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9

Cohomology on Symmetric Products, Syzygies of Canonical Curves, and a Theorem of Kempf

Robert Lazarsfeld*
Department of Mathematics
U.C.L.A.
Los Angeles, CA 90024

Let C be a compact connected Riemann surface of genus $g \geq 2$, denote by C_m the m^{th} symmetric product of C, and consider the Abel-Jacobi map

$$u : C_m \longrightarrow J(C),$$

where $J(C)$ is the Jacobian of C. The derivative of u determines a homomorphism

$$du : \Theta_{C_m} \longrightarrow u^* \Theta_{J(C)}$$

of coherent sheaves on C_m, where as usual Θ_X denotes the tangent sheaf of a complex manifold X. In the course of his celebrated investigation [K1] of the deformation theory of symmetric products, Kempf computed $H^1(C_m, \Theta_{C_m})$ and analyzed the map $H^1(du)$ induced by du. His result is that $H^1(C_m, \Theta_{C_m}) = H^1(C, \Theta_C)$, and that $H^1(du)$ is identified with the canonical homomorphism $H^1(C, \Theta_C) \longrightarrow H^1(C, \mathcal{O}_C) \otimes H^1(C, \mathcal{O}_C)$ dual to the multiplication $H^0(C, \Omega) \otimes H^0(C, \Omega) \longrightarrow H^0(C, \Omega^2)$, Ω being the canonical bundle on C. In particular, it then follows from Noether's theorem that $H^1(du)$ fails to be injective if and only if C is hyperelliptic.

The purpose of this note is to carry out the analogous computations for higher cohomology groups. Surprisingly, we find that the answer involves the syzygies of canonical curves.

To give precise statements, we start with some notation. Let $M = M_\Omega$ denote the kernel of the evaluation homomorphism $H^0(C, \Omega) \otimes \mathcal{O}_C \longrightarrow \Omega$, and write $Q = Q_\Omega = M^*$, so that Q is a vector bundle of rank $g-1$ on X. Making the identification $H^1(C, \mathcal{O}_C) = H^0(C, \Omega)^*$, one thus has an exact sequence

$$0 \longrightarrow \Theta_C \longrightarrow H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C \longrightarrow Q \longrightarrow 0$$

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of vector bundles on C, which in turn gives rise to

$$(*)_k \qquad 0 \longrightarrow \Lambda^{k-1}Q \otimes \mathcal{O}_C \longrightarrow \Lambda^k H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C \longrightarrow \Lambda^k Q \longrightarrow 0.$$

In technical terms, we may summarize the result of our computation in the following

Theorem. If $m \geq k$ and $k < g-1$, then there is a canonical isomorphism

$$H^k(C_m, \Theta_{C_m}) = H^1(C, \Lambda^{k-1}Q \otimes \Theta_C),$$

and the map $H^k(du) : H^k(C_m, \Theta_{C_m}) \longrightarrow H^k(C_m, u^* \Theta_{J(C)})$ is identified with the homomorphism $H^1(C, \Lambda^{k-1}Q \otimes \Theta_C) \longrightarrow \Lambda^k H^1(C, \mathcal{O}_C) \otimes H^1(C, \Theta_C)$ determined by $(*)_k$.

A more picturesque formulation of this result involves the syzygies of the canonical embedding $C \subset \mathbb{P}^{g-1}$ of C. Recall (c.f. [GL1] or [L1]) that one says that the canonical bundle Ω satisfies **property (N_p)** if roughly speaking the first p steps in the minimal graded free resolution of the homogeneous ideal of $C \subset \mathbb{P}^{g-1}$ are as simple as possible. Referring to [GL1] or [L1] for the precise definition, suffice it to say here that Ω satisfies (N_0) iff Ω is normally generated; (N_1) holds iff Ω is normally generated and in addition the homogeneous ideal $I_{C/\mathbb{P}^{g-1}}$ is generated by quadrics; (N_2) holds iff (N_1) does, and the module of syzygies among quadratic generators $Q_i \in I_{C/\mathbb{P}^{g-1}}$ is generated by relations of the form $\Sigma L_i Q_i = 0$ where the L_i are linear polynomials; and so on. A well-known conjecture of Green's [G1] asserts that least value of p for which (N_p) fails for Ω is equal to the Clifford index Cliff(C) of C. (See [L2] for a recent survey of this and related conjectures.)

