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A Barth-Type Theorem for Branched Coverings of Projective Space

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Introduction

Let X be a non-singular connected complex projective variety of dimension n. In 1970, Barth [B1] discovered that if X admits an embedding $X^n \hookrightarrow \mathbb{P}^{n+e}$ of codimension e, then the restriction mappings $H^i(\mathbb{P}^{n+e}, \mathbb{C}) \to H^i(X, \mathbb{C})$ are isomorphisms for $i \leqq n-e$. Our main result is an analogue of Barth's theorem for branched coverings of projective space:

Theorem 1. *Let $f: X^n \to \mathbb{P}^n$ be a finite mapping of degree d. Then the induced maps $f^*: H^i(\mathbb{P}^n, \mathbb{C}) \to H^i(X, \mathbb{C})$ are isomorphisms for $i \leqq n+1-d$.*

Observe that the conclusion is vacuous for $d > n+1$. On the other hand, as the degree d becomes small compared to n, one obtains progressively stronger topological obstructions to expressing a variety as a d-sheeted covering of $\mathbb{P}^n$.

The proof of the theorem relies on a basic construction which clarifies somewhat the connection between subvarieties and branched coverings. Canonically associated to a finite morphism $f: X^n \to \mathbb{P}^n$ of degree d, there exists a vector bundle $E \to \mathbb{P}^n$ of rank $d-1$ having the property that f factors through an embedding of X in the total space of E (Sect. 1). An important fact about coverings of projective space is that these bundles are always ample. This leads one to consider quite generally a smooth n-dimensional projective variety Y, an ample vector bundle $E \to Y$ of rank e, and a non-singular projective variety X of dimension n embedded in the total space of E:

$$\begin{array}{ccc} X & \hookrightarrow & E \\ & \searrow \quad \swarrow & \\ & Y. & \end{array}$$

Inspired by Hartshorne's proof [H2, H3] of the Barth theorem, we show in Sect. 2 that under these circumstances one has isomorphisms $H^i(Y, \mathbb{C}) \to H^i(X, \mathbb{C})$ for $i \leqq n-e$. This yields Theorem 1. And in fact, by taking E to be the direct sum of e copies of the hyperplane line bundle on $\mathbb{P}^n$, one also recovers Barth's theorem for embeddings $X^n \hookrightarrow \mathbb{P}^{n+e}$.

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In Sect. 3 we give two applications to low degree branched coverings of projective space by non-singular varieties. First we prove

Proposition 3.1. *If* $f: X^n \to \mathbb{P}^n$ *has degree* $\leqq n-1$, *then* f *gives rise to an isomorphism* $\operatorname{Pic}(\mathbb{P}^n) \xrightarrow{\cong} \operatorname{Pic}(X)$.

Proposition 3.1 allows us to analyze the rank two vector bundle associated to a triple covering, and we deduce

Proposition 3.2. *If* $f: X^n \to \mathbb{P}^n$ *has degree three, and if* $n \geqq 4$, *then* f *factors through an embedding of* X *in a line bundle over* $\mathbb{P}^n$.

This generalizes the familiar fact that a non-singular subvariety of projective space having degree three and dimension at least four is necessarily a hypersurface.

It was shown in [G–L], where Theorem 1 was announced, that if $f: X^n \to \mathbb{P}^n$ is a covering of degree $\leqq n$, then X is algebraically simply connected. Deligne [D] and Fulton [F] subsequently proved that in fact the topological fundamental group of X is trivial. This result, plus the analogy with Larsen's extension of the Barth theorem [L, B2], lead one to conjecture that in the situation of Theorem 1 the homomorphisms $f_*: \pi_i(X) \to \pi_i(\mathbb{P}^n)$ are bijective for $i \leqq n+1-d$. Deligne [D] has recently stated a conjecture which – at least in certain cases – would imply this homotopy version of Theorem 1.

