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On the projective normality of complete linear series on an algebraic curve

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Introduction

Let X be a smooth irreducible complex projective curve of genus $g \geqq 2$, and let L be an ample line bundle on X, generated by its global sections. Then L defines a morphism

$$\phi_L\colon X \to \mathbb{P}(H^0(L)) = \mathbb{P}^r,$$

and following Mumford [18] one says that L is *normally generated* if L is very ample and if ϕ_L embeds X as a projectively normal curve. Equivalently, L is normally generated if and only if the natural maps $S^m H^0(L) \to H^0(L^m)$ are surjective for all $m \geqq 0$. Several classical results address the question of when a given line bundle is normally generated. Most notably, a well-known theorem of Noether (c.f. [18] or [16]) asserts that the canonical bundle defines a projectively normal embedding unless X is hyperelliptic, and a result of Castelnuovo [3], Mattuck [16] and Mumford [18] states that any line bundle of degree at least $2g+1$ is normally generated. More recently, Lange and Martens [10] have shown that a general line bundle of degree $2g$ on a non-hyperelliptic curve is normally generated, and Arbarello, Cornalba, Griffiths and Harris [1]

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have proven that if X is a sufficiently general curve of genus g, then a general line bundle of degree $[\frac{3}{2}g+2]$ or greater defines a projectively normal embedding. The diversity of these results leads one to ask if one can obtain a unified and strengthened statement by taking into account in some quantitative way the intrinsic geometry of the curve. Our first purpose here is to show that this is indeed the case, the invariant in question being the Clifford index of X.

Recall that the Clifford index of a line bundle A on X is defined by

$$\operatorname{Cliff}(A)=\deg(A)-2\cdot r(A),$$

where $r(A)=h^0(A)-1$. The Clifford index of X itself is taken to be

$$\operatorname{Cliff}(X)=\min\{\operatorname{Cliff}(A)\,|\,h^0(A)\geqq 2,\ h^1(A)\geqq 2\}$$

(c.f. [15, 13, 14], and (0.3) below). This gives a rough indication of how special X is in the sense of moduli. Thus Clifford's theorem says that $\operatorname{Cliff}(X)\geqq 0$ with equality if and only if X is hyperelliptic, and similarly $\operatorname{Cliff}(X)=1$ if and only if X is either trigonal or a smooth plane quintic. At the other extreme, if X is a general curve of genus g then $\operatorname{Cliff}(X)=\left[\frac{g-1}{2}\right]$, and in any event $\operatorname{Cliff}(X)\leqq\left[\frac{g-1}{2}\right]$.

Our first result, which applies when $h^1(L)\leqq 1$, generalizes the various theorems stated above:

Theorem 1. *Let L be a very ample line bundle on X, with*

$$\deg(L)\ \geqq\ 2g+1-2\cdot h^1(L)-\operatorname{Cliff}(X)$$

(and hence $h^1(L)\leqq 1$). Then L is normally generated.

To deduce Noether's theorem, for instance, one uses the fact that X is non-hyperelliptic if and only if $\operatorname{Cliff}(X)\geqq 1$. Or again, if X is a general curve then $\operatorname{Cliff}(X)=\left[\frac{g-1}{2}\right]$, and so one recovers the result of Arbarello et al.

The inequality above is in general the best possible. In fact, let us say that a very ample line bundle L on X is *extremal* if

$$\deg(L)=2g-2\cdot h^1(L)-\operatorname{Cliff}(X),$$

and if L fails to be normally generated. To get a picture of the situations in which Theorem 1 is optimal, and in the hope of finding further geometry in these questions, it is natural to look for examples of curves having a given Clifford index, but arbitrarily large genus, on which there exist extremal line bundles. We classify all such infinite families, and at least for extremal bundles with $h^1=1$ the conclusion seems somewhat amusing:

Theorem 2. *There exists an explicit constant $N(e)$ such that if X is a curve of Clifford index e and genus $g>N(e)$, then:*

(a) *X always carries an extremal line bundle L with $h^1(L)=0$, but never one with $h^1(L)\geqq 2$;*

(b) *X carries an extremal line bundle L with $h^1(L)=1$ if and only if $e=2f\geqq 4$ is even, and X is a two-sheeted branched covering*

$$\pi\colon X\to Y\subseteq \mathbb{P}^2$$

of a smooth plane curve Y of degree $f+2$.

In fact, although we shall not go into all the details here, one can give a quite precise description of the extremal bundles.

Theorems 1 and 2 are consequences of a general result on the projective normality of very ample complete linear series of degree roughly $\frac{3}{2}g$ or greater. The theme is that the failure of normal generation is accounted for by the existence of special configurations of points on $\phi_L(X)\subseteq \mathbb{P}^r$. Specifically, consider a very ample line bundle L on X with $\deg(L)=2g+1-k$. Assume that $2k+1\leqq g$ if $h^1(L)=0$, or that $2k-3\leqq g$ if $h^1(L)\neq 0$, and consider the embedding

$$X\subseteq \mathbb{P}(H^0(L))=\mathbb{P}^r$$

defined by L.

