A Simple Proof of Petri's Theorem on Canonical Curves

by

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Introduction

Let X be a smooth irreducible complex projective curve of genus g, which we assume throughout is non-hyperelliptic. Then the canonical bundle Ω on X defines an embedding

$$\phi_{\Omega} : X \longrightarrow \mathbb{P}^{g-1},$$

and a famous theorem of Petri [Ptri] states that the homogeneous ideal $I_{X/\mathbb{P}^{g-1}}$ of X in $\mathbb{P}^{g-1}$ is generated by quadrics unless X is trigonal or a smooth plane quintic. Petri's proof — which has been given modern expositions in [StD], [Mfd2] and [ACGH] — involves explicitly writing down all the quadrits through X. The argument, although elementary, is long and (as Mumford puts it) "unavoidably a bit messy". A cleaner and more conceptual proof was recently given by the first author in [Grn], where Petri's result is recovered as a special case of the "$K_{p,1}$ theorem". In both proofs the exceptional curves are characterized by the fact that they lie on surfaces of minimal degree in $\mathbb{P}^{g-1}$.

Our purpose here is to give a new proof of Petri's theorem, not involving surfaces of minimal degree. While somewhat ad hoc, the argument is simple and -- once some elementary Koszul-theoretic facts and secant plane constructions have been reviewed — very quick. The idea is that to determine whether or not the canonical image of X is cut out by quadrics, it is enough to compute the number of sections of a certain vector bundle on X. By taking a suitable filtration of the bundle in question, one arrives at an approach in which the exceptional curves are recognized by the absence of certain base-point free pencils

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of degree $g-1$. Specifically, we will prove the following

THEOREM. Assume that $g \geq 5$, and that X carries a line bundle A of degree $g-1$, with $h^0(A) = 2$, such that both

$$A \quad \text{and} \quad \Omega \otimes A^*$$

are generated by their global sections. Then the homogeneous ideal of X in its canonical embedding is generated by forms of degree two.

It is a more or less standard consequence of the Mumford-Martens theorems that any non-trigonal curve other than a plane quintic satisfies the hypotheses of the theorem, and so we recover Petri's result.

The cohomological interpretation of Petri's theorem is given in §1. In an attempt to keep the exposition self-contained, we prove what we need here from scratch. We indicate in §2 how secant planes to a canonical curve may be used to construct filtrations of the vector bundle that governs the curve's syzygies. By way of warming up for Petri, we use the simplest of these filtrations to deduce Noether's theorem on the projective normality of canonical curves. The proof of the theorem occupies §3. Finally, in §4 we outline how the theorem implies Petri's statement. We work throughout over $\mathbb{C}$, and if V is a vector space we denote by $\mathbb{P}(V)$ the projective space of one-dimensional quotients of V.

§1. Cohomological Interpretation of Petri's Theorem.

Our goal in this section is to show how Petri's theorem reduces to a purely cohomological statement (Corollary 1.7). This reduction is in fact an immediate consequence of general Koszul-theoretic results (see Remark 1.4 below). The arguments that follow are intended for the benefit of the reader not versed in such matters: we give an elementary derivation of the facts we use, without however attempting to put them into a more natural general context (for which c.f. [Grn]).

We start with some notation. Let X be a smooth irreducible projective curve, and let L be an ample line bundle on X, generated by its global sections. Thus L defines a morphism

$$\Phi_L : X \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}^r,$$

where $r = h^0(L) - 1$. Set

$$M_L = \Phi_L^*(\Omega^1_{\mathbb{P}^r}(1)),$$

so that M_L fits into an exact sequence

$$(1.1) \qquad 0 \longrightarrow M_L \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

of vector bundles on X, obtained by pulling back the Euler sequence on $\mathbb{P}^r$. Taking wedge products in (1.1) and twisting by L^{k-1}, one obtains for each $k \in \mathbb{Z}$ an exact sequence

