and if $f : \mathbb{A}^1 \to \mathbb{A}^1$ is defined by $t \mapsto t^{\ell+1}$, then $R_\ell(f) = \{0\}$.

In order to state the main lemma upon which the proof of (2.2) is based, we introduce some notation. Specifically, if V is a variety with irreducible components $V_1, \dots, V_r$, then we set

$$C_V(V_i) = \left(\bigcup_{j \neq i} V_j \right) \cap V_i .$$

In other words, $C_V(V_i)$ is the subvariety of V_i consisting in all points of V_i that lie also in some other irreducible component of V.

<u>LEMMA 2.3</u>. <u>Let</u> V <u>be an irreducible normal variety of dimension</u> n, <u>and let</u> $W \subsetneq V$ <u>be a subvariety which is locally defined (set-theoretically) by</u> e <u>equations. Fix an irreducible component</u> W' <u>of</u> W. <u>Then for any</u> $x \in C_W(W')$,

$$\dim_x C_W(W') \geq n - e - 1.$$

The lemma is derived at the end of this section as a consequence of a local theorem of Grothendieck's.

We will prove Theorem 2.2 using an approach which in a general way goes back at least as far as the work of Artin and Nagata [A-N]. The strategy is to interpret the question as a residual intersection problem, and then to invoke

Grothendieck's theorem in the form of Lemma 2.3. The construction is most readily visible when $\ell = 1$, in which case $R_1(f)$ is just the intersection of the diagonal $\Delta_X \subsetneq X \times_Y X$ with the other irreducible components of $X \times_Y X$. Observing that $X \times_Y X$ is locally cut out by n equations in the normal variety $X \times X$, Lemma 2.3 implies that every irreducible component of $R_1(f) = C_{X \times_Y X}(\Delta_X) \subsetneq \Delta_X$ has dimension $\geqslant 2n - n - 1 = n - 1$. This is precisely the assertion of Theorem 2.2 in the case $\ell = 1$. Although for technical reasons we shall phrase the argument somewhat differently, the idea in general is to proceed inductively on ℓ, applying Lemma 2.3 in a similar manner to the fibre product $X \times_Y T_{\ell-1}$.

Turning to the proof of Theorem 2.2, we start by observing that it is sufficient to establish the following statement:

(2.4) <u>Let</u> $S \subsetneq X$ <u>be any irreducible subvariety, of dimension</u> e, <u>and let</u> $\ell = e_f(S)$. <u>Suppose that</u> $x \in S$ <u>is a point at which</u> $e_f(x) > \ell$. <u>Then there exists an irreducible set</u> $S' \subsetneq S$ <u>passing through</u> x, <u>of dimension</u> $e - 1$, <u>such that</u> $e_f(S') > \ell$.

That (2.4) implies (2.2) is quite clear: given $T_i \subsetneqq T_{i-1}$ as in (2.2), we must check that T_i has codimension one in T_{i-1}. To this end, fix a point $x \in T_i$ not contained in any other irreducible component of $R_i(f)$, and apply (2.4) with

$S = T_{i-1}$, $\ell = e_f(T_{i-1}) = i$, and the chosen point x. We conclude that x lies in an irreducible set $S' \subseteq T_{i-1} \cap R_i(f)$ of codimension one in T_{i-1}. But T_i is the only irreducible component of $R_i(f)$ passing through x, and hence $S' = T_i$.

The next step is to verify that <u>it suffices to prove</u> (2.4) <u>under the additional assumption that</u> x <u>is the only point in</u> X <u>mapping to its image</u> f(x). This reduction is made by using (1.4) to pass to suitable étale neighborhoods of x and f(x). The details are routine and will be omitted.

Assuming, then, that $\{x\} = \{f^{-1}f(x)\}$, we now prove (2.4). Let $\pi : S^* \longrightarrow S$ be the normalization of S, and consider the following diagram of fibre squares:

$$(2.5) \qquad \begin{tikzcd} Z^* \arrow[r, "p"] \arrow[d, "g=pr_1"'] & Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S^* \arrow[r, "\pi"'] & S \arrow[r, "f|S"'] & Y \end{tikzcd}$$

We think of Z (resp. Z*) as a subvariety of $S \times X$ (resp. $S^* \times X$). By way of notation, we set:

$$\Delta \subseteq Z \ := \text{diagonal } \Delta_S \subseteq S \times_Y X \ ;$$

$$\Delta^* \subseteq Z^* := \text{image of natural embedding } S^* \hookrightarrow Z^*$$
$$= \{(s^*, s) \mid \pi(s^*) = s\} \quad .$$

Finally, we fix a point $x^* \in S^*$ lying over the given point

$x \in S$, and we put $z^* = (x^*,x) \in \Delta^*$. One sees from (2.5) that since x is the only point in X lying over f(x), z^* is the only point mapping to $x^* \in S^*$ under the projection $g = pr_1$.

We assert:

$$z^* \in C_{Z^*}(\Delta^*). \tag{2.6}$$

Indeed, since $e_f(S) < e_f(x) = \deg_s f$, it follows that $\#\{f^{-1}f(s)\} \geqslant 2$ for almost every $s \in S$. Hence also $\#\{g^{-1}(s^*)\} \geqslant 2$ for almost all points $s^* \in S^*$, i.e. the projection $g: Z^* \longrightarrow S^*$ is generically at least two-to-one. As $\Delta^* \subsetneq Z^*$ projects isomorphically to S^*, this means that there exists at least one other irreducible component of Z^*, say T^*, which maps onto S^*. But since z^* is the only preimage of x^*, it follows that T^* must pass through z^*, which gives (2.6).

