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Special divisors on curves on a $K3$ surface

Mark Green* and Robert Lazarsfeld**

Department of Mathematics, University of California, Los Angeles, CA 90024, USA

Introduction

Our purpose is to prove a conjecture of Harris and Mumford, as modified by the first author in [G], to the effect that all smooth curves in a given linear series on a complex projective $K3$ surface X have the same Clifford index. Loosely speaking, this means that if one curve $C \subset X$ has an exceptional g^r_d, then every $C' \in |C|$ carries an "equally exceptional" linear series.

Let C be a smooth irreducible complex projective curve of genus $g \geqq 2$. Recall that the *Clifford index* of a line bundle A on C is the integer

$$\operatorname{Cliff}(A) = \deg(A) - 2 \cdot r(A),$$

where $r(A) = h^0(A) - 1$. The Clifford index of C itself is defined to be

$$\operatorname{Cliff}(C) = \min\{\operatorname{Cliff}(A) \mid h^0(A) \geqq 2, h^1(A) \geqq 2\}$$

(cf. [M1], [M2], and (0.4) below). This gives a rough measure, from the point of view of special linear series, of how general C is in the sense of moduli. Thus Clifford's theorem states that $\operatorname{Cliff}(C) \geqq 0$ with equality if and only if C is hyperelliptic, and similarly $\operatorname{Cliff}(C) = 1$ if and only if C is trigonal or a smooth plane quintic. At the other extreme, if C is a general curve of genus g then $\operatorname{Cliff}(C) = [(g-1)/2]$, and in any event $\operatorname{Cliff}(C) \leqq [(g-1)/2]$. We say that a line bundle A on C *contributes to the Clifford index* of C if A satisfies the inequalities in the definition of $\operatorname{Cliff}(C)$; it *computes the Clifford index* of C if in addition $\operatorname{Cliff}(C) = \operatorname{Cliff}(A)$.

Our main result is the following:

Theorem. *Let X be a complex projective $K3$ surface, and let $C \subset X$ be a smooth irreducible curve of genus $g \geqq 2$. Then*

$$\operatorname{Cliff}(C') = \operatorname{Cliff}(C)$$

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for every smooth curve $C' \in |C|$. Furthermore, if Cliff(C) *is strictly less than the generic value* $[(g-1)/2]$, *then there is a line bundle L on X whose restriction to any smooth $C' \in |C|$ computes the Clifford index of C'.*

So for example one recovers the well-known fact that a curve $C \subset X$ is hyperelliptic if and only if every curve $C' \in |C|$ is. We remark that the line bundle L in the second statement corresponds to a divisor arising as a sum of components of a reducible curve in $|C|$, i.e. $H^0(X, L^*(C)) \neq 0$.

One might hope that if a given curve $C \subset X$ carries a g^r_d then so too does every smooth $C' \in |C|$, but Donagi pointed out that this is not the case. In fact, let $\pi\colon X \to \mathbb{P}^2$ be a genus two $K3$ surface, i.e. a double covering of $\mathbb{P}^2$ branched along a smooth sextic. If $E \subset \mathbb{P}^2$ is a non-singular cubic, then $C = \pi^{-1}(E)$ has a g^1_4; on the other hand, a general curve $C' \in |C|$ maps isomorphically by π to a smooth plane sextic, and hence does not carry such a pencil. The first author proposed in [G, (5.8)] that one should aim for the constancy of Clifford index, and he noted that the first statement of the Theorem would follow immediately - at least for ample C - from his conjecture [G, (5.1)] on the syzygies of canonical curves. (This conjecture remains open, however, and the results here may be seen as added evidence for its truth.) The second statement of the Theorem was suggested by Donagi, Morrison and Reid.

Special divisors on curves on a $K3$ surface X have been considered by a number of authors starting with Reid [R], who showed that under suitable numerical hypotheses, a pencil on a curve $C \subset X$ is cut out by on X. Drawing on Reid's geometric techniques, Donagi and Morrison, and independently Reid himself, proved the Theorem when the Clifford index of C is computed by a pencil. By studying certain vector bundles on X, the second author showed in [L] that if every member of the linear series $|C|$ is reduced and irreducible, then the curves in this series behave generically from the point of view of Brill-Noether theory. Tjurin [T] subsequently gave different proofs of some of the results of [L], and he found infinitesimal criteria for a g^r_d on C to extend to a general member of the linear series $|C|$. (Similar criteria had been obtained by Donagi and Morrison.) Along somewhat different lines, it is interesting to ask whether various specific curves can lie on a $K3$ surface; Wahl [W] has recently shown that with a few exceptions, complete intersection curves in $\mathbb{P}^r$ $(r \geqq 3)$ cannot. Some of Reid's results have been extended to surfaces other than $K3$'s by Serrano-Garcia [S].

The proof of the Theorem uses the techniques of [L] and (to a lesser extent) [GL]. The idea is quite simple, but the possible presence of rational or elliptic curves leads to technical complications which may obscure the motivation for some of the steps. So it seems worthwhile to sketch here in some detail how the argument would proceed under the simplifying assumption that X doesn't contain any smooth curves of genus $\leqq 1$. To begin with, among all smooth curves in a given linear series on X, fix one - say C - of minimal Clifford index, and let A be a line bundle computing the Clifford index of C, with $\deg(A) \leqq g(C)-1$. We may assume that Cliff$(C) < [(g(C)-1)/2]$, and we are required to produce a line bundle L on X, contributing to the Clifford index of every curve $C' \in |C|$, such that Cliff$(L \otimes \mathcal{O}_{C'}) \leqq$ Cliff(C).

As in [L] one can canonically associate to the pair (C, A) a vector bundle $E(C, A)$ of rank $h^0(A)$ on X; this bundle is generated by its global sections, and one has $c_1(E(C,A)) = [C]$, $c_2(E(C,A)) = \deg(A)$, and $h^0(E(C,A)) = g(C)+1-\mathrm{Cliff}(A)$. We say that a bundle E on X, of rank $\geqq 2$, is a *reduction* of $E(C,A)$ if

(a) There is a map $E(C,A) \to E$ surjective off a finite set;
(b) $h^0(E) \geqq h^0(E(C,A))$ and $h^i(E) = h^i(E(C,A))$ for $i \geqq 1$; and
(c) $\det E = \det E(C,A)$ and $2 \cdot rk(E) - c_2(E) \geqq 2 \cdot rk(E(C,A)) - c_2(E(C,A))$.