It is standard and elementary (consult [GL2], [L1] or [PR]) that the syzygies of Ω are governed by the exact sequences $(*)_k$. Specifically, (N_k) holds for Ω if and only if the homomorphism $H^1(C, \Lambda^k Q \otimes \Theta_C) \longrightarrow \Lambda^{k+1} H^1(C, \mathcal{O}_C) \otimes H^1(C, \Theta_C)$ determined by $(*)_{k+1}$ is injective. Hence one has the

Corollary. If $k \leq m$ and $k < g-1$, then $H^k(du)$ fails to be injective if and only if property (N_{k-1}) fails for the canonical bundle Ω.

So for example, $H^2(du)$ fails to be injective if and only if C is Petri-exceptional. The Corollary helps to explain some of the computations appearing in [K2] and [Muk]. We hope that it may eventually open the door to some progress on computing the syzygies of generic canonical curves. Some other, more geometric, variants of the Theorem and its Corollary appear in §3.

The reader will recognize my debt to Kempf's paper [K1]. I am also grateful to L. Ein, M. Green, G. Kempf, M. Schacher and C. Voisin for valuable discussions.

§1. A Lemma on Cohomology and Galois Coverings

The purpose of this section is to record an elementary result (Proposition 1.1) concerning invariant cohomology classes on Galois coverings in characteristic zero. The fact in question is certainly known in much greater generality (c.f. [Tohuku]), but we have been unable to find a suitable reference for the particular statement we need. Therefore, for the benefit of the reader, we give here an elementary direct argument.

Let $f : X \longrightarrow Y$ be a finite surjective mapping of complex algebraic varieties. Recall that one says that f is Galois with (finite) group G if G acts on X by automorphisms commuting with f, and if the sheaf $\mathcal{O}_Y$ of germs of functions on Y consists of the G-invariant germs of sections of $\mathcal{O}_X$. In other words, $Y = X/G$.

Proposition 1.1. Let $f : X \longrightarrow Y$ be a Galois covering of complex algebraic varieties with group G, and set $A = \Gamma(Y, \mathcal{O}_Y)$. Let L be a line bundle on Y, so that f^*L is a G-bundle on X, and consider the corresponding action of G on the A-module $H^i(X, f^*L)$. Then there is a canonical isomorphism

$$H^i(Y, L) \xrightarrow{\ \simeq\ } H^i(X, f^*L)^G,$$

where the space on the right is the submodule of invariants of the G-module $H^i(X, f^*L)$.

Proof. Since f is finite (and hence affine) we have using the projection formula a canonical identification

$$(*) \qquad H^i(X, f^*L) = H^i(Y, L \otimes f_*\mathcal{O}_X),$$

with G- module structure arising from the natural action of G on the sheaf $f_*\mathcal{O}_X$ on Y. Choose an affine open covering $\underline{U} = \{U_\alpha\}$ of Y, and let $C = C^\cdot(\underline{U}, L \otimes f_*\mathcal{O}_X)$ be the corresponding Cech complex of A-modules computing the groups on the right in (*). G acts on this complex by chain homomorphisms. Since $\mathcal{O}_Y = f_*\mathcal{O}_X{}^G$, we see that $H^i(Y, L) = H^i(C^G)$, where $C^G \subset C$ denotes the subcomplex of G-invariants. Therefore the assertion of the Proposition boils down to the statement that the canonical map $H^i(C^G) \longrightarrow H^i(C)^G$ is an isomorphism. But this is the content of the following Lemma.∎

Lemma 1.2. Let k be a field of characteristic zero, let A be a commutative k-algebra, let G be a finite group, and let $C = C^\cdot$ be a complex of $A[G]$-modules. (I.e. C is a complex of A-modules on which G acts by chain homomorphisms.) Denote by $C^G \subset C$ the subcomplex of G-invariants. Then the natural A-module homomorphism

$$H^i(C^G) \longrightarrow H^i(C)^G$$

is an isomorphism.