(1.2)

$$0 \longrightarrow \Lambda^2 M_L \otimes L^{k-1} \longrightarrow \Lambda^2 H^0(L) \otimes_{\mathbb{C}} L^{k-1} \longrightarrow M_L \otimes L^k \longrightarrow 0 .$$

<u>Lemma 1.3</u>. <u>Assume that</u> L <u>is normally generated, i.e. that the natural homomorphisms</u>

$$\rho_k : S^k H^0(L) \longrightarrow H^0(L^k)$$

<u>are surjective for all</u> $k \geq 0$ <u>(so that in particular</u> Φ_L <u>is an embedding). Let</u> k_0 <u>be an integer such that the maps</u>

$$\sigma_k : \Lambda^2 H^0(L) \otimes H^0(L^{k-1}) \longrightarrow H^0(M_L \otimes L^k)$$

<u>determined by (1.2) are surjective for all</u> $k \geq k_0$. <u>Then every minimal generator of the homogeneous ideal of</u> X <u>in</u> $\mathbb{P}^r$ <u>has degree</u> k_0 <u>or less.</u>

<u>Proof</u>. Letting $I_k = \ker(\rho_k)$ denote the degree k piece of the homogeneous ideal of X, one has the exact commutative diagram shown at the top of the following page. The vertical maps are defined in the evident way by multiplication, while R_k and P_k are the kernels of μ_k and ν_k respectively; surjectivity on the right follows from the normal generation of L. The statement of the Lemma is equivalent to the assertion that the maps $H^0(L) \otimes I_k \longrightarrow I_{k+1}$ are surjective for $k \geq k_0$. By the snake lemma, it is in turn equivalent to verify that the indicated homomorphism $\alpha_k : R_k \longrightarrow P_k$ is surjective.

To this end, note first that the homomorphism

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & R_k \arrow[r, "\alpha_k"] \arrow[d] & P_k \arrow[d] & \\
0 \arrow[r] & H^0(L) \otimes I_k \arrow[rr] \arrow[d] & & H^0(L) \otimes S^k H^0(L) \arrow[r] \arrow[d, "\mu_k"] & H^0(L) \otimes H^0(L^k) \arrow[r] \arrow[d, "\nu_k"] & 0 \\
0 \arrow[r] & I_{k+1} \arrow[rr] & & S^{k+1} H^0(L) \arrow[r, "\rho_{k+1}"] \arrow[d] & H^0(L^{k+1}) \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd}
$$

Diagram to accompany proof of Lemma 1.3

$$\beta_k : \Lambda^2 H^0(L) \otimes S^{k-1} H^0(L) \longrightarrow H^0(L) \otimes S^k H^0(L)$$

determined by

$$(v_1 \wedge v_2) \otimes \alpha \longmapsto v_1 \otimes (v_2 \cdot \alpha) - v_2 \otimes (v_1 \cdot \alpha)$$

evidently maps to the subspace $R_k = \ker(\mu_k) \subseteq H^0(L) \otimes S^k H^0(L)$ (in fact β_k surjects onto R_k). On the other hand, twisting (1.1) by L^k and taking global sections, one sees that $P_k = H^0(M_L \otimes L^k)$. Furthermore, one has the commutative diagram

$$
\begin{tikzcd}
\Lambda^2 H^0(L) \otimes S^{k-1} H^0(L) \arrow[r, "\beta_k"] \arrow[d, "1 \otimes \rho_{k-1}"] & R_k \arrow[d, "\alpha_k"] \\
\Lambda^2 H^0(L) \otimes H^0(L^{k-1}) \arrow[r, "\sigma_k"'] & H^0(M_L \otimes L^k) = P_k .
\end{tikzcd}
$$

But the vertical map on the left is surjective thanks to the normal generation of L. Thus the surjectivity of σ_k implies (in fact is equivalent to) the surjectivity of α_k , and this proves the lemma.