Theorem 3. *Under the hypotheses just stated, L fails to be normally generated if and only if there exists an integer $1\leqq n\leqq r-2$, and an effective divisor D on X of degree at least $2n+2$ such that*

(a) *$H^1(X, L^2(-D))=0$ and*

(b) *D spans an n-plane $\Lambda\subseteq \mathbb{P}^r$ in which D fails to impose independent conditions on quadrics.*

Concerning assertion (b), we mean that $H^1(\Lambda, I_{D/\Lambda}(2))\neq 0$, where $I_{D/\Lambda}$ is the ideal sheaf of D in Λ. Note that when $n=1$ this condition is automatic, but for $n\geqq 2$ it is not enough that X simply have a $(2n+2)$-secant n-plane. Theorem 1 follows immediately by computing the Clifford index of the linear series cut out on X by hyperplanes through Λ. As for Theorem 2, the essential point is to combine a slight strengthening of the result just stated with Castelnuovo's bound on the genus of a space curve; our use of Castelnuovo's bound here was inspired by the techniques of Martens in [13]. The proof of Theorem 3, which occupies § 1, is simple but perhaps a bit surprising. The idea is to interpret the failure of normal generation in terms of extensions of line bundles. One arrives at a rank two vector bundle E on X, with determinant Ω^1_X, having a large number of sections. The required secant plane then pops out from the existence of a line sub-bundle of E of suitably high degree.

It seems very likely that Theorem 3 and its corollary Theorem 1 should be the first cases of a much more general picture of how the geometry of X influences the whole chain of syzygies associated to a complete linear series. For instance, we conjecture that if L is a very ample line bundle with $\deg(L)\geqq 2g+2-2\cdot h^1(L)-\mathrm{Cliff}(X)$, then the homogeneous ideal of X in $\mathbb{P}(H^0(L))$ is generated by quadrics unless ϕ_L embeds X with a tri-secant line; if $\deg(L)\geqq 2g+3-2\cdot h^1(L)-\mathrm{Cliff}(X)$, then the first module of syzygies among these quadrics should be generated by relations with *linear* coefficients unless X has a 4-secant 2-plane; and so on. Precise statements are given in § 3. They generalize an earlier conjecture of the first author [7] to the effect that one should be able to read off the Clifford index of a curve from the minimal resolution of its canonical ring. In view of the role of the Clifford index in these conjectural generalizations of several classical results, it seems of interest to understand this invariant more clearly than one does at the moment. Hence we have also included in § 3 some open problems pertaining to the Clifford index.

§ 0. Notation and conventions

(0.1) We work throughout over the complex numbers. However the results of § 1 remain valid over an algebraically closed ground field of arbitrary characteristic.

(0.2) X will always be a smooth irreducible projective curve of genus $g \geqq 2$. We denote by Ω the canonical bundle on X. If F is a coherent sheaf on X we write $H^i(F)$ instead of $H^i(X, F)$ so long as no confusion can arise; as usual, $h^i(F)$ is the dimension of the corresponding cohomology group. We freely use Riemann-Roch and Serre duality on X without explicit mention.

(0.3) The definition of $\mathrm{Cliff}(X)$ presupposes that $W^1_{g-1}(X) \neq \emptyset$, and hence that $g \geqq 4$. Our results remain valid for curves of genus $g \geqq 2$ if we adopt the convention that $\mathrm{Cliff}(X)=0$ for X of genus 2 or hyperelliptic of genus 3, and that $\mathrm{Cliff}(X)=1$ if X is non-hyperelliptic of genus 3. Concerning the inequalities in the definition of $\mathrm{Cliff}(X)$, observe that the existence of a line bundle A of small degree says nothing about the curve X unless $h^0(A) \geqq 2$. On the other hand, $\mathrm{Cliff}(A) = \mathrm{Cliff}(\Omega \otimes A^*)$ for any line bundle A. So the inequalities in the definition may be explained as simply a device for preventing such examples from contributing to the computation of $\mathrm{Cliff}(X)$.

(0.4) We will several times use the observation that if A is any line bundle on X, then

$$h^0(A) + h^0(\Omega \otimes A^*) = g + 1 - \mathrm{Cliff}(A).$$

In fact, suppose that A has degree d and that $h^0(A) = r+1$. Then $h^0(\Omega \otimes A^*) = h^1(A) = g - d + r$, whence the stated formula.

(0.5) We say that a short exact sequence

$$(*) \qquad 0 \to F' \to F \to F'' \to 0$$

of sheaves on X is *exact on global sections* if $(*)$ induces an exact sequence $0 \to H^0(F') \to H^0(F) \to H^0(F'') \to 0$, or equivalently if the connecting homomorphism $H^0(F'') \to H^1(F')$ determined by $(*)$ is zero.

§ 1. Normal generation of complete linear series

This section is devoted to the proofs of Theorems 1 and 3 stated in the Introduction. Assuming the statement of Theorem 3, we start with the

Proof of Theorem 1. Suppose to the contrary that L fails to be normally generated. If A is any line bundle of degree $2g-e$, then $\mathrm{Cliff}(A)=e-2\cdot h^1(A)$. Thus the inequality in the hypothesis is equivalent to the assumption that

$$(*) \qquad \mathrm{Cliff}(L)<\mathrm{Cliff}(X).$$

Furthermore, $h^0(L)\geqq 2$ since L is very ample, and in view of the definition of $\mathrm{Cliff}(X)$ it follows from $(*)$ that $h^1(L)\leqq 1$. Recalling that $\mathrm{Cliff}(X)\leqq(g-1)/2$, one then finds that L satisfies the numerical hypotheses of Theorem 3. Let D be the divisor of degree $\geqq 2n+2$ whose existence is guaranteed by that theorem. Since D spans an n-plane in $\mathbb{P}(H^0(L))$, one has

$$\mathrm{Cliff}(L(-D))\leqq\mathrm{Cliff}(L).$$

Moreover $h^0(L(-D))\geqq 2$ and $h^1(L(-D))\geqq 2$ thanks to the fact that $1\leqq n\leqq r(L)-2$, and consequently $\mathrm{Cliff}(X)\leqq\mathrm{Cliff}(L(-D))$. Thus we have a contradiction to $(*)$, and this proves the theorem. □

We turn next to Theorem 3. Recall the assumptions: L is a very ample line bundle of degree $2g+1-k$, where $2k+1\leqq g$ if $h^1(L)=0$ or $2k-3\leqq g$ if $h^1(L)\neq 0$.