These conditions mean that E looks numerically and cohomologically as though it is of the form $E(C_1, A_1)$ for some curve $C_1 \in |C|$, and some line bundle A_1 on C_1 with $\mathrm{Cliff}(A_1) \leqq \mathrm{Cliff}(A)$ and $r(A_1) \leqq r(A)$.

There are now two main steps to the argument, the first being:

(1) Let E be a reduction of $E(C,A)$ of minimal rank $\geqq 2$. Then there exists a line bundle L on X, with $h^0(L) \geqq 2$, plus a non-zero map $L \to E$.

When $rkE = 2$, this is proved as in [L, (1.3)]; when $rk(E) \geqq 3$, one starts by observing that there is a section $s\colon \mathcal{O}_X \to E$ vanishing at two or more points. But then in fact s must vanish along a divisor Δ, for otherwise the reflexive hull of coker(s) would be a reduction of E, contradicting minimality. By construction Δ is non-zero, and so long as X does not contain any rational curves, it suffices to take $L = \mathcal{O}_X(\Delta)$.

It is not hard to check that $h^0(L \otimes \mathcal{O}_{C'}) \geqq 2$ and $h^1(L \otimes \mathcal{O}_{C'}) \geqq 2$ for all smooth $C' \in |C|$, and so it is enough to show that $\mathrm{Cliff}(L \otimes \mathcal{O}_{C'}) \leqq \mathrm{Cliff}(A) = \mathrm{Cliff}(C)$. To this end consider the exact sequence

$$0 \to L \to E \to F \to 0$$

deduced from (1), and assume for simplicity that F is locally free. Then F is generated by its global sections away from finitely many points, and $L \otimes \det F = \mathcal{O}_X(C)$. Furthermore, if X doesn't contain any elliptic pencils, the following general fact automatically applies:

(2) Let F be a vector bundle on X which is generated by its global sections away from finitely many points. If $c_1(F)^2 > 0$, then $h^0(\det F) \geqq h^0(F)$.

(This holds for globally generated bundles on any regular surface.) But then

$$\begin{aligned} g(C)+1-\mathrm{Cliff}(A) \leqq h^0(E) &\leqq h^0(L) + h^0(F) \\ &\leqq h^0(L) + h^0(\det F) \\ &\leqq h^0(L \otimes \mathcal{O}_{C'}) + h^0(L^* \otimes \omega_{C'}), \end{aligned}$$

while by Riemann-Roch on C' one has

$$h^0(L \otimes \mathcal{O}_{C'}) + h^0(L^* \otimes \omega_{C'}) = g(C') + 1 - \mathrm{Cliff}(L \otimes \mathcal{O}_{C'}).$$

The required inequality follows.

Our exposition proceeds in three parts. In § 1 we prove the basic inequality (2); when $c_1(F)^2=0$ the inequality in question fails, but in this case one can describe the bundles that arise. We define the bundles $E(C,A)$ and their reductions in § 2, and carry out step (1) above. The proof of the theorem occupies § 3.

§ 0. Notation and conventions

(0.1) We work with varieties defined over the complex numbers.

(0.2) Throughout the paper, X denotes a smooth projective $K3$ surface. We say that a coherent sheaf F on X is *generated by its global sections away from finitely many points* if the canonical map $H^0(F)\otimes_{\mathbb{C}}\mathcal{O}_X\to F$ is surjective off a finite set. The Riemann-Roch formula for F states that:

$$\chi(F)=\frac{c_1(F)^2}{2}-c_2(F)+2\cdot rk F.$$

We freely use Serre duality and Riemann-Roch on X and on curves $C\subset X$ without explicit mention.

(0.3) If τ is a coherent sheaf supported on a finite subset of X, then $c_1(\tau)=0$ and it follows by Riemann-Roch that $c_2(\tau)=-\mathrm{length}(\tau)$. Hence if

$$0\to U\to V\to W\to \tau\to 0$$

is an exact sequence of sheaves on X, with U, V, and W locally free, and $\dim\operatorname{supp}(\tau)=0$, then $c_2(V)=c_2(U)+c_1(U)\cdot c_1(W)+c_2(W)+\mathrm{length}(\tau)$.

(0.4) Let C be a smooth irreducible curve of genus g. The inequalities in the definition of $\mathrm{Cliff}(C)$ presuppose that C carries a pencil of degree $\leqq g-1$, and hence that $g\geqq 4$. Our results remain valid for all $g\geqq 2$ if we take $\mathrm{Cliff}(C)=0$ for C of genus 2 or hyperelliptic of genus 3, and $\mathrm{Cliff}(C)=1$ for C non-hyperelliptic of genus 3. Note that if A is any line bundle on C, then $\mathrm{Cliff}(\omega\otimes A^*)=\mathrm{Cliff}(A)$. Consequently $\mathrm{Cliff}(C)=\min\{\mathrm{Cliff}(A)\,|\deg(A)\leqq g-1,\ r(A)\geqq 1\}$. Finally, observe that for every line bundle A on C, one has

$$h^0(A)+h^1(A)=g+1-\mathrm{Cliff}(A).$$

§ 1. A Clifford-type inequality for vector bundles on a regular surface

Our purpose in this section is to prove the inequality (2) from the Introduction. The result in question is not specific to $K3$ surfaces, and it it no harder to work in a more general context.

(1.1) **Proposition.** *Let S be a smooth irreducible complex projective surface with $H^1(S,\mathcal{O}_S)=0$, and let E be a vector bundle of* rank n *on S. Assume that E is generated by its global sections, and that $H^0(E^*)=0$.*

(i) *If* $c_1(E)^2>0$, *then* $h^0(E)\leqq h^0(\det E)$.

