CONNECTIVITY AND ITS APPLICATIONS IN ALGEBRAIC GEOMETRY

by

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INTRODUCTION

A recent connectedness theorem [19] has been applied to a number of questions in algebraic geometry. It was originally used to prove several surprising results about subvarieties of, and mappings to, projective space [19, 20]. Shortly thereafter, it led to the solution of Zariski's problem on coverings of the plane branched along a node curve [16]. Deligne [10, 11] then generalized the connectedness theorem to a statement about the topological fundamental group π_1 as well as π_0, and used this to prove Zariski's assertion that the fundamental group of the complement of a nodal plane curve is abelian. He later showed [12] how a conjecture since proved by Goresky and MacPherson [23] leads to the introduction of higher homotopy groups into the setting of the connectedness theorem. Along different lines, F. L. Zak [65] has used the connectedness theorem to establish a striking result on tangencies to subvarieties of projective space, from which he obtains a proof of Hartshorne's conjecture [34] on linear normality.

Our purpose is to give an exposition of this circle of ideas, especially the new contributions of Deligne. Drawing on his techniques, we also extend several of the corollaries of [19] and [20] to the topological case. For example:

(1) __If__ $X \subseteq \mathbf{P}^m_{\mathbf{C}}$ __is a closed irreducible subvariety of dimension__ n, __and if__ $2n > m$, __then__ X __is simply connected__. (§5)

(2) __Any__ n-__dimensional normal variety which can be expressed as a branched covering of__ $\mathbf{P}^n$ __with no more than__ n __sheets is simply connected__. (§6)

(3) __If__ $X, Y \subseteq \mathbf{P}^m$ __are irreducible subvarieties, with__ X __normal, and if__ $\dim X + \dim Y > m$, __then__ $X \cap Y$ __is__

connected, and the natural map $\pi_1(X \cap Y) \to \pi_1(X)$ *is surjective.* (§4)

Using in addition the theorem of Goresky and MacPherson [23], we prove

(4) *Let* X *be a compact local complete intersection variety of pure dimension* n , *let* $f : X \to \mathbf{P}^m$ *be a finite morphism, and let* $Y \subseteq \mathbf{P}^m$ *be a closed local complete intersection of pure codimension* d . *Then the induced homomorphisms*

$$f_* : \pi_i(X , f^{-1}(Y)) \to \pi_i(\mathbf{P}^m , Y)$$

are bijective for $i \leqslant n - d$, *and surjective if* $i = n - d + 1$. (§9)

When Y is a hyperplane, this yields the Lefschetz hyperplane theorem for local complete intersections. Taking $Y = X \subseteq \mathbf{P}^m$, one finds that $\pi_i(\mathbf{P}^m , X) = 0$ for $i \leqslant 2n - m + 1$, which strengthens results of Barth, Larsen, and Ogus [5, 7, 44, 52].

Here is an overview of the contents and organization of these notes. The first two sections are devoted to Bertini-type theorems on the connectedness of linear sections of an irreducible variety: §1 treats intersections with a generic linear space, and in §2 we pass to the limiting situation of an arbitrary linear section. Over the complex numbers, one obtains information on fundamental groups by applying these connectivity results to covering spaces of the varieties in question.

The connectedness theorem, proved in §3, asserts in its simplest form that if X is a complete irreducible variety, and if

$$f : X \to \mathbf{P}^m \times \mathbf{P}^m$$

is a morphism such that $\dim f(X) > m$, then the inverse image $f^{-1}(\Delta)$

of the diagonal $\Delta \subseteq \mathbb{P}^m \times \mathbb{P}^m$ is connected. Moreover in the complex case, the homomorphism $\pi_1(f^{-1}(\Delta)) \to \pi_1(X)$ is surjective provided that X is locally irreducible in the classical topology. The proof we present is due to Deligne, and simplifies previous arguments. In brief, one uses a basic construction to pass from the given map f to a morphism $f^* : X^* \to \mathbb{P}^{2m+1}$, X^* being a $\mathbb{C}_m$-bundle over X. The assertions on $f^{-1}(\Delta)$ reduce to the analogous statements for $f^{*-1}(L)$ where $L \subseteq \mathbb{P}^{2m+1}$ is a certain linear space of dimension m, and here the Bertini theorems apply.

In §§4 - 7 we discuss applications of the connectedness theorem to the geometry of projective space. The simplest of these (§4) concerns the intersection of two irreducible subvarieties $X, Y \subseteq \mathbb{P}^m$: by considering the natural embedding $X \times Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^m$, one finds that $X \cap Y$ is connected if $\dim X + \dim Y > m$. We turn in §5 to the singularities of a finite mapping $f : X^n \to \mathbb{P}^m$, the basic result being that if $2n > m$, then f must ramify unless it is an embedding. The proof involves an application of the connectedness theorem to the map $F = f \times f : X \times X \to \mathbb{P}^m \times \mathbb{P}^m$; roughly speaking, if $p, q \in X$ are distinct points with the same image, the connectivity of $F^{-1}(\Delta)$ allows one to degenerate (p, q) through double points of f to a pair (p^*, q^*) with $p^* = q^*$, where f will ramify. This result is used to study the singularities and fundamental groups of subvarieties of small codimension in projective space, and to investigate the degeneration of secant and tangent varieties. Section 6 centers around a theorem to the effect that if $f : X \to \mathbb{P}^n$ is a branched covering of degree at least $n + 1$, then there exist points on X at which $n + 1$ or more sheets of the covering come together. We also discuss a generalization of this result, due to Deligne, to possibly infinite-sheeted coverings. In §7 we give an exposition of the work of F. L. Zak on tangencies to smooth subvarieties $X^n \subseteq \mathbb{P}^m$. Zak's result bounds the dimension of the locus along which a fixed linear space is tangent to X. Using this, he is able to deduce that X is linearly normal if $3n > 2(m-1)$, as conjectured by Hartshorne. We present a new proof of Zak's theorem on linear normality which emphasizes the role of the connectedness theorem.

In §8, we turn to Zariski's problem on the fundamental group of the complement of a nodal curve $C \subseteq \mathbf{P}^2$. In the algebraic case, one wants to show that every (tamely ramified) covering of $\mathbf{P}^2$ branched along C is abelian. An argument of Abhyankar reduces this to the assertion that the inverse image of every component of C is irreducible, which is proved using the connectedness theorem. A generalization of the argument yields Deligne's result, in the complex case, that $\pi_1(\mathbf{P}^2 - C)$ is abelian.

Deligne's extension of the connectedness theorem to higher homotopy groups is described in §9. The basic ingredient here is a noncompact generalization of the Lefschetz hyperplane theorem due to Goresky and MacPherson. As an application, one obtains a strengthened and unified formulation of several well-known results on the topology of projective varieties.

Finally, we list in §10 some open questions.

For accounts of related work, and historical remarks, we refer the reader to the notes at the end of each section.

One of the pleasant features of the connectedness theorem is its elementary nature, and we have tried to reflect this in our presentation. Except in §9, the exposition is largely self-contained. While theorems are stated in reasonably full generality, further hypotheses - sufficient for the main applications - are made in some of the proofs. In addition, many of the arguments are given here only for the complex case, with references indicated for the extensions to arbitrary ground fields.

Preliminary versions of a number of sections were circulated in [17]. These have been revised and updated in the present notes.

Besides the crucial correspondence from Deligne, and the letter from Zak, this work and its presentation have benifitted from communication with many others, including: M. Artin, S. Bloch, T. Gaffney, N. Goldstein, M. Goresky, P. Griffiths, H. Hamm, J. Hansen, R. Hardt, J. Harris, R. Hartshorne, K. Johnson, J. P. Jouanolou, A. Landman, Lê Dũng-Tráng, R. MacPherson, R. Mandelbaum, B. Moishezon, D. Mumford, C. Peskine, R. Piene, D. Prill, J. Roberts, K. Saito, A. Sommese, J. Stormes, B. Teissier, K. Vilonen, and O. Zariski.

§0. Notation, Conventions, and Preliminary Facts.

0.1. Unless otherwise stated, a variety is an irreducible algebraic variety. The assertion that a space is connected includes the statement that it is non-empty.

0.2. $\mathrm{Grass}_d(\mathbf{P}^m)$ denotes the Grassmannian of codimension d linear spaces in the projective m-space $\mathbf{P}^m$.

0.3. Given maps $f : Y \to X$ and $g : Z \to X$ of topological spaces, $Y \times_X Z$ denotes the fibre product of Y and Z over X. Recall that, by definition,

$$Y \times_X Z = \{(y,z) \in Y \times Z \mid f(y) = g(z)\} .$$

0.4. Any statement labeled (A) refers to varieties over an arbitrary algebraically closed field, and to the Zariski topology. In assertions labeled (B), the ground field is $\mathbb{C}$, and the topology is the classical one unless otherwise indicated. We will refer to these respectively as the "algebraic" and "topological" settings.

0.5. All spaces that occur when we are working in the classical topology are Hausdorff and locally path connected, and when connected will possess universal covering spaces; we let $\pi : \tilde{X} \to X$ denote the universal covering of X. If X is a complex analytic space, $\tilde{X}$ carries a natural analytic structure, defined by the requirement that π be a local analytic isomorphism.

0.6. In the topological setting, if $f : Y \to X$ is a continuous map, with X connected, we write

$$\pi_1(Y) \twoheadrightarrow \pi_1(X)$$

to mean that the induced homomorphism f_* from $\pi_1(Y,y)$ to $\pi_1(X,f(y))$ is surjective for <u>some</u> y in Y. When Y is connected, this is independent of the choice of y. We will frequently use two elementary facts:

(1) If $g : Z \to Y$ and $f : Y \to X$ are given, and if $\pi_1(Z) \twoheadrightarrow \pi_1(X)$, then $\pi_1(Y) \twoheadrightarrow \pi_1(X)$.

(2) Given $f : Y \to X$, with X and Y connected, the following are equivalent:

(i) $\pi_1(Y) \twoheadrightarrow \pi_1(X)$;

(ii) for any connected topological covering $X' \to X$, the induced covering $Y \times_X X' \to Y$ is connected;

(iii) $Y \times_X \tilde{X}$ is connected.

0.7. Algebraic varieties enjoy the following connectivity property:

(A) <u>If</u> X <u>is an (irreducible) variety, and</u> $Z \subsetneq X$ <u>a closed algebraic subset,</u> $X - Z$ <u>is connected</u>.

In fact, $X - Z$ is an irreducible variety. The same is true when X is an irreducible analytic space, and Z is a closed analytic subspace; this follows easily from the corresponding local statement proved in [28, p. 115].

(B) If X is an irreducible complex variety whose universal covering $\tilde{X}$ is an irreducible analytic space, then for any closed analytic subspace $Z \subsetneq X$,

$$\pi_1(X-Z) \twoheadrightarrow \pi_1(Z)$$

Indeed, by the previous remark $(X-Z) \times_X \tilde{X}$ is irreducible.