Excellent accounts of Barth's theorem and related work may be found in Hartshorne's survey articles [H2] and [H3]. Sommese [S] emphasizes the role played by ampleness in Barth-type results. Along different lines, Berstein and Edmonds [B-E] have obtained an inequality relating the degree of a branched covering $f: X \to Y$ of topological manifolds to the lengths of the cohomology algebras of X and Y. They sketch some applications to branched coverings of $\mathbb{P}^n$ by algebraic varieties in Sect. 4 of their paper.

0. Notation and Conventions

0.1. Except when otherwise indicated, we deal with *non-singular* irreducible complex algebraic varieties. By a branched covering, we mean a finite surjective morphism.

0.2. $H^*(X)$ denotes the cohomology of X with complex coefficients.

0.3. If E is a vector bundle on X, $\mathbb{P}(E)$ denotes the bundle whose fibre over $x \in X$ is the projective space of one-dimensional *subspaces* of $E(x)$. We follow Hartshorne's definition [H1] of an ample vector bundle.

1. The Vector Bundle Associated to a Branched Covering

Consider a branched covering $f: X \to Y$ of degree d. As we are assuming that X and Y are non-singular, f is flat, and consequently the direct image $f_* \mathcal{O}_X$ is locally free of rank d on Y. The trace $\operatorname{Tr}_{X/Y}: f_* \mathcal{O}_X \to \mathcal{O}_Y$ gives rise to a splitting

$$f_* \mathcal{O}_X = \mathcal{O}_Y \oplus F,$$

where $F=\ker(\mathrm{Tr}_{X/Y})$. We shall be concerned with the rank $d-1$ vector bundle

$$E=F^{\vee}$$

on Y. We refer to E as the vector bundle associated to the covering f. Recall that as a variety, E can be identified with $\mathrm{Spec}(\mathrm{Sym}_Y(F))$.

Lemma 1.1. *The covering $f: X \to Y$ factors canonically as the composition*

$$X \hookrightarrow E \to Y,$$

where $E \to Y$ is the bundle projection, and $X \hookrightarrow E$ is a closed embedding.

Proof. The natural inclusion $F \to f_* \mathcal{O}_X$ of $\mathcal{O}_Y$-modules determines a surjection $\mathrm{Sym}_Y(F) \to f_* \mathcal{O}_X$ of $\mathcal{O}_Y$-algebras. Taking spectra, we obtain a canonically defined embedding $X \hookrightarrow E$ over Y. QED

When $f: X \to Y$ is a double covering, for example, the lemma yields the familiar representation of X as subvariety of a line bundle over Y.

A basic property of coverings of projective space is that the vector bundles obtained by this construction are ample:

Proposition 1.2. *Let E be the vector bundle on $\mathbb{P}^n$ associated to a branched covering $f: X^n \to \mathbb{P}^n$. Then $E(-1)$ is generated by its global sections. In particular, E is ample.*

Proof. It suffices to show that $E(-1)$ is 0-regular, i.e. that

$$H^i(\mathbb{P}^n, E(-i-1))=0 \quad \text{for} \quad i>0$$

(cf. [M1, Lecture 14]). It is equivalent by Serre duality to verify

$$(*) \quad H^{n-i}(\mathbb{P}^n, F(i-n))=0 \quad \text{for } i>0,$$

where as above $F=E^{\vee}$. When $i=n$, $(*)$ is clear, since

$$H^0(X, \mathcal{O}_X)=H^0(\mathbb{P}^n, f_* \mathcal{O}_X)=H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \oplus H^0(\mathbb{P}^n, F),$$

and $H^0(X, \mathcal{O}_X)=H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})=\mathbb{C}$. In the remaining cases $0<i<n$, we note similarly that

$$\begin{aligned} H^{n-i}(\mathbb{P}^n, F(i-n)) &= H^{n-i}(\mathbb{P}^n, f_* \mathcal{O}_X(i-n)) \\ &= H^{n-i}(X, f^* \mathcal{O}_{\mathbb{P}^n}(i-n)). \end{aligned}$$

But for $0<i<n$, $f^* \mathcal{O}_{\mathbb{P}^n}(i-n)$ is the dual of an ample line bundle on X, whence $H^{n-i}(X, f^* \mathcal{O}_{\mathbb{P}^n}(i-n))=0$ by the Kodaira vanishing theorem. QED.