Proof of Theorem 3. Suppose first that a divisor D as described exists. Then thanks to (a) the homomorphism

$$H^1(\mathbb{P}^r, I_{X/\mathbb{P}^r}(2))\to H^1(\mathbb{P}^r, I_{D/\mathbb{P}^r}(2))$$

is surjective. Since in any event $h^1(\mathbb{P}^r, I_{D/\mathbb{P}^r}(2))=h^1(\Lambda, I_{D/\Lambda}(2))$, it follows from (b) that $X\subseteq\mathbb{P}^r$ is not projectively normal.

Assume conversely that L fails to be normally generated. Note to begin with that the multiplication maps

$$\mu_m\colon\ H^0(L)\otimes H^0(L^m)\to H^0(L^{m+1})$$

are surjective for $m\geqq 2$. If L is non-special this is a well-known general fact (c.f. [17, Lecture 14]), while for special L it follows in the degree range at hand from [7, Theorem (4.e.1)]. Hence μ_1 – which we call simply μ – cannot be surjective. Equivalently, its transpose

$$\mu^*\colon\ H^0(L^2)^*\to H^0(L)^*\otimes H^0(L)^*$$

has a non-zero kernel.

On the other hand, $H^0(L^2)^*=\mathrm{Ext}^1(L,\Omega\otimes L^*)$ classifies extensions of L by $\Omega\otimes L^*$, and μ^* is identified (up to multiplication by a non-zero scalar) with the map

$$\mathrm{Ext}^1(L,\Omega\otimes L^*)\to\mathrm{Hom}(H^0(L), H^1(\Omega\otimes L^*))$$

which sends an extension of L by $\Omega\otimes L^*$ to the connecting homomorphism it determines. Hence there exists a non-trivial extension

$$(1.1) \qquad 0\to\Omega\otimes L^*\to E\to L\to 0$$

which is exact on global sections. Note that $\det E = \Omega$, and that

$$h^0(E) = h^0(L) + h^0(\Omega \otimes L^*) = g + 1 - \mathrm{Cliff}(L)$$

(c.f. (0.4)).

The next step is to produce a sub-bundle of E of suitably large degree. To this end we use a theorem of Ghione [5, 6], which implies that a rank two vector bundle F on X of degree d has a line sub-bundle $A \subseteq F$ of degree $\geqq a$ provided that $2a \leqq d - g + 1$.[1] In view of the numerical hypotheses of the theorem, it follows that E has a line sub-bundle A with

$$\deg(A) \geqq k \quad \text{if} \quad h^1(L) = 0$$

or

$$\deg(A) \geqq k-2 \quad \text{if} \quad h^1(L) \neq 0.$$

Fixing such a sub-bundle, the situation is summarized by the following diagram, in which (1.1) appears as the horizontal sequence:

(1.2)

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & A \arrow[d] \arrow[dr, "\alpha_D"] & & \\
0 \arrow[r] & \Omega \otimes L^* \arrow[r] & E \arrow[r] \arrow[d] & L \arrow[r] & 0. \\
& & \Omega \otimes A^* \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

Since $\deg(A) > \deg(\Omega \otimes L^*) = k - 3$, the indicated homomorphism $A \to L$ is non-zero. Hence

$$A = L(-D)$$

for some effective divisor D on X, and the map α_D in (1.2) is just the natural inclusion $L(-D) \to L$ determined by D. The goal now is to show that D has the properties asserted in the statement of the theorem.

Let n be the dimension of the linear space $\Lambda \subseteq \mathbb{P}(H^0(L)) = \mathbb{P}^r$ spanned by D. We begin by proving that $\deg(D) \geqq 2n + 2$ and that $1 \leqq n \leqq r - 2$. Recalling that $A = L(-D)$, it is equivalent (as in the proof of Theorem 1) to verify that

(i) $\mathrm{Cliff}(A) \leqq \mathrm{Cliff}(L) \quad [= k - 1 - 2 \cdot h^1(L)]$

and

(ii) $2 \leqq h^0(A) \leqq h^0(L) - 2$.

[1] The assumption in [5] that F be "general" is unnecessary (c.f. [12, § 2], but note the misprint in the definition of $\rho_d^r(C, M)$). For our purposes here, results of Nagata [19] are also applicable

But (i) follows immediately from the vertical sequence in (1.2), which gives:

$$g+1-\mathrm{Cliff}(L)=h^0(E)\leqq h^0(A)+h^0(\Omega\otimes A^*)=g+1-\mathrm{Cliff}(A).$$

In particular, the lower bounds on $\deg(A)$ then force $\deg(A)>\mathrm{Cliff}(A)$, and consequently $h^0(A)\geqq 2$. For the other inequality in (ii) observe to begin with that $D\neq 0$, or else the inclusion $A=L(-D)\to E$ would split (1.1). Since L is generated by its global sections, it follows that $h^0(A)=h^0(L(-D))\leqq h^0(L)-1$. Thus thanks to (i), D must have degree at least two. But this being so, if $h^0(L(-D))=h^0(L)-1$ then L fails to be very ample. Therefore $h^0(L(-D))\leqq h^0(L)-2$, and (ii) is proved.