To make use of (B), we will frequently need to know that $\tilde{X}$ is irreducible. The simplest way of arranging this is to require that X be locally irreducible in the classical topology (unibranch in the terminology of algebraic geometry): for then $\tilde{X}$ is likewise locally irreducible, and being connected, it is irreducible. For example, any normal variety has this property. The reader will note that whenever we assume a variety is locally irreducible, it would actually be enough to suppose that its universal covering space is irreducible.

§1. Generic Linear Sections

Statement (A) of the following result is a classical theorem of Bertini; the second assertion is due to Deligne [11].

THEOREM 1.1. Let X be a variety, and $f : X \to \mathbf{P}^m$ a morphism. Assume as always that X is irreducible, and in (B) suppose in addition that X is locally irreducible as a complex analytic space. Fix an integer $d < \dim \overline{f(X)}$. Then there is a non-empty Zariski-open set $U \subseteq \mathrm{Grass}_d(\mathbf{P}^m)$ such that for all $(m-d)$-planes L in U:

(A) $f^{-1}(L)$ is irreducible;

and

(B) $\pi_1(f^{-1}(L)) \twoheadrightarrow \pi_1(X)$.

Proof. We will make the additional assumption that f is generically finite-to-one (See [39] and [11] for the general case.)

Note that it suffices to prove the theorem for any non-empty Zariski-open subset $X - Z$ of X . For (A) this follows from the fact that for generic L every irreducible component of $f^{-1}(L)$ has dimension strictly greater than that of $Z \cap f^{-1}(L)$. For (B) it follows from (0.7).

Let $n = \dim X$. We will first prove the theorem when $m = n$, so that $f : X \to \mathbf{P}^n$ is dominating. This special case contains the heart of the argument. Replacing X if necessary by an open subset, we may assume that there exists a hypersurface $B \subseteq \mathbf{P}^n$ such that f realizes X as a (connected) topological covering space of $\mathbf{P}^n - B$.

CLAIM. For any line $\ell \subseteq \mathbf{P}^n$ meeting B transversely, $f^{-1}(\ell - \ell \cap B)$ is connected.

The claim is proved much as in [8, p. 192] and [48, p. 68] . Choose a point $O \in \mathbf{P}^n - B$. Then (via projection from O) the lines through O are naturally parametrized by a projective (n-1)-space $\mathbf{P}^{n-1}$. Consider the sets

$$\begin{aligned} X^* &= \{(x,\lambda) \mid f(x) \in \ell_\lambda\} \subseteq X \times \mathbf{P}^{n-1} \\ P &= \{(y,\lambda) \mid y \in \ell_\lambda\} \subseteq (\mathbf{P}^n - B) \times \mathbf{P}^{n-1} \end{aligned} .$$

X^* is irreducible - it is the blow-up of X at the points $\{f^{-1}(O)\}$ - and the map $f^* = f \times \mathrm{id} : X^* \to P$ is a topological covering. Now over the open set $T \subseteq \mathbf{P}^{n-1}$ of lines through O meeting B transversely, the projection $pr_2 : P \to \mathbf{P}^{n-1}$ restricts to a topologically locally trivial fibre space: the fibres are spheres with $\deg(B)$ points removed. Hence $h = pr_2 \circ f^* : X^* \to \mathbb{P}^{n-1}$ is likewise locally trivial over T. Moreover $h^{-1}(T) \subseteq X^*$ is irreducible since X^* is. On the other hand, for a fixed point $O' \in f^{-1}(O)$, the map $\lambda \to (O',\lambda)$ defines a section of h. Since a locally trivial fibration between connected spaces which admits a section necessarily has connected fibres, we conclude that $f^{-1}(\ell_\lambda - \ell_\lambda \cap B)$ is connected for all $\lambda \in T$, which proves the claim.

It follows that if $L' \subseteq \mathbb{P}^n$ is a linear space of codimension d which contains a line meeting B transversely, then $f^{-1}(L' - L' \cap B)$ is connected; being non-singular, it is irreducible. This proves assertion (A) of the theorem in the case $m = n$. Note that we have not used the fact that the covering f of $\mathbb{P}^n - B$ is finite sheeted. In view of (0.6), statement (B) thus follows by applying the above argument to the composition $f \circ \pi : \tilde{X} \to X \to \mathbb{P}^n - B$.

Next consider the general case $f : X \to \mathbb{P}^m$. Let $Y = f(X)$. After possibly replacing X by an open subset, we may assume that Y is an irreducible, locally closed subvariety of $\mathbb{P}^m$ of dimension n. Fix a linear space $M \subseteq \mathbb{P}^m$ of codimension $n + 1$ disjoint from the Zariski-closure $\bar{Y}$ of Y. We assert:

(*) For almost every $L \subseteq \mathbb{P}^m$ of codimension d containing M, $f^{-1}(L)$ is irreducible, and $\pi_1(f^{-1}(L)) \twoheadrightarrow \pi_1(X)$.

Indeed, consider the linear projection $p_M : \mathbb{P}^m - M \to \mathbb{P}^n$ centered at M. There is a natural isomorphism

$$\mathrm{Grass}_d(\mathbb{P}^n) \xrightarrow{\sim} \{L \in \mathrm{Grass}_d(\mathbb{P}^m) \mid L \supseteq M\}$$

given by $L' \to \overline{p_M^{-1}(L')}$. So the assertion follows from the case $m = n$ of theorem, applied to the generically finite map $p_M \circ f : X \to \mathbb{P}^n$.

It follows from (*) that there is a dense set of $\mathrm{Grass}_d(\mathbf{P}^m)$ for which statements (A) and (B) of the theorem hold. This suffices for later purposes. To produce a Zariski-open set $U \subseteq \mathrm{Grass}_d(\mathbf{P}^m)$ with the desired properties, choose a Zariski-open $V \subseteq \mathrm{Grass}_{n+1}(\mathbf{P}^m)$ consisting of linear spaces $M \subseteq \mathbf{P}^m$ disjoint from $\overline{Y}$, and such that the universal quotient bundle on $\mathrm{Grass}_{n+1}(\mathbf{P}^m)$ trivializes over V. Then the projections from M considered above fit together to form a finite morphism $\overline{Y} \times V \to \mathbf{P}^n \times V$. Choose a divisor $\mathcal{B} \subseteq \mathbf{P}^n \times V$ such that the composition

$$X \times V \xrightarrow{f \times \mathrm{id}} \overline{Y} \times V \to \mathbf{P}^n \times V$$

is a topological covering over the complement of $\mathcal{B}$. After shrinking V if necessary, $\mathcal{B}$ gives rise to a family $\{B_M \subseteq \mathbf{P}^n\}_{M \in V}$ of hypersurfaces of some degree b, parametrized by V. Then we may take U to consist of all $L \subseteq \mathbf{P}^m$ containing an M in V such that the corresponding $L' \in \mathrm{Grass}_d(\mathbf{P}^n)$ contains a line meeting B_M transversely in b points. ∎

<u>NOTES</u>. (1). Proofs of the Bertini theorem (A) over abstract ground fields were given by Akizuki, Matsusaka, and Zariski. Jouanolou has recently written a complete, modern, and elementary account of these Bertini theorems. [39].

(2) Deligne [11] deduces (B) from the "Zariski hyperplane theorem," which has been given modern proofs by Hamm and Lê [31] and by Cheniot [9].

(3) Examples show that the local irreducibility hypothesis cannot be dropped in (B). (Cf. §6, Note 2).

§2. Arbitrary Linear Sections.

While it is certainly not the case that an arbitrary linear section of an irreducible variety remains irreducible, one does have the following result:

THEOREM 2.1. *Let* X *be a variety,* $f : X \to \mathbb{P}^m$ *a morphism, and* $L \subseteq \mathbb{P}^m$ *an arbitrary linear subspace of codimension* d *, with* $d < \dim \overline{f(X)}$.

(A) *If* X *is complete, then* $f^{-1}(L)$ *is connected. More generally, if* f *is proper over some open set* $V \subseteq \mathbb{P}^m$ *, and if* $L \subseteq V$ *, then* $f^{-1}(L)$ *is connected.*

(B) *If* X *is locally irreducible, then for any neighborhood* U *of* L *in* $\mathbb{P}^m$ *one has*

$$\pi_1(f^{-1}(U)) \twoheadrightarrow \pi_1(X) \quad .$$

Proof. For (A), let $W \subseteq \mathrm{Grass}_d(\mathbb{P}^m)$ be the open subset consisting of linear spaces contained in V , and let

$$Z = \{(x,L') \in X \times W \mid x \in f^{-1}(L')\} \quad .$$

Z arises as an open subset of a Grassmannian bundle over X , and hence is irreducible. Moreover, since f is proper over V , the projection $pr_2 : Z \to W$ is likewise proper. Consider its Stein factorization

$$Z \xrightarrow{q} W' \xrightarrow{r} W$$

(cf [27, III.4.3.3]); q has connected fibres and r is finite. By Theorem 1.1(A), r is generically one-to-one. But r is surjective and W is normal, so it follows that in fact r is everywhere one-to-one. Hence $f^{-1}(L')$ is connected for every L' in W.

By Theorem 1.1(B), any neighborhood of U of L contains linear spaces L' for which $\pi_1(f^{-1}(L')) \twoheadrightarrow \pi_1(X)$, and (B) follows. ∎

<u>REMARK 2.2</u>. In case $f^{-1}(L)$ is non-empty, a sharper form of (B) will be needed:

(B'). <u>For any</u> $x \in f^{-1}(L)$, <u>the homomorphism</u> $\pi_1(f^{-1}(U),x) \to \pi_1(X,x)$ <u>is surjective</u>.

In fact, by Theorem 1.1(B) we may choose a point $x' \in f^{-1}(U)$ in the same path component as x, such that $f(x')$ lies on a linear space L' with $f^{-1}(L')$ irreducible and $\pi_1(f^{-1}(L'),x') \to \pi_1(X,x')$ surjective. Then $\pi_1(f^{-1}(U),x') \twoheadrightarrow \pi_1(X,x')$, and (B') follows.

<u>NOTES</u>. (1). The proof of (A) follows Jouanolou [39], who simplified considerably the argument in [19]. It turns out that the result was known previously. The earliest reference we are aware of is [26, XIII.2.3], where Grothendieck deduces it from an analogous local theorem, proved using the machinery of local cohomology. Grothendieck's method has been applied and extended by others, including Hartshorne, Ogus, Speiser, and recently by Faltings [14].

(2). By a more careful application of Theorem 1.1(B), Deligne [11] shows that there are in fact arbitrarily small neighborhoods U of L such that $f^{-1}(U)$ is connected.

(3). Goresky and MacPherson [23] have proved a conjecture of Deligne's which extends (B) to higher homotopy groups. (See §9.)

§3. The Connectedness Theorem

The following theorem expresses a basic property of projective space. As subsequent sections will show, it has numerous geometric and topological consequences.