We remark that the ampleness of the vector bundle associated to a branched covering $f: X \to Y$ has a striking geometric consequence, concerning the ramification of f. Specifically, consider the local degree

$$e_f(x)=\dim_{\mathbb{C}}(\mathcal{O}_x X / f^* \mathfrak{m}_{f(x)})$$

of f at $x \in X$, which counts the number of sheets of the covering that come together at x (cf. [M2, Appendix to Chap. 6]).

Proposition 1.3. *If the vector bundle associated to a branched covering $f:X^n \to Y^n$ of projective varieties is ample, then there exists at least one point $x \in X$ at which*

$$e_f(x) \geqq \min(\deg f, n+1).$$

So for instance if $\deg f \geqq n+1$, then $n+1$ or more branches of the covering must come together at some point of X. For coverings of $\mathbb{P}^n$, the existence of such higher ramification points was proved with Gaffney [G–L] as a consequence of the Fulton-Hansen connectedness theorem [F–H]. (The definition of $e_f(x)$ adopted in the more general setting of [G–L] reduces to the one stated above thanks to the fact that we are dealing with non-singular complex varieties.)

Sketch of Proof of (1.3). The argument given in [G–L, Sect. 2] goes over with only minor changes once we know the following:

If S is a possibly singular integral projective variety of dimension $\geqq 1$, and if $g: S \to Y$ is a finite morphism, then $Z = X \times_Y S$ is connected.

We will show that in fact $h^0(Z, \mathcal{O}_Z) = 1$. To this end, let $f': Z \to S$ denote the projection. Then

$$f'_* \mathcal{O}_Z = g^* f_* \mathcal{O}_X = g^* \mathcal{O}_Y \oplus g^* F,$$

where F is the dual of the vector bundle associated to f. Since g is finite and $F^\vee$ is ample, g^*F is the dual of an ample vector bundle on the positive-dimensional integral projective variety S. Therefore $h^0(S, g^*F) = 0$, and

$$h^0(Z, \mathcal{O}_Z) = h^0(S, f'_* \mathcal{O}_Z) = h^0(S, \mathcal{O}_S) = 1. \quad \text{QED}$$

2. A Barth-Type Theorem

Our object in this section is to prove the following theorem. Recall that we are dealing with irreducible nonsingular varieties.

Theorem 2.1. *Let Y be a projective variety of dimension n, and let $E \to Y$ be an ample vector bundle of rank e on Y. Suppose that $X \subsetneqq E$ is an n-dimensional projective variety embedded in E. Denote by f the composition $X \hookrightarrow E \to Y$. Then the induced maps*

$$f^*: H^i(Y) \to H^i(X)$$

are isomorphisms for $i \leqq n-e$.

Note that f, being affine and proper, is finite.

In view of (1.1) and (1.2), Theorem 1 stated in the introduction follows immediately. More generally, we see that if Y^n is projective, and if $f:X^n \to Y^n$ is a branched covering of degree d such that the vector bundle associated to f is ample, then the homomorphisms $f^*: H^i(Y) \to H^i(X)$ are bijective for $i \leqq n+1-d$. For example, if $f:X^n \to Y^n$ is a double cover branched along an ample divisor on Y, then $H^i(Y) \xrightarrow{\cong} H^i(X)$ for $i \leqq n-1$.