(ii) *If* $c_1(E)^2=0$ *and* $H^1(E^*)=0$, *then* $E=\overset{n}{\bigoplus}\mathcal{O}_S(\Sigma)$, *where* Σ *is a smooth irreducible curve on* S *which moves in a base-point free pencil.*

Remark. In the next section we will recall how under suitable hypotheses one can associate a vector bundle $E(C,A)$ to a line bundle A on a curve C on a $K3$ surface. One has $h^0(E(C,A))=g(C)+1-\mathrm{Cliff}(A)$ and $h^0(\det E(C,A))=g(C)+1$. Hence in this case, the inequality in (i) is precisely Clifford's theorem.

Proof of Proposition (1.1). Let $V\subset H^0(E)$ be a general subspace of dimension n, so that V determines a vector bundle map $e_V\colon V\otimes_{\mathbb{C}}\mathcal{O}_S\to E$. Since E is globally generated, we may assume by a standard general position argument (cf. [F], Appendix B) that e_V is generically an isomorphism, and that e_V has rank exactly $n-1$ along the curve $\Gamma\subset S$ defined by the vanishing of $\det(e_V)$. Then $\mathrm{coker}(e_V)$ is a line bundle B on Γ, and one has the exact sequence

(1.2) $$0\to V\otimes_{\mathbb{C}}\mathcal{O}_S\to E\to B\to 0$$

of $\mathcal{O}_S$-modules. Dualizing (1.2) one obtains

(1.3) $$0\to E^*\to V^*\otimes_{\mathbb{C}}\mathcal{O}_S\to N_\Gamma\otimes B^*\to 0,$$

where N_Γ is the normal bundle to Γ in S. Evidently $\mathcal{O}_S(\Gamma)=\det E$, and so from the exact sequence $0\to\mathcal{O}_S\to\mathcal{O}_S(\Gamma)\to N_\Gamma\to 0$ we see that what is required for (i) is to show that $h^0(E)\leqq 1+h^0(N_\Gamma)$. But since $H^0(E^*)=0$, it follows from (1.2) and (1.3) that

$$h^0(E)=n+h^0(B)\leqq h^0(N_\Gamma\otimes B^*)+h^0(B).$$

So for (i) we are reduced to proving the inequality

(*) $$h^0(N_\Gamma\otimes B^*)+h^0(B)\leqq 1+h^0(N_\Gamma).$$

To this end, note from (1.2) and (1.3) that both B and $N_\Gamma\otimes B^*$ are generated by their global sections. Hence the rank two vector bundle $B\oplus(N_\Gamma\otimes B^*)$ on Γ is globally generated, and so a general section determines an exact sequence $0\to\mathcal{O}_\Gamma\to B\oplus(N_\Gamma\otimes B^*)\to N_\Gamma\to 0$ of bundles on Γ. Taking cohomology yields

(**) $$h^0(B)+h^0(N_\Gamma\otimes B^*)\leqq h^0(\mathcal{O}_\Gamma)+h^0(N_\Gamma).$$

Now Γ moves in a base-point free linear system thanks to the fact that E is generated by its global sections (cf. (3.1)(i) below), and if $\Gamma\cdot\Gamma=c_1(E)^2>0$, then it follows from the Mumford-Ramanujam vanishing theorem (or directly from Bertini's theorem) that $H^1(\mathcal{O}_S(-\Gamma))=0$. Consequently $h^0(\mathcal{O}_\Gamma)=1$, and (*) then follows from (**). This proves statement (i) of the Proposition.

Turning to assertion (ii), assume that $c_1(E)^2=\Gamma\cdot\Gamma=0$. Since Γ moves in a base-point free linear system, and since S is regular, it follows that the morphism $\Phi_{|\Gamma|}\colon S\to\mathbb{P}^m$ determined by Γ is composed of a rational pencil; i.e. there is a smooth irreducible curve $\Sigma\subset X$ defining a map $\Phi_{|\Sigma|}\colon S\to\mathbb{P}^1$ with connected fibres, and $\Gamma\equiv m\Sigma$ for some $m\geqq 0$. Keeping the notation introduced above, we claim:

(1.4) There exists a subspace $V \subset H^0(E)$ such that the divisor $\Gamma \subset S$ defined by the vanishing of $\det(e_V)$ consists of the disjoint union of m smooth irreducible curves $\Sigma_1, \ldots, \Sigma_m \in |\Sigma|$.

Granting this, the line bundle B on Γ is the direct sum $B = \oplus B_i$ of line bundles B_i on Σ_i. But both B and $B^* \otimes N_\Gamma = B^*$ are generated by their global sections, and it follows that $B_i = \mathcal{O}_{\Sigma_i}$ for all i. Then (1.3) takes the form:

$$0 \to E^* \to V^* \otimes_{\mathbb{C}} \mathcal{O}_S \xrightarrow{\rho} \bigoplus_{i=1}^{m} \mathcal{O}_{\Sigma_i} \to 0.$$

Since $H^0(E^*) = H^1(E^*) = 0$, ρ must induce an isomorphism on global sections; in particular, $m = n$. But then ρ decomposes as the direct sum of the canonical maps $\rho_i\colon \mathcal{O}_S \to \mathcal{O}_{\Sigma_i}$, and hence $E^* = \oplus \ker \rho_i = \oplus \mathcal{O}_S(-\Sigma_i)$, as required.