THEOREM 3.1. *Let* X *be a variety, and let* $f : X \to \mathbb{P}^m \times \mathbb{P}^m$ *be a morphism with* $\dim \overline{f(X)} > m$. *Denote by* Δ *the diagonal in* $\mathbb{P}^m \times \mathbb{P}^m$.

(A) ([19]) *If* X *is complete, then* $f^{-1}(\Delta)$ *is connected.*

(B) ([10, 11]) *If* X *is locally irreducible, and if* U *is any neighborhood of* Δ *in* $\mathbb{P}^m \times \mathbb{P}^m$, *then*

$$\pi_1(f^{-1}(U)) \twoheadrightarrow \pi_1(X) \quad .$$

Proof. (Deligne [12]). The idea is to pass from the diagonal embedding $\Delta \subseteq \mathbb{P}^m \times \mathbb{P}^m$ to a linear embedding $L^m \subseteq \mathbb{P}^{2m+1}$. To this end, let $[x] = [x_0,\dots,x_m]$ and $[y] = [y_0,\dots,y_m]$ be coordinates on the two factors of $\mathbb{P}^m \times \mathbb{P}^m$, and introduce the coordinates $[x,y] = [x_0,\dots,x_m,y_0,\dots,y_m]$ on $\mathbb{P}^{2m+1}$. Denote by V the complement in $\mathbb{P}^{2m+1}$ of the two linear spaces $x_0 = \dots = x_m = 0$ and $y_0 = \dots = y_m = 0$. Then there is a natural map

$$p : V \to \mathbb{P}^m \times \mathbb{P}^m \quad ,$$

given by $[x , y] \to ([x] , [y])$, which realizes V as a $\mathbb{C}^*$-bundle over $\mathbb{P}^m \times \mathbb{P}^m$. Let $L \subseteq V$ be the m-dimensional linear space defined by $x_i = y_i$ $(0 \le i \le m)$; observe that L maps isomorphically to the diagonal $\Delta \subseteq \mathbb{P}^m \times \mathbb{P}^m$. Given $f : X \to \mathbb{P}^m \times \mathbb{P}^m$, set

$$X^* = X \times_{\mathbb{P}^m \times \mathbb{P}^m} V \quad ,$$

and let $q : X^* \to X$ and $f^* : X^* \to V$ denote the projections. The situation is summarized in the following diagram:

$$\begin{tikzcd}
 & X^* \arrow[r, "q"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\mathbf{P}^{2m+1} \supseteq & V \arrow[r, "p"] & \mathbf{P}^m \times \mathbf{P}^m \\
L \arrow[ur, hook] \arrow[r, "\simeq", "p|L"'] & \Delta \arrow[ur, hook] &
\end{tikzcd}$$

Note that X^* is irreducible, and that $\dim \overline{f^*(X^*)} > m+1$.

Since L maps isomorphically to Δ , q gives rise to an isomorphism $f^{*-1}(L) \xrightarrow{\approx} f^{-1}(\Delta)$. If X is complete, then f^* is proper, and the Bertini theorem (2.1(A)) for arbitrary linear sections applies to give the connectivity of $f^{*-1}(L)$. This proves (A).

Turning to (B), let $U^* = p^{-1}(U)$, so that U^* is a neighborhood of L in $\mathbf{P}^{2m+1}$. Consider the commutative square

$$\begin{tikzcd}
\pi_1(f^{*-1}(U^*)) \arrow[r] \arrow[d] & \pi_1(f^{-1}(U)) \arrow[d] \\
\pi_1(X^*) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

The bottom horizontal map is surjective since X^* is a $\mathbb{C}^*$-bundle over X , and the vertical homomorphism on the left is surjective by Theorem 2.1(B). Hence $\pi_1(f^{-1}(U)) \twoheadrightarrow \pi_1(X)$, as desired. ∎

<u>REMARK 3.2.</u> In (B), if $f^{-1}(\Delta)$ is non-empty, then the homomorphism $\pi_1(f^{-1}(U),x) \to \pi_1(X,x)$ is surjective for any $x \in f^{-1}(\Delta)$. In view of Remark 2.2, this follows immediately from the proof of the theorem.

<u>COROLLARY 3.3.</u> (B). <u>In the situation of Theorem</u> 3.1(B), <u>assume in</u>

<u>addition that</u> X <u>is complete. Then</u>

$$\pi_1(f^{-1}(\Delta)) \twoheadrightarrow \pi_1(X) \quad .$$

<u>Proof</u>. Choose a neighborhood V of $f^{-1}(\Delta)$ in X such that $f^{-1}(\Delta)$ is a deformation retract of V . Since f is proper, there exists a neighborhood U of the diagonal Δ with $f^{-1}(U) \subseteq V$. Fix $x \in f^{-1}(\Delta)$, and consider the homomorphisms

$$\begin{tikzcd} \pi_1(f^{-1}(\Delta),x) \arrow[r] \arrow[d] & \pi_1(V,x) \arrow[d] \\ \pi_1(f^{-1}(U),x) \arrow[r] \arrow[ur] & \pi_1(X,x) \end{tikzcd}$$

induced by inclusions. The top horizontal map is an isomorphism, while the bottom is surjective by Remark 3.2, and the corollary follows. ∎

<u>REMARK 3.4.</u> The connectedness theorem extends to more than two factors: one considers a morphism $f : X \to (\mathbf{P}^m)^r$ with $\dim f(X) > (r-1)m$, and the small diagonal $\Delta \approx \mathbf{P}^m$ in $(\mathbf{P}^m)^r$. Then statements (A) and (B) of Theorem 3.1 and the assertion of Corollary 3.3 hold (cf. [19]). This may be proved as before by passing from the diagonal $\Delta \subseteq (\mathbf{P}^m)^r$ to a linear space $L^m \subseteq \mathbf{P}^{r(m+1)-1}$.

<u>NOTES</u>. (1) A different proof of Theorem 3.1(A) was given by Barth in 1969 [4], although he only stated a special case of the theorem; Barth's argument was rediscovered in [19]. It depends upon constructing a birational correspondence between $\mathbf{P}^m \times \mathbf{P}^m$ and $\mathbf{P}^{2m}$ which reduces the assertion for the diagonal to the corresponding statement for a linear space $L^m \subseteq \mathbf{P}^{2m}$. (B) is due to Deligne [10, 11], who originally proved it using the birational correspondence. The proof of the connectedness theorem presented above was given by Deligne [12]

in the course of extending the theorem to higher homotopy groups (cf §9).

(2) In the situation of Theorem 3.1(B), Deligne shows as before that there are arbitrarily small neighborhoods U of the diagonal such that $f^{-1}(U)$ is connected [11].

(3) Observe that Deligne's construction in the proof of the connectedness theorem reduces a general intersection in projective space to an intersection with a linear space. This can be used to give a simple proof of the refined Bézout theorem announced at the end of [19]. See [18, §6] for details.

(4) Mumford [49] has given an alternative proof of Theorem 3.1(A) in characteristic zero. He shows that if a simply connected algebraic group G acts transitively on a complete variety Z with connected stabilizer H, and if $Y \subseteq Z$ is a closed subvariety such that the H-orbit of the tangent space $T_p Y$ at a generic $p \in Y$ is Zariski-open in $\mathrm{Grass}(T_p Z)$, then for any proper morphism $f : X \to Z$ with X irreducible and $\dim f(X) > \mathrm{codim}(Y,Z)$, $f^{-1}(Y)$ is connected. Applying this to $G = SL(m+1) \times SL(m+1)$ acting on $\mathbf{P}^m \times \mathbf{P}^m$ gives 3.1(A). Mumford proves his result by considering the variety

$$V = \{(x,y,\sigma) \in X \times Y \times G \mid \sigma f(x) = y\} \quad .$$

He shows that the projection $p : V \to G$ is surjective, and that if $V' \subseteq V$ is the set where p is not smooth, and $S \subseteq G$ is the locus over which some component of the fibre is contained in V', then $\mathrm{codim}(S,G) \geqslant 2$. Using a Stein factorization and purity of the branch locus, it follows easily that all fibres of p are connected, in particular $p^{-1}(e) = f^{-1}(Y)$.

(5) Hansen [32] has extended Theorem 3.1(A) to Grassmannians. He proves in fact that if F is any flag manifold of subspaces of $\mathbf{P}^m$, and if $f : X \to F \times F$ is a proper morphism with

$\operatorname{codim}(f(X), F\times F) < m$, then $f^{-1}(\Delta_F)$ is connected. Examples show that this result is sharp. Hansen's argument uses the geometry of Grassmannians to reduce to the connectedness theorem for projective space.

(6) Faltings [14] generalized a local theorem from [26], and used this to give a new proof of Theorem 3.1(A). More recently, he has generalized the connectedness theorem to other homogeneous spaces [15]. Specifically, let $Z = G/P$, where G is a connected semi-simple linear algebraic group over a field k of characteristic zero, and $P \subseteq G$ is a parabolic subgroup. Denote by ℓ the minimum rank of the simple factors of G when k is extended to its algebraic closure. Faltings shows that if X is a proper irreducible k-scheme, and if $f : X \to Z \times Z$ is a morphism, then $f^{-1}(\Delta)$ is non-empty if $\operatorname{codim}(f(Z), X\times X) \leqslant \ell$, and connected if $\operatorname{codim}(f(Z), X\times X) < \ell$. In particular, he recovers Hansen's result on flag manifolds (at least in characteristic zero).

§4. Intersections

We present in this section the simplest applications of the connectedness theorem.

THEOREM 4.1. *Let* X *and* Y *be complete varieties, and let* $f : X \to \mathbf{P}^m$, $g : Y \to \mathbf{P}^m$ *be morphisms such that* $\dim f(X) + \dim g(Y) > m$. *Then*

(A) ([19]) $X \times_{\mathbf{P}^m} Y$ *is connected.*

If also X *and* Y *are locally irreducible, then*

(B) $\pi_1(X \times_{\mathbf{P}^m} Y) \twoheadrightarrow \pi_1(X \times Y)$.

In particular, if $X, Y \subseteq \mathbf{P}^m$ are closed irreducible subvarieties with $\dim X + \dim Y > m$, then $X \cap Y$ is connected.

Proof. Apply Theorem 3.1 and Corollary 3.3 to the morphism $F = f\times g : X\times Y \to \mathbf{P}^m \times \mathbf{P}^m$ noting that $F^{-1}(\Delta) = X \times_{\mathbf{P}^m} Y$. ∎

REMARK 4.2. Like Theorem 3.1, this result extends to more than two factors: if $f_i : X_i \to \mathbf{P}^m$ $(1\le i\le r)$ are proper morphisms, and if $\sum \dim f_i(X_i) > (r-1)m$, then $X_1 \times_{\mathbf{P}^m} \cdots \times_{\mathbf{P}^m} X_r$ is connected. If in addition each X_i is locally irreducible, then

$$\pi_1(X_1 \times_{\mathbf{P}^m} \cdots \times_{\mathbf{P}^m} X_r) \twoheadrightarrow \pi_1(X_1 \times \cdots \times X_r) .$$

COROLLARY 4.3. *Let* X *be a complete variety, let* $f : X \to \mathbf{P}^m$ *be a morphism, and let* $Y \subseteq \mathbf{P}^m$ *be a closed subvariety. If* $\dim f(X) > \operatorname{codim}(Y, \mathbf{P}^m)$, *then*

(A) ([19]) $f^{-1}(Y)$ *is connected.*

If in addition X *is locally irreducible; then*

(B) $\pi_1(f^{-1}(Y)) \twoheadrightarrow \pi_1(X)$.