Remark 2.2. Theorem 1 is sharp "on the boundary of its applicability", i.e. there exists for every $n \geqq 1$ a covering $f:X^n \to \mathbb{P}^n$ of degree $n+1$ with $H^1(X) \neq 0$. Assuming $n \geqq 2$, for example, start with an elliptic curve $C \subsetneqq \mathbb{P}^n$ of degree $n+1$, with C not contained in any hyperplane, and consider the incidence correspondence

$$X = \{(p, H) | p \in H\} \subseteq C \times \mathbb{P}^{n*}.$$

X is a $\mathbb{P}^{n-1}$-bundle over C, whence $H^1(X) \neq 0$, and the second projection gives a covering $f: X^n \to \mathbb{P}^{n*}$ of degree $n+1$. (The reader may find it amusing to check that the vector bundle associated to this covering is isomorphic to the tangent bundle of $\mathbb{P}^n$.) Similarly, if $C \subseteq \mathbb{P}^n$ is a rational normal curve of degree n, we obtain an n-sheeted covering $f: X \to \mathbb{P}^{n*}$ with $\dim H^2(X) = 2$. On the other hand, Proposition 3.2 and Theorem 2.1 show that as one would expect, Theorem 1 is not sharp for all d and n.

Remark 2.3. It follows from Theorem 1 that if $f: X^n \to \mathbb{P}^n$ is a branched covering of degree d, and if $S, T \subseteq X$ are (possibly singular) subvarieties such that $\operatorname{codim} S + \operatorname{codim} T \leqq n+1-d$, then S meets T. (The first non-trivial case is when $d = n-1$, the assertion then being that any two divisors on X must meet.) This result remains true even if X is singular. For by [G–L, Theorem 1], there exists a subvariety $R \subseteq X$ of codimension $\leqq d-1$ such that f is one-to-one over $f(R)$. And $f(R) \cap f(S) \cap f(T)$ is non-empty for dimensional reasons.

Remark 2.4. Theorem 2.1 implies the Barth theorem for embeddings $X^n \hookrightarrow \mathbb{P}^{n+e}$. In fact, choose a linear space $L \subseteq \mathbb{P}^{n+e}$ of dimension $e-1$, with L disjoint from X, and consider the projection $(\mathbb{P}^{n+e} - L) \to \mathbb{P}^n$ centered along L. The variety $\mathbb{P}^{n+e} - L$ is isomorphic over $\mathbb{P}^n$ to the total space of $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ (e summands), and we conclude from (2.1) that $H^i(\mathbb{P}^n) \xrightarrow{\cong} H^i(X)$ for $i \leqq n-e$. But this is equivalent to Barth's assertion.

The remainder of Sect. 2 is devoted to the proof of (2.1). The argument is inspired by Hartshorne's simple proof of the Barth theorem [H2, p. 1020; H3, p. 147] and by Sommese's demonstration of a related result [S, Proposition 2.6].

We assume henceforth that $e \leqq n$. Let $\pi: \bar{E} = \mathbb{P}(E \oplus 1) \to Y$ be the projective completion of E. One has the commutative diagram

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \bar{E} \arrow[dl, "\pi"] \\ & Y, & \end{tikzcd}$$

where j denotes the composition of the given embedding $X \hookrightarrow E$ with the natural inclusion $E \subseteq \bar{E}$. Let $\xi = c_1(\mathcal{O}_{\bar{E}}(1)) \in H^2(\bar{E})$, and let $\eta_X \in H^{2e}(\bar{E})$ be the cohomology class defined by X. The class ξ represents the divisor at infinity in $\bar{E}$ [i.e. $\mathbb{P}(E) \subseteq \mathbb{P}(E \oplus 1)$], and X does not meet this divisor. Hence