To verify (1.4), we may argue e.g. as follows. Choose a general subspace $W \subset H^0(E)$ of dimension $n-1$. Then W determines an exact sequence

$$(*) \qquad 0 \to W \otimes_{\mathbb{C}} \mathcal{O}_S \to E \to I_Z \otimes \mathcal{O}_S(m\Sigma) \to 0,$$

where $Z \subset S$ is a finite or empty subscheme representing $c_2(E)$, and I_Z is its ideal sheaf. Now since E is generated by its global sections, one has the inequalities $c_2(E) \geqq 0$ and $c_1(E)^2 - c_2(E) \geqq 0$; and since $c_1(E)^2 = 0$, it follows that $Z = \emptyset$. With this in mind, let s_0 be *any* section of $\mathcal{O}_S(m\Sigma)$. Then one sees from $(*)$ that s_0 lifts to a section $s \in H^0(E)$. On the other hand, if $V \subset H^0(E)$ is the subspace spanned by s and W, then the divisor defined by the vanishing of $\det(e_V)$ is precisely the divisor of zeroes of s_0. This implies (1.4) and completes the proof of statement (ii). □

We will have occasion to draw on the following slight strengthening of (1.1) for bundles on a $K3$ surface:

(1.5) **Proposition.** *Let X be a projective $K3$ surface, and E a vector bundle on X with $H^2(E) = 0$. Then Proposition* (1.1) *remains valid if one assumes only that E is generated by its global sections away from finitely many points.*

Note that the curve Σ occuring in case (ii) is elliptic, since these are the only curves on X which move in a pencil (cf. [St.-D.]). As for (1.5), the main point to observe is the following:

(1.6) **Lemma.** *Let E be a vector bundle on the $K3$ surface X which is generated by its global sections away from finitely many points. Then there exists a globally generated vector bundle F on X with* $\det F = \det E$, *$h^0(F) \geqq h^0(E)$, and $h^i(F) = h^i(E)$ for $i \geqq 1$.*

Remark. Note that this result generalizes – and gives a new proof of – the classical fact ([St.-D.], Theorem 3.1) that a linear system on a $K3$ surface without fixed components is actually base-point free. (Take E to have rank 1, and observe that if F is generated by its global sections, then so too is $\det F$.)

Proof of Lemma (1.6). We prove the lemma first under the additional hypothesis that $H^0(E^*) = 0$. Denote by V and S_E respectively the kernel and cokernel of the canonical evaluation map e: $H^0(E)\otimes_{\mathbb{C}}\mathcal{O}_X\to E$, so that one has the exact sequence

$$0\to V\to H^0(E)\otimes_{\mathbb{C}}\mathcal{O}_X\to E\to S_E\to 0 \tag{1.7}$$

of $\mathcal{O}_X$-modules. By assumption S_E is supported on a finite set, and hence $Ext^i(S_E,\mathcal{O}_X)=0$ for $i\leqq 1$. Furthermore, V – being a second syzygy sheaf – is locally free. Thus one deduces from (1.7) the exact sequence:

$$0\to E^*\to H^0(E)^*\otimes_{\mathbb{C}}\mathcal{O}_X\to V^*\to R_E\to 0, \tag{1.8}$$

where $R_E=Ext^2(S_E,\mathcal{O}_X)$. By Serre duality the map $H^2(E^*)\to H^0(E)^*$ determined by (1.8) is an isomorphism, and it follows that $H^0(V^*)$ surjects onto $H^0(R_E)$. Since $H^0(E^*)=0$, one may thus choose a subspace $W\subset H^0(V^*)$, fitting into an exact sequence $0\to H^0(E)^*\to W\to H^0(R_E)\to 0$, such that W generates V^*. Let F be the dual of the kernel of the surjective map $W\otimes_{\mathbb{C}}\mathcal{O}_X\to V^*$. Thus F is generated by its global sections, and one has an exact sequence

$$0\to E^*\to F^*\to H^0(R_E)\otimes_{\mathbb{C}}\mathcal{O}_X\to R_E\to 0. \tag{1.9}$$

One sees from (1.9) that $\det(F)=\det(E)$, $h^i(E^*)=h^i(F^*)$ for $i\leqq 1$, and that $h^2(E^*)\leqq h^2(F^*)$. The assertions of the Lemma then follow by duality. Finally, if $H^0(E^*)\neq 0$, then $E=E_0\oplus H$, where H is a trivial vector bundle and $H^0((E_0)^*)=0$; it suffices to take $F=F_0\oplus H$. □

Proof of Proposition (1.5). If $c_1(E)^2>0$, the required inequality follows immediately from (1.1) and (1.6). So assume that $c_1(E)^2=0$, and consider the vector bundle F constructed in Lemma 1.6. Then Proposition (1.1) applies, and $F=\bigoplus\mathcal{O}_X(\Sigma)$ for some elliptic curve $\Sigma\subset X$. The exact sequence dual to (1.9) then takes the form

$$0\to H\to\bigoplus\mathcal{O}_X(\Sigma)\to E\to\tau\to 0,$$

where H is a trivial vector bundle, and $\dim\operatorname{supp}(\tau)=0$. Now the support of τ is precisely the set of points at which the vector bundle map v: $H\to\oplus\mathcal{O}_X(\Sigma)$ on the left drops rank. But an element in $\ker v(x)$ gives rise to a section $s\in H^0(\bigoplus\mathcal{O}_X(\Sigma))$ vanishing at x, and since Σ moves in a base-point free pencil, it follows that s must then vanish on a divisor. Hence v either drops rank on a divisor or not at all, and therefore $\tau=0$. But then the vanishing of $H^1(E)=H^1(E^*)$ forces $H=0$. □

§ 2. The bundles $E(C,A)$ and their reductions

In this section we study the vector bundles $E(C,A)$, and carry out step (1) of the argument outlined in the Introduction.