Proof. As A is an algebra over a field of characteristic zero, one can average over G to see that the left exact functor $M \longrightarrow M^G$ on $A[G]$-modules is right exact. Since evidently $B^i(C^G) = B^i(C)^G$ and $Z^i(C^G) = Z^i(C)^G$, taking invariants in the exact sequence $0 \longrightarrow B^i(C) \longrightarrow Z^i(C) \longrightarrow H^i(C) \longrightarrow 0$ yields the required isomorphism. ∎

We will apply Proposition 1.1 in the following relative setting:

Corollary 1.3. Let $f : X \longrightarrow Y$ be a Galois covering of complex algebraic varieties, with group G. Consider the commutative diagram

$$\begin{tikzcd} X \times S \arrow[rr, "f \times 1"] \arrow[dr, "a"'] & & Y \times S \arrow[dl, "b"] \\ & S & \end{tikzcd}$$

where S is some (complex algebraic) variety and a and b are the projections on the first factors. If L is a line bundle on $Y \times S$, then one has a canonical isomorphism

$$R^i b_*(L) \;=\; R^i a_*((f\times 1)^* L)^G ,$$

where the term on the right is the G - invariant subsheaf of the G-sheaf $R^i a_*((f\times 1)^* L)$.

Proof. The assertion is local on S, so we may suppose that $S = \mathrm{Spec}(A)$ is affine. Then $R^i b_*(L)$ is the sheaf associated to the A - module $H^i(Y\times S , L)$ and similarly for $R^i a_*((f\times 1)^* L)$. Therefore the Corollary follows by applying Proposition 1.1 to the Galois covering $f \times 1 : X\times S \longrightarrow Y\times S$. ∎

§2. The Computation

We keep notation as in the Introduction: thus C is a compact connected Riemann surface of genus $g \geq 2$, and Q is the vector bundle occurring in the statement of the Theorem.

Lemma 2.1. (Compare [Kl], p. 326].) Let $\Delta \subset C \times C$ be the diagonal, and denote by $p : C \times C \longrightarrow C$ the projection onto the first factor. Then

$$R^0 p_* \mathcal{O}_{C\times C}(\Delta) = \mathcal{O}_C \qquad \text{and} \qquad R^1 p_* \mathcal{O}_{C\times C}(\Delta) = Q .$$

Proof. This follows from the exact sequence $0 \longrightarrow \mathcal{O}_{C\times C} \longrightarrow \mathcal{O}_{C\times C}(\Delta) \longrightarrow \mathcal{O}_C \longrightarrow 0$ upon taking direct images. ∎

Consider now the universal divisor $D_m \subset C \times C_m$ of degree m, so that $D_m \simeq C \times C_{m-1}$, and let $p : C \times C_m \longrightarrow C$ be projection onto the first factor.

Proposition 2.2. One has a canonical isomorphism

$$R^k p_* (\mathcal{O}_{C\times C_m}(D_m)) = \Lambda^k Q.$$

Proof The idea is to apply Corollary 1.3. To this end, denote by $C^m = C \times \ldots \times C$ the m- fold Cartesian product of C, and write $r : C^m \longrightarrow C_m$ for the canonical map, so that r is a Galois covering, with group the symmetric group S_m on m objects. Then one has a commutative diagram

$$\begin{tikzcd} C \times C^m \arrow[rr, "1 \times r"] \arrow[dr, "q"'] & & C \times C_m \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

where p and q are projections to C. Corollary 1.3 yields first of all:

(2.3) $$R^k p_* (\mathcal{O}_{C\times C_m}(D_m)) \;=\; R^k q_*((1\times r)^* \mathcal{O}_{C\times C_m}(D_m))^{S_m} .$$