We claim next that

$$\dim_{z^*} C_{Z^*}(\Delta^*) = e - 1. \tag{2.7}$$

In fact, $Z^* \subseteq S^* \times X$ is the inverse image of the diagonal $\Delta_Y \subsetneq Y \times Y$ under the natural map $S^* \times X \longrightarrow Y \times Y$. Y being non-singular, Z^* is thus locally cut out by n equations in $S^* \times X$. On the other hand, $S^* \times X$ is the product of two normal varieties, and is hence itself normal, of dimension e + n. So (2.7) follows from Lemma 2.3.

But now we're done. For by (2.7) there exists an irreducible component T^* of Z^*, distinct from Δ^*, such that $\dim_{z^*}(T^* \cap \Delta^*) = e - 1$. Let $T = p(T^*) \subseteq Z$, p being as in (2.5). T is an irreducible component of Z, distinct from Δ, and $T \cap \Delta$ contains an $(e - 1)$-dimensional irreducible set S' passing through $(x,x) = p(z^*)$. Identifying Δ with S, we think of S' as a subvariety of S containing x. The projections of T onto the factors of $S \times X$ determine distinct morphisms $c_1 : T \longrightarrow S \subseteq X$, $c_2 : T \longrightarrow X$ such that $f \circ c_1 = f \circ c_2$. We have $e_f(c_1(t)) \geq \ell$ and $e_f(c_2(t)) \geq 1$ for all $t \in T$, and it follows from (1.3) that $e_f(s') \geq \ell + 1$ for any $s' \in S'$. This completes the proof of (2.4).

The remainder of this section is devoted to the derivation of Lemma 2.3 from Grothendieck's theorem.

Recall that a noetherian topological space X is said to be connected in dimension $\geq \ell$ if every irreducible component of X has dimension $\geq \ell + 1$, and if $X - F$ is connected for every closed set $F \subseteq X$ of dimension $< \ell$.

LEMMA 2.8. *Let V be a variety, let V' be an irreducible component of V, and let x be a (closed) point in $C_V(V')$. Suppose that the scheme $Y = \mathrm{Spec}(\mathcal{O}_x V)$ is connected in dimension $\geq \ell$. Then*

$$\dim_x C_V(V') \geq \ell .$$

Proof. In view of the natural dimension preserving correspondence between the irreducible subvarieties of V passing through x, and the irreducible closed subsets of Y, the lemma is equivalent to the assertion that given an irreducible component Y' of Y, there exists some other irreducible component Y" such that $\dim(Y' \cap Y") \geq \ell$. But if Y is connected in dimension $\geq \ell$, the existence of such a component is immediate. QED.

The following result of Grothendieck's lies at the heart of things:

(2.9) ([SGA2, XIII.2.1]) Let A be a complete local noetherian domain of dimension n. Fix elements $f_1, \ldots, f_e \in \underline{m}_A \subsetneq A$ $(e \leq n-1)$. Then

$$\operatorname{Spec} A/(f_1, \ldots, f_e)$$

is connected in dimension $\geq n - e - 1$.

Finally, we give the

Proof of Lemma 2.3. It is enough to show that $\operatorname{Spec}(\mathcal{O}_x W)$ is connected in dimension $\geq n - e - 1$ for any $x \in W$. This in turn will follow if we show that $\operatorname{Spec}(\widehat{\mathcal{O}_x W})$ is connected in dimension $\geq n - e - 1$, where $\widehat{\mathcal{O}_x W}$ is the completion of $\mathcal{O}_x W$ at its maximal ideal. We may choose e functions $f_1, \ldots, f_e \in \mathcal{O}_x V$ which cut out W near x. Then

$$\widehat{\mathcal{O}_x W} = \widehat{\mathcal{O}_x V} / (f_1, \ldots, f_e) \cdot \widehat{\mathcal{O}_x V} .$$

But as V is normal, the completion of its local ring at x is a domain, and Grothendieck's theorem applies to give the required connectivity of $\mathrm{Spec}(\widehat{\mathcal{O}_x W})$. QED.

3. Applications

In this section we use Theorem 2.2 to deduce the existence of higher-order ramification points in various circumstances. We start by giving a new proof of the main result of [G-L], a generalization of the classical fact that any non-trivial covering of projective space must ramify. The present argument is inspired by Serre's observation in [Sr] that the algebraic simple-connectedness of $\mathbb{P}^n$ can be deduced from purity of the branch locus.

PROPOSITION 3.1. *Let X be an irreducible variety, and let $f : X \to \mathbb{P}^n$ be a branched covering of geometric degree d. Then the ramification locus $R_\ell(f) \subseteq X$ is non-empty provided that $\ell \leq \min(d-1,n)$.*

Proof. In view of (1.6), we may assume that X is normal. The technique is to construct a normal covering $g : Y \to \mathbb{P}^{n+1}$ which plays the role of the "cone" over the given covering f. Specifically, $g : Y \to \mathbb{P}^{n+1}$ will satisfy two properties:

(i) There exists a point $c \in \mathbb{P}^{n+1}$ such that only one point $c' \in Y$ lies over c.

(ii) For any hyperplane $\mathbb{P}^n \subseteq \mathbb{P}^{n+1}$ not containing c, the restriction $\mathrm{res}(g) : g^{-1}(\mathbb{P}^n) \to \mathbb{P}^n$ is isomorphic to the given covering $f : X \to \mathbb{P}^n$.

Note that (ii) implies that $\#\{g^{-1}(y)\} = d$ for almost all $y \in \mathbb{P}^{n+1}$, and hence that $\deg_s g = d$.