We next show that D fails to impose independent conditions on quadrics in Λ or – what is the same thing – in $\mathbb{P}(H^0(L))$. Denoting by μ_D the composition of the natural maps

$$H^0(L)\otimes H^0(L)\xrightarrow{\ \mu\ } H^0(L^2)\xrightarrow[D]{\text{evaluation on}} H^0(L^2\otimes\mathcal{O}_D),$$

it is equivalent to show that μ_D fails to be surjective. But α_D gives rise to an inclusion $H^0(L(-D))\subseteq H^0(L)$; and setting $W_D=H^0(L)/H^0(L(-D))$, evaluation on D yields a homomorphism

$$\rho_D\colon\ H^0(L)\otimes W_D\to H^0(L^2\otimes\mathcal{O}_D)$$

having the same image as μ_D. So the question is in turn equivalent to proving that ρ_D is not surjective.

Consider to this end the following commutative diagram of exact sequences:

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
H^0(L^2\otimes\mathcal{O}_D)^* & \xrightarrow{\ \rho_D^*\ } & H^0(L)^*\otimes W_D^* \\
\downarrow & & \downarrow \\
H^0(L^2)^* & \xrightarrow{\ \mu^*\ } & H^0(L)^*\otimes H^0(L)^* \\
\downarrow & & \downarrow \\
\mathrm{H}^0(L^2(-D))^* & \longrightarrow & H^0(L)^*\otimes H^0(L(-D))^*. \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Let $e\in\ker(\mu^*)$ be the element corresponding to the extension (1.1). We claim that e maps to zero in $H^0(L^2(-D))^*=\mathrm{Ext}^1(L(-D),\Omega\otimes L^*)$. In fact, by construction α_D lifts to a homomorphism $\tilde{\alpha}_D\colon L(-D)\to E$ and hence (1.1) induces the trivial extension of $L(-D)$ by $\Omega\otimes L^*$. By the diagram, there thus exists a non-zero element $\tilde{e}\in\ker(\rho_D^*)$ mapping to e. Consequently ρ_D is not surjective, and this completes the proof of assertion (b) of the Theorem.

It remains only to show that $H^1(L^2(-D))=0$. But this is clear, since $L^2(-D)=A\otimes L$, and

$$\deg(A\otimes L)\geqq(k-2)+(2g+1-k)=2g-1.\quad\square$$

(1.3) *Remark.* Observe for later reference that by varying the numerical hypotheses of Theorem 3 one can obtain somewhat stronger bounds in the conclusion. Suppose, for instance, that e is a fixed non-negative integer and that L is a very ample line bundle of degree $2g+1-k$ which fails to be normally generated. If

$$(*) \qquad 2k+4e+1 \leqq g,$$

then the conclusion of the theorem holds now with

$$(**) \qquad 1 \leqq n \leqq r(L)-2-e.$$

In fact, by virtue of $(*)$ one can assume that the line bundle A constructed in the proof of Theorem 3 has degree $\geqq k+2e$. Then

$$k+2e-2\cdot r(A) \leqq \operatorname{Cliff}(A) \leqq \operatorname{Cliff}(L) \leqq k-1,$$

and it follows that $h^0(A) \geqq e+2$. But $n = r(L) - h^0(A)$, so this gives $(**)$.

We conclude this section with two simple applications of these results and techniques.

(1.4) **Corollary.** *Let L be a very ample line bundle of degree $2g$ on X, defining an embedding $X \subseteq \mathbb{P}^g$. Then L fails to be normally generated if and only if X is hyperelliptic.*

Proof. In view of Theorem 1, we need only check that if X is hyperelliptic, then L cannot be normally generated. But the hyperelliptic pencil sweeps out a two-dimensional rational normal scroll $S \subseteq \mathbb{P}^g$ containing X, and S lies on enough quadrics to force $H^1(\mathbb{P}^g, I_{X/\mathbb{P}^g}(2)) \neq 0$. (Compare [10, §3] for a more general result proved along these lines.) $\square$

It was suggested by Lange and Martens [10] that the result of the Corollary should hold. (C.f. also [8] and [9].)

(1.5) *Remark.* If L is a *general* line bundle of degree $2g$ on a hyperelliptic curve X, giving rise to an embedding $X \subseteq \mathbb{P}^g$, then X cannot have a $(2n+2)$-secant n-plane for any $n<(g-2)/2$. Hence the least value of n for which Theorem 3 applies can be arbitrarily large.

(1.6) **Corollary.** *Let L be a very ample line bundle on X of degree $2g-1$, with $g \geqq 4$. Then L fails to be normally generated if and only if either:*

(a) *X is hyperelliptic;*

(b) *X is trigonal and $L = \Omega \otimes B^*(D)$, where $B \in W_3^1(X)$ and D is an effective divisor of degree 4; or*

(c) *$X \subseteq \mathbb{P}^2$ is a smooth plane quintic with embedding line bundle $B \in W_5^2(X)$, and $L = B \otimes N$ for some line bundle $N \in W_6^1(X)$ generated by its global sections.*

In case (b) ϕ_L embeds X with a 4-secant line; in case (c) L defines an embedding $X \subseteq \mathbb{P}^5$ in which X has ∞^1 6-secant conics. We leave the proof of (1.6) to the interested reader.