Remark 1.4. Lemma 1.3 is a special case of a general theorem to the effect that syzygies can be computed as Koszul cohomology groups (c.f. [Grn]). Always assuming that L is normally generated, the result implies in the case at hand that the number of minimal generators in degree $k+1$ of the homogeneous ideal of X is given by the dimension of $\ker \delta_{k+1} / \operatorname{im} \gamma_k$, where δ_{k+1} and γ_k are the maps appearing in the Koszul complex

$$\Lambda^2 H^0(L) \otimes H^0(L^{k-1}) \xrightarrow{\gamma_k} H^0(L) \otimes H^0(L^k) \xrightarrow{\delta_{k+1}} H^0(L^{k+1}) \quad .$$

But evidently

$$\ker \delta_{k+1} \; / \; \operatorname{im} \gamma_k = \operatorname{coker} \sigma_k \quad ,$$

and so one recovers the lemma. (It also follows from the proof of (1.3) that $\dim(\operatorname{coker} \sigma_k)$ indeed computes the number of degree $k+1$ generators of I_X. Note however that the lemma as stated remains valid if one assumes only that ρ_k is surjective for $k \geq k_0 - 1$.)

Remark 1.5. The statement and proof of Lemma 1.3, as well as the facts quoted in (1.4), hold without change for a projective variety X of any dimension defined over an algebraically closed field of arbitrary characteristic.

For the application to Petri's theorem, it will be convenient to rephrase the hypotheses of the lemma. To this end, let Q_Ω denote the dual of the rank $g-1$ vector bundle M_Ω:

$$Q_\Omega = M_\Omega^* \, .$$

Then one obtains from (1.2) exact sequences

(1.6)

$$0 \longrightarrow Q_\Omega \otimes \Omega^{-\ell-1} \longrightarrow \Lambda^2 H^0(\Omega)^* \otimes_{\mathbb{C}} \Omega^{-\ell} \longrightarrow \Lambda^2 Q_\Omega \otimes \Omega^{-\ell} \longrightarrow 0 \quad .$$

Corollary 1.7. Assume that X is non-hyperelliptic, and that $H^0(\Lambda^2 Q_\Omega \otimes \Omega^{-\ell}) = 0$ for $\ell \geq 1$. Suppose furthermore that the map

(1.8) $$\Lambda^2 H^0(\Omega)^* \longrightarrow H^0(\Lambda^2 Q_\Omega)$$

deduced from (1.6) is surjective. Then the homogeneous ideal of X in its canonical embedding is generated by quadrics.

Remark 1.9. It will emerge from (2.4) that the vanishings in the hypothesis are automatic, as is the injectivity of (1.8). Hence in order to apply the corollary, it is enough to show that

$$\dim H^0(\Lambda^2 Q_\Omega) = \dim \Lambda^2 H^0(\Omega)^* \quad [= \tbinom{g}{2}] .$$

Proof of Corollary 1.7. Since X is non-hyperelliptic, Ω is normally generated thanks to Noether's theorem, and thus Lemma 1.3 applies. The surjectivity of the maps σ_k in the hypotheses of (1.3) is equivalent to the injectivity of the homomorphisms

$$\tau_k : H^1(\Lambda^2 M_\Omega \otimes \Omega^{k-1}) \longrightarrow \Lambda^2 H^0(\Omega) \otimes H^1(\Omega^{k-1})$$

coming from (1.2). But by Serre duality τ_k may be identified with the transpose of the maps

$$\Lambda^2 H^0(\Omega)^* \otimes H^0(\Omega^{2-k}) \longrightarrow H^0(\Lambda^2 Q_\Omega \otimes \Omega^{2-k})$$

determined by (1.6) with $\ell = k-2$. And the surjectivity of these homomorphisms for $k \geq 2$ follows immediately from the hypotheses of the Corollary.

§2. Filtrations of M_Ω and a Proof of Noether's Theorem.