To explicate the sheaf on the right, consider the maps

$$\pi_i = 1 \times pr_i : C \times C^m \longrightarrow C \times C ,$$

pr_i being projection onto the i^{th} factor. Then

$$(2.4) \qquad (1\times r)^* \mathcal{O}_{C\times C_m}(D_m) = \bigotimes_{i=1}^{m} \pi_i^* \mathcal{O}_{C\times C}(\Delta) .$$

Set

$$R^0 = \mathcal{O}_C, \quad R^1 = Q, \quad \text{and} \quad R^j = 0 \quad \text{for } j \neq 0 \text{ or } 1,$$

so that the R^j are the sheaves appearing in Lemma 2.1. Since the direct images R^j are locally free, the Kunneth formula [EGA III.6.7.8] applies to (2.4) to give

$$(2.5) \qquad R^k q_*((1\times r)^* \mathcal{O}_{C\times C_m}(D_m)) = \bigoplus_{j_1+\ldots+j_m=k} \left(\bigotimes_{i=1}^{m} R^{j_i} \right).$$

Let F denote the direct sum of sheaves appearing on the right in (2.5). Each summand of F is naturally isomorphic to $T^k(R^1) \otimes T^{m-k}(R^0)$; let $\delta : T^k(R^1) \otimes T^{m-k}(R^0) \longrightarrow F$ denote the diagonal map. The action of S_m on F is such that $\sigma \in S_m$ carries the summand $R^{j_1} \otimes \ldots \otimes R^{j_m}$ of F to $R^{j_{\sigma(1)}} \otimes \ldots \otimes R^{j_{\sigma(m)}}$. Hence the S_m invariant subsheaf F^{S_m} of F is the image under δ of $T^k(R^1)^{S_k} \otimes T^{m-k}(R^0)^{S_{m-k}}$. Now $R^0 = \mathcal{O}_C$, so we may identify each summand of F with $T^k(R^1)$, and then $F^{S_m} = T^k(R^1)^{S_k}$. But the resulting diagonal map

$$\delta : T^k(R^1) = R^1 \otimes \ldots \otimes R^1 \longrightarrow F = R^k q_* \bigotimes_{i=1}^{m} \pi_i^* \mathcal{O}_{C\times C}(\Delta)$$

is alternating, since it is given by a sum of fibre-wise cup products. Hence

$$F^{S_m} = \Lambda^k R^1 = \Lambda^k Q,$$

proving the Proposition. ∎

Proof of the Theorem. As above, let $D_m \subset C\times C_m$ be the universal divisor of degree m, so that $D_m \simeq C\times C_{m-1}$ via the map

$$\begin{aligned} \gamma : C \times C_{m-1} &\longrightarrow C \times C_m \\ (P, D) &\longmapsto (P, D+P) . \end{aligned}$$

Let $f = f_m : C\times C_m \longrightarrow C_m$ denote projection onto the second factor. Then it is well known (c.f. [K1]) that

$$\mathcal{O}_{C_m} = f_* N$$

where $N = N_{D_m/C\times C_m}$ is the normal bundle to D_m in $C\times C_m$. Furthermore, the sheaf homomorphism du is identified with the connecting map

$$f_* N \longrightarrow R^1 f_* \mathcal{O}_{C\times C_m} = H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$$

determined by the exact sequence

$$(2.6) \qquad 0 \longrightarrow \mathcal{O}_{C\times C_m} \longrightarrow \mathcal{O}_{C\times C_m}(D_m) \longrightarrow N \longrightarrow 0 .$$

But $f \mid D_m$ is finite, and therefore

$$(2.7) \qquad H^k(C_m, \mathcal{O}_{C_m}) = H^k(D_m, N) .$$

Moreover, $H^k(\, du\,)$ is identified with the composition

(2.8) $$H^k(D_m\, , N) \longrightarrow H^{k+1}(C \times C_m\, , \mathcal{O}_{C\times C_m}) \longrightarrow H^1(C\, , \mathcal{O}_C) \otimes H^k(C_m\, , \mathcal{O}_{C_m})$$

arising from (2.6) and the Kunneth decomposition.