§ 2. Classification of extremal line bundles on curves of large genus

Let X be a smooth curve of genus g and Clifford index e, and let L be a very ample line bundle on X of degree

$$(*) \qquad \deg(L)=2g-2i-e,$$

where $i=h^1(L)$ is the index of speciality of L. Motivated by Theorem 1, we say as in the Introduction that such a line bundle is *extremal* if it fails to be normally generated. This section is devoted to the classification of extremal line bundles on curves for which $g>N(e)$, where

$$N(e)\underset{\text{def}}{=}\max\left\{\binom{e+3}{2},\ 10e+6\right\}.$$

Specifically, we will prove

(2.1) **Theorem.** *Let L be a very ample line bundle satisfying* $(*)$. *Assume that $g>N(e)$ and that X is neither hyperelliptic nor elliptic-hyperelliptic. Then L fails to be normally generated if and only if the pair (X, L) falls into one of the three families listed in the following Table* (2.2):

(2.2) **Table**

	X	$h^1(L)$	ϕ_L
I.	Has a g^1_{e+2}	0	Embeds X with a 4-secant line
II.	$e=2f\geqq 4$; X is a double covering $\pi: X\to Y\subseteq \mathbb{P}^2$ of a smooth plane curve Y of degree $f+2$.	1	Embeds X with a 4-secant line
III.	as in II	0	Embeds X with a 6-secant conic, but no 4-secant line

Concerning (I) we note that according to a theorem of Martens [13], any curve of Clifford index e and genus $g>(e+3)(e+2)/2$ carries a g^1_{e+2}. We remark also that each of the families described in (2.2) does in fact occur for every $g>N(e)$; the construction of the extremal bundles is sketched in (2.5) below. As for the two classes of curves excluded from the theorem, the hyperelliptic case is covered by Corollary 1.4 and the observation that no special line bundle on a hyperelliptic curve is very ample. The situation on elliptic-hyperelliptic curves is discussed in (2.6). Taken together, these results constitute the proof of Theorem 2.

Theorem (2.1) depends on the following elementary consequence of Castelnuovo's bound on the geometric genus of a non-degenerate space curve. Martens has used this bound in a similar way in [13]. Recall that a line bundle B on X is *birationally very ample* if B is generated by its global sections and the corresponding map ϕ_B is birational onto its image.

(2.3) **Lemma.** *Let X be a curve of genus g, and let B be a line bundle on X with $r(B)=r$ and $\deg(B)=2r+e$ for some $e\geqq 0$. If B is birationally very ample, then either*

$$r\geqq g-2e-1 \quad \text{or} \quad g\leqq (e+3)(e+2)/2.$$

We omit the proof.

Proof of Theorem (2.1). It is immediate from Theorem 3 that if L is a very ample bundle of the stated degree for which ϕ_L has one of the properties described, then L cannot be normally generated.

Assume conversely that L fails to define a projectively normal embedding. Since L is very ample and $g>(e+3)(e+2)/2$, it follows from Lemma (2.3) that

$$r(L)=g-i-e\geqq g-2e-1,$$

i.e. that $i\leqq e+1$. Then thanks to the fact that $g>N(e)\geqq 10e+6$ we are in the situation of Remark (1.3). Thus Theorem 3 gives the existence of an integer

$$1\leqq n\leqq r(L)-2-e,$$

and an effective divisor D of degree $\geqq 2n+2$ which spans an n-plane in $\mathbb{P}(H^0(L))$.

Since $\mathrm{Cliff}(L)=\mathrm{Cliff}(X)$, one must in fact have $\deg D=2n+2$, and hence $h^1(L(-D))=n+i+1$. Setting

$$B=\Omega\otimes L^*(D), \tag{2.4}$$

this means that $\deg B=2n+2i+e$ and $r(B)=n+i$. In particular $\mathrm{Cliff}(X)=\mathrm{Cliff}(B)$, and so B is generated by its global sections.

If $r(B)=1$ then $i=0$, B is a g^1_{e+2}, and we are in case (I). Assuming that $r(B)\geqq 2$, consider the map

$$\phi_B\colon X\to\mathbb{P}^{n+i}.$$

We claim, and this is the main point, that ϕ_B cannot be birational onto its image. Indeed, if on the contrary B were birationally very ample, then since $g>(e+3)(e+2)/2$ Lemma (2.3) would give the inequality

$$n+i=r(B)\geqq g-2e-1.$$

But this contradicts our upper bound on n.

The upshot is that ϕ_B factors through a branched covering $\pi\colon X\to Y$ of degree $m\geqq 2$, where Y is a smooth curve mapped birationally onto its image in $\mathbb{P}^{n+i}$ by a line bundle B_0 with $r(B_0)=r(B)=n+i$, and $B=\pi^*(B_0)$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi_B"] & \\ Y \arrow[r, "\phi_{B_0}"'] & \mathbb{P}^{n+i}. \end{tikzcd}$$

If $m \geqq 3$ then $\mathrm{Cliff}(\pi^*(B_0(-y))) < \mathrm{Cliff}(B) = \mathrm{Cliff}(X)$ for any $y \in Y$. But $h^0(\pi^*(B_0(-y))) \geqq 2$ and $h^1(\pi^*(B_0(-y))) \geqq 2$, so this is impossible. Therefore $m=2$, $e=2f$ is even, and $\deg(B_0)=n+i+f$. In fact, for the same reasons of Clifford index, B_0 must be very ample and $Y \subseteq \mathbb{P}^{n+i}$ cannot have any $(s+2)$-secant s-planes if $s \leqq n+i-2$. But for $s \geqq 1$, any smooth non-degenerate curve $Y \subseteq \mathbb{P}^{s+2}$ of degree at least $s+4$ has an $(s+2)$-secant s-plane; this is standard if $s=1$, and one may reduce to this case by projecting from general points of Y. Hence if $n+i \geqq 3$, then Y must be rational or elliptic normal, and X in turn is hyperelliptic or elliptic-hyperelliptic. So we must have $n+i=2$, and

$$Y \subseteq \mathbb{P}^2$$

is a smooth plane curve of degree $f+2 \geqq 4$. If $n=1$ then $i=1$ and we are in case (II); case (III) occurs if $n=2$ and $i=0$. In either event it follows from Theorem 3 that ϕ_L must have the stated properties, except that in (III) it might happen that $D=F+D'$ for some effective divisors F and D' of degree respectively 2 and 4, where ϕ_L maps D' to a line. But then $B'=B(-F)$ is a g^1_{e+2}, and we revert to case I. □