We deal in this section with a non-hyperelliptic curve X and its canonical embedding

$$X \subseteq \mathbb{P}(H^0(\Omega)) = \mathbb{P}^{g-1} .$$

As noted in Remark 1.9, in order to prove that X is cut out by quadrics, the essential point is to compute $h^0(\Lambda^2 Q_\Omega)$. To this end the strategy will be to construct a suitable filtration of Q_Ω — or equivalently of M_Ω — with line bundle quotients. Such filtrations arise most simply from secant planes to the canonical curve, and we give in this section some general remarks on this construction. These are applied to deduce the projective normality of non-hyperelliptic canonical curves by an argument that serves as a model for our proof of Petri's theorem. The material in this section is largely motivational; the only things really essential for the proof of the theorem in §3 are the definitions and notation summarized in diagram (2.1).

Let D be a non-zero effective divisor on X such that the line bundle $\Omega(-D)$ is generated by its global sections, and such that

$$s \underset{\text{def}}{=\!=} h^0(\Omega(-D)) \geq 2 .$$

Thus D spans a $(g-1-s)$- plane $\Lambda_D \subset \mathbb{P}^{g-1}$. Intrinsically, Λ_D is the subspace $\mathbb{P}(W_D) \subset \mathbb{P}(H^0(\Omega))$, where

$$W_D = H^0(\Omega) / H^0(\Omega(-D)) .$$

Since D is naturally a subscheme of $\mathbb{P}(W_D)$, there is a canonical surjection

$$u_D : W_D \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \Omega \otimes \mathcal{O}_D$$

of sheaves on X. Concretely, if D is reduced — which is the only case we use — to describe u_D it is essentially equivalent to specify for each $x \in D$ a non-zero linear functional $u_D(x) : W_D \longrightarrow \mathbb{C}$ defined up to scalars; one takes $u_D(x)$ to be the quotient corresponding to the point $x \in \mathbb{P}(W_D)$. Set

$$\Sigma_D = \ker(u_D) ,$$

so that Σ_D is a vector bundle on X of rank $g-s$. All these pieces are tied together by the commutative diagram (2.1) of exact sequences on X shown at the top of the next page. (Note that one could have used the top two rows of (2.1) to define u_D and Σ_D.) Our filtations of M_Ω arise by filtering — or even decomposing — Σ_D. We refer to Remark 2.6 below for a statement of the general construction. For the moment we will content ourselves with illustrating how the procedure works in the simplest case.

Example 2.2. (Compare [GLP], end of §1) Take D to be the sum of $g-2$ general points on X :

$$D = x_1 + x_2 + \ldots + x_{g-2} ,$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_{\Omega(-D)} \arrow[r] \arrow[d] & H^0(\Omega(-D)) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d] & \Omega(-D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{\Omega} \arrow[r] \arrow[d] & H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d] & \Omega \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma_D \arrow[r] \arrow[d] & W_D \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, "u_D"] \arrow[d] & \Omega \otimes \mathcal{O}_D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram (2.1)

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where the x_i are distinct and linearly independent in $\mathbb{P}^{g-1}$, and $\Lambda_D \cap X = D$ (as schemes). Then $h^0(\Omega(-D)) = 2$ and hence $M_{\Omega(-D)} = \Omega^*(D)$. Furthermore, we claim that

$$(*) \qquad \Sigma_D \simeq \bigoplus_{i=1}^{g-2} \mathcal{O}_X(-x_i) .$$

In fact, let W_i $(1 \leq i \leq g-2)$ be the one-dimensional quotient of W_D defined by the point $x_i \in \mathbb{P}(W_D)$. Since the x_i are linearly independent, we may identify W_D with the direct sum of the W_i. Then u_D decomposes as the direct sum of the natural maps $u_i : W_i \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \Omega \otimes \mathcal{O}_{\{x_i\}}$. But $\ker(u_i) \simeq \mathcal{O}_X(-x_i)$, and (*) follows. The left-hand column of (2.1) thus gives an exact sequence

$$(2.3) \quad 0 \longrightarrow \Omega^*(x_1+\ldots+x_{g-2}) \longrightarrow M_\Omega \longrightarrow \bigoplus_{i=1}^{g-2} \mathcal{O}_X(-x_i) \longrightarrow 0 .$$

In particular M_Ω has a filtration whose quotients are the line bundles appearing in (2.3), and in practice we do not use the more precise information embodied by this sequence.