Let X be a projective $K3$ surface, and $C\subset X$ a smooth irreducible curve of genus g. If A is a line bundle on C such that both A and $\omega_C\otimes A^*$ are generated by their global sections, then as in [L] one can associate to the pair (C,A) a vector bundle $E(C,A)$ on X, of rank $h^0(A)$, as follows. Thinking of A as a sheaf on X, there is a canonical surjective evaluation map

$$e(C,A)\colon\ H^0(A)\otimes_{\mathbb{C}}\mathcal{O}_X \twoheadrightarrow A$$

of $\mathcal{O}_X$-modules. Since A is locally isomorphic to $\mathcal{O}_C$, it has homological dimension one over $\mathcal{O}_X$. Therefore $\ker e(C,A)$ is locally free, and we take $E(C,A)$ to be the dual of this kernel:

$$E(C,A) =_{\mathrm{def}} (\ker e(C,A))^*.$$

We recall the basic properties of these bundles. To begin with, one has by construction the exact sequence

(2.1) $$0 \to E(C,A)^* \to H^0(A)\otimes_{\mathbb{C}}\mathcal{O}_X \to A \to 0$$

of sheaves on X. Since $\omega_X = \mathcal{O}_X$, dualizing (2.1) gives:

(2.2) $$0 \to H^0(A)^* \otimes_{\mathbb{C}}\mathcal{O}_X \to E(C,A) \to \omega_C \otimes A^* \to 0,$$

and from (2.1) and (2.2) one deduces:

(2.3) $$\det E(C,A) = \mathcal{O}_X(C), \qquad c_2(E(C,A)) = \deg(A);$$

(2.4) $E(C,A)$ is generated by its global sections [recall: $h^1(\mathcal{O}_X)=0$];

(2.5) $$H^1(E(C,A)) = H^2(E(C,A)) = 0;$$
$$h^0(E(C,A)) = h^0(A) + h^1(A) = g(C) + 1 - \mathrm{Cliff}(A) \quad [\text{cf. } (0.4)].$$

Furthermore, one has an exact sequence

(2.6) $$0 \to E(C,\omega_C\otimes A^*)^* \to H^0(E(C,A))\otimes_{\mathbb{C}}\mathcal{O}_X \to E(C,A) \to 0$$

(cf. [T], Lemma I.2.6).

Ideally, one would like to produce the bundle L required by the main theorem (or a related bundle) as a subsheaf of $E(C,A)$, where $C\subset X$ is a curve of minimal Clifford index in its linear series, and A is a suitable bundle computing the Clifford index of C. However, we do not know whether this is possible. So we try to "cut down" $E(C,A)$ by modding out by sections vanishing at two or more points; eventually one arrives at a bundle that does contain a suitable rank one subsheaf.

We formalize this reduction procedure in the following definition. It differs slightly from the one given in the Introduction due to the possible presence here of rational curves.

(2.7) **Definition.** *Let E_0 be a vector bundle on X, of* rank $\geqq 2$. *We will say that a bundle E of* rank $\geqq 2$ *on X is a* reduction *of E_0 if it satisfies the following properties:*

(R 1) *There is a map $E_0 \to E$ which is surjective off a finite set;*

(R 2) *$h^0(E) \geqq h^0(E_0)$, and $h^i(E) = h^i(E_0)$ for $i>0$;*

(R 3) *$\det E = \det E_0 \otimes \mathcal{O}_X(-D)$ for some effective or zero divisor D;*

(R 4) $c_1(E)^2 - 4c_2(E) + 8\cdot rk(E) \geqq c_1(E_0)^2 - 4c_2(E_0) + 8\cdot rk(E_0)$.

We say that a reduction E of E_0 is *minimal* if there is no reduction E_1 of E_0 with $rk(E_1) < rk(E)$. (In particular, if $rk(E)=2$, then E is already minimal.)

Note that if E_0 is generated by its global sections away from finitely many points, then so too is any reduction thanks to (R1). Observe also that if E_1 is a reduction of E_0, and if E is a reduction of E_1, then E is a reduction of E_0. Finally, remark that E_0 is a reduction of itself, so any bundle of rank $\geqq 2$ has a minimal reduction.

The point of this definition is manifested in

(2.8) **Proposition.** *Let $C \subset X$ be a smooth irreducible curve, and let A be a line bundle on C, such that both A and $\omega_C \otimes A^*$ are generated by their global sections. Assume that*

$$r(A) \geqq 1, \quad \deg(A) \leqq g(C)-1 \quad \textit{and} \quad \mathrm{Cliff}(A) < [(g(C)-1)/2].$$

If E is a minimal reduction of $E(C,A)$, then there exists a line bundle N on X, with $h^0(N) \geqq 2$, which admits a non-zero map $N \to E$.

Proof. Set $E_0 = E(C,A)$. To begin with, note that by (R2) and the computations of $H^i(E_0)$ above, one has:

$$h^0(E) \geqq g(C)+1-\mathrm{Cliff}(A), \quad \text{and} \quad H^1(E)=H^2(E)=0.$$

Furthermore, E is generated by its global sections away from finitely many points since E_0 is.

We assume first $rk(E)=2$, and argue much as in [L, Lemma (1.3)]. Specifically, we claim that:

$$h^0(E \otimes E^*) \geqq 2. \tag{2.9}$$

In fact, by (R4) and the hypothesis on Cliff(A) one has:

$$\begin{aligned} c_1(E)^2 - 4 \cdot c_2(E) + 8 &\geqq c_1(E_0)^2 - 4 \cdot c_2(E_0) + 8r \\ &= 2g(C) - 2 - 4 \cdot \mathrm{Cliff}(A) \\ &\geqq 4. \end{aligned}$$

But

$$\chi(E \otimes E^*) = 2 \cdot h^0(E \otimes E^*) - h^1(E \otimes E^*) = c_1(E)^2 - 4 \cdot c_2(E) + 8$$

by Serre duality and Riemann-Roch, and this proves (2.9).

It follows from (2.9) by a well-known argument that there is a non-zero map $v\colon E \to E$ which drops rank everywhere on X. [Take any endomorphism w of E, $w \neq (\text{const}) \cdot 1$, and set $v = w - \lambda \cdot 1$, where λ is an eigenvalue of $w(x)$ for some $x \in X$, Then $\det(v) \in H^0(\det(E^*) \otimes \det(E)) = H^0(\mathcal{O}_X)$ vanishes at x, and hence is identically zero.] Let $N = (\mathrm{im}\, v)^{**}$ be the reflexive hull of $\mathrm{im}(v)$. Then N is a line bundle which sits as a subsheaf of E, whence $\mathrm{Hom}(N, E) \neq 0$. Furthermore, v gives a map $E \to N$ which is surjective off a finite set. But E is generated by its global sections away from finitely many points, and hence so too is N. Therefore $h^0(N) \geqq 2$ unless $N = \mathcal{O}_X$; but this possibility is ruled out since $H^2(E) = \mathrm{Hom}(E, \mathcal{O}_X)^* = 0$. This proves the Proposition when $rk(E)=2$.