The next step is to push the exact sequence (2.6) down to C. To this end, we use from [K1, p. 321] the fact that

$$\gamma^* N \;=\; p^*\mathcal{O}_C \otimes f^*\mathcal{O}_{C_{m-1}}(\, D_{m-1}\,),$$

where (somewhat abusively) we are writing p and f for the projections of $C \times C_{m-1}$ onto its first and second factors. Recalling that $H^k(C_m\, , \mathcal{O}_{C_m}) = \Lambda^k H^1(C\, , \mathcal{O}_C)$, and applying Proposition 2.2, the push-forward of (2.6) becomes

(2.9) $$\begin{array}{ccccc} R^{k-1}p_*N & \longrightarrow & R^k p_* \mathcal{O}_{C\times C_m} & \longrightarrow & R^k p_* \mathcal{O}_{C\times C_m}(\, D_m\,) \\ \| & & \| & & \| \\ \mathcal{O}_C \otimes \Lambda^{k-1} Q & & \Lambda^k H^1(C\, , \mathcal{O}_C) \otimes \mathcal{O}_C & & \Lambda^k Q \end{array}$$

The reader may check that under the indicated identifications, this is nothing but the exact sequence $(*)_k$ from the Introduction. [One analyzes the map $R^k p_* \mathcal{O}_{C\times C_m} \longrightarrow R^k p_* \mathcal{O}_{C\times C_m}(\, D_m\,)$ using the argument of Proposition 2.2.] In particular, (2.9) is a short exact sequence.

As $g(C) \geq 2$, one easily verifies that $H^0(C\, , \mathcal{O}_C \otimes \Lambda^k Q) = 0$ provided that $k < g-1$ (c.f. [GL2] or [L, §1]). Therefore, by the Leray spectral sequence:

$$H^k(D_m\, , N) \;=\; H^1(C\, , \mathcal{O}_C \otimes \Lambda^{k-1} Q)$$

when $k < g-1$, which proves the first statement of the Theorem. The analysis of $H^k(\, du\,)$ follows from (2.8) and (2.9). ∎

§3. Variants

In conclusion, we state some variants of the the theorem and its corollary.

To begin with, fix an integer $m \leq g-1$, and consider the set

$$C_m \times C_{2g-2-m} \;\supset\; Z_m \;=\; \{\, (D\, , E) \mid D + E \in |K|\, \},$$

where K denotes a canonical divisor on C. There is a tautologous branched covering

$$f = f_m \;:\; Z_m \longrightarrow |K| = \mathbb{P}^{g-1}$$

which takes the pair of divisors $(D\, , E)$ to the canonical divisor $D+E$. One may think of Z_m as the scheme parametrizing all possible ways of writing a canonical divisor as a sum of two effective divisors, of degrees m and $2g-2-m$ respectively. Let $L = f^*\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ be the line bundle on Z_m defining the covering.

Proposition 3.1. Fix an integer $k \le m-1$, and assume that property (N_{k-1}) holds for the canonical bundle Ω. Then (N_k) is satisfied if and only if

$$\dim H^k(Z_m, L) = g\cdot\binom{g}{k} + \binom{g-1}{k-1}\cdot(2k+1-3g)$$

Proof. Let N denote the cokernel of the sheaf homomorphism du, so that one has an exact sequence

$$0 \longrightarrow \mathcal{O}_{C_m} \xrightarrow{du} H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{C_m} \longrightarrow N \longrightarrow 0$$

of sheaves on C_m. Then $Z_m = \mathbb{P}(N)$, and the map $f : Z_m \longrightarrow \mathbb{P}H^1(C, \mathcal{O})$ arises in the evident way from this sequence. We claim that:

$$(*) \qquad R^i\pi_*\mathcal{O}_{\mathbb{P}(N)}(1) = 0 \quad \text{for} \quad i \ge 1,$$

where $\pi : \mathbb{P}H^1(C, \mathcal{O}_C) \times C_m \longrightarrow C_m$ is the projection. In fact, write $\mathbb{P} = \mathbb{P}H^1(C, \mathcal{O}_C)$. Then $\mathbb{P}(N) \subset \mathbb{P} \times C_m$ is defined scheme-theoretically by the vanishing of the natural map

$$(**) \qquad \pi^*\mathcal{O}_{C_m} \longrightarrow f^*\mathcal{O}_{\mathbb{P}}(1).$$