(2.5) $\quad j^*(\xi) = 0.$

We claim next that $j^*(\eta_X) \in H^{2e}(X)$ is given by

(2.6) $\quad j^*(\eta_X) = (\deg f) c_e(f^* E).$

Indeed, in view of (2.5) it suffices to verify the formula

$(*)$ $\quad \eta_X = (\deg f) \sum_{i=0}^{e} c_i(\pi^* E) \xi^{e-i}.$

To this end, note that the fundamental class $[X]$ of X is homologous in E to $(\deg f)[Y]$, where $Y \subsetneqq E$ is the zero section. Hence $\eta_X = (\deg f)\eta_Y$, $\eta_Y \in H^{2e}(\bar{E})$ being the cohomology class defined by Y. Now if $Q = \pi^*(E \oplus 1)/\mathcal{O}_{\bar{E}}(-1)$ denotes the universal quotient bundle on $\bar{E}$, then one has $\eta_Y = c_e(Q)$, and $(*)$ follows.

The key to the argument is having some control over the effect on $H^*(X)$ of multiplication by $j^*(\eta_X)$. The requisite fact is provided by Sommese's formulation of a result of Bloch and Geiseker [B–G] on the top Chern class of an ample vector bundle:

(2.7) *Let F be an ample vector bundle of rank $e \leqq n$ on a (non-singular, irreducible) projective variety X of dimension n. Then multiplication by $c_e(F)$ gives surjections*

$$H^{n-e+l}(X) \twoheadrightarrow H^{n+e+l}(X)$$

for $l \geqq 0$.

See [S, Proposition 1.17] for the proof, which ultimately depends on the Hard Lefschetz theorem.

These preliminaries out of the way, we conclude the proof of Theorem 2.1. Note that it suffices to prove

$$(*) \quad \begin{array}{l} f^*: H^{n+e+l}(Y) \to H^{n+e+l}(X) \text{ is} \\ \text{surjective for } l \geqq 0. \end{array}$$

Indeed, $H^*(Y)$ injects into $H^*(X)$ for any generically finite morphism $X^n \to Y^n$, and so (2.1) is equivalent to $(*)$ by Poincaré duality.

Consider the commutative diagram

$$\begin{tikzcd} H^{n-e+l}(X) \arrow[r, "j_*"] \arrow[dr, "j^*(\eta_X)"'] & H^{n+e+l}(\bar{E}) \arrow[d, "j^*"] & H^{n+e+l}(Y) \arrow[l, "\pi^*"'] \arrow[dl, "f^*"] \\ & H^{n+e+l}(X) & \end{tikzcd}$$

where j_* is the Gysin map defined by Poincaré duality from $H_{n+e-l}(X) \to H_{n+e-l}(\bar{E})$. Since f is finite, f^*E is an ample vector bundle on X, and it follows from (2.6) and (2.7) that $H^{n-e+l}(X) \to H^{n+e+l}(X)$ is surjective. Hence so also is j^*. But $H^*(\bar{E})$ is generated over $H^*(Y)$ by $\xi \in H^2(\bar{E})$, and j^* kills ξ. The surjectivity of j^* therefore implies the surjectivity of f^*. This completes the proof.

Remark 2.8. We mention some additional results concerning the geometry of an ample vector bundle $E \to \mathbb{P}^n$ of rank e. First, if $X \subseteqq E$ is a (non-singular) projective variety of dimension a, then the maps $H^i(\mathbb{P}^n) \to H^i(X)$ are isomorphisms for $i \leqq 2a - n - e$. The proof is similar to that just given, except that formula (2.6) is replaced by the observation that the normal bundle of X in E is ample. Along somewhat different lines, the connectedness theorem of Fulton and Hansen [F–H] can be used to prove an analogous result for ample bundles on $\mathbb{P}^n$, from which one deduces the following:

If S and T are irreducible but possibly singular projective subvarieties of E, then

(i) *$S \cap T$ is connected and non-empty if* $\dim S + \dim T \geqq n + e + 1$;

(ii) *S is algebraically simply connected if*

$$2\dim S \geqq n+e+1 .$$

In particular, if $f:X^n \to \mathbb{P}^n$ is a branched covering of degree d, with X non-singular, then assertions (i) and (ii) apply with $e=d-1$ to subvarieties $S, T \subsetneqq X$. Details appear in [Lz].