(2.5) *Remark*. The three families described in Theorem (2.1) do in fact occur for all $g > N(e)$, and we indicate here how the extremal bundles arise. To this end, note that the previous proof shows exactly what the line bundle

$$B \in W^{n+i}_{2n+2i+e}(X)$$

of (2.4) must be in each of the three cases, and then

$$(*) \qquad L = \Omega \otimes B^*(D)$$

for some effective divisor D of degree $2n+2$ on X. The extremal bundles are determined via $(*)$ by the following recipes:

Case I. Take X to be a curve of Clifford index e and genus $g > N(e)$ with a g^1_{e+2} but no g^2_{e+4}; for instance X could be a general $(e+2)$-gonal curve of the appropriate genus. Then take

$$B \in W^1_{e+2}(X) \quad \text{and} \quad D = x_1 + x_2 + x_3 + x_4,$$

where the x_i are distinct points of X, no two of which lie in the same fibre of ϕ_B.

For cases II and III one starts with a smooth plane curve

$$Y \subseteq \mathbb{P}^2,$$

of degree $f+2 \geqq 4$, with embedding line bundle $B_0 \in W^2_{f+2}(Y)$. Set $e=2f$, and let X be any curve of genus $g > N(e)$ arising as a double covering

$$\pi\colon X \to Y \subseteq \mathbb{P}^2;$$

one checks that $\mathrm{Cliff}(X)=e$.

Case II. Take

$$B=\pi^*(B_0) \quad \text{and} \quad D=x_1+x_2+x_3+x_4$$

where the x_i are distinct points on X mapping via π to four distinct *collinear* points of Y.

Case III. Take

$$B=\pi^*(B_0) \quad \text{and} \quad D=x_1+\ldots+x_6$$

where the x_i are distinct points on X mapping to six distinct points of Y which *lie on a conic* in $\mathbb{P}^2$, but with no four collinear.

In each case the resulting line bundle

$$L=\Omega\otimes B^*(D)$$

is very ample, with the appropriate index of speciality; the divisor D spans a 4-secant line in cases I and II, and a 6-secant conic in case III. Moreover, if D is reduced these are the only possibilities. (For general D the descriptions become a bit more involved, and in case I if X has a g^2_{e+4} then there are some additional restrictions on D.)

(2.6) *Remark.* To complete the picture, it remains to describe extremal line bundles on an elliptic-hyperelliptic curve X, i.e. a curve which admits a degree two covering $\pi\colon X\to Y$, with Y elliptic. Assume that X has genus $g>N(2)=26$. First of all, arguing from the proof of Theorem (2.1), one easily shows:

(a) There is no extremal line bundle L on X with $h^1(L)\neq 0$.

By contrast, there are at least two irreducible families of extremal non-special line bundles on X:

(b) Let L be a very ample line bundle of degree $2g-2$ on X, with $h^1(L)=0$. Then L fails to be normally generated if and only if either ϕ_L embeds X with a four-secant line, or else

$$(*) \qquad \det(\pi_* L)=\det(\pi_* \Omega).$$

In the former case $(*)$ is not generally satisfied, and hence the family of extremal non-special line bundles on X has one irreducible component of dimension 5 (parametrizing embeddings with 4-secant lines), and also one or more $(g-1)$-dimensional components defined by $(*)$.

In an earlier version of this paper we included a rather lengthy proof of (b) based on the results and techniques of §1; however Lange and Martens pointed out to us that the result can be deduced (for a wider range of genera) from the proofs in their paper [11].

§ 3. Conjectures and open problems

We discuss in this section a number of open questions related to the above circle of ideas.

§ 3a. Higher syzygies

Results on normal generation frequently go hand in hand with analogous statements giving conditions under which a curve is cut out by quadrics (c.f. [18] and [22]). Until recently, however, the corresponding questions for higher syzygies seem not to have been addressed. But as some of the results and conjectures of [7] suggest, it is in fact very reasonable to expect that "lower order" results will extend in a natural way to higher syzygies, and that moreover these extensions may shed new light on the classical situations. We present and discuss here some conjectures along these lines.

We start with some notation. With X and L as in the Introduction, set

$$R=R(L)=\bigoplus_n H^0(L^n).$$

Thus $R(L)$ is a Cohen-Macaulay module of dimension two over the homogeneous coordinate ring $S=\mathrm{Sym}(H^0(L))$. Let $E_{.}=E_{.}(L)$ be a minimal graded free resolution of $R(L)$ over S:

$$0\to E_{r-1}\to E_{r-2}\to \dots \to E_1\to E_0\to R\to 0, \tag{3.1}$$

where each $E_i=E_i(L)$ is a direct sum of twists of S, and $r=r(L)$. As we are dealing with complete linear series, one has:

(i) $E_0=S\oplus E_0'$, with $E_0'=\oplus S(-a_{0j})$ and all $a_{0j}\geqq 2$;

and, for $i\geqq 1$,

(ii) $E_i=\oplus S(-a_{ij})$, where all $a_{ij}\geqq i+1$.

