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On the Ramification of Branched Coverings of $\mathbb{P}^n$

Terence Gaffney* and Robert Lazarsfeld

Department of Mathematics, Brown University, Providence, RI 02912, USA

Introduction

Our purpose is to prove a rather surprising result concerning the ramification of branched coverings of $\mathbb{P}^n$.

Let X be an irreducible projective variety of dimension n over an algebraically closed field k, and let $f: X \to \mathbb{P}^n$ be a finite morphism. Denote by d the geometric degree of f, i.e., the separable degree of the extension $k(X)/k(\mathbb{P}^n)$ of function fields. Recall that this degree is characterized by the fact that almost all points of $\mathbb{P}^n$ have precisely d preimages in X. For each $x \in X$, one can define the local degree $e_f(x)$ of f at x, which may be thought of as the number of sheets of the covering $X \to \mathbb{P}^n$ that come together at x. When X is a non-singular complex variety, $e_f(x)$ coincides with the usual topological local degree.

Our main result generalizes the classical fact that any non-trivial irreducible covering of $\mathbb{P}^n$ ramifies:

Theorem 1. *There exists at least one point $x \in X$ at which*

$$e_f(x) \geqq \min(d, n+1).$$

The theorem asserts, for example, that if $f: X \to \mathbb{P}^n$ has geometric degree $\geqq n+1$, then $n+1$ or more branches of the covering must come together at some point of X.

As a simple application of Theorem 1, we deduce

Theorem 2. *If X is normal, and admits a branched covering $f: X \to \mathbb{P}^n$ of geometric degree $\leqq n$, then X is algebraically simply connected.*

It follows for instance that if X is an n-dimensional abelian variety, then any finite morphism $f: X \to \mathbb{P}^n$ must have geometric degree at least $n+1$. For complex tori, this fact was noticed by Sommese et al. ([BE, §4]).

The paper is divided into two parts. The first is devoted to the definition and formal properties of the local degree. Much of this section can be skipped by the

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reader interested only in the case of complex varieties. Proofs of the theorems appear in §2, where our basic tool is the remarkable connectedness theorem of Fulton and Hansen ([FH]).

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§1. The Local Degree

By way of motivation, we review briefly the definition of local degrees in the complex case, referring to [M, Appendix to Chap. 6] for details. Consider a finite surjective morphism $f\colon X \to Y$ of irreducible complex varieties, and assume that Y is normal. Given $x \in X$, let $y = f(x)$. For a sufficiently small connected neighborhood $U(y)$ of y in the complex topology, the inverse image $f^{-1}(U(y))$ will split up into a disjoint union of neighborhoods of the preimages of y:

$$f^{-1}(U(y)) = \coprod_{f(x)=y} U(x). \tag{1.1}$$

The local degree $e_f(x)$ of f at x can be defined as the "degree" of the map $\operatorname{res}(f)\colon U(x) \to U(y)$, i.e., as the number of preimages in $U(x)$ of a generic point of $U(y)$. This integer has also a simple analytic interpretation. Namely, if K denotes the field of fractions of $\mathcal{O}_y^{\mathrm{an}} Y$, then $e_f(x) = \dim_K(\mathcal{O}_x^{\mathrm{an}} X \otimes_{\mathcal{O}_y^{\mathrm{an}} Y} K)$.

In the abstract case, one can proceed quite similarly; implicit use of the étale topology substitutes for the classical topology. Throughout the remainder of this section, we deal except when otherwise indicated with separated noetherian schemes, and call them simply schemes.

Let $f\colon X \to Y$ be a finite morphism of schemes, with Y integral. Denote by y the generic point of Y. We define the *geometric degree* $\deg_s f$ of f by setting

$$\deg_s f = \sum_{f(x)=y} [k(x)\colon k(y)]_s,$$

where the sum is taken over all $x \in \{f^{-1}(y)\}$. (If f is not dominating, put $\deg_s f = 0$.) Observe that if $Y' \to Y$ is a dominating morphism, with Y' integral, and if $f'\colon X \times_Y Y' \to Y'$ is the induced map, then $\deg_s f' = \deg_s f$.

Let Ω be an algebraically closed field, and let $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ be a geometric point of X located at $x \in X$. $\bar{x}$ determines an embedding of $k(x)$ into its separable closure in Ω, and we denote by $\mathcal{O}_{\bar{x}} X$ the corresponding strict henselization of $\mathcal{O}_x X$ (c.f. [EGA IV.18.8], [SGA 4, VIII] or [R, Chap. 8]). We set $\bar{X}(\bar{x}) = \operatorname{Spec}(\mathcal{O}_{\bar{x}} X)$, so that there is a natural map $\bar{X}(\bar{x}) \to X$. Recall that $\bar{X}(\bar{x})$ is noetherian (since X is).