Granting the existence of g, we complete the proof. For a fixed hyperplane $\mathbb{P}^n \subseteq \mathbb{P}^{n+1} - \{c\}$, the covering $g^{-1}(\mathbb{P}^n) \to \mathbb{P}^n$ is identified with $f : X \to \mathbb{P}^n$. Thanks to the fact that c' is the only preimage of c, $R_\ell(g)$ is non-empty for $\ell \leq d - 1$. According to Theorem 2.2, then, $\dim R_\ell(g) \geq 1$ provided that $\ell \leq \min(d-1, n)$. But $g^{-1}(\mathbb{P}^n)$ is an ample divisor on Y, and $R_\ell(f) = R_\ell(g) \cap g^{-1}(\mathbb{P}^n)$. Since an ample divisor meets any curve, the proposition follows.

It is in the construction of the covering $g : Y \to \mathbb{P}^{n+1}$ that the geometry of projective space comes into play. Fix a point $c \in \mathbb{P}^{n+1}$, and let $\pi : P \to \mathbb{P}^{n+1}$ be the blowing-up of $\mathbb{P}^{n+1}$ at c. There is a natural map $P \to \mathbb{P}^n$ which realizes P as a $\mathbb{P}^1$-bundle over $\mathbb{P}^n$. Set $W = X \times_{\mathbb{P}^n} P$. W is normal and irreducible, and the composition $W \to P \to \mathbb{P}^{n+1}$ is surjective. The desired covering $g : Y \to \mathbb{P}^{n+1}$ is obtained as the normalization of $\mathbb{P}^{n+1}$ in the function field $k(W)$ of W:

$$\begin{tikzcd}
Y \arrow[d, "g"'] & W \arrow[l, dashed] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathbb{P}^{n+1} & P \arrow[l, "\pi"] \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

Bearing in mind that $W \to \mathbb{P}^{n+1}$ factors through Y, the connectivity of $g^{-1}(c)$ follows from the fact that the preimage of c in W, being a copy of X, is connected. Property (ii) is equally easy to check. QED.

The original proof of (3.1) with T. Gaffney used the Fulton-Hansen connectedness theorem. It was realized that the argument would give information about coverings of low co-dimensional subvarieties of projective space provided that one had an a priori estimate on the dimensions of the ramification loci R_ℓ. (This, in fact, was one of the motivations for proving Theorem 2.2.) Being in posession of (2.2), one now has the following generalization of Proposition 3.1:

PROPOSITION 3.2. *Let* Y *be a non-singular n-dimensional sub-variety of* $\mathbb{P}^{n+e}$, *and let* X *be an irreducible variety of dimension* n. *Let* $f: X \to Y$ *be a branched covering of geometric degree* d. *If* $\ell \leq \min(d-1, n-e)$, *then* $R_\ell(f)$ *is non-empty, and contains at least one irreducible component of codimension* $\leq \ell$ *in* X.

The argument is substantially the same as the proof of [G-L, Thm. 1], which the reader may consult for details. In brief, after reducing to the case when X is normal, one proceeds inductively on ℓ. Assuming that $R_\ell(f)$ is non-empty for some $\ell < \min(d-1, n-e)$, Theorem 2.2 guarentees that $R_\ell(f)$ contains an irreducible component S of dimension $> e$. Denoting by F the natural map $X \times S \to Y \times Y \subseteq \mathbb{P}^{n+e} \times \mathbb{P}^{n+e}$, the Fulton-Hansen theorem [F-H] implies the connectivity of $F^{-1}(\Delta_Y)$. It then follows using (1.3) that $R_{\ell+1}(f)$ is non-empty.

Exactly as in [G-L, Theorem 2] one obtains

<u>COROLLARY 3.3</u>. <u>Let</u> $Y \subseteq \mathbb{P}^{n+e}$ <u>be as in Proposition 3.2. If</u> X <u>is a normal variety which admits a branched covering</u> $f : X \longrightarrow Y$ <u>of geometric degree</u> $\leq n - e$, <u>then</u> X <u>is algebraically simply connected</u>.

Deligne has remarked that over $\mathbb{C}$ the methods of Fulton give the topological simple connectivity of X in (3.3). In fact, in the situation of the corollary, there exists by Proposition 3.2 a subvariety $S \subsetneq X$ of dimension $\geqslant e + 1$ such that f is one-to-one over f(S). If $S^* \longrightarrow S$ is the normalization of S, it follows that $X \times_{\mathbb{P}^{n+e}} S^* \approx S^*$. Then ([F, § 4, Cor. B]) the natural inclusion $S^* \hookrightarrow S^* \times X$ determines a surjection

$$\pi_1(S^*) \longrightarrow \pi_1(S^*) \times \pi_1(X) \longrightarrow \{1\} \quad .$$

But this is only possible if $\pi_1(X)$ is trivial.

Finally, we remark that Hansen's connectedness theorem for flag manifolds [Hn] allows one to generalize (3.2) and (3.3) to branched coverings of non-singular subvarieties of small codimension in such manifolds. For instance, if F is any manifold of flags in $\mathbb{P}^n$, then the statement of (3.1) holds with $\mathbb{P}^n$ replaced by F.

PART II

We henceforth deal with complex algebraic varieties. By a local complete intersection, we mean a variety which is (set-theoretically) locally a complete intersection in some smooth variety. The reader will note that assertions about π_i for $i \geqslant 1$ are always accompanied by π_0 statements implying that the spaces in question are connected. Therefore, except in statements of results, base-points are ignored.