Remark 2.9. It is natural to ask whether in the situation of Theorem 2.1 the relative homotopy groups $\pi_i(E,X)$ vanish for $i \leqq n-e+1$. At least when $Y=\mathbb{P}^n$, it seems reasonable to conjecture that this is so. Assertion (ii) of the previous remark, applied with $S=X$, points in this direction. Larsen's theorem [L] provides additional evidence.

3. Applications to Coverings of $\mathbb{P}^n$ of Low Degree

We give two applications of the results and techniques of the previous sections to branched coverings $f:X^n \to \mathbb{P}^n$ of low degree. We continue to assume that X is irreducible and non-singular. The first result deals with Picard groups:

Proposition 3.1. *If $f:X^n \to \mathbb{P}^n$ has degree $\leqq n-1$, then $f^*:\mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is an isomorphism.*

Proof. A well-known argument (cf. [H3, p. 150]) shows that the proposition is equivalent to the assertion that $f^*:H^2(\mathbb{P}^n,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an isomorphism. [Briefly: one looks at the exponential sequences on $\mathbb{P}^n$ and on X, noting that Theorem 1, and the Hodge decomposition yield $H^1(X,\mathcal{O}_X)=H^2(X,\mathcal{O}_X)=0$.] Theorem 1 implies that $H_2(X,\mathbb{Z})$ has rank one. On the other hand, X is algebraically simply connected ([G–L, Theorem 2]), whence $H_1(X,\mathbb{Z})=0$. It follows from the universal coefficient theorem that $H^2(X,\mathbb{Z})=\mathbb{Z}$. Finally, as f has degree $\leqq n-1<2^n$, f^* must map the generator of $H^2(\mathbb{P}^n,\mathbb{Z})$ to the generator of $H^2(X,\mathbb{Z})$. QED

As a second application, we derive a fairly explicit description of all degree three coverings $f:X^n \to \mathbb{P}^n$ with $n \geqq 4$. Specifically, we will prove

Proposition 3.2. *Let $f:X^n \to \mathbb{P}^n$ be a triple covering. Denote by b the degree of the branch divisor of f.*

(i) *If $\omega_X = f^*\mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$, then f factors through an embedding of X in a line bundle $L \to \mathbb{P}^n$, and conversely. In this case,*

$$6\deg(L)=b .$$

(ii) *The condition in* (i) *always holds if $n \geqq 4$.*

By the branch divisor of a covering $f:X^n \to \mathbb{P}^n$ we mean the push-forward to $\mathbb{P}^n$ of the ramification divisor of f.

Statement (ii) is a consequence of Proposition 3.1, so only (i) needs proof. The method is to focus on the rank two vector bundle E on $\mathbb{P}^n$ associated to f (Sect. 1). Lemmas 3.3 and 3.4 show that if $\omega_X = f^*\mathcal{O}_{\mathbb{P}^n}(k)$, then E at least has the form that it should if X is to embed in a line bundle. Finally we show that this implies that f actually admits the indicated factorization.

Lemma 3.3. *Let $L \to \mathbb{P}^n$ be an ample line bundle, and let $Z \subsetneqq L$ be a possibly singular projective variety of dimension n embedded in L. Denoting by d the degree of the natural map $g: Z \to \mathbb{P}^n$, one has*

$$g_* \mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^n} \oplus L^{-1} \oplus \ldots \oplus L^{1-d}.$$

Proof. Let $\pi: \bar{L} = \mathbb{P}(L \oplus 1) \to \mathbb{P}^n$ be the projective completion of L. Considering Z as a divisor on $\bar{L}$, and noting that Z does not meet the divisor at infinity $\mathbb{P}(L) \subsetneqq \mathbb{P}(L \oplus 1)$, we see that $\mathcal{O}_{\bar{L}}(-Z) = \mathcal{O}_{\bar{L}}(-d) \otimes \pi^* L^{-d}$. Using [H4, Exercise III.8.4] to calculate $R^1 \pi_* \mathcal{O}_{\bar{L}}(-d)$, the assertion follows from the exact sequence $0 \to \mathcal{O}_{\bar{L}}(-Z) \to \mathcal{O}_{\bar{L}} \to \mathcal{O}_Z \to 0$ upon taking direct images. QED