We henceforth consider a finite morphism $f\colon X \to Y$, with Y integral and normal. Let $\bar{x}$ be a geometric point of X, and let $\bar{y} = f \circ \bar{x}$. Then f induces a finite map $f_{\bar{x}}\colon \bar{X}(\bar{x}) \to \bar{Y}(\bar{y})$, and since Y is normal, $\bar{Y}(\bar{y})$ is integral. Set

$$e_f(\bar{x}) = \deg_s(f_{\bar{x}}).$$

The integer $e_f(\bar{x})$ depends only on the image of $\bar{x}$, and given $x \in X$ we put

$$e_f(x) = e_f(\bar{x}), \tag{1.2}$$

where $\bar{x}$ is any geometric point located at x. Provided that x lies in some irreducible component of X dominating Y, we have $e_f(x) > 0$. If f_{red} denotes the composition $X_{\text{red}} \hookrightarrow X \to Y$, then $e_{f_{\text{red}}}(x) = e_f(x)$. When x is a closed point of a complex variety X, one sees by passing to completions that (1.2) agrees with the analytic definition indicated above.

If $\bar{y}$: Spec$(\Omega) \to Y$ is a geometric point of Y, then

$$X \times_Y \bar{Y}(\bar{y}) = \coprod_{f \circ \bar{x} = \bar{y}} \bar{X}(\bar{x})$$

as $\bar{Y}(\bar{y})$-schemes, the sum being taken over all geometric points $\bar{x}$: Spec$(\Omega) \to X$ such that $f \circ \bar{x} = \bar{y}$ (c.f. [SGA 4, VIII.5.4]). Since the natural map $\bar{Y}(\bar{y}) \to Y$ is dominating, one obtains

$$\sum_{f \circ \bar{x} = \bar{y}} e_f(\bar{x}) = \deg_s f. \tag{1.3}$$

Hence

$$\sum_{f(x) = y} [k(x) : k(y)]_s \, e_f(x) = \deg_s f \tag{1.4}$$

for any $y \in Y$.

We shall need one more preliminary fact, which is the étale analogue of (1.1). Specifically, let $\bar{y}$ be a geometric point of Y. Then there exists a connected (and hence normal and integral) étale neighborhood $q: V \to Y$ of $\bar{y}$ having the following property:

There is a commutative diagram of cartesian squares:

$$\begin{tikzcd}
X \times_Y \bar{Y}(\bar{y}) \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[r, "p'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\bar{Y}(\bar{y}) \arrow[r, "s"'] & V \arrow[r, "q"'] & Y
\end{tikzcd} \tag{1.5}$$

with $q \circ s$ the natural map, such that the map r establishes a bijection between the connected components of $X \times_Y \bar{Y}(\bar{y})$ and those of U.

This is proved by a standard limit argument using [EGA IV.8.4.2], the construction of the strict henselization of a local ring, and [R, p. 55]. We omit details.

Lemma 1. *The function $x \mapsto e_f(x)$ is upper semicontinuous on X.*

Proof. Given $x_0 \in X$, we will show that $e_f(x) \leq e_f(x_0)$ for all x in some neighborhood of x_0. Choose a geometric point $\bar{x}_0$ located at x_0, and let $\bar{y}_0 = f \circ \bar{x}_0$. Apply the construction (1.5) to the geometric point $\bar{y}_0$. $\bar{X}(\bar{x}_0)$ is a connected component of $X \times_Y \bar{Y}(\bar{y}_0)$, and we denote by W the connected component of U

containing the image of $\bar{X}(\bar{x}_0)$. We then obtain a commutative diagram

$$\begin{tikzcd} \bar{X}(\bar{x}_0) \arrow[r] \arrow[d, "f_{\bar{x}_0}"'] \arrow[dr, phantom, "\square"] & W \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \bar{Y}(\bar{y}_0) \arrow[r] & V \arrow[r, "q"'] & Y \end{tikzcd}$$

the left-hand square being cartesian, with p and q étale, g finite, and V integral and normal.