4. A Construction of Deligne's

In this section we review how the non-compact Lefschetz hyperplane theorem of Goresky and MacPherson is used to deduce Deligne's generalization of the Fulton-Hansen connectedness theorem. These results play a role in §§ 5 and 6. This section is purely expository; it is included only because there are as yet no accounts of Deligne's theorem in the literature.

The following deep generalization of the Lefschetz hyperplane theorem was conjectured by Deligne for smooth varieties in [D1], and proved by Goresky and MacPherson. An announcement with indications of proofs appears in [G-M, § 4].

THEOREM 4.1 *Let X be a connected, local complete intersection* *of pure dimension* n. *Let*

$$f : X \longrightarrow \mathbb{P}^r$$

be a quasi-finite (i.e. finite-to-one) morphism, and let $L \subseteq \mathbb{P}^r$ *be a linear space of codimension* c. *Denote by* L_ε *an* ε *-neighborhood of* L *with respect to some Riemannian metric on* $\mathbb{P}^r$, *and fix* $x \in f^{-1}(L_\varepsilon)$. *Then for sufficiently small* ε , *one has*

$$\pi_i(X, f^{-1}L_\varepsilon, x) = 0$$

for $i \leqslant n - c$.

If X is compact, then $f^{-1}(L)$ is a deformation retract of $f^{-1}(L_\varepsilon)$ for small ε . Hence (4.1) reduces to the classical Lefschetz hyperplane theorem when X is a smooth projective variety and L is a hyperplane.

Fulton and Hansen proved in [F-H] that the inverse image $f^{-1}(\Delta)$ of the diagonal $\Delta \subseteq \mathbb{P}^r \times \mathbb{P}^r$ under a finite mapping $f : X \to \mathbb{P}^r \times \mathbb{P}^r$ is connected if X is complete and irreducible of dimension at least $r + 1$. They showed that this assertion has many surprising geometric and topological consequences. (See [F] for a survey.) In [D1], Deligne extended the Fulton-Hansen theorem to a statement about π_1 as well as π_0. Somewhat later, he observed that Theorem 4.1 could be used to bring higher homotopy groups into the picture [D2]. We outline Deligne's beautiful construction.

Deligne's idea is to pass from the diagonal embedding $\Delta \subsetneq \mathbb{P}^r \times \mathbb{P}^r$ to a linear embedding $\mathbb{P}^r \subsetneq \mathbb{P}^{2r+1}$. To this end, fix two disjoint linear spaces

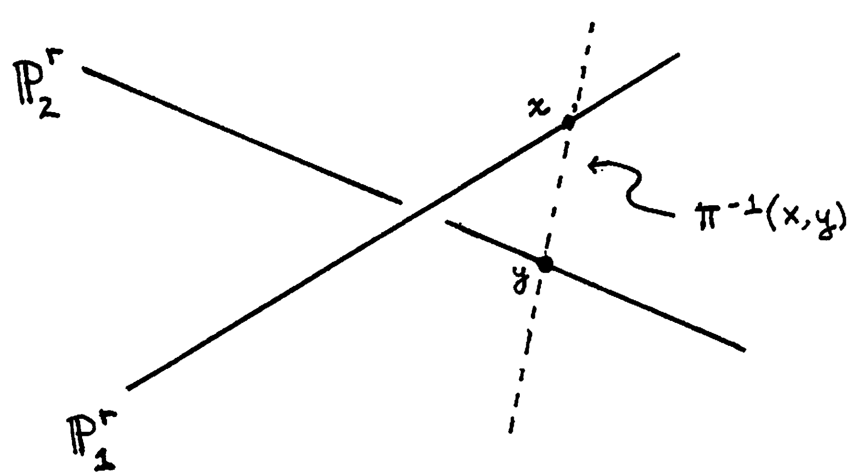
$$\mathbb{P}^r_1 \; , \; \mathbb{P}^r_2 \subsetneq \mathbb{P}^{2r+1} \quad ,$$

which one identifies with the factors of $\mathbb{P}^r \times \mathbb{P}^r$, and set

$$V = \mathbb{P}^{2r+1} - (\mathbb{P}^r_1 \cup \mathbb{P}^r_2).$$

There is a natural map $\pi : V \to \mathbb{P}^r \times \mathbb{P}^r$ which realizes V as a $\mathbb{C}^*$-bundle over $\mathbb{P}^r \times \mathbb{P}^r$: the fibre over $(x,y) \in \mathbb{P}^r \times \mathbb{P}^r$ is the punctured line $\overline{xy} - \{x,y\} = \mathbb{C}^*$. (See Figure 1). There exists moreover a linear space $L \subsetneq V$ of dimension r which maps isomorphically to the diagonal $\Delta \subsetneq \mathbb{P}^r \times \mathbb{P}^r$.

Figure 1



Now suppose that X is a complete, connected, local complete intersection of pure dimension n, and that $f : X \rightarrow \mathbb{P}^r \times \mathbb{P}^r$ is a finite mapping. Set $U = X \times_{\mathbb{P}^r \times \mathbb{P}^r} V$, and let $\pi' : U \rightarrow X$ and $g : U \rightarrow V$ denote the projections. The situation is summarized in the following diagram:

$$\begin{tikzcd}
& U \arrow[r, "\pi'"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& V \arrow[r, "\pi"] & \mathbb{P}^r \times \mathbb{P}^r \\
L \arrow[ur, hook] \arrow[rr, "\simeq", "\pi|L"'] & & \Delta \arrow[u, hook]
\end{tikzcd}$$

Since L maps isomorphically to Δ, π' gives rise to an isomorphism

$$(4.2) \qquad g^{-1}(L) \xrightarrow{\simeq} f^{-1}(\Delta).$$

U is a connected (but non-compact) local complete intersection of pure dimension $n + 1$, and so Theorem 4.1 applies to the composition $U \rightarrow V \subseteq \mathbb{P}^{2r+1}$, and to the linear space $L \subseteq V$. For sufficiently small ε, then, one has

$$\pi_i(U , g^{-1}L_\varepsilon) = 0$$

for $i \leqslant \dim U - \operatorname{codim} L = (n + 1) - (r + 1) = n - r$. But since g is proper and L is closed, $g^{-1}(L)$ is a deformation retract of $g^{-1}(L_\varepsilon)$ provided that ε is small enough. Hence:

$$(4.3) \qquad \pi_i(U , g^{-1}L) = 0$$

for $i \leqslant n - r$.