Assume henceforth that $rk(E) \geqq 3$, and fix once and for all a point $x_0 \in X$ not lying on any smooth rational curve; this is possible since there are at most countably many such curves on X. We claim then that:

(2.10) There is a point $y \in X$, $y \neq x_0$, plus a section $s_0 \in H^0(E)$ vanishing at x_0 and y.

In fact, fix an arbitrary point $x \in X$. By (R2), (2.5) and the hypothesis that $\deg(A) \leqq g(C) - 1$, one has:

$$h^0(E) \geqq h^0(E_0) = g(C) + 1 - \deg(A) + 2r(A) \geqq 2 \cdot rk(E_0) \geqq 2 \cdot rk(E).$$

Therefore $h^0(E \otimes I_x) \geqq rk(E)$, where I_x denotes the ideal sheaf of x. If either $h^0(E \otimes I_x) > rk(E)$, or the subspace $H^0(E \otimes I_x) \subset H^0(E)$ maps by evaluation to a subsheaf F of E with $rk(F) < rk(E)$, then $H^0(E \otimes I_x \otimes I_y) \neq 0$ for a general point $y \in X$. The remaining possibility is that the natural map v: $H^0(E \otimes I_x) \otimes_{\mathbb{C}} \mathcal{O}_X \to E$ is generically an isomorphism. But $H^2(E) = 0$, and in particular E is non-trivial. Thus v drops rank along a curve $\Gamma \subset X$, and it suffices to take $y \in \Gamma$, and $s_0 \in \ker v(y) \subset H^0(E \otimes I_x)$.

Let $\Delta \subset X$ be the largest effective (or zero) divisor along which the section s_0 in (2.10) vanishes. Then s_0: $\mathcal{O}_X \to E$ factors through a map s: $\mathcal{O}_X(\Delta) \to E$ which vanishes on a finite (or empty) subscheme $Z \subset X$, and one has an exact sequence

$$0 \to G \to E^* \xrightarrow{s^*} \mathcal{O}_X(-\Delta) \to \mathcal{O}_Z(-\Delta) \to 0 \tag{2.11}$$

defining a sheaf G. Note that G – being a second syzygy sheaf – is locally free, and that $\operatorname{im} s^* = I_Z(-\Delta)$, where I_Z is the ideal sheaf of Z in X. Dualizing (2.11) gives the exact sequence

$$0 \to \mathcal{O}_X(\Delta) \xrightarrow{s} E \to F \to \tau \to 0, \tag{2.12}$$

where $F = G^*$ and $\tau = \mathcal{E}xt^2(\mathcal{O}_Z(-\Delta), \mathcal{O}_X)$. There are now three possibilities:

(i) $\Delta = 0$;

(ii) $\Delta \neq 0$, but $h^0(\mathcal{O}_X(\Delta)) = 1$;

(iii) $h^0(\mathcal{O}_X(\Delta)) \geqq 2$.

In case (iii), the bundle $N = \mathcal{O}_X(\Delta)$ satisfies the assertion of the Proposition, and we are done. In cases (i) and (ii), the strategy is to show that the bundle F, which has rank $\geqq 2$, is a reduction of E. This will contradict the minimality of E, and will complete the proof.

Assume then that $\Delta = 0$, and let us check that F is a reduction of E. Properties (R1) and (R3) are evident from (2.12). Turning to (R2), note that since $\Delta = 0$, Z contains at least the two points x_0 and y appearing in (2.10). Therefore

$$h^0(I_Z) = 0, \quad h^2(I_Z) = 1 \quad \text{and} \quad h^1(I_Z) \geqq 1.$$

Recalling that $h^0(E^*) = h^1(E^*) = 0$, it follows from (2.11) that

$$h^0(G) = h^1(G) = 0, \quad \text{and} \quad h^2(G) \geqq h^2(E^*).$$

Since $G=F^*$, (R2) is then a consequence of Serre duality. Finally, in view of the fact that $c_1(E)=c_1(F)$, it is enough for (R4) to show that $c_2(F)\leqq c_2(E)-2$. Calculating from (2.11), one finds using (0.3):

$$c_2(E^*)=c_2(G)+\operatorname{length}(Z)\geqq c_2(G)+2.$$

But $c_2(E^*)=c_2(E)$ and $c_2(G)=c_2(F)$, and thus F is indeed a reduction of E. Hence case (i) is ruled out by the minimality of E.

Finally, suppose that $\Delta\neq 0$ and $h^0(\mathcal{O}_X(\Delta))=1$. Then $H^2(\mathcal{O}_X(\Delta))=0$, $\Delta\cdot\Delta\leqq -2$ by Riemann-Roch, and Δ is supported on a union of smooth rational curves (cf. [St.-D.]). In particular, the point x_0 does not lie on Δ. But $s_0(x_0)=0$, and therefore s must vanish at x_0, i.e. $x_0\in Z=\operatorname{supp}(\tau)$. Turning to the verification that F is a reduction of E, (R1) and (R3) are again clear from (2.12). Referring to (2.11), since Z is non-empty one has

$$h^0(I_Z(-\Delta))=0,\quad h^2(I_Z(-\Delta))=1\quad\text{and}\quad h^1(I_Z(-\Delta))\geqq 1,$$

and (R2) follows as above. It remains to check (R4). To this end, let Γ be a divisor representing $c_1(E)$. Since E is generated by its global sections away from finitely many points, Γ moves in a linear system without fixed components (cf. (3.1)(i)), and hence $\Gamma\cdot\Delta\geqq 0$. Computing e.g. from (2.12) one finds:

$$c_1(F)^2=c_1(E)^2-2(\Gamma\cdot\Delta)+\Delta\cdot\Delta$$

and

$$c_2(F)=c_2(E)-\Delta\cdot(\Gamma-\Delta)-\operatorname{length}(\tau).$$

But $\operatorname{length}(\tau)\geqq 1$ since $x_0\in\operatorname{supp}(\tau)$, and consequently

$$\begin{aligned}c_1(F)^2-4c_2(F)&=c_1(E)^2-4c_2(E)+2(\Gamma\cdot\Delta)-3(\Delta\cdot\Delta)+4\cdot\operatorname{length}(\tau)\\&\geqq c_1(E)^2-4c_2(E)+0+6+4.\end{aligned}$$

This verifies (R4), and thus F is a reduction of E. Again we have a contradiction to the minimality of E, and this completes the proof of the Proposition. □

§ 3. Proof of the theorem

In this section we complete the proof of the theorem stated in the Introduction. As always, X is a smooth projective $K3$ surface.