Proof. Statement (A) is an immediate consequence of Theorem 4.1. For (B), let Y^* be the normalization of Y , and $g : Y^* \to \mathbf{P}^m$ the induced map. Then one has the commutative diagram

$$\begin{array}{ccc} \pi_1(X \times_{\mathbf{P}^m} Y^*) & \longrightarrow & \pi_1(X\times Y^*) \\ \downarrow & & \downarrow \\ \pi_1(f^{-1}(Y)) & \longrightarrow & \pi_1(X) \end{array} .$$

Since X and Y^* are locally irreducible, Theorem 4.1(B) applies to show that the top horizontal homomorphism is surjective. But the projection $\pi_1(X\times Y^*) \to \pi_1(X)$ is also surjective, and the Corollary follows. ∎

NOTES. (1) Hironaka and Matsumura [36] had proved assertion (A) of the Corollary when f is surjective. The theorem stated by Barth in

[4] was the special case of the corollary in which f is the normalization of a subvariety of $\mathbb{P}^m$. A related result was proved by Rossi [58].

(2) The Corollary and the connectedness theorem suggested the conjecture in [19] that if $f : X \to Z$ is a morphism between complete varieties, and if $Y \hookrightarrow Z$ is a subvariety with ample normal bundle, then $f^{-1}(Y)$ would be connected (but possibly empty) provided that $\dim f(X) > \operatorname{codim}(Y,Z)$. However Hansen observed that examples of Hironaka and Hartshorne (cf. [33, p. 199]) give counter-examples to this conjecture.

§5. Singularities of Mappings to Projective Space.

In this section we apply the connectedness theorem to study singularities of mappings $f : X \to \mathbb{P}^m$. The philosophy here is that under mild hypotheses, such singularities must occur.

THEOREM 5.1. ([19]) (A). *Let X be a complete variety of dimension n, and let $f : X \to \mathbb{P}^m$ be an unramified morphism. If $2n > m$, then f is a closed embedding.*

Recall that a morphism $f : X \to Y$ is *unramified* if the sheaf of relative differentials $\Omega^1_{X/Y}$ is zero. The exact sequence

$$f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

shows that f is unramified if and only if the canonical map $f^*\Omega^1_Y \to \Omega^1_X$ is surjective. When X and Y are non-singular, this is equivalent to requiring that the induced maps on tangent spaces be injective, i.e. that f be an immersion in the sense of differential topology.

Proof of Theorem 5.1. Given a morphism $f : X \to Y$, the diagonal map $X \to X \times_Y X$ embeds X as a closed subscheme Δ_X of $X \times_Y X$. If I is the ideal sheaf of Δ_X in $X \times_Y X$, then $\Omega^1_{X/Y}$ can be identified with I/I^2 . It follows that f is unramified if and only if Δ_X is an open (as well as a closed) subscheme of $X \times_Y X$ (cf [27, IV. 17. 4.2]).

The theorem now follows by applying Theorem 4.1(A) to the product of f with itself: since $2n > m$, $X \times_{\mathbf{P}^m} X$ is connected. On the other hand, Δ_X is a connected component of $X \times_{\mathbf{P}^m} X$ thanks to the fact that f is unramified, and hence $\Delta_X = X \times_{\mathbf{P}^m} X$. Therefore f is one-to-one. But a one-to-one unramified morphism is a closed embedding (cf.[27, IV.8.11.5 and IV.17.2.6]). ∎

As a first consequence, one has

COROLLARY 5.2. ([19]) (A) *Let* $X \subseteq \mathbf{P}^m$ *be a closed subvariety of dimension* n , *with* $2n > m$. *If* X *is not normal, then the normalization map* $X^* \to X$ *must be ramified*. ∎

For example, let X be a singular surface in $\mathbf{P}^3$ with ordinary singularities, i.e. a double curve C (along which a local analytic equation for X is $xy = 0$), a finite number of triple points (with local equation $xyz = 0$), and a finite number of pinch-points (with local equation $z^2 = xy^2$). The normalization X^* of X is non-singular, and the map $X^* \to X$ ramifies precisely over the pinch-points of X . Hence X *must have pinch-points*. In fact, every connected component of C must contain pinch points, as one sees by normalizing over individual components of C .

Similarly, if X is a non-singular three-fold, and if $f : X \to \mathbf{P}^4$ is a generic projection, then the curve of triple points (if non-empty) must meet the curve of pinch points. This follows from the connectedness of $F^{-1}(\Delta)$ for $F = f \times f \times f : X \times X \times X \to \mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4$.

<u>COROLLARY 5.3.</u> <u>Let</u> $X \subset \mathbb{P}^m$ <u>be a closed (irreducible) subvariety of dimension</u> n . <u>If</u> $2n > m$, <u>then</u>

(A) ([19]) X <u>has no non-trivial finite étale coverings</u>, <u>i.e.</u> $\pi_1^{alg}(X) = 1$.

(B) X <u>is simply connected</u>.

<u>Proof</u>. (A) Suppose that $p : Y \to X$ is a connected étale covering. Let Y' be an irreducible component of Y . The composition $Y' \to X \hookrightarrow \mathbb{P}^m$ is unramified, and hence $Y' \xrightarrow{\cong} X$ by Theorem 5.1. This provides a section of p , so it is trivial.

(B) Let $f : X^* \to X$ be the normalization of X . We claim first that

(*) The homomorphism $f_* : \pi_1(X^*) \to \pi_1(X)$ is trivial.

To see this, consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(X^* \times_{\mathbb{P}^m} X^*) & \longrightarrow & \pi_1(X^* \times X^*) & = & \pi_1(X^*) \times \pi_1(X^*) \\ \downarrow & & \downarrow{\scriptstyle f_* \times f_*} & & \\ \pi_1(X) & \xrightarrow{\delta_*} & \pi_1(X \times X) & = & \pi_1(X) \times \pi_1(X) \end{array} ,$$

where δ_* is the homomorphism induced by the diagonal embedding $\delta : X = X \times_{\mathbb{P}^m} X \hookrightarrow X \times X$. Since X^* is locally irreducible, the top horizontal map is surjective by Theorem 4.1(B), which implies that $\mathrm{Im}(f_* \times f_*) \subseteq \mathrm{Im}(\delta_*)$. But this is only possible if f_* is trivial: for if $\alpha \in \mathrm{Im}(f_*)$ then $(\alpha, 1) \in \mathrm{Im}(\delta_*)$, i.e. $\alpha = 1$.

The remainder of the argument is similar to the proof of (A). Specifically, it follows from (*) that the universal covering $\pi : \tilde{X} \to X$ induces a trivial covering $X^* \times_X \tilde{X} \to X^*$ on X^* . The normalization map f therefore factors through π . The image $X' \subseteq \tilde{X}$ of X^* is an irreducible closed subvariety of $\tilde{X}$. The map $X' \to X$ is unramified, and thus an embedding by Theorem 5.1. Hence π sections, and so is trivial. ∎

By way of application, suppose that X is a non-singular n-dimensional variety $(n\geq 2)$, and that $f : X \to \mathbf{P}^{2n-1}$ is a finite morphism with only ordinary singularities, so that f is an isomorphism onto its image $\overline{X} \subseteq \mathbf{P}^{2n-1}$ except along a double curve $C \subseteq X$. For example f might be a generic projection. Let $\overline{C}$ be the image of C in $\overline{X}$. Then the fact that $H_1(\overline{X}) = 0$ (by Corollary 5.3), plus the observation that $H_*(X,C) \xrightarrow{\cong} H_*(\overline{X},\overline{C})$, imply that the homomorphism

$$\ker(H_1 C \to H_1 \overline{C}) \longrightarrow H_1(X)$$

is surjective. Thus if the irregularity of X is large, C must have large genus.

Especially when X is singular, it can be difficult to determine from geometric hypotheses whether a morphism $f : X \to \mathbf{P}^m$ is ramified, which limits the applicability of Theorem 5.1. However for many purposes a somewhat more flexible notion is sufficient. Let us say that a morphism $f : X \to Y$ is <u>weakly unramified</u> if Δ_X is a connected component of $X \times_Y X$, ignoring scheme structures; f is <u>weakly ramified</u> in the contrary case. Loosely speaking, $f : X \to Y$ is weakly ramified if there exists a pair (p,q) of distinct points on X with the same image in Y , such that (p,q) can be continuously degenerated through pairs of distinct double points of f to a pair (p^*,q^*) with $p^* = q^*$. For example, any generically one-to-one morphism from a curve to a surface will be weakly unramified, although it may well be ramified. As an immediate consequence of

Theorem 4.1(A) and this definition, one has

PROPOSITION 5.4. *Let X be a complete variety of dimension n, and $f : X \to \mathbf{P}^m$ a weakly unramified morphism. If $2n > m$, then f is one-to-one.* ∎

We will apply this result to study tangent and secant varieties.

Let $X \subseteq \mathbf{P}^r$ be a closed subvariety of dimension n, and denote by G the Grassmannian of lines in $\mathbf{P}^r$. The morphism from $X \times X - \Delta_X$ to G which takes a pair (x,y) to the line through x and y extends to a morphism

$$\phi : \widetilde{X \times X} \to G ,$$

where $\widetilde{X \times X}$ is the blow-up of $X \times X$ along the diagonal (see [38]). Let $P(X)$ be the exceptional divisor of this blowing-up; set $S = \phi(\widetilde{X \times X})$, and $T = \phi(P(X))$. Then $S \subseteq G$ is an irreducible variety of dimension $\leqslant 2n$, consisting of all secant lines and their limits. These limits are parametrized by the closed algebraic subset $T \subseteq G$, which has dimension $\leqslant 2n - 1$. When X is non-singular, $P(X)$ is the projectivized tangent bundle of X, and T consists of all tangent lines; when X is singular, not all Zariski tangent lines in general belong to T.

Let $F = \{(p,\ell) \in \mathbf{P}^r \times G \mid p \in \ell\}$ be the natural incidence correspondence, and let $p_1 : F \to \mathbf{P}^r$, $p_2 : F \to G$ denote the projections. The secant variety

$$\mathrm{Sec}(X) = p_1 p_2^{-1}(S)$$

is the closure of the union of all secant lines. Similarly, we set

$$\mathrm{Tan}(X) = p_1 p_2^{-1}(T) .$$

It is evident from this construction that $Sec(X) \subseteq \mathbb{P}^r$ is an irreducible variety of dimension $\leqslant 2n + 1$, and that $Tan(X)$ is a closed algebraic subset of $Sec(X)$ of dimension $\leqslant 2n$.