Lemma 3.4. *Under the assumption of* (i) *of Proposition 3.2, the vector bundle E associated to f has the form $E = L \oplus L^2$, where $6 \deg(L) = b$. Equivalently, $f_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus L^{-1} \oplus L^{-2}$.*

Proof. By duality for f, one has $f_* \omega_X = \omega_{\mathbb{P}^n} \otimes (f_* \mathcal{O}_X)^\vee$, while the hypothesis on ω_X yields $f_* \omega_X = (f_* \mathcal{O}_X)(k)$. Writing $l = k + n + 1$, we conclude the existence of an isomorphism

$$(*) \quad \mathcal{O}_{\mathbb{P}^n}(-l) \oplus E(-l) = \mathcal{O}_{\mathbb{P}^n} \oplus E^\vee.$$

Now the ramification divisor of f represents the first Chern class of $f^* \mathcal{O}_{\mathbb{P}^n}(l)$, and one deduces the relation $b = 3l$. Note that in particular, l is positive. With this in mind, it is a simple exercise to show using $(*)$ that $E = \mathcal{O}_{\mathbb{P}^n}(l/2) \oplus \mathcal{O}_{\mathbb{P}^n}(l)$. QED

Proof of 3.2. If f factors as stated, then X is a divisor on L, and hence $\omega_X = f^* \mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$. Conversely, suppose that ω_X is of this form. By (1.1) and (3.4) there is then an embedding $X \hookrightarrow L \oplus L^2$ over $\mathbb{P}^n$. Let $Z \subsetneqq L$ denote the image of X under the natural projection $\pi: L \oplus L^2 \to L$, and consider the resulting factorization of f:

$$\begin{array}{ccc} X & \hookrightarrow & L \oplus L^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \hookrightarrow & L \\ & {\scriptstyle g}\searrow \;\; \swarrow & \\ & \mathbb{P}^n & \end{array} \qquad f = g \circ p.$$

We will show that p is an isomorphism.

To this end, note first that p is birational. For if on the contrary $\deg p = 3$, then g would be an isomorphism and f would factor through an embedding of X in $\pi^{-1}(Z)$, i.e. in the line bundle $L^2 \to Z = \mathbb{P}^n$. But then using (3.3) to compute $f_* \mathcal{O}_X$, we would arrive at a contradiction to (3.4). Hence g has degree 3, and upon comparing the calculations of (3.3) and (3.4), one finds that

$$(*) \quad g_* \mathcal{O}_Z \simeq f_* \mathcal{O}_X.$$

But this implies that p is an isomorphism. In fact, let $\mathcal{F}$ be the cokernel of the natural inclusion $\mathcal{O}_Z \hookrightarrow p_* \mathcal{O}_X$. It follows from $(*)$ that $H^0(Z, \mathcal{F} \otimes g^* \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $d \gg 0$, and hence $\mathcal{F} = 0$. QED