We start by collecting together some elementary observations:

(3.1) **Lemma.** *Let E be a vector bundle of rank n on X which is generated by its global sections away from finitely many points. Suppose given an exact sequence*

$$0\to N\to E\to F\to\tau\to 0,$$

where N is a line bundle with $h^0(N)\geqq 2$, F is locally free, and τ is supported on a finite set. Assume that $H^2(E)=0$, and let Γ be a divisor representing $c_1(E)$. Then

(i) *Γ moves in a linear system without fixed components, and* $h^0(\mathcal{O}_X(\Gamma)) \geqq 2$.

(ii) *F is generated by its global sections off a finite set, and* $N \otimes \det F = \mathcal{O}_X(\Gamma)$. *Furthermore,* $H^2(F)=0$, $h^1(F) \leqq h^1(E)$, *and* $h^0(\det F) \geqq 2$.

(iii) *If $C \subset X$ is any curve such that* $H^0(\mathcal{O}_X(C-\Gamma)) \neq 0$, *then*

$$h^0(N \otimes \mathcal{O}_C) \geqq h^0(N) \geqq 2 \quad \textit{and} \quad h^1(N \otimes \mathcal{O}_C) \geqq h^0((\det F)) \geqq 2.$$

Proof. (i) The first statement follows from the observation that the evident map $\Lambda^n H^0(E) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Lambda^n E = \mathcal{O}_X(\Gamma)$ is surjective off a finite set. For the second, it is then enough to show that $\det E \neq \mathcal{O}_X$. If $V \subset H^0(E)$ is a general subspace of dimension n, then the natural homomorphism u: $V \otimes_{\mathbb{C}} \mathcal{O}_X \to E$ is injective. But u cannot be an isomorphism since $h^2(E)=0$, and consequently $\Lambda^n u$ determines a section of $\det E$ vanishing on a non-zero effective divisor.

(ii) The first assertion is clear. One has $c_1(\tau)=0$ thanks to the fact that τ is supported on a finite set, and the second statement follows. Now $h^2(N) = h^0(N^*)=0$ since $h^0(N) \geqq 2$, and evidently $H^1(\tau)=H^2(\tau)=0$. Therefore the homomorphism $H^i(E) \to H^i(F)$ induced by the given sequence is surjective for $i=1$ and bijective when $i=2$. The last assertion follows from (i).

(iii) Let $D \in |C-\Gamma|$ be an effective (or zero) divisor. By (ii), $N^*(C) = (\det F) \otimes \mathcal{O}_X(D)$ is the line bundle associated to non-zero effective divisor, and hence $H^0(N(-C))=0$. The first statement follows. For the second, the vanishing of $H^2(N)$ (noted above) gives

$$h^1(C, N \otimes \mathcal{O}_C) \geqq h^2(X, N(-C)) = h^2(X, (\det F)^*(-D)) = h^0(X, (\det F)(D)).$$

But $h^0((\det F)(D)) \geqq h^0(\det F)$ since D is effective. □

We now turn to the

Proof of the theorem. Among all smooth irreducible curves in a given linear series on X choose one - say C - of minimal Clifford index. We may assume that $\mathrm{Cliff}(C) < [(g(C)-1)/2]$, for otherwise there is nothing to prove. It suffices to produce a line bundle L on X, whose restriction to every smooth curve $C' \in |C|$ contributes to the Clifford index of C', such that

$$\mathrm{Cliff}(L \otimes \mathcal{O}_{C'}) \leqq \mathrm{Cliff}(C). \tag{3.2}$$

Let A be a line bundle on C, with $\deg(A) \leqq g(C)-1$, computing the Clifford index of C. Then both A and $\omega_C \otimes A^*$ are generated by their global sections: for if not, by removing base-points one would arrive at a bundle with strictly smaller Clifford index. Consequently we are in the situation of Proposition (2.8), and there exists a reduction E of $E(C, A)$ and a line bundle N on X, with $h^0(N) \geqq 2$, such that $\mathrm{Hom}(N, E) \neq 0$. Replacing N if necessary by $N(\Delta)$ for some effective divisor Δ, we can assume as in (2.11) and (2.12) that one has an exact sequence

$$0 \to N \to E \to F \to \tau \to 0 \tag{3.3}$$

of $\mathcal{O}_X$-modules, where F is locally free and τ is supported on a finite set. Recall that E - begin a reduction $E(C, A)$ - is generated by its global sections away from finitely many points, and that $\det E = \mathcal{O}_X(C-D)$ for some effective (or zero) curve $D \subset X$. Furthermore, $h^0(E) \geqq g(C)+1-\mathrm{Cliff}(A)$, and $H^1(E)=H^2(E)$

$=0$. In particular Lemma (3.1) applies, and it follows from statement (iii) of that lemma that the restriction of N to any smooth $C' \in |C|$ contributes to the Clifford index of C'. We now proceed in two cases.