COROLLARY 5.5. ([19]) (A) Either

(i) $\dim Tan(X) = 2n$ and $\dim Sec(X) = 2n + 1$

or

(ii) $Tan(X) = Sec(X)$

Proof. Suppose to the contrary that $Tan(X) \subsetneq Sec(X)$, but that $\dim Tan(X) < 2n$. Then we may choose a linear space $L \subseteq \mathbb{P}^r$ of codimension $\leqslant 2n$ such that L meets $Sec(X)$ but not $Tan(X)$. Projection from L gives a finite map $f : X \to \mathbb{P}^m$, with $m \leq 2n - 1$. Since L does not meet $Tan(X)$, f is weakly unramified. On the other hand, since L meets $Sec(X) - Tan(X)$, f cannot be one-to-one. But this contradicts Proposition 5.4. ∎

NOTES. (1) The special case of Corollary 5.5 when X^n is a subvariety of $\mathbb{P}^{2n}$ was discovered by K. Johnson in 1976 [38]. Johnson proved the expected formulas for the obstruction cycles to projections being unramified or one-to-one, generalized to singular varieties. A formal calculation showed that for projections from $\mathbb{P}^{2n}$, the vanishing of these two homology (or rational equivalence) classes is equivalent. Since positive algebraic cycles on a projective variety cannot be homologous to zero without vanishing, it followed that if the projection is weakly unramified, then it must be one-to-one. The connectedness theorem grew out of the attempt to extend Johnson's remarkable result.

Other cases of Corollary 5.5 had been proved before [19] by B. Moishezon and D. Mumford (n=2), and J. Harris (n=2,3). For non-singular X, the corollary was discovered independently by F. Zak [65];

secant varieties play a crucial role in his proof of Hartshorne's conjecture on linear normality (cf. §7). P. Griffiths and J. Harris [25] have given an illuminating local analysis of varieties with degenerate secant and tangent varieties.

(2) Special cases of Corollary 5.3 go back to Picard, who proved that a non-singular surface in $\mathbb{P}^3$ is simply connected. From the contemporary point of view, it is a standard consequence of the Lefschetz hyperplane theorem ([47]; cf. also §9) that <u>any</u> hypersurface of dimension at least two is simply connected. In the algebraic setting, Abhyankar [1, II p. 75] proved the (algebraic) simple-connectivity of certain singular hypersurfaces, and Grothendieck [26] subsequently extended the Lefschetz theorem to the abstract case.

Corollary 5.3(B) was proved by Barth and Larsen [7] for non-singular X. 5.3(A) was established by Ogus [53] when X is a local complete intersection in characteristic zero, and by Hartshorne and Speiser [35] when X is Cohen-Macaulay in characteristic p.

Our proof of Corollary 5.3(B) follows suggestions of Deligne for simplifying the argument in [17]. Higher homotopy analogues of this result are described in §9.

§6. Branched Coverings of Projective Space.

In this section, the connectedness theorem is used to study the ramification of branched coverings of projective space.

Let X be a complete variety of dimension n, and let

$$f : X \to \mathbb{P}^n$$

be a finite morphism. Denote by d the (geometric) degree of f, i.e. the number of preimages of a general point of $\mathbb{P}^n$. For each $x \in X$, let $e_f(x)$ be the local degree, or ramification index, of f at x : if f is locally* e-to-one near x , then $e_f(x) =_{\text{def}} e$. Thus $e_f(x)$ counts the number of sheets of the covering that come together at x . One has

$$\sum_{x \in f^{-1}(y)} e_f(x) = d$$

for each $y \in \mathbb{P}^n$.

The following theorem generalizes the classical fact that every irreducible covering of projective space must ramify.

THEOREM 6.1. ([20]) (A) There exists at least one point $x \in X$ at which $e_f(x) \geq \min(d, n+1)$.

The proof will yield a stronger statement. Namely, consider the sets

$$R_\ell = \{x \in X \mid e_f(x) > \ell\} .$$

These ramification loci are closed algebraic subsets of X : for if $U \subseteq X^{\ell+1}$ is the set of $(\ell+1)$-tuples of distinct points with the same image in $\mathbb{P}^n$, then $R_\ell \approx \overline{U} \cap \Delta_X$. We will show that in fact

$$\operatorname{codim}(R_\ell , X) \leqslant \ell$$

for $\ell \leqslant \min(d-1, n)$.

Proof of Theorem 6.1. The argument is by induction on n , the case $n = 1$ being the fact that $\mathbb{P}^1$ is algebraically simply connected. If $n \geq 2$, the inverse image $X' = f^{-1}(L)$ of a generic hyperplane $L \subseteq \mathbb{P}^n$ is irreducible (Theorem 1.1(A)). By induction, the theorem

* In the classical topology over $\mathbb{C}$, in the étale topology otherwise.

is valid for the covering $f' : X' \to L = \mathbb{P}^{n-1}$, and $e_{f'}(x) = e_f(x)$ for $x \in X'$ and generic L. It follows that $\operatorname{codim}(R_\ell, X) \leqslant \ell$ when $\ell \leqslant \min(d-1, n-1)$. It remains to show that R_n is non-empty if $d \geqslant n+1$.

To this end, pick an irreducible component S of R_{n-1} of dimension at least one, and apply the connectedness theorem to the map

$$F = f \times f|S : X \times S \to \mathbb{P}^n \times \mathbb{P}^n \quad .$$

Note that $\Delta_S \subseteq S \times S$ embeds in $X \times S$ as an irreducible component of $F^{-1}(\Delta)$. If $F^{-1}(\Delta) = \Delta_S$, then $e_f(x) = d \geqslant n+1$ for all $x \in S$. So we may assume that $\Delta_S \subsetneq F^{-1}(\Delta)$, in which case the connectivity of $F^{-1}(\Delta)$ implies that there is an irreducible component $T \neq \Delta_S$ of $F^{-1}(\Delta)$ which meets Δ_S. Choose a path $(\alpha(t), \beta(t))$ in $T \subseteq X \times S$ from a point not in Δ_S to a point $(x,x) \in \Delta_S$. Then since f is locally at least n-to-one at each point $\beta(t)$, $n+1$ or more sheets of the covering must come together at the limit point x. ∎

Deligne has given a topological generalization of this result:

THEOREM 6.2. ([12]) (B) *Let $H \subseteq \mathbb{P}^n$ be a closed algebraic set, and let*

$$f : X \to \mathbb{P}^n - H$$

be a connected topological covering of degree d, with d possibly infinite. Set $e = \min(d, n+1)$. Then there exists a point $y \in H$ over which at least e sheets of the covering come together. More precisely, if $B_\varepsilon(y)$ is an arbitrarily small ε-neighborhood of y with respect to some metric on $\mathbb{P}^n$, then there is a connected component V of $f^{-1}B_\varepsilon(y)$ such that the covering $f|V : V \to B_\varepsilon(y) - B_\varepsilon(y) \cap H$ has degree $\geqslant e$.

<u>Sketch of Proof</u>. Induction on n, the case $n = 1$ being clear. So assume that $d \geq n+1$, that over a generic hyperplane n sheets come together, and - by contradiction - that more never do. This last hypothesis allows one to extend f to a ramified covering $\bar{f} : \bar{X} \to \mathbf{P}^n$; over a small open set in $\mathbf{P}^n$, $\bar{f}$ is a disjoint union of finite branched coverings. Let $R \subseteq \bar{X}$ be the locus where n sheets come together. One shows that $Y = \bar{f}(R)$ is analytic (hence algebraic), of dimension at least one, and that $R \to Y$ is a topological covering over some smooth irreducible Zariski-open subset $Y_o \subseteq Y$.

Fix a connected component R_o of $\bar{f}^{-1}(Y_o)$, and consider the topological covering

$$F = f \times \bar{f}|R_o : X \times R_o \to (\mathbf{P}^n - H) \times Y_o .$$

By Theorem 3.1(B) and Note 2 in §3, there are neighborhoods U of the diagonal $\Delta \subseteq \mathbf{P}^n \times \mathbf{P}^n$, contained in arbitrarily small ε-neighborhoods Δ_ε of Δ, such that this covering remains connected over $(\mathbf{P}^n - H) \times Y_o \cap U$. Choose points $b \in R_o$, and $a', b' \in \bar{f}^{-1}B_\varepsilon(\bar{f}(b)) \cap X$, such that (a',b), $(b',b) \in F^{-1}(U)$, and such that a' and b (resp. b' and b) lie in different (resp. the same) connected components of $\bar{f}^{-1}B_\varepsilon(\bar{f}(b))$. Then we can find a path $(\alpha(t), \beta(t))$ in $X \times R_o$ joining (a',b) and (b',b), such that

$$(*) \qquad \operatorname{dist}(\bar{f}(\alpha(t)), \bar{f}(\beta(t))) < \varepsilon$$

for all t. On the other hand, using again the assumption that no more than n sheets come together, one verifies that if $(\alpha(t), \beta(t))$ is a path in $X \times R$ satisfying $(*)$, then for sufficiently small ε the set

$$\left\{ t \;\middle|\; \begin{array}{c} \alpha(t) \text{ and } \beta(t) \text{ lie in distinct} \\ \text{connected components of } \bar{f}^{-1}(B_\varepsilon(\bar{f}\beta(t))) \end{array} \right\}$$

is both open and closed. (It suffices to take ε small enough so that for every $x \in R$, $\bar{f}^{-1}B_{2\varepsilon}(\bar{f}(x))$ contains as a connected component a neighborhood $\bar{V}(x)$ of x such that the covering $\bar{V}(x) \to B_{2\varepsilon}(\bar{f}(x))$ has degree n.) Thus we arrive at a contradiction. ∎

COROLLARY 6.3. *Let* X *be a locally unibranch (e.g. normal) projective variety of dimension* n *which admits a branched covering* $f : X \to \mathbb{P}^n$ *of (geometric) degree* $d \leq n$. *Then:*

(A) ([20]) X *has no non-trivial étale coverings, i.e.* $\pi_1^{alg}(X) = 1$.

(B) X *is simply connected.*

Proof. (A) Suppose to the contrary that $g : Y \to X$ is a connected étale covering of degree at least two. Then Y is irreducible since X is locally unibranch, and $f \circ g : Y \to \mathbb{P}^n$ has (geometric) degree $> d$. Hence by Theorem 6.1, and the hypothesis that $d \leq n$, there exists a point $y \in Y$ at which $e_{fg}(y) \geq d + 1$. But $e_{fg}(y) = e_f(g(y))$, since g is étale, whereas $e_f(x) \leq d$ for all $x \in X$, a contradiction.

(B) Following a suggestion of Deligne's, consider the universal covering $\pi : \tilde{X} \to X$. There is an algebraic set $H \subseteq \mathbb{P}^n$ such that the composition $\tilde{X} \to X \to \mathbb{P}^n$ restricts to a connected topological covering of $\mathbb{P}^n - H$. Arguing as in the proof of (A), it follows that π is trivial.