The natural way to extend the various classical results alluded to above is to ask when the first few terms in $E_{.}(L)$ are as simple as possible. Specifically, we ask whether, for a given integer $p\geqq 0$, L enjoys the following property:

$$(\mathrm{N_p})\qquad \begin{array}{l} E_0(L)=S \quad \text{and} \\ E_i(L)=\oplus S(-i-1) \quad (\text{i.e. all } a_{ij}=i+1) \text{ for } 1\leqq i\leqq p. \end{array}$$

Thus, very concretely:

$(\mathrm{N_0})$ holds for L iff L is normally generated;

$(\mathrm{N_1})$ holds for L iff L is normally generated, and the homogeneous ideal of X in $\mathbb{P}(H^0(L))$ is generated by quadrics;

$(\mathrm{N_2})$ holds for L iff $(\mathrm{N_0})$ and $(\mathrm{N_1})$ do, and the module of syzygies among quadratic generators $Q_i\in I_{X/\mathbb{P}^r}$ is spanned by relations of the form

$$\sum L_i Q_i=0,$$

where the L_i are *linear* polynomials;

and so on. Property $(\mathrm{N_1})$ is what Mumford calls "normal presentation" in [18]; in the terminology of [7], $(\mathrm{N_p})$ is equivalent to the vanishing of the Koszul cohomology groups $K_{i,j}(L)$ for $i\leqq p$ and $j\geqq 2$.

The theorem of Castelnuovo-Mattuck-Mumford on projective normality was generalized to higher syzygies by the first author in [7]:

(3.2) If $\deg(L) \geqq 2g+1+p$, then L satisfies property $(\mathrm{N_p})$.

In particular, one recovers a result of Fujita [4] and St. Donat [23] to the effect that if $\deg(L) \geqq 2g+2$, then X is cut out by quadrics in $\mathbb{P}(H^0(L))$. The examples for which (3.2) is optimal have been classified by the authors, and one finds:

(3.3) If $\deg(L) = 2g+p$, then property $(\mathrm{N_p})$ *fails* for L if and only if either:

(a) X is hyperelliptic; or

(b) ϕ_L embeds X with a $(p+2)$-secant p-plane, i.e. $H^0(L \otimes \Omega^*) \neq 0$.

(When $p=0$, the statement in (b) should be interpreted to mean that ϕ_L is not, in fact, an embedding: compare Corollary (1.4) above.) The proof will appear elsewhere.

In light of Theorem 1, the natural generalization of (3.2) and (3.3) is the following:

(3.4) **Conjecture.** *Assume that L is very ample, with*

$$\deg(L) \geqq 2g+1+p-2 \cdot h^1(L) - \mathrm{Cliff}(X).$$

Then property $(\mathrm{N_p})$ *holds for L unless* ϕ_L *embeds X with a* $(p+2)$*-secant p-plane.*

In fact, one would hope for a more precise statement along the lines of Theorem 3.

A particularly interesting case of (3.4) is when $L = \Omega$. Then the conjecture states that the canonical bundle satisfies $(\mathrm{N_p})$ for $p < \mathrm{Cliff}(X)$. On the other hand, it was shown in the appendix to [7] that $(\mathrm{N_p})$ does not hold for Ω if $p \geqq \mathrm{Cliff}(X)$. Hence (3.4) contains as a special case the "Noether-Enriques-Babbage-Petri" conjecture of the first author:

(3.5) **Conjecture** ([7, Conjecture (5.1)]. *The Clifford index of X is equal to the least integer p for which property* $(\mathrm{N_p})$ *fails for the canonical bundle* Ω.

In other words, knowing the Clifford index of X should be equivalent to knowing the degrees of the terms in the minimal resolution of its canonical ring. The conjecture would generalize - and clarify - Petri's theorem [20] that the homogeneous ideal of a non-hyperelliptic canonical curve X is generated by quadrics unless X is trigonal or a smooth plane quintic, i.e. unless $\mathrm{Cliff}(X) = 1$. The conjecture has been verified by Schreyer [21] for $g \leqq 8$.

In a somewhat different direction, there is a conjecture which would give a complete picture of the grading of the resolution (3.1) when $\deg(L)$ is large compared to $2g$. In particular L will be non-special, and then it is elementary that there are only two degrees in which generators of $E_i(L)$ might occur. Specifically, so long as $h^1(L)=0$ one has:

$$E_0 = \begin{matrix} S \\ \oplus \\ \oplus S(-2) \end{matrix} \quad \text{and} \quad E_i = \begin{matrix} \oplus S(-i-1) \\ \oplus \\ \oplus S(-i-2) \end{matrix} \quad \text{if } i \geqq 1,$$

although it is not necessarily the case that summands of both degrees are actually present. In fact, always assuming that $h^1(L)=0$, one finds from (3.2) that

$$E_0=S \text{ if } 0\leqq(r-1)-g \quad\text{and}\quad E_i=\bigoplus S(-i-1) \text{ for } 1\leqq i\leqq(r-1)-g,$$

where as usual $r=r(L)$. On the other hand, Schreyer observed that if $\deg(L)\geqq 3g-2$ then for every $i\geqq r-g$ the module $E_i(L)$ must have some $S(-i-2)$ summands (this follows, e.g., from (3.3)). This leaves undetermined only whether or not, for a given integer $1\leqq q\leqq g$, $E_{r-q}(L)$ has generators in more than one degree. More precisely, we ask whether L satisfies the following property for some $1\leqq q\leqq g$:

$$(\mathrm{M_q}) \qquad E_i(L)=\bigoplus S(-i-2) \quad \text{for } i\geqq r-q.$$

In the terminology of [7], $(\mathrm{M_q})$ is equivalent to the vanishing of $K_{i,j}(L)$ for $i\geqq r-q$ and $j\neq 2$.