Consider finally the long exact homotopy sequence of the C^*-bundle $\pi' : U \to X$:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i(C^*) \arrow[r] & \pi_i(U) \arrow[r] & \pi_i(X) \arrow[r] & \pi_{i-1}(C^*) \arrow[r] & \cdots \\
 & & \pi_i(g^{-1}L) \arrow[u] \arrow[r, "\simeq"] & \pi_i(f^{-1}\Delta) \arrow[u] & &
\end{tikzcd}$$

In view of (4.3), there arises for $i \leq n - r$ an exact sequence:

$$(4.4) \qquad \pi_i(f^{-1}\Delta) \longrightarrow \pi_i(X) \longrightarrow \pi_{i-1}(C^*) \longrightarrow \pi_{i-1}(f^{-1}\Delta) \longrightarrow \cdots$$

(Note that there is no assertion about the kernel of $\pi_{n-r}(f^{-1}\Delta) \longrightarrow \pi_{n-r}(X)$.) In summary, we rephrase (4.4) as

THEOREM 4.5 (Deligne, [D2]) *Let* X *be a complete, connected, local complete intersection of pure dimension* n, *and let*

$$f : X \to \mathbb{P}^r \times \mathbb{P}^r$$

be a finite mapping. Let $\Delta \subseteq \mathbb{P}^r \times \mathbb{P}^r$ *be the diagonal, and fix* $x \in f^{-1}\Delta$.

a) *If* $i \leq n - r$, *and* $i \neq 2$, *then* $\pi_i(X, f^{-1}\Delta, x) = 0$.

b) *If* $2 \leq n - r$, *then one has an exact sequence:*

$$\pi_2(f^{-1}\Delta, x) \to \pi_2(X,x) \xrightarrow{\beta} \mathbb{Z} \longrightarrow \pi_1(f^{-1}\Delta, x) \longrightarrow \pi_1(X,x) \longrightarrow \{1\}.$$

Observe that the exact sequence in (b) is compatible with the corresponding exact sequence for the identity map on

$\mathbb{P}^r \times \mathbb{P}^r$; i.e. if $n - r \geqslant 2$, the diagram of exact sequences

$$
(4.6) \qquad \begin{array}{ccccccc}
\pi_2(f^{-1}\Delta) & \longrightarrow & \pi_2(X) & \xrightarrow{\beta} & \mathbb{Z} & \rightarrow & \pi_1(f^{-1}\Delta) \\
\downarrow (f|f^{-1}\Delta)_* & & \downarrow f_* & & \downarrow \wr & & \downarrow \\
\pi_2(\Delta) & \longrightarrow & \pi_2(\mathbb{P}^r \times \mathbb{P}^r) & \longrightarrow & \mathbb{Z} & \longrightarrow & \{1\}
\end{array}
$$

commutes. This follows from the construction, plus the naturality properties of the long exact sequence of a fibre bundle. Using (4.6), one sees that the homomorphism β occuring in (b) is given as follows. Denoting by f_1 and f_2 the compositions of f with the projections of $\mathbb{P}^r \times \mathbb{P}^r$ onto its factors, one obtains two homomorphisms

$$f_{1*} \, , \quad f_{2*} \quad : \pi_2(X) \longrightarrow \pi_2(\mathbb{P}^r) = \mathbb{Z} \; .$$

Then $\beta : \pi_2(X) \rightarrow \mathbb{Z}$ may be identified with their difference: $\beta = f_{1*} - f_{2*}$.

We remark that Fulton ([F']) has observed that Theorem 4.5 leads to a very quick proof of the results of Barth, Larsen, and Ogus [B1, B2, L, O] on the topology of low codimensional subvarieties of projective space:

(4.7) _If_ X _is a complete, connected, local complete intersection of pure dimension_ n _which admits an embedding_

$$X \subseteq \mathbb{P}^{n+e}$$

of codimension e, _then_ $\pi_i(\mathbb{P}^{n+e}, X) = 0$ _for_ $i \leqslant n - e + 1$.

One applies Deligne's theorem (4.5) to the embedding $X \times X \hookrightarrow \mathbb{P}^{n+e} \times \mathbb{P}^{n+e}$, observing that $\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X) \subsetneq X \times X$ is the diagonal embedding of X. We shall give the details of the argument in a slightly different setting in the next section.

5. An Analogue of the Barth-Larsen Theorem

THEOREM 5.1. *Let* X *be an irreducible, non-singular, projective variety of dimension* n, *and let* $f : X \to \mathbb{P}^n$ *be a finite mapping of degree* d. *Fix* $x \in X$. *Then the induced homomorphisms*

$$f_* : \pi_i(X , x) \longrightarrow \pi_i(\mathbb{P}^n , f(x))$$

are isomorphisms for $i \leq n + 1 - d$, *and surjective if* $i = n + 2 - d$.