Remark 3.5. As a special case of Proposition 3.2 [with $L=\mathcal{O}_{\mathbb{P}^n}(1)$] one recovers the well known fact that the only non-singular subvarieties of projective space having degree three and dimension at least four are hypersurfaces. For coverings $f: X^n \to \mathbb{P}^n$ of larger degree, however, the analogy with subvarieties does not hold as directly. For instance, a non-singular projective subvariety of degree five and dimension $\geqq 7$ is a hypersurface. On the other hand, one may construct in the following manner five-sheeted coverings $f: X^n \to \mathbb{P}^n$, with n arbitrarily large, that do not factor through line bundles. Let $L=\mathcal{O}_{\mathbb{P}^n}(1)$, and consider the vector bundle $\pi: E = L^2 \oplus L^3 \to \mathbb{P}^n$. Then there are canonical sections $S \in \Gamma(E, \pi^* L^2)$, $T \in \Gamma(E, \pi^* L^3)$ which serve as global coordinates on E. Choose forms $A \in \Gamma(\mathbb{P}^n, L^5)$, $B \in \Gamma(\mathbb{P}^n, L^6)$, and consider the subscheme $X \subsetneqq E$ defined by the common vanishing of the sections

$$ST + \pi^* A \in \Gamma(E, \pi^* L^5)$$

$$S^3 + T^2 + \pi^* B \in \Gamma(E, \pi^* L^6).$$

One checks that the natural map $f: X^n \to \mathbb{P}^n$ is finite of degree five. X is connected (at least when $n \geqq 2$), and for generic choices of A and B, X is non-singular. Finally, the scheme-theoretic fibre of X over a point in $V(A, B) \subsetneqq \mathbb{P}^n$ has a two-dimensional Zariski tangent space, which shows that f cannot factor through an embedding of X in a line bundle over $\mathbb{P}^n$. [Alternately, this follows by (3.3) from a computation of $f_* \mathcal{O}_X$:

$$f_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus L^{-2} \oplus L^{-3} \oplus L^{-4} \oplus L^{-6}.]$$

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References

[B1] Barth, W.: Transplanting cohomology classes in complex-projective space. Amer. J. Math. **92**, 951–967 (1970)

[B2] Barth, W.: Larsen's theorem on the homotopy groups of projective manifolds of small embedding codimension. Proc. Symp. Pure Math. **29**, 307–313 (1975)

[B–E] Berstein, I., Edmonds, A.: The degree and branch set of a branched covering. Invent. Math. **45**, 213–220 (1978)

[B–G] Bloch, S., Gieseker, D.: The positivity of the Chern classes of an ample vector bundle. Invent. Math. **12**, 112–117 (1971)

[D] Deligne, P.: Letter to W. Fulton, 18 Nov. 1979

[F] Fulton, W.: Notes on connectivity in algebraic geometry (preprint)

[F–H] Fulton, W., Hansen, J.: A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings. Ann. of Math. **110**, 159–166 (1979)

[G–L] Gaffney, T., Lazarsfeld, R.: On the ramification of branched coverings of $\mathbb{P}^n$. Invent. Math. (to appear)

[H1] Hartshorne, R.: Ample vector bundles. Publ. Math. IHES **29**, 63–94 (1966)

[H2] Hartshorne, R.: Varieties of small codimension in projective space. Bull. AMS **80**, 1017–1032 (1974)

[H3] Hartshorne, R.: Equivalence relations on algebraic cycles and subvarieties of small codimension. Proc. Symp. Pure Math. **29**, 129–164 (1975)

[H4] Hartshorne, R.: Algebraic geometry. In: Graduate Texts in Mathematics 52. Berlin, Heidelberg, New York: Springer 1977
[L] Larsen, M.: On the topology of complex projective manifolds. Invent. Math. **19**, 251–260 (1973)
[Lz] Lazarsfeld, R.: Thesis, Brown University 1980
[M1] Mumford, D.: Lectures on curves on an algebraic surface. Ann. of Math. Stud. **59** (1966)
[M2] Mumford, D.: Algebraic geometry. I. complex projective varieties. In: Grundlehren der mathematischen Wissenschaften, Vol. 221, Berlin, Heidelberg, New York: Springer 1976
[S] Sommese, A.: Submanifolds of abelian varieties. Math. Ann. **233**, 229–256 (1978)

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Note added in proof. Using results of Goresky-MacPherson and Deligne, we have proved the homotopy analogue of Theorem 1. The argument appears in [Lz], and will be published elsewhere.