To interpret property $(\mathrm{M_q})$ somewhat more geometrically, recall that by Serre duality $\check{E}.(-r-1)$ is a minimal resolution of the S-module

$$W=W(L)=\bigoplus_n H^0(\Omega\otimes L^n).$$

(This module was studied by Arbarello and Sernesi in [2].) Hence if L is non-special, then:

(M_1) holds for $L\Leftrightarrow$ the map $H^0(L)\otimes H^0(\Omega)\to H^0(L\otimes\Omega)$ is surjective;

(M_2) holds for $L\Leftrightarrow(\mathrm{M}_1)$ does, and if $\omega_1,\ldots,\omega_g\in H^0(\Omega)$ is a basis, then all relations of the form

$$\sum P_i\omega_i=0\in W(L), \qquad \text{where } P_i\in S^k H^0(L),$$

are already generated by those relations in which the P_i are *linear* polynomials;

and so on. The "$K_{p,1}$ Theorem" of [7] gives a geometric criterion, in a more general setting, for the failure of (M_1) or (M_2). In particular, one finds:

(3.6) If $\deg(L)\geqq 2g+1$, then (M_1) fails for L if and only if $X=\mathbb{P}^1$;
If $\deg(L)\geqq 2g+2$, then (M_2) fails for L if and only if X carries a g_2^1.

This suggests:

(3.7) **Conjecture.** *If* $\deg(L)$ *is sufficiently large compared to* $2g$, *then property* $(\mathrm{M_q})$ *fails for L if and only if X carries a* g_q^1, i.e. *a line bundle A with* $\deg(A)=q$ *and* $r(A)=1$.

Ideally $\deg(L)\geqq 2g+q$ (and hence also $\deg(L)\geqq 3g$) would do; one certainly wants a bound independent of q. Then (3.7) would have the surprising consequence that one could read off the "gonality" of a curve from the minimal resolution of any one line bundle of sufficiently large degree. One implication in the statement is known: it is elementary that if X carries a g_q^1, then no line bundle of degree $\geqq 2g+q$ can satisfy $(\mathrm{M_q})$. However we remark that of the three conjectures presented here, this is the one of which we are the least convinced.

§ 3b. The Clifford index

Our results for projective normality, and the conjectures just stated, suggest that the Clifford index is an important invariant of a curve for questions of an algebraic nature involving complete linear series. Hence it seems worthwhile to understand this invariant as closely as possible. Martens [13, 14] has taken some basic steps in this direction, but much remains to be learned. To begin with, it is not known that there even exist curves of arbitrary Clifford index $0 \leqq e \leqq \left[\frac{g-1}{2}\right]$ for a given genus g. The obvious thing to expect here is:

(3.8) **Conjecture.** *Given an integer* $0 \leqq e \leqq \left[\frac{g-1}{2}\right]$, *if* X *is a sufficiently general* $(e+2)$*-gonal curve of genus* g, *then* $\mathrm{Cliff}(X) = e$.

The issue of course is to rule out the possibility that $\mathrm{Cliff}(X) < e$; this is elementary if g is reasonably large compared to e (e.g. $g > 4e$).

Define the *Clifford dimension* of a curve X to be the integer

$$r(X) = \min \left\{ r(A) \;\middle|\; \begin{array}{l} A \text{ is a line bundle on } X, \\ \text{with } h^0(A) \geqq 2,\ h^1(A) \geqq 2, \\ s.t.\ \mathit{Cliff}(X) = \mathrm{Cliff}(A) \end{array} \right\}.$$

In other words, the Clifford dimension of X is the least dimension of a linear series which computes the Clifford index of X. For instance Martens' theorem [13] states that if X has Clifford index e and genus $g > (e+3)(e+2)/2$ then $r(X) = 1$, while if X is a smooth plane curve then $r(X) = 2$. Martens has pointed out that it is very rare to find a curve with Clifford dimension $\geqq 3$. For if $r(X) = r \geqq 3$, and if $X \subseteq \mathbb{P}^r$ is embedded by a line bundle computing $\mathrm{Cliff}(X)$, then evidently X cannot have any $(2r-2)$-secant $(r-2)$-planes. But a formula of Castelnuovo counting the number of such planes (c.f. [14] or [1]) then puts severe restrictions on the possible genus and Clifford index of X. Martens uses these ideas to prove the remarkable fact [14]:

(3.9) $r(X) = 3$ if and only if X is the complete intersection of two cubic surfaces in $\mathbb{P}^3$.

In a letter to the authors, Martens also points out that the example in (3.9) should be merely the first in an infinite family:

(3.10) *Problem* (G. Martens). *Show that for each* $r \geqq 3$ *there exist smooth curves*

$$X \subseteq \mathbb{P}^r$$

with:

$$\begin{array}{ll} \deg(X) = 4r-3 & r(X) = r, \\ g(X) = 4r-2, & \mathrm{Cliff}(X) = 2r-3. \end{array}$$

As Martens notes, one would expect such curves to be half-canonically embedded and projectively normal. In fact, it seems very likely that the desired curves will exist on suitable $K3$ surfaces $S \subseteq \mathbb{P}^r$, but it's not immediately clear how to verify that the candidates have the stated Clifford index and dimension.

(3.11) *Problem. Is it true that any curve X with Clifford dimension* $r(X) \geqq 3$ *is in one of the families specified in* (3.10)?

This might seem like too much to expect, but in view of (3.9) there doesn't appear to be any reason not to hope for a simple picture. Furthermore, a home computer was used to make numerical calculations based on Castelnuovo's secant-plane formula. It turned out that for $r=4$ and 5, and $g \leqq 100$, there were no other possibilities except for a few with $\deg(X)=g-1$, some of which could be ruled out for non-numerical reasons. An affirmative solution to (3.11) together with a proof of (3.5) would yield an essentially complete list of all curves for which the minimal resolution of the canonical ring has a given grading. Presumably a positive solution to (3.11) would also imply Conjecture (3.8), at least if one had a good enough understanding of the linear series on the curves in (3.10).

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Note added in proof

Ballico has proven conjecture (3.8) (to appear). Schreyer informs us that conjecture (3.5) fails for the generic curve of genus 7 in characteristic 2.