Since $\bar{Y}(\bar{y}_0) \to V$ is dominating we get $\deg_s(g) = \deg_s(f_{\bar{x}_0}) = e_f(x_0)$, while it follows from (1.4) that $e_g(w) \leqq \deg_s(g)$ for all $w \in W$. On the other hand, since p and q are étale, one has $e_g(w) = e_f(p(w))$ for all $w \in W$. In short, $e_f(x) \leqq e_f(x_0)$ for all $x \in \mathrm{Im}(p)$. But $\mathrm{Im}(p)$ is an open neighborhood of x_0. QED.

One further property of the local degree will be needed in §2. For simplicity, we assume that all schemes involved in the following lemma are separated and of finite type over an algebraically closed field. As usual, we consider a finite morphism $f: X \to Y$ where Y is integral and normal.

Lemma 2. *Let T be an integral variety with generic point t. Let $c_1, c_2: T \to X$ be distinct morphisms, with $f \circ c_1 = f \circ c_2$, such that $c_1(t_0) = c_2(t_0) = x_0$ for some closed point $t_0 \in T$. Then*

$$e_f(x_0) \geqq e_f(c_1(t)) + e_f(c_2(t)).$$

Proof. Identifying closed points with the natural geometric points they determine, consider the set-up of (1.5) applied to $y_0 = f(x_0)$. Pick a point $v_0 \in V$ over y_0, and let $u_0 = (v_0, x_0) \in U$. It is enough to exhibit distinct geometric points $\bar{u}_1$, $\bar{u}_2: \mathrm{Spec}(\Omega) \to U$ located at generizations of u_0, with $g' \circ \bar{u}_1 = g' \circ \bar{u}_2$, such that $p' \circ \bar{u}_i$ is situated at $c_i(t)$ $(i=1,2)$. For then the assertion follows from (1.3) much as in the proof of Lemma 1.

Let $h = f \circ c_1$. Replacing T first by its normalization and then by a suitable connected component of $T \times_Y V$, we may assume that there is a map $j: T \to V$ with $j(t_0) = v_0$ and $q \circ j = h$. Then there exist morphisms $d_i: T \to U$ $(i=1,2)$ such that $p' \circ d_i = c_i$ and $g' \circ d_i = j$. Note that we have $d_1(t_0) = d_2(t_0) = u_0$. If $\bar{t}$ is a geometric point centered at t, then the hypothesis that the morphisms c_i are distinct implies that $c_1 \circ \bar{t} \neq c_2 \circ \bar{t}$. Taking $\bar{u}_i = d_i \circ \bar{t}$ thus gives the required geometric points of U. QED.

§2. Proofs of Theorems 1 and 2

In this section we shall deal with varieties over an algebraically closed field k, and with closed points. Keeping this convention in mind, consider a finite morphism $f: X \to Y$ of geometric degree d, where Y is irreducible and normal, and every irreducible component of X dominates Y. Then (1.4) reads simply

$$\sum_{f(x)=y} e_f(x) = \deg_s f, \tag{2.1}$$

and all of the local degrees are positive. Hence $\#\{f^{-1}(y)\} \leqq d$ for every $y \in Y$, and indeed the geometric degree is characterized by the fact that $\#\{f^{-1}(y)\} = d$ for almost all $y \in Y$. In particular, if Y' is an irreducible normal subvariety of Y, and if $f'\colon X' = f^{-1}(Y') \to Y'$ denotes the induced map, then $\deg_s f' \leqq \deg_s f$. Observe finally that for $x \in X'$, one has $e_{f'}(x) \leqq e_f(x)$, with equality if and only if $\deg_s f' = \deg_s f$. (The inequality follows from (2.1) if x is the only point in X lying over $f(x)$, and in general one uses (1.5) to reduce to this case.)

Now let X be an *irreducible* projective variety of dimension n, and let $f\colon X \to \mathbb{P}^n$ be a finite morphism of geometric degree d. For $\ell \geqq 0$, we introduce the sets

$$R_\ell = \{x \in X \mid e_f(x) \geqq \ell + 1\}.$$

By Lemma 1, these ramification loci are closed in X.

Theorem 1. *If $\ell \leqq \min(d-1, n)$, then R_ℓ is non-empty and has at least one irreducible component of codimension $\leqq \ell$ in X.*

Proof. Induction on n, the case $n=1$ being treated at the end of the proof. So take $n \geqq 2$, and consider $f\colon X \to \mathbb{P}^n$ as above. For a generic hyperplane $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$, the inverse image $X' = f^{-1}(\mathbb{P}^{n-1})$ is irreducible (cf. [J, Cor. 6.11]), and $f' = \operatorname{res}(f)\colon X' \to \mathbb{P}^{n-1}$ has geometric degree d. Since then $e_{f'}(x) = e_f(x)$ for all $x \in X'$, it follows by induction that R_ℓ has an irreducible component of codimension $\leqq \ell$ in X for each $\ell \leqq \min(d-1, n-1)$. So we may assume that $d \geqq n+1$, and we need only show the existence of a point $x \in X$ at which $e_f(x) \geqq n+1$.