<u>Corollary 2.4</u>. <u>One has</u>

$$\text{(a)} \quad H^0(Q_\Omega \otimes \Omega^{-\ell}) = 0 \quad \underline{\text{for all}} \quad \ell \geq 1$$

<u>and</u>

$$\text{(b)} \quad H^0(\Lambda^2 Q_\Omega \otimes \Omega^{-\ell}) = 0 \quad \underline{\text{for all}} \quad \ell \geq 1 .$$

<u>Proof</u>. Both assertions follow immediately from (2.3).

Application 2.5: Noether's Theorem. The exact sequence (2.3) -- which was derived only under the assumption that Ω is very ample -- leads to a simple proof of the projective normality of non-hyperelliptic canonical curves. Specifically, as in the proof of Corollary 1.7, note first that Ω is normally generated if and only if the maps

$$H^1(M_\Omega \otimes \Omega^k) \longrightarrow H^0(\Omega) \otimes H^1(\Omega^k)$$

defined by twisting (1.1) are injective for all $k \geq 1$. In view of Corollary 2.4 (a) it is equivalent by duality to show that the necessarily injective homomorphism

$$H^0(\Omega)^* \longrightarrow H^0(Q_\Omega)$$

is surjective. But from (2.3) we get

$$\begin{aligned} h^0(Q_\Omega) &\leq h^0(\Omega(-x_1-\dots-x_{g-2})) + \sum_{i=1}^{g-2} h^0(\mathcal{O}_X(x_i)) \\ &= 2 + (g-2) \\ &= h^0(\Omega) , \end{aligned}$$

which proves Noether's theorem. We remark that this and other classical results on projective normality have been generalized in [GL] by taking into account the intrinsic geometry of the curve X .

Remark 2.6. The filtration obtained in Example 2.2 is a special case of a general construction. Specifically, with notation as above consider a flag of linear spaces

$$\Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_{g-1-s} = \Lambda_D$$

with $\dim \Lambda_i = i$, and let D_i be the divisor on X defined as the (scheme-theoretic) intersection

$$D_i = X \cap \Lambda_i \qquad (0 \leq i \leq g-1-s).$$

Set $E_0 = D_0$, and for $i > 0$ let E_i denote the (possibly zero) effective divisor on X determined by the relation

$$D_i = D_{i-1} + E_i .$$

Then there is a decreasing filtration

$$\Sigma_D = F_0 \supset F_1 \supset \quad \cdots \quad \supset F_{g-1-s} \supset F_{g-s} = 0$$

of Σ_D by vector bundles such that

$$F_i \,/\, F_{i+1} \cong \mathcal{O}_X(\, -E_i \,) \,.$$

We leave the proof to the interested reader; it seems easiest to treat the one case we need in §3 directly.

Remark 2.7. Identical constructions can of course be made with Ω replaced by any very ample line bundle L .

§3. Proof of the Theorem.

We suppose now that X is a non-hyperelliptic curve of genus $g \geq 5$ satisfying the hypotheses of the theorem stated in the Introduction. Note to begin with that Corollary 2.4(b) gives the vanishings required in (1.7), while statement (a) of (2.4) shows that the homomorphism (1.8) is injective. Hence (1.7) will apply to show that $I_{X/\mathbb{P}^{g-1}}$ is generated by quadrics as soon as we verify that

$$h^0(\, \Lambda^2 Q_\Omega \,) \;\leq\; \binom{g}{2} \,.$$

Let $A \in W^1_{g-1}(X)$ be the line bundle given in the hypotheses, and let D be the divisor of a general section of A . Since A is generated by its global sections, and since we are in characteristic zero, we may first of all assume that D consists of $g-1$ distinct points:

$$D = x_1 + x_2 + \ldots + x_{g-1} \,.$$

Furthermore, no effective divisor properly contained in D can move in in a non-trivial linear series, for otherwise $|D|$ would either have base-points or dimension at least two. The picture in canonical space $\mathbb{P}^{g-1} = \mathbb{P}(H^0(\Omega))$, then, is that D spans a $(g-3)$-plane Λ_D , whereas any proper subset of the x_i are linearly independent in $\mathbb{P}^{g-1}$. [Recall: an effective divisor of degree k spans a $(k-r-1)$-plane in $\mathbb{P}^{g-1}$ if and only if it moves in a linear system of dimension r .]

On the other hand, $\Omega(-D) = \Omega \otimes A^*$ is generated by its global sections, so the discussion in §2 applies and one has the exact sequence

$$(3.1) \qquad 0 \longrightarrow M_{\Omega(-D)} \longrightarrow M_{\Omega} \longrightarrow \Sigma_D \longrightarrow 0$$

from the left-hand column of (2.1). Note that

$$M_{\Omega(-D)} = \Omega^*(D)$$

thanks to the fact that $h^0(\Omega(-D)) = 2$.

We claim next that Σ_D fits into an exact sequence

$$(3.2) \qquad 0 \longrightarrow \mathcal{O}_X(-x_{g-2}-x_{g-1}) \longrightarrow \Sigma_D \longrightarrow \bigoplus_{i=1}^{g-3} \mathcal{O}_X(-x_i) \longrightarrow 0$$

(compare Remark 2.6). To see this, consider the divisors

$$D' = x_1 + \dots + x_{g-3} \quad \text{and} \quad E = x_{g-2} + x_{g-1} \quad .$$

Then $\Omega(-D')$ is generated by its global sections: the only potential base-points are x_{g-2} and x_{g-1}, and if either actually occured then $g-2$ of the $\{x_i\}$ would lie in the $(g-4)$-plane $\Lambda_{D'}$ spanned by D' . Thus the discussion in §2 also applies to D' , and with notation as introduced there one constructs in the evident way the commutative diagram of exact sequences:

$$(3.3) \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{v_E} & \Omega \otimes \mathcal{O}_E & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_D & \longrightarrow & W_D \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{u_D} & \Omega \otimes \mathcal{O}_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_{D'} & \longrightarrow & W_{D'} \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{u_{D'}} & \Omega \otimes \mathcal{O}_{D'} & \longrightarrow & 0 \, , \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $V = H^0(\Omega(-D')) \,/\, H^0(\Omega(-D))$, and where v_E is the homomorphism induced from the natural map

$$\tilde{v}_E : H^0(\Omega(-D')) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \Omega \otimes \mathcal{O}_E$$

obtained by evaluating a section of $\Omega(-D')$ on E . But if

$s \in H^0(\Omega(-D'))$ is a section which doesn't vanish on D, then s can't vanish at <u>either</u> of the points x_{g-2} or x_{g-1}. Thus $\tilde{v}_E$ -- and hence also v_E — are surjective maps of sheaves. Since V is one dimensional, it follows that $\ker(v_E) \simeq \mathcal{O}_X(-E)$, and consequently (3.3) gives the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-x_{g-2}-x_{g-1}) \longrightarrow \Sigma_D \longrightarrow \Sigma_{D'} \longrightarrow 0 \quad .$$

On the other hand, D' consists of $g-3$ linearly independent points spanning the $(g-4)$-plane $\Lambda_{D'}$, and so just as in Example 2.2 one has $\Sigma_{D'} \simeq \bigoplus_{i=1}^{g-3} \mathcal{O}_X(-x_i)$. This proves (3.2).