COROLLARY 5.2. *Under the hypotheses of Theorem 5.1, the induced maps*

$$f_* : H_i(X , \mathbb{Z}) \longrightarrow H_i(\mathbb{P}^n , \mathbb{Z})$$

and

$$f^* : H^i(\mathbb{P}^n , \mathbb{Z}) \longrightarrow H^i(X , \mathbb{Z})$$

are isomorphisms for $i \leq n + 1 - d$. *When* $i = n + 2 - d$, f_* *is surjective, and* f^* *is injective.* ■

It is shown in [Lz, § 1] that canonically associated to a branched covering $f : X \to \mathbb{P}^n$ satisfying the hypotheses of Theorem 5.1, there exists a vector bundle $E \to \mathbb{P}^n$ of rank d-1 having the property that f factors through an embedding of X in the total space of E. Moreover-- and this is the important fact-- $E(-1)$ is generated by its global sections. Hence (5.1) is a consequence of

THEOREM 5.3. *Let* $E \to \mathbb{P}^n$ *be a vector bundle of rank* e *such that* $E(-1)$ *is generated by its global sections. Suppose that* X *is a complete, connected, local complete intersection of pure dimension* n *embedded in the total space of* E:

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & E \arrow[dl] \\ & \mathbb{P}^n & \end{tikzcd}.$$

Fix $x \in X$, *and denote by* f *the composition* $X \hookrightarrow E \to \mathbb{P}^n$. *Then*

$$f_* : \pi_i(X, x) \to \pi_i(\mathbb{P}^n, f(x))$$

is an isomorphism for $i \leq n - e$, *and surjective for* $i = n - e + 1$ (*i.e.* $\pi_i(E,X,x) = 0$ *for* $i \leq n - e + 1$).

Note that f, being affine and proper, is finite. When E is the direct sum of e copies of the hyperplane line bundle, the theorem is equivalent to the Barth-Larsen-Ogus theorem (4.7) for embeddings $X^n \subseteq \mathbb{P}^{n+e}$ (c.f. [Lz, Rmk. 2.4]). We leave it to the reader to formulate the corresponding results for integral homology and cohomology implied by (5.3). Note that the latter in turn imply that if X is smooth, and if $e \leq n - 2$, then $f^* : \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is an isomorphism.

Turning to the proof of (5.3), the strategy is to use Deligne's theorem to derive an analogous statement for the diagonal embedding $X \hookrightarrow X \times X$, which is interpreted as the intersection of $X \times X \hookrightarrow E \times E$ with the diagonal $\Delta_E \subsetneq E \times E$. Then we can transcribe Fulton's proof of the Barth-Larsen-Ogus theorem for embeddings $X \subseteq \mathbb{P}^{n+e}$. The one additional ingredient we shall need is the following lemma, which is proved in § 6.

LEMMA 5.4. *Let X be a complete, connected, local complete intersection of pure dimension n, and let $\mathcal{O}_X(1)$ be an ample line bundle on X which is generated by its global sections. Suppose that E is a vector bundle of rank e on X having the property that $E(-1)$ is generated by its global sections. Let $s \in \Gamma(X,E)$ be a section of E, and let $Z \subseteq X$ be the zero-locus of s. Then, fixing $x \in Z$, one has*

$$\pi_i(X , Z , x) = 0$$

for $i \leq n - e$.

Proof of (5.3). Put $Y = (f \times f)^{-1}(\Delta_{\mathbb{P}^n})$, so that the diagonal embedding $X \hookrightarrow X \times X$ factors through Y. The set-up we shall deal with is summarized in diagram (5.5). Each of the three squares is cartesian, and we henceforth make free use of the natural identifications indicated in (5.5). The inclusion $E \hookrightarrow E \oplus E = E \times_{\mathbb{P}^n} E$ is the obvious diagonal homomorphism over $\mathbb{P}^n = \Delta_{\mathbb{P}^n}$.

$$
(5.5) \qquad
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & Y \arrow[r, hook] \arrow[d, hook] & X \times X \arrow[d, hook] \\
E = \Delta_E \arrow[r, hook] \arrow[dr] & E \oplus E = E \times_{\mathbb{P}^n} E \arrow[r, hook] \arrow[d] & E \times E \arrow[d] \\
& \mathbb{P}^n = \Delta_{\mathbb{P}^n} \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^n
\end{tikzcd}
$$

Note that Y is a complete, connected, local complete intersection of pure dimension n. Indeed, Y is locally cut out in $X \times X$ by n equations, and maps finitely to $\mathbb{P}^n$. It follows that Y is pure n-dimensional, and hence a local complete intersection. The connectedness of Y follows, for instance, from (4.5).

Consider next the inclusion $X = \Delta_X \hookrightarrow Y$. We claim that

$$(5.6) \qquad \pi_i(Y, X) = 0$$

for $i \leq n - e$. Letting h denote the composition $Y \hookrightarrow E \oplus E \longrightarrow \mathbb{P}^n$, the point to observe is that X is defined in Y as the zero-locus of a section of h^*E. In fact, the embedding $Y \hookrightarrow E \oplus E$ over $\mathbb{P}^n$ determines a "tautological" section of $h^*(E \oplus E)$, i.e. two sections $s_1, s_2 \in \Gamma(Y, h^*E)$, and $X = \text{zeroes}(s_1 - s_2) \subsetneq Y$. The assumption on E implies that $h^*E(-1)$ is generated by its global sections, and so (5.6) is a consequence of Lemma 5.4.