For an alternative argument, note that by what was proved in

Theorem 6.1, there exists an irreducible set $S \subseteq X$ of dimension at least one such that f is one-to-one over $f(S)$; one may take S to be an irreducible component of $R_{d-1} = \{x \in X | e_f(x) = d\}$. Hence if S^* is the normalization of S , one has $S^* \times_{\mathbb{P}^n} X \approx S^*$. Theorem 4.1(B) then implies that the natural homomorphism

$$\pi_1(S^*) \to \pi_1(S^* \times X) = \pi_1(S^*) \times \pi_1(X)$$

is surjective. But this is only possible if $\pi_1(X) = 1$; for if $\alpha \in \pi_1(X)$ were a non-trivial element, then $(1,\alpha) \in \pi_1(S^*) \times \pi_1(X)$ would not be in the image of $\pi_1(S^*)$. ∎

<u>NOTES.</u> (1) Concerning the ramification loci R_ℓ , it is shown in [46] that if $f : X \to Y$ is a branched covering with X normal and Y non-singular, and if $R_\ell \subseteq X$ is non-empty, then every irreducible component of R_ℓ has codimension $\leqslant \ell$ in X . This generalizes Zariski's theorem on the purity of the branch locus, and can be used to give an alternative proof of Theorem 6.1.

(2) In contrast to the corresponding statement (Corollary 5.3) for subvarieties of projective space, it is not true that an arbitrarily singular variety which admits a branched covering $f : X \to \mathbb{P}^n$ of degree $\leqslant n$ is simply connected. This is shown by an example due to A. Landman. Start with a covering $f' : X' \to \mathbb{P}^n$ of degree $d \leqslant n$, with X' normal, and let X be the variety obtained by identifying two points of X' lying in the same fibre of f' . Then f' induces a covering $f : X \to \mathbb{P}^n$ of degree d , but $\pi_1(X)$ is non-trivial. If $d \leqslant n - 1$, this also gives an example of a situation where the homomorphism $\pi_1(f^{-1}(L)) \to \pi_1(X)$ fails to be surjective for a generic hyperplane $L \subseteq \mathbb{P}^n$ (compare Theorem 1.1(B)).

(3) Gaffney (unpublished) and Hansen [32] have given extensions

of Theorem 6.1 and Corollary 6.3 to the case of finite maps $f : X^n \to \mathbf{P}^m$. The complications that arise when $m > n$ are indicated by the fact that there are several different notions which generalize the degree and local degrees of a branched covering.

§7. Zak's Theorem on Tangencies and Hartshorne's Conjecture

In a recent letter [65], the Soviet mathematician F. L. Zak sketched a remarkable result concerning linear spaces tangent to a subvariety $X \subseteq \mathbf{P}^m$, and indicated how it leads to a proof of Hartshorne's conjecture on linear normality. This section is devoted to an exposition of Zak's work. J. Roberts and independently J. Hansen have reconstructed the arguments suggested in Zak's letter. Pending Zak's publication of his results, we refer the reader to Roberts' notes [57] for detailed arguments.

We adopt the convention that when we deal with a subvariety X of projective space, it is assumed to be <u>non-degenerate</u>, i.e. not contained in any hyperplane.

Zak's theorem on tangencies

Let $X \subset \mathbf{P}^m$ be a smooth projective variety of dimension n. For $x \in X$, $T_x \subseteq \mathbf{P}^m$ denotes the projective tangent space to X at x. One says that a linear space $L \subseteq \mathbf{P}^m$ is tangent to X at x if $T_x \subseteq L$. When L is a hyperplane, it is equivalent to require that x be a singular point of the divisor $X \cap L$.

Zak's first main theorem bounds the dimension of the locus on X along which a given linear space is tangent:

<u>THEOREM 7.1.</u> ([65]) (A) <u>Fix a linear space</u> $L \subseteq \mathbf{P}^m$ <u>of dimension</u> k $(n \leqslant k \leqslant m - 1)$. <u>Then</u> $\{x \in X | T_x \subseteq L\}$ <u>has dimension</u> $\leqslant k - n$.

Before proceeding to the proof, we give several striking corollaries, also due to Zak [65].

COROLLARY 7.2. (A) The Gauss map $X \to \mathrm{Grass}(\mathbf{P}^n, \mathbf{P}^m)$ defined by $x \to T_x$ is finite.

COROLLARY 7.3. (A) Let Y be an arbitrary hyperplane section of X. Then Y is non-singular in codimension $2n - m - 1$.

Since Y is Cohen-Macaulay, it follows for instance that if $2n - m \geqslant 2$, then every hyperplane section of X is normal (and in particular, being connected, is irreducible).

COROLLARY 7.4. (A) Let $X^* \subseteq \mathbf{P}^{m*}$ be the dual variety of $X \subseteq \mathbf{P}^m$. Then $\dim X^* \geqslant n$.

Recall that X^* is by definition the set of hyperplanes tangent to X at some point.

Proofs of Corollaries. The first two corollaries are immediate consequences of the theorem in the cases $k = n$ and $k = m - 1$ respectively. For (7.4), consider the incidence correspondence

$$P = \{(x,L) \mid T_x \subseteq L\} \subseteq X \times \mathbf{P}^{m*} .$$

The first projection realizes P as a $\mathbf{P}^{m-n-1}$-bundle over X, and hence $\dim P = m - 1$. The dual variety $X^* \subseteq \mathbf{P}^{m*}$ is the image of P under the second projection. But according to the theorem, the fibres of $P \to X^*$ all have dimension $\leqslant m - n - 1$, and the result follows. ∎

We remark that the bounds in Theorem 7.1 and Corollary 7.4 are achieved for the Segre variety $\mathbf{P}^1 \times \mathbf{P}^{n-1} \subseteq \mathbf{P}^{2n-1}$.

Proof of Theorem 7.1. Suppose to the contrary that there is an irreducible component $S \subseteq \{x \in X | T_x \subseteq L\}$ of dimension $> k - n$. We claim first that there exists a linear space $V \subseteq \mathbb{P}^m$ of codimension $k+1$, disjoint from X and L , such that the projection $\pi_V : X \to L$ centered at V is not one-to-one over $\pi_V(S)$. To verify this, observe that $X \nsubseteq L$, since X spans $\mathbb{P}^m$, and choose points $x \in X - L$, $s \in S$. Because T_s does not contain the line $\overline{xs}$, $\overline{xs}$ cannot lie in X . Then fixing a point $p \in \overline{xs}$, with $p \notin X$, one may take V to be a generic $(m-k-1)$-plane through p .

As $\dim X \times S > k$, the connectedness theorem applies to the finite map

$$F = \pi_V \times \pi_V | S : X \times S \to L \times L = \mathbb{P}^k \times \mathbb{P}^k \quad .$$

The choice of V guarantees that $F^{-1}(\Delta)$ does not consist only of the diagonal $\Delta_S \subseteq X \times S$. Then since $F^{-1}(\Delta)$ is connected, there exists a smooth curve T , plus a morphism $T \to F^{-1}(\Delta)$ whose image meets, but is not contained in, Δ_S . This gives rise to a family of pairs

$$\{ (x_t , s_t) \}_{t \in T} \subseteq X \times_{\mathbb{P}^k} S$$

parametrized by T , such that $x_t \neq s_t$ for almost all $t \in T$, but with $x_{t*} = s_{t*} = s*$ for some $t* \in T$. As $t \to t*$ the secant lines $\overline{x_t s_t}$ degenerate to a tangent line $\ell * \subseteq T_{s*}X$. On the other hand, when $x_t \neq s_t$ the secants $\overline{x_t s_t}$ meet the center of projection V , and hence so too must $\ell*$. But $\ell* \subseteq T_{s*} \subseteq L$, and L is disjoint from V , a contradiction. ∎

<u>REMARK 7.5.</u> It is amusing to note that when $X \subseteq \mathbb{P}^m$ is a non-degenerate complete intersection (i.e. the transversal intersection of $m-n$ hypersurfaces of degrees $\geqslant 2$), then one has a stronger result:

(*) A hyperplane can be tangent to X at only finitely many points.

If X is a hypersurface, this is Theorem 7.1; the point is that the same statement holds for complete intersections of any codimension. To verify (*), consider the incidence correspondence $P \subseteq X \times \mathbf{P}^{m*}$ arising in the proof of Corollary 7.4. (*) is equivalent to the assertion that the second projection $\pi : P \to \mathbf{P}^{m*}$ is finite. Now $P = \mathbf{P}(N^*(1))$, where N is the normal bundle to X, and π is the morphism defined by $\mathcal{O}_P(1)$ on P (cf.[13, Exp. XVII]). But the hypotheses on X imply that $N(-1)$ is an ample vector bundle. Hence $\mathcal{O}_P(1)$ is an ample line bundle, and so π is finite.

Hartshorne's conjecture on linear normality.

Recall that a non-singular variety $X \subseteq \mathbf{P}^m$ is linearly normal if $X \subseteq \mathbf{P}^m$ is not the projection of a (non-degenerate) embedding of X in $\mathbf{P}^{m+1}$. Alternatively, $X \subseteq \mathbf{P}^m$ is linearly normal if and only if the natural map

$$H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is surjective. From this second description one shows, for example, that complete intersections are linearly normal. Motivated by the conjecture that a subvariety of projective space of sufficiently small codimension is a complete intersection, Hartshorne [34] was led on the basis of a few examples to make the following conjecture:

If $X \subseteq \mathbf{P}^m$ is a non-singular closed subvariety of dimension n, and if $3n > 2(m-1)$, then X is linearly normal.

Zak succeeded in using his theorem on tangencies to give a proof of Hartshorne's conjecture. Equivalently, setting $r = m + 1$, we may state Zak's result as

THEOREM 7.6. ([65]) (A) *If* $X \subseteq \mathbb{P}^r$ *is a smooth, closed, (non-degenerate) subvariety of dimension* n , *and if* $3n > 2(r-2)$, *then* $\mathrm{Sec}(X) = \mathbb{P}^r$.

We will present below a proof (in the complex case) based on the connectedness theorem. However we cannot resist first sketching the beautiful argument outlined in Zak's letter. We omit proofs, which are given by Roberts [57].

The connection between Hartshorne's conjecture and Zak's theorem on tangencies comes from

PROPOSITION 7.7. (A) *Let* $X \subseteq \mathbb{P}^r$ *be a smooth projective variety of dimension* n . *If* $\dim \mathrm{Sec}(X) = \ell < r$, *then there exists a hyperplane* $L \subseteq \mathbb{P}^r$ *which is tangent to* X *along an algebraic subset* $Z \subseteq X$ *of dimension* $\geqslant 2n+2-r$.