Since $Z_m = \mathbb{P}(N) \longrightarrow \mathbb{P}$ is evidently finite, every component of $\mathbb{P}(N)$ has dimension $\le g-1$. Therefore (**) exhibits $\mathbb{P}(N) \subset \mathbb{P} \times C_m$ as a local complete intersection, and in particular the Koszul complex determined by (**) is exact. The assertion (*) then follows by chasing through that complex.

It follows from (*) that $H^k(Z_m, L) = H^k(C_m, N)$ for all k. By the hypothesis and the main theorem, one has an exact sequence

$$0 \longrightarrow H^k(\mathcal{O}_{C_m}) \longrightarrow H^1(\mathcal{O}_C) \otimes H^k(\mathcal{O}_{C_m}) \longrightarrow H^k(N) \longrightarrow H^{k+1}(\mathcal{O}_{C_m}) \longrightarrow H^1(\mathcal{O}_C) \otimes H^{k+1}(\mathcal{O}_{C_m}).$$

Hence again invoking the theorem one finds that (N_k) holds for Ω if and only if

$$\dim H^k(X, N) = \dim H^1(\mathcal{O}_C) \otimes \Lambda^k H^1(\mathcal{O}_C) - \dim H^k(\mathcal{O}_{C_m}).$$

Recalling that $H^k(C_m, \mathcal{O}_{C_m}) = H^1(C, \Lambda^{k-1}Q \otimes \mathcal{O}_C)$, the assertion now follows with a computation. ∎

Finally, when $m = g-1$, the theorem ties up in an amusing way with the geometry of the theta divisor on $J(C)$, and in particular with some of the ideas used by Green in his analysis [G2] of quadrics of rank four containing the canonical curve. We follow the notation of [ACGH, Chapter VI, §4]. Assume henceforth that C is non-hyperelliptic, and let $D \subset C_{g-1}$ be the locus over which the Abel-Jacobi map fails to be finite. Denote by L the pull-back to C_{g-1} the pull-back of the principal polarization on $J(C)$, so that $L = K_{C_{g-1}}$. Green noted that one can view the second derivatives $\partial^2\theta/\partial z_i\partial z_j$ of the Riemann theta function as sections of $L \mid D$, thereby defining a map

$$f : \mathrm{Sym}^2 H^1(C, \mathcal{O}_C) \longrightarrow H^0(C_{g-1}, L \otimes \mathcal{O}_D).$$

Corollary 3.2. If C is non-hyperelliptic, then f fails to be surjective if and only if C is Petri-exceptional, i.e (N_1) fails for C.

Sketch of Proof. One has an exact sequence

$$0 \longrightarrow \Theta_{C_{g-1}} \longrightarrow u^* \Theta_{J(C)} \longrightarrow I_D \otimes L \longrightarrow 0$$

where I_D denotes the ideal sheaf of D in C_{g-1}, and by the Theorem (N_1) holds for C if and only if the map

$$H^1(u^* \Theta_{J(C)}) = H^1(C, \mathcal{O}_C) \otimes H^1(C, \mathcal{O}_C) \longrightarrow H^1(C_{g-1}, I_D \otimes L)$$

is surjective. But referring to diagram (4.4) on p. 258 of [ACGH], one sees that this is equivalent to the surjectivity of f. ∎

Remark. We suspect that the Corollary generalizes as follows. Set

$$V^k = \ker\{ H^1(C, \mathcal{O}_C) \otimes \Lambda^k H^1(C, \mathcal{O}_C) \longrightarrow \Lambda^{k+1} H^1(C, \mathcal{O}_C) \}.$$

Then presumably there is a map $f_k \colon V^k \longrightarrow H^{k-1}(C_{g-1}, \mathcal{O}_D \otimes L)$, and if (N_{k-1}) holds, then (N_k) should if and only if f_k is surjective.

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