The argument now proceeds as in [F']. Specifically, consider the diagonal homomorphism $\pi_i(X) \longrightarrow \pi_i(X \times X)$. It factors as the composition of the two homomorphisms

$$\pi_i(X) \longrightarrow \pi_i(Y) \longrightarrow \pi_i(X \times X) \tag{5.7}$$

induced by the inclusions on the top line of (5.5). (5.6) applies to the first map in (5.7) when $i \leq n - e$, Deligne's theorem (4.5) applies to the second for $i \leq n$, and their composition is always injective. We conclude to begin with that

$$\pi_i(X \times X\ ,\ X) = 0 \quad \text{for} \quad i \leq n - e, \quad i \neq 2,$$

which gives $\pi_i(X) = 0 = \pi_i(\mathbb{P}^n)$ for the indicated values of i. If $2 \leq n - e$, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(X \times X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 = \pi_1(X) \\ & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_* \times f_*} & & \downarrow{\scriptstyle \wr} & & \\ 0 \longrightarrow & \pi_2(\mathbb{P}^n) & \longrightarrow & \pi_2(\mathbb{P}^n \times \mathbb{P}^n) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array} \tag{5.8}$$

of exact sequences [(4.6)]. Hence ker. $f_* = \ker(f_* \times f_*)$, and coker $f_* = \operatorname{coker}(f_* \times f_*)$. But this forces

$$\ker f_* = \operatorname{coker} f_* = 0,$$

and so f_* is an isomorphism on π_2. Finally, the surjectivity of $\pi_{n-e+1}(X) \longrightarrow \pi_{n-e+1}(\mathbb{P}^n)$ is non-trivial only if $n - e = 1$. In this case X is simply connected, and (5.8) remains valid on the right, i.e. we have

$$(5.9)\qquad \begin{array}{ccccc} \pi_2(X \times X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \Big\downarrow f_* \times f_* & & \Big\downarrow \wr & & \\ \pi_2(\mathbb{P}^n \times \mathbb{P}^n) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array} \quad .$$

Recalling that the map $\pi_2(\mathbb{P}^n \times \mathbb{P}^n) = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ is given by $(a,b) \mapsto a - b$, it follows that $f_* : \pi_2(X) \to \pi_2(\mathbb{P}^n)$ is surjective. QED.

<u>REMARK 5.10</u> . For π_0, one can prove the following analogue of the Fulton - Hansen connectedness theorem:

> Let $E \to \mathbb{P}^n$ be an ample vector bundle of rank e, let X be a complete irreducible variety, and let $f : X \to \cancel{E}\,(E \times E)$ be a morphism. If $\dim f(X) \geq n + e + 1$, then $f^{-1}(\Delta_E)$ is connected.

The proof is sketched in the appendix.

<u>6. Proof of Lemma 5.4</u>

This section is devoted to the proof of Lemma 5.4. A construction somewhat reminiscent of Deligne's (§4) reduces the question to intersecting a non-compact variety with a linear subspace of projective space, so that the theorem of Goresky and MacPherson (4.1) applies. We start with the preliminary

<u>LEMMA 6.1</u>. <u>Let</u> X <u>be a complete variety, and</u> $\mathcal{O}_X(1)$ <u>an ample line bundle on</u> X <u>which is generated by its global sections</u>. <u>Let</u>

U *be the direct sum of* c *copies of* $\mathcal{O}_X(1)$, *and denote by* $X_0 \hookrightarrow U$ *the zero section of* U. *Suppose that* T *is a connected, local complete intersection of pure dimension* n, *and that* $g : T \longrightarrow U$ *is a finite map. Then fixing* $t \in g^{-1}(X_0)$, *one has*

$$\pi_i(T , g^{-1}(X_0), t) = 0$$

for $i \leq n - c$.

Proof. The assumption on $\mathcal{O}_X(1)$ means that there is a finite map $f : X \longrightarrow \mathbb{P}^r$ such that $\mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}^r}(1)$. Let V be the direct sum of c copies of $\mathcal{O}_{\mathbb{P}^r}(1)$. In a well-known manner, we can represent V as a Zariski-open subset of $\mathbb{P}^{r+c}$. To wit, fix disjoint linear spaces L , $L' \subsetneq \mathbb{P}^{r+c}$ of dimensions r and $c - 1$ respectively. Then $V = \mathbb{P}^{r+c} - L'$, the bundle map $V \longrightarrow L = \mathbb{P}^r$ being the projection from L' onto L. The natural inclusion $L \hookrightarrow V$ is identified with the zero section. Hence we can realize the bundle U on X as the fibre product $X \times_{\mathbb{P}^r} V$. The projection $F : U \rightarrow V$ is finite, and $X_0 = F^{-1}(L)$:

$$\begin{array}{ccc} X_0 & \hookrightarrow & U \\ \downarrow & & \downarrow F \\ L & \hookrightarrow & V \subsetneq \mathbb{P}^{r+c} \end{array} .$$

Now let h be the composition $F \circ g : T \longrightarrow V$, which is finite. Then $g^{-1}(X_0) = h^{-1}(L)$. We apply the theorem of Goresky-MacPherson to the composition $T \longrightarrow V \subsetneq \mathbb{P}^{r+c}$, and to the linear space $L \subseteq V$. It asserts that for sufficiently small ϵ,

$$\pi_i(T\ ,\ h^{-1}L_{\varepsilon}) = 0$$

for $i \leqslant \dim T - \operatorname{codim} L = n - c$. As L is closed, and h is proper, $h^{-1}(L)$ is a deformation retract of $h^{-1}(L_\varepsilon)$ when ε is small, and the lemma follows. QED.