Theorem 7.6 follows (with a small argument) from the Proposition and from Theorem 7.1. The Proposition, in turn, revolves around a differential study of the secant variety $\mathrm{Sec}(X)$. The main point, which as Roberts observes goes back to Terracini [64], is the following

LEMMA 7.8. *Let* $X \subseteq \mathbb{P}^r$ *be a smooth n-dimensional variety, and let* $x, y \in X$ *be distinct points.*

(A) *If* p *lies on the secant line* $\overline{xy}$, *and* $p \neq x, y$, *then*

$$(*) \qquad \mathrm{Span}(T_x, T_y) \subseteq T_p\mathrm{Sec} ,$$

where $T_p\mathrm{Sec} \subseteq \mathbb{P}^r$ *denotes the Zariski tangent space to* $\mathrm{Sec}(X)$ *at* p .

(B) *For generic points* $x, y \in X$, *and* $p \in \overline{xy}$, *equality*

<u>holds in</u> (*).

(Compare [25, (6.2)].) Returning to the situation of the Proposition, the idea is that any point $p \in \mathrm{Sec}(X) - X$ lies on a family of secant lines having dimension at least $(2n+1) - \ell \geq 2n + 2 - r$. By the lemma, one obtains a subset $Z_p \subseteq X$ of dimension $\geq 2n + 2 - r$ such that $T_x \subseteq T_p\mathrm{Sec}$ for every $x \in Z_p$. But $\dim \mathrm{Sec}(X) < r$, and hence for generic $p \in \mathrm{Sec}(X)$, $T_p\mathrm{Sec}$ is contained in a hyperplane, yielding (7.7).

The remainder of this section is devoted to an alternative approach to Theorem 7.6. The method is to focus on the variety $\mathrm{Trisec}(X)$ of trisecant 2-planes to $X \subseteq \mathbf{P}^r$, and its relation with $\mathrm{Sec}(X)$. We verify first that $\mathrm{Sec}(X) = \mathbf{P}^r$ if and only if $\mathrm{Sec}(X) = \mathrm{Trisec}(X)$. Then we use the connectedness theorem to show that this latter criterion holds if $3n > 2(r-2)$. We henceforth work over $\mathbf{C}$.

Given a smooth projective variety $X \subseteq \mathbf{P}^r$ of dimension n, let $\mathrm{Trisec}(X) \subseteq \mathbf{P}^r$ be the Zariski closure of the union of all trisecant planes $\overline{xyz}$, where $x, y, z \in X$ are distinct non-collinear points. $\mathrm{Trisec}(X)$ is irreducible, of dimension $\leq 3n + 2$, and $\mathrm{Sec}(X) \subseteq \mathrm{Trisec}(X)$. We start with a Terracini-type lemma, analogous to the one stated above.

<u>LEMMA 7.9.</u> <u>Let</u> $x_1, x_2, x_3 \in X$ <u>be distinct non-collinear points of</u> X, <u>and let</u> p <u>be a point on the plane</u> $\overline{x_1x_2x_3}$. <u>Assume that</u> p <u>does not lie on any of the lines</u> $\overline{x_ix_j}$. <u>Then</u>

$$(*) \qquad \mathrm{Span}(T_{x_1}, T_{x_2}, T_{x_3}) \subseteq T_p\mathrm{Trisec},$$

<u>and for generic</u> $x_1, x_2, x_3 \in X$ <u>and</u> $p \in \overline{x_1x_2x_3}$, <u>equality holds in</u> (*).

Proof. It is enough to treat the affine situation in which X and p are contained in $\mathbb{C}^r$. For $1 \le i \le 3$, choose local analytic parametrizations $f_i : U_i \to X \subseteq \mathbb{C}^r$ about the points x_i, with $U_i \subseteq \mathbb{C}^n$ neighborhoods of the origin, and $x_i = f_i(0)$. Consider the map $\phi : U_1 \times U_2 \times U_3 \times \mathbb{C} \times \mathbb{C} \to \mathrm{Trisec}(X) \subseteq \mathbb{C}^r$ defined by $\phi = (1-s-t)f_1 + sf_2 + tf_3$. As long as $st(1-s-t) \neq 0$, the translate of $\mathrm{Im}\, d\phi(0,0,0,s,t)$ by the vector $p = (1-s-t)x_1 + sx_2 + tx_3$ is precisely the linear span of T_{x_1}, T_{x_2} and T_{x_3} in $\mathbb{C}^r$. This proves the first assertion. The second follows from the fact that ϕ is generically submersive. ∎

We remark that this is of course the sort of argument used to prove Lemma 7.8, at least over $\mathbb{C}$. (In the abstract case [57], the proof is more involved.)

LEMMA 7.10. *If $X \subseteq \mathbb{P}^r$ is a smooth (non-degenerate) projective variety, then $\mathrm{Sec}(X) = \mathbb{P}^r$ if and only if $\mathrm{Sec}(X) = \mathrm{Trisec}(X)$.*

Proof. Since in any event $\mathrm{Sec}(X) \subseteq \mathrm{Trisec}(X)$, it suffices to prove that if $\dim \mathrm{Sec}(X) = \ell < r$, then $\dim \mathrm{Trisec}(X) > \ell$. By Lemma 7.9, it is in turn sufficient to show that for one - and hence for a generic - triple $x, y, z \in X$ of distinct points, $\mathrm{Span}(T_x, T_y, T_z)$ has dimension $> \ell$. But the description (7.8(B)) of the tangent spaces to $\mathrm{Sec}(X)$ shows that for a generic pair of points $x, y \in X$, $\dim \mathrm{Span}(T_x, T_y) = \ell$. Since $\ell < r$, and since X spans $\mathbb{P}^r$, we may then take $z \in X - \mathrm{Span}(T_x, T_y)$. ∎

Theorem 7.6 now follows from

PROPOSITION 7.11. *Let $X \subseteq \mathbb{P}^r$ be a smooth projective variety of dimension n. If $3n > 2(r-2)$, then $\mathrm{Sec}(X) = \mathrm{Trisec}(X)$.*

Proof. Suppose to the contrary that $\mathrm{Sec}(X) \neq \mathrm{Trisec}(X)$, so that we may choose distinct, non-collinear points $x_0, y_0, z_0 \in X$ such that the trisecant plane $\overline{x_0 y_0 z_0}$ is not contained in $\mathrm{Sec}(X)$. A generic line $\ell \subseteq \overline{x_0 y_0 z_0}$ is then disjoint from X , and meets $\mathrm{Sec}(X)$ at only finitely many points. Fix such a line ℓ , and consider the finite map $\pi : X \to \mathbf{P}^{r-2}$ obtained by projection from ℓ to a complementary $\mathbf{P}^{r-2}$. Since $3n > 2(r-2)$, the connectedness theorem for three factors (3.4) applies to the map

$$F = \pi \times \pi \times \pi : X \times X \times X \to \mathbf{P}^{r-2} \times \mathbf{P}^{r-2} \times \mathbf{P}^{r-2} .$$

Then as in the proof of Theorem 7.1, we can find a family of triple-points

$$\{ (x_t , y_t , z_t) \}_{t \in T} \subseteq X \times_{\mathbf{P}^{r-2}} X \times_{\mathbf{P}^{r-2}} X$$

parametrized by a smooth irreducible curve T , containing (x_0 , y_0 , z_0) , such that x_t , y_t , and z_t are distinct for $t \in T - \{t^*\}$, while two or more members of the limiting triple $(x^* , y^* , z^*) = (x_{t^*} , y_{t^*} , z_{t^*})$ coincide.*

The key to the argument is the observation that since x_0 , y_0 , z_0 are non-collinear, and since $\ell \cap \mathrm{Sec}(X)$ is finite, <u>the points of intersection</u>

$$a = \overline{x_t y_t} \cap \ell , \ b = \overline{x_t z_t} \cap \ell , \ c = \overline{y_t z_t} \cap \ell$$

<u>are distinct and independent of</u> t so long as $t \neq t^*$ (see Figure 1).

---

* We are tacitly assuming here that (x_0 , y_0 , z_0) lies on an irreducible component W of $F^{-1}(\Delta)$ which meets the set $D_X = \{(x_1 , x_2 , x_3) \in X \times X \times X \mid$ two or more of the x_i coincide$\}$. In reality it will be necessary to choose a sequence $W = W_0 , \ldots , W_s$ of components such that W_i meets W_{i+1} , and W_s meets D_X . We leave it to the reader to carry out the small additional argument required in this general case.

Hence if ℓ^*_{xy} , ℓ^*_{xz} , and ℓ^*_{yz} denote the limiting positions of the secants $\overline{x_t y_t}$, $\overline{x_t z_t}$ and $\overline{y_t z_t}$ as $t \to t^*$, then $\ell^*_{xy} \cap \ell = a$, $\ell^*_{xz} \cap \ell = b$, and $\ell^*_{yz} \cap \ell = c$. In particular, these lines are distinct. But then all three of the limiting points x^* , y^* and z^* must coincide: for if e.g. $y^* = z^* \neq x^*$, then the secants $\overline{x_t y_t}$ and $\overline{x_t z_t}$ would degenerate as $t \to t^*$ to a common line. On the other hand, if $(x_t , y_t , z_t) \to (x^* , x^* , x^*)$ as $t \to t^*$, then ℓ^*_{xy} , ℓ^*_{xz} and ℓ^*_{yz} are tangent lines to X at x^* . In particular, the center of projection ℓ meets the tangent space T_{x^*} in more than one point. But then $\ell \subseteq T_{x^*} \subseteq Sec(X)$, contradicting the choice of ℓ . ∎

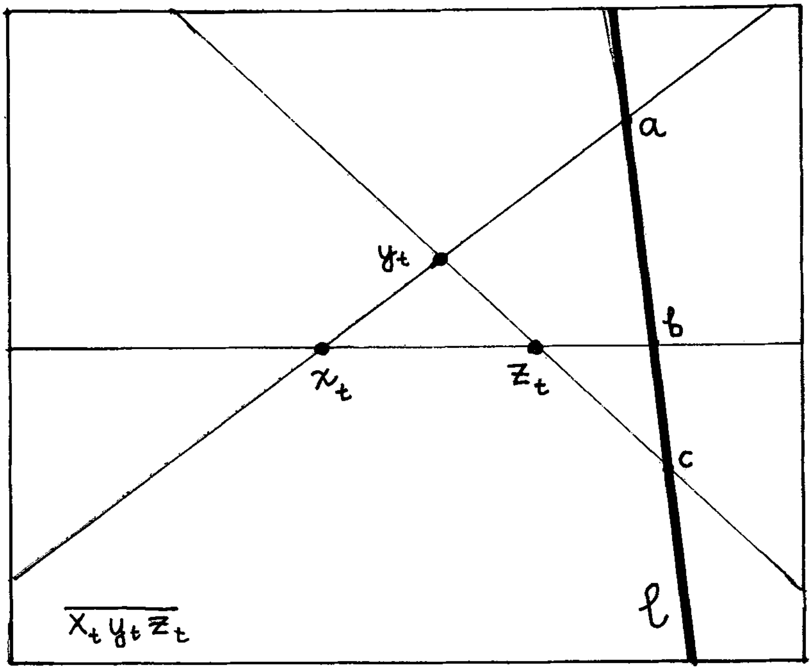

Figure 1.