Case 1. $c_1(F)^2 > 0$. Then Proposition (1.1)(i) applies by virtue of (1.5), and for any smooth $C' \in |C|$:

$$\begin{aligned} g(C)+1-\mathrm{Cliff}(A) &\leqq h^0(E) & \\ &\leqq h^0(N)+h^0(F) & [\text{by } (3.3)] \\ &\leqq h^0(N)+h^0(\det F) & [\text{by } (1.1)(\text{i})] \\ &\leqq h^0(N\otimes \mathcal{O}_{C'})+h^1(N\otimes \mathcal{O}_{C'}) & [\text{by } (3.1)(\text{iii})] \\ &= g(C')+1-\mathrm{Cliff}(N\otimes \mathcal{O}_{C'}) & [\text{by } (0.4)]. \end{aligned}$$

So it suffices for (3.2) to take $L=N$.

Case 2. $c_1(F)^2=0$. Then we assert to begin with:

(3.4) There is a line bundle N_0 on X, with $h^0(N_0) \geqq 2$, plus a map $N_0 \to E(C, \omega_C \otimes A^*)$ vanishing on a finite or empty set.

In fact, $H^1(F)=0$ thanks to (3.1)(ii), and therefore Propositions (1.1)(ii) and (1.5) show that $F = \bigoplus \mathcal{O}_X(\Sigma)$ for some elliptic curve $\Sigma \subset X$. Since E is a reduction of $E(C,A)$, it follows from (3.3) and property (R1) in Definition (2.7) that there is a map μ: $E(C,A) \to \mathcal{O}_X(\Sigma)$ which is surjective away from finitely many points. On the other hand, since Σ moves in a base-point free pencil, the kernel of the natural map $H^0(\mathcal{O}_X(\Sigma)) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X(\Sigma)$ is just $\mathcal{O}_X(-\Sigma)$. Therefore, recalling the exact sequence (2.6), one obtains an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & E(C,\omega_C\otimes A^*)^* & \longrightarrow & H^0(E(C,A))\otimes_{\mathbb{C}}\mathcal{O}_X & \longrightarrow & E(C,A) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mu} & \\ 0 \to & \mathcal{O}_X(-\Sigma) & \longrightarrow & H^0(\mathcal{O}_X(\Sigma))\otimes_{\mathbb{C}}\mathcal{O}_X & \longrightarrow & \mathcal{O}_X(\Sigma) & \to 0. \end{array}$$

The vertical map on the left cannot vanish, for otherwise μ would factor through a non-zero map $E(C,A) \to H^0(\mathcal{O}_X(\Sigma)) \otimes_{\mathbb{C}} \mathcal{O}_X$. But this is impossible since $\mathrm{Hom}(E(C,A), \mathcal{O}_X)=0$ by (2.5). Thus there exists a non-zero homomorphism

$$v\colon \ \mathcal{O}_X(\Sigma) \to E(C, \omega_C \otimes A^*),$$

and if Δ is the largest effective or zero divisor along which v vanishes, then (3.4) follows by taking $N_0 = \mathcal{O}_X(\Sigma+\Delta)$.

From (3.4) one deduces an exact sequence

$$0 \to N_0 \to E(C, \omega_C \otimes A^*) \to F_0 \to \tau_0 \to 0, \tag{3.5}$$

where F_0 is locally free and τ_0 is supported on a finite set. Note that Lemma (3.1) applies to this sequence, and in particular the restriction of N_0 to any smooth $C' \in |C|$ contributes to the Clifford index of C'. If $c_1(F_0)^2 > 0$, then just as in Case 1 one finds that $\mathrm{Cliff}(N_0 \otimes \mathcal{O}_{C'}) \leqq \mathrm{Cliff}(\omega_C \otimes A^*)$; but $\mathrm{Cliff}(\omega_C \otimes A^*) = \mathrm{Cliff}(A)$, so in this case it suffices to take $L=N_0$ in (3.2).

There remains the possibility that $c_1(F_0)^2=0$. Then $F_0=\bigoplus\mathcal{O}_X(\Sigma_0)$ for some elliptic curve $\Sigma_0\subset X$ by Proposition (1.1)(ii), and we claim that in this case (3.2) holds with $L=\mathcal{O}_X(\Sigma_0)$. In fact, let D_0 be a curve representing $c_1(N_0)$, and write

$$e=\deg(\omega_C\otimes A^*) \quad\text{and}\quad s=h^0(\omega_C\otimes A^*)-1.$$

Computing first Chern classes in (3.5) one finds

$$C\equiv D_0+s\cdot\Sigma_0,$$

and as in the proof of Lemma (3.1)(iii) it follows first of all that

$$h^0(\mathcal{O}_X(\Sigma_0)\otimes\mathcal{O}_{C'})\geqq 2 \quad\text{and}\quad h^1(\mathcal{O}_X(\Sigma_0)\otimes\mathcal{O}_{C'})\geqq 2$$

for all smooth $C'\in|C|$. So we need only check that $\operatorname{Cliff}(\mathcal{O}_X(\Sigma_0)\otimes\mathcal{O}_{C'})\leqq\operatorname{Cliff}(A)$. Now $c_2(F_0)=0$, and so computing from (3.5) and (0.3) one finds:

$$e=c_2(E(C,\omega_C\otimes A^*))\geqq D_0\cdot(s\,\Sigma_0).$$

Then for any $C'\in|C|$:

$$\begin{aligned}\operatorname{Cliff}(\omega_C\otimes A^*)=e-2s&\geqq s\cdot\{(D_0\cdot\Sigma_0)-2\}\\&=s\cdot\{(C'\cdot\Sigma_0)-2\} &&[\text{since } C'\cdot\Sigma_0=D_0\cdot\Sigma_0]\\&\geqq s\cdot\operatorname{Cliff}(\mathcal{O}_X(\Sigma_0)\otimes\mathcal{O}_{C'}) &&[\text{since } h^0(\mathcal{O}_{C'}(\Sigma_0))\geqq 2].\end{aligned}$$

But $s=h^1(A)-1\geqq 1$ since A contributes to the Clifford index of C, and it follows that

$$\operatorname{Cliff}(\mathcal{O}_X(\Sigma_0)\otimes\mathcal{O}_{C'})\leqq\operatorname{Cliff}(\omega_C\otimes A^*)=\operatorname{Cliff}(A)$$

as desired. This completes the proof of the Theorem. □

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