<u>Proof of Lemma 5.4</u>. The hypothesis on E implies that there is a surjective homomorphism $p : U \twoheadrightarrow E$, where U is the direct sum of c copies of $\mathcal{O}_X(1)$ for some c. Let $X_s \hookrightarrow E$ be the image of the given section $s \in \Gamma(X,E)$, and set $T = X_s \times_E U$. Denote by $X_0 \hookrightarrow U$ and $X \hookrightarrow E$ the sero sections:

$$\begin{tikzcd}
& T \arrow[rr] \arrow[d, hook] & & X_s \arrow[d, hook] \\
& U \arrow[rr, "p"] & & E \\
X_0 \arrow[ur, hook] \arrow[rr, "\simeq", "p|X_0"'] & & X \arrow[ur, hook] & \text{zero section}
\end{tikzcd}$$

Since p restricts to an isomorphism on zero-sections, we have

$$X_0 \cap T \xrightarrow{\ \simeq\ } X \cap X_s = \left\{ \begin{matrix} \text{zero locus} \\ Z \text{ of } s \end{matrix} \right\}.$$

Bearing in mind that T is a $\mathbb{C}^{c-e}$-bundle over X_s, on the level of homotopy groups one gets:

$$\begin{array}{ccc} \pi_i(T) & \xrightarrow{\simeq} & \pi_i(X_s) \\ \uparrow & & \uparrow \\ \pi_i(X_0 \cap T) & \xrightarrow{\simeq} & \pi_i(Z) \end{array} .$$

But T is a connected, local complete intersection of pure dimension $n + c - e$, and the inclusion $T \hookrightarrow U$ is a closed embedding, hence finite. By (6.1), then, $\pi_i(T , X_0 \cap T) = 0$ when $i \leqslant (n + c - e) - c = n - e$, and the lemma follows. QED.

<u>REMARK 6.2</u>. Lemma 5.4 should be thought of as a "Lefschetz hyperplane theorem" for the zero locus of a section of a vector bundle satisfying a strong ampleness condition. For similar results along these lines, see [G] and [S].

<u>Appendix</u>.

We sketch the proof of the result stated in (5.10).

To begin with, by taking its Stein factorization as in [F-H], we may assume that f is finite. Observe that then X is finite over $\mathbb{P}^n \times \mathbb{P}^n$. Let $\ell = \dim X$, so that $\ell \geqslant n + e + 1$. Let $Y \subsetneq X$ be the inverse image of $\Delta_{\mathbb{P}^n}$ under the composition $X \longrightarrow E \times E \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$, and denote by Z the inverse image $f^{-1}(\Delta_E)$. The situation is summarized by a diagram analogous to (5.5):

$$
\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] & Y \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\
\Delta_E = E \arrow[r, hook] \arrow[dr] & E \oplus E = E \times_{\mathbb{P}^n} E \arrow[r, hook] \arrow[d] & E \times E \arrow[d] \\
& \mathbb{P}^n = \Delta_{\mathbb{P}^n} \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^n
\end{tikzcd}
$$

If h denotes the composition $Y \longrightarrow E \oplus E \longrightarrow \mathbb{P}^n$, then as on page 31 one sees that Z is defined in Y as the zero-locus of a section of h^*E. On the other hand, by Hansen's strengthening [Hn] of the Fulton-Hansen connectedness theorem, Y is non-empty and connected in dimension $\geqslant \ell - n - 1 \geqslant e$. Hence any two irreducible components Y' , Y'' of Y can be joined by a chain $Y' = Y_0 , \ldots , Y_r = Y''$ of components having the property that $\dim(Y_i \cap Y_{i+1}) \geqslant e$ for each $0 \leqslant i \leqslant r-1$. To prove the connectedness of Z, it is therefore enough to show that $Z \cap (Y_0 \cup \ldots \cup Y_r)$ is connected for any such chain. Bearing in mind that each irreducible component of Y has dimension $\geq e + 1$, this follows at once from the

LEMMA. *Let* V *be a (possibly singular) irreducible projective variety of dimension* m, *and let* F *be an ample vector bundle of rank* e *on* V. *Let* $s \in \Gamma(V,F)$ *be a section of* F, *and denote by* Z *the zero-locus of* s. *If* $m \geq e$, *then* Z *is non-empty, and if* $m \geq e + 1$, *then* Z *is connected*.

At least in the non-singular case, the lemma is well known. For lack of a reference in the general situation, we sketch the

Proof. Arguing as in [S], we observe that $H_i(V-Z)=0$ for $i \geq m + e$. Indeed, let $P = \mathbb{P}(F^*)$. The section $s \in \Gamma(V,F)$ determines in the natural way a section $s^* \in \Gamma(P,\mathcal{O}_P(1))$. Denoting by $Z^* \subsetneq P$ the zero-locus of s^*, one sees that $P - Z^*$ is locally trivial over $V - Z$, with fibre $\mathbb{C}^{e-1}$. As F is ample, $P - Z^*$ is an affine variety of dimension $m + e - 1$, and hence $H_i(V-Z) = H_i(P - Z^*) = 0$ for $i \geq m + e$ (c.f. [V]). In particular, if $e \leq m$, then $H_{2m}(V-Z) = 0$; it follows that $Z \neq \emptyset$. For the second assertion, we may assume that V is normal. Then as Vilonen observed, the Zeeman spectral sequence (c.f. [McC]) gives rise to an injection $0 \longrightarrow H^1(V,Z) \longrightarrow H_{2m-1}(V-Z)$. But $H_{2m-1}(V-Z) = 0$ when $e \leq m - 1$, and the connectedness of Z follows from the vanishing of $H^1(V,Z)$. QED.

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