But now the theorem follows at once. In fact, from (3.1) we get an exact sequence

$$(3.4) \qquad 0 \longrightarrow \Lambda^2(\Sigma_D^*) \longrightarrow \Lambda^2 Q_\Omega \longrightarrow \Sigma_D^* \otimes \Omega(-D) \longrightarrow 0 ,$$

while (3.2) yields

(3.5)

$$0 \longrightarrow \Lambda^2\Big(\bigoplus_{i=1}^{g-3} \mathcal{O}_X(x_i)\Big) \longrightarrow \Lambda^2 \Sigma_D^* \longrightarrow \bigoplus_{i=1}^{g-3} \mathcal{O}_X(x_i+x_{g-2}+x_{g-1}) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{i=1}^{g-3} \Omega(-D+x_i) \longrightarrow \Sigma_D^* \otimes \Omega(-D) \longrightarrow \Omega(-D+x_{g-2}+x_{g-1}) \longrightarrow 0 .$$

Since $g \geq 5$, all of the divisors appearing in (3.5) are properly contained in D, and consequently each of the corresponding line bundles has only one section. Thus

$$h^0(\Lambda^2 \Sigma_D^*) \leq \binom{g-3}{2} + (g-3).$$

On the other hand, $h^0(\Omega(-D+x_i)) = 2$ for all i, whereas $h^0(\Omega(-D+x_{g-2}+x_{g-1})) = h^0(\Omega(-D')) = 3$, and hence

$$h^0(\Sigma_D^* \otimes \Omega(-D)) \leq 2\cdot(g-3) + 3 .$$

All told, thanks to (3.4) one has

$$h^0(\Lambda^2 Q_\Omega) \leq \binom{g-3}{2} + 3(g-3) + 3 = \binom{g}{2} ,$$

and we are done.

§4. The Exceptional Curves.

Finally, we recall the proof of the following

Proposition 4.1. *If X is a non-hyperelliptic curve of genus $g \geq 5$ which fails to satisfy the hypotheses of the Theorem stated in the Introduction, then X is either trigonal or a smooth plane quintic.*

Proof. (Compare [ACGH, p. 373].) Suppose that $W \subsetneq W^1_{g-1}(X)$ is an irreducible component such that a general point of W corresponds to a line bundle which fails to be generated by its global sections. Denote by k (≥ 1) the degree of the corresponding divisor of fixed points. Since $\dim W = g-4$, there exists a component

$$W' \subsetneq W^1_{g-1-k}(X)$$

with $\dim W' \geq g-4-k$. But a theorem of Mumford [Mfdl] states that if $W^1_d(X)$ has an irreducible component of dimension $d-3$ for some $3 \leq d \leq g-2$, then X is either trigonal, bielliptic or a smooth plane quintic (see [ACGH, Chapter IV §5] for an exposition).

It remains to treat the case of bielliptic curves. So assume that X admits a degree two mapping

$$\pi : X \longrightarrow E,$$

with E elliptic. Then Mumford's proof shows that an irreducible component $W \subsetneq W^1_{g-1}(X)$ as above exists if and only if $g \geq 6$, in which case there is only one such component, parametrizing line bundles of the form

$$\pi^* B \otimes \mathcal{O}_X(x_1+\ldots+x_{g-5}), \text{ with } B \in W^1_2(E), \ x_i \in X.$$

Recalling that $\Omega = \pi^* N$ for some line bundle N of degree $g-1$ on E [viz. $N = \det(\pi_*\Omega)$], it follows that W is mapped to itself by the involution of $\mathrm{Jac}_{g-1}(X)$ taking A to $\Omega \otimes A^*$. On the other hand, a pleasant geometric argument of Shokurov [Shkrv, Prop. 2.5.2] shows that W is not the only irreducible component of $W^1_{g-1}(X)$. (Shokurov assumes that $g \geq 7$, but his proof works just as well when $g = 6$. An alternative enumerative argument is suggested in [ACGH, p. 373].) Hence there exists a line bundle A on X satisfying the hypotheses of the theorem, and this completes the proof.

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