This will follow from the Fulton-Hansen connectedness theorem ([FH]), which asserts that if V is an irreducible projective variety of dimension $> n$, and if $F\colon V \to \mathbb{P}^n \times \mathbb{P}^n$ is a finite morphism, then the inverse image $F^{-1}(\Delta_{\mathbb{P}^n}) \subseteq V$ of the diagonal $\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is connected. We choose an irreducible component S of R_{n-1} of dimension $\geqq 1$ and apply the connectedness theorem to the map

$$F = f \times f \mid S\colon X \times S \to \mathbb{P}^n \times \mathbb{P}^n.$$

Observe that the diagonal $\Delta_S \subseteq S \times S$ embeds in $X \times S$ as an irreducible component of $F^{-1}(\Delta_{\mathbb{P}^n})$.

If $F^{-1}(\Delta_{\mathbb{P}^n}) = \Delta_S$ we are done, for in this case $e_f(x) = d \geqq n+1$ for every $x \in S$. So we may assume that $\Delta_S \subsetneqq F^{-1}(\Delta_{\mathbb{P}^n})$. The connectedness of $F^{-1}(\Delta_{\mathbb{P}^n})$ then implies that there exists an irreducible component T of $F^{-1}(\Delta_{\mathbb{P}^n})$, $T \neq \Delta_S$, such that T meets Δ_S. Choose a point $t_0 = (x_0, x_0) \in \Delta_S \cap T$. Projection onto the factors of $X \times S$ yields distinct morphisms $c_1\colon T \to X$, $c_2\colon T \to S \subseteq X$, with $f \circ c_1 = f \circ c_2$. We have $e_f(c_1(t)) \geqq 1$ and $e_f(c_2(t)) \geqq n$ for all $t \in T$, and so we conclude from Lemma 2 that $e_f(x_0) \geqq n+1$, as desired.

Finally, to prove the theorem in the case $n=1$, one applies the connectedness theorem to the map $f \times f\colon X \times X \to \mathbb{P}^1 \times \mathbb{P}^1$, and argues as above. QED.

Remark. Concerning the dimensions of the sets R_ℓ, the second author has established the following result, which generalizes Zariski's theorem on the purity of the branch locus.

Let $f: X \to Y$ be a finite surjective morphism of irreducible varieties, with X normal and Y non-singular. Define ramification loci $R_\ell \subseteq X$ as above. If $R_\ell \neq \emptyset$, then every irreducible component of R_ℓ has codimension $\leqq \ell$ in X.

Details will appear elsewhere.

As a corollary of Theorem 1, we prove

Theorem 2. *Let X be an irreducible normal projective variety of dimension n. Assume that X admits a finite mapping $f: X \to \mathbb{P}^n$ of geometric degree $d \leqq n$. Then X is algebraically simply connected.*

Proof. Suppose to the contrary that $g: Y \to X$ is a connected étale covering of degree at least two. As X is normal, Y is irreducible, and the composition $h = f \circ g: Y \to \mathbb{P}^n$ has geometric degree $> d$. Then since $d \leqq n$, one has $\min(\deg_s(h) - 1, n) \geqq d$. Therefore, by Theorem 1, we may choose a point $y \in Y$ at which $e_h(y) \geqq d+1$. Since g is étale, $e_h(y) = e_f(g(y))$. But $e_f(g(y)) \leqq \deg_s f = d$, a contradiction. QED.

Remark. Theorem 2 is an indication of the fact that the existence of a finite morphism $f: X^n \to \mathbb{P}^n$ of low degree places topological restrictions on X analogous to those imposed on small codimensional subvarieties of projective space (c.f. [B], [H]). Indeed, the second author has proved the following Barth-type theorem for branched coverings of $\mathbb{C}\mathbb{P}^n$:

Let X be a connected complex projective manifold of dimension n, and let $f: X \to \mathbb{P}^n$ be a finite holomorphic mapping of degree d. Then the induced maps

$$f^*: H^i(\mathbb{P}^n, \mathbb{C}) \to H^i(X, \mathbb{C})$$

on cohomology are isomorphisms for $i \leqq n+1-d$.

The proof will appear in a forthcoming paper.

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