NOTES. (1) Zak reports in his letter that he originally proved Theorem 7.1 in January, 1979 (before [19] appeared) using methods of formal geometry. He became aware of the connectedness theorem at the end of 1979, and realized that it led to a simple proof of his theorem on tangencies. In the situation of (7.1), Zak obtained additional information when the ground field has characteristic zero. Specifically, he has shown that for a general k-plane $L \subseteq \mathbf{P}^m$ tangent to X, the set $\{x \in X | T_x \subseteq L\} \subseteq \mathbf{P}^m$ is a linear space.

(2) Zak's corollary (7.2) on the finiteness of the Gauss mapping strengthens a result of Griffiths and Harris [25], who had proved that the Gauss map is generically finite.

(3) Concerning the dimension $n^* = \dim X^*$ of the dual of a smooth n-dimensional variety $X \subseteq \mathbf{P}^m$, Zak and independently A. Landman established that $n^* \geqslant m - n + 1$ provided that $n \geqslant 2$. Equality holds here, as it does in Corollary 7.4, for the Segre varieties $\mathbf{P}^1 \times \mathbf{P}^{n-1} \subseteq \mathbf{P}^{2n-1}$, which are self-dual. Landman [43] has used Picard-Lefschetz theory to prove the remarkable results that if X^* is degenerate (i.e. $n^* < m - 1$), then (i) the "defect" $m - 1 - n^*$ is determined by the Betti numbers of X, and (ii) $m - 1 - n^* \equiv n \pmod 2$. Holme [37] shows that n^* can be computed in terms of characteristic classes associated to the embedding $X \subseteq \mathbf{P}^m$. We recommend Kleiman's survey [42], especially Section D of Chapter IV, for an overview of other facts about dual varieties.

(4) Zak indicates in his letter that he now has several proofs of Hartshorne's conjecture. He also reports that he has classified all n-dimensional smooth varieties $X \subseteq \mathbf{P}^r$, with $3n = 2(r-2)$, which project isomorphically onto their image in $\mathbf{P}^{r-1}$.

§8. The Fundamental Group of the Complement of a Node Curve

Let C be a curve in the projective plane whose only singularities are nodes, i.e. ordinary double points with distinct tangents (given in local analytic coordinates by the equation $zw = 0$).

THEOREM 8.1. (A) ([16]) Any (tamely ramified) branched covering of $\mathbb{P}^2$ with branch locus contained in C is abelian.

(B) ([10, 11]) $\pi_1(\mathbb{P}^2 - C)$ is abelian.

Let $C_1,\ldots,C_r$ be the irreducible components of C, and set $d_i = \deg(C_i)$.

COROLLARY 8.2. One has:

(A) $\pi_1^{\text{tame}}(\mathbb{P}^2 - C) = (\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/(d_1,\ldots d_r))^{(p)}$

(B) $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/(d_1,\ldots,d_r)$.

Proof of Corollary. For (B), since the fundamental group is abelian, it is isomorphic to the homology group $H_1(\mathbb{P}^2 - C)$, and

$$H_1(\mathbb{P}^2 - C) = H^3(\mathbb{P}^2, C) = \operatorname{coker}(H^2(\mathbb{P}^2) \to H^2(C)).$$

Now $H^2(C) = \oplus H^2(C_i)$, $H^2(\mathbb{P}^2) = \mathbb{Z}$, $H^2(C_i) = \mathbb{Z}$, and the induced map $H^2(\mathbb{P}^2) \to H^2(C_i)$ is multiplication by d_i.

It follows from (B) and the Riemann-Enriques-Grauert-Remmert existence theorem (see[59]) that finite coverings of $\mathbb{P}^2$ branched along C are determined (up to isomorphism) by subgroups of finite index in $\mathbb{Z}^r/(d_1,\ldots d_r)$ (up to conjugacy). The isomorphism in (A) is the corresponding assertion for tamely ramified coverings over a general ground field; the group on the right is the profinite completion of $\mathbb{Z}^r/(d_1,\ldots d_r)$ using all subgroups of finite index prime to the characteristic p . Any finite abelian covering is a composition of cyclic coverings, and these can be analyzed by Kummer theory; the isomorphism in (A) follows easily (see [1], I, p. 83). The characteristic p version can also be deduced from the complex case by specialization [11].

Proof of Theorem 8.1. We start with some general remarks about a finite ramified covering $f : X \to \mathbf{P}^2$, with X normal, whose branch locus is contained in a curve $C \subseteq \mathbf{P}^2$. Set $X_o = X - f^{-1}(C)$, and let $f_o : X_o \to \mathbf{P}^2 - C$ be the induced topological covering. We assume that the covering is Galois, or regular, i.e. that the deck transformations act transitively on the fibres over a point in $\mathbf{P}^2 - C$.

Let

$$G = \mathrm{Aut}(X/\mathbf{P}^2) = \mathrm{Aut}(X_o/\mathbf{P}^2 - C) \quad .$$

One says that the covering is abelian if G is.

Each irreducible component D' of $f^{-1}(C)$ determines an inertia group

$$I(D') = \{\sigma \in G : \sigma|D' = \text{identity}\} \quad .$$

The inertia groups of the components of $f^{-1}(C)$ which map to the same irreducible component of C are conjugate subgroups in G , since G acts transitively on the set of such components.

The local analytic structure of the branched covering f may be analyzed in terms of the local topology of C in $\mathbf{P}^2$. Fix points $y \in C$, and $x \in f^{-1}(y)$. Let B be a small ε-neighborhood of y in $\mathbf{P}^2$ with respect to some metric, and denote by B' the connected component of $f^{-1}(B)$ containing x . Let $B_o = B - B\cap C$, and $B'_o = B' - B'\cap f^{-1}(C)$. The possibilities for the branched covering

$B' \to B$, or the unbranched covering $B_o' \to B_o$, depend on the local fundamental group $\pi_1(B_o)$.

If y is a <u>simple point</u> of C, then $\pi_1(B_o) = \mathbb{Z}$, and $B' \to B$ is given in local analytic coordinates by $(z,w) \to (z^e,w)$. In particular, x is a simple point of $f^{-1}(C)$ (regarded as a reduced curve). If D' is the irreducible component of $f^{-1}(C)$ containing x, then $I(D')$ is canonically isomorphic to the group of covering transformations of $B_o' \to B_o$. Since any irreducible component of $f^{-1}(C)$ contains points whose images on C are simple, <u>all of the inertia groups are cyclic</u>.

Note also that $I(D')$ is trivial only when the covering is unramified at x. It follows that <u>the inertia groups of the irreducible components of</u> $f^{-1}(C)$ <u>generate</u> G. For if H is the subgroup they generate, then H is normal in G; the corresponding covering $X/H \to \mathbb{P}^2$ has trivial inertia groups, and is therefore unramified over any curve. Since the complement of a finite set of points in $\mathbb{P}^2$ is simply connected, $X/H = \mathbb{P}^2$, i.e. $H = G$.

Suppose next that y is a <u>node</u> of C. Then B_o is homeomorphic to the product of two punctured disks, so $\pi_1(B_o) = \mathbb{Z} \oplus \mathbb{Z}$, and the covering $B' \to B$ is dominated by one given in local analytic coordinates by $(z,w) \to (z^d,w^e)$. In particular, $f^{-1}(C)$ has at most two branches at x, and distinct branches of $f^{-1}(C)$ at x do not map to the same branch of C at y. Moreover, if two irreducible components D_i' and D_j' of $f^{-1}(C)$ meet at x, then $I(D_i')$ and $I(D_j')$ may be identified with subgroups of the group of deck transformations of $B_o' \to B_o$, which is abelian. Therefore $I(D_i')$ and $I(D_j')$ <u>commute</u>.

When C is a node curve, these are the only possibilities that arise. Hence to prove (A), it suffices to show that $f^{-1}(D)$ is irreducible for every irreducible component D of C. For then any two irreducible components of $f^{-1}(C)$ must meet (since any two components of C do), and so G is generated by a collection of cyclic subgroups, any two of which commute with each other. The irreducibility of $f^{-1}(D)$ is the assertion of Lemma 8.3(A) below.

For (B), consider any regular topological covering $f_o : X_o \to \mathbb{P}^2 - C$, with group G of deck transformations. (It is enough to consider simply the universal covering, where $G = \pi_1(\mathbb{P}^2 - C)$.) Denote by S the set of singular points of C. Let V be the complement of $C - S$ in a tubular neighborhood of $C - S$ in $\mathbb{P}^2 - S$; V is the disjoint union of punctured tubular neighborhoods V_D of the irreducible components $D - S$ of $C - S$. Each connected component V' of $f_o^{-1}(V)$ determines an inertia group $I(V')$, as follows. Suppose that $f_o(V') = V_D$, and take an ε-neighborhood B of a simple point y of D. Let B_o' be a connected component of $f_o^{-1}(B)$ which is contained in V'. Then set

$$I(V') = \{\sigma \in G \mid \sigma(B_o') = B_o'\} .$$

One sees that this is independent of the choice of y and B_o' by joining two such choices by a chain where successive open sets B_o' overlap. (When the covering is finite, and therefore the restriction of a branched covering $f : X \to \mathbb{P}^2$, the connected components of $f_o^{-1}(V)$ correspond to the irreducible components of $f^{-1}(C)$, and the two notions of inertia groups coincide.) As before, the inertia groups are cyclic - but possibly infinite - and they generate G. If B is a small neighborhood of a node y of C, and B_o' is a connected component of $f_o^{-1}(B)$, and if two components V_i' and V_j' of $f_o^{-1}(V)$ meet B_o', then $I(V_i')$ and $I(V_j')$ commute. It suffices therefore to prove that $f_o^{-1}(V_D)$ is connected for each irreducible component D of C, which is Lemma 8.3(B).

LEMMA 8.3. *Let* D *be an irreducible component of a plane curve* C. *Assume that all the singularities of* C *which lie on* D *are nodes.*

(A) *If* $f : X \to \mathbf{P}^2$ *is a (tamely ramified) finite covering branched along* C, *then* $f^{-1}(D)$ *is irreducible.*

(B) *If* $f_o : X_o \to \mathbf{P}^2 - C$ *is a topological covering, and* V_D *is a punctured tubular neighborhood of* $D\text{-Sing}(C)$ *as above, then* $f_o^{-1}(V_D)$ *is connected.*

Proof. Let $\eta : \tilde{D} \to D$ be the normalization of D. In case (A), one sees from the local description of the covering at simple points and nodes that the fibre product $X \times_{\mathbf{P}^2} \tilde{D} = f^{-1}(D) \times_D \tilde{D}$ has only one branch at any point, i.e. it is locally unibranch. In fact, as Deligne and Zariski point out, $(X \times_{\mathbf{P}^2} \tilde{D})_{red}$ is non-singular. Now $X \times_{\mathbf{P}^2} \tilde{D}$ projects onto $f^{-1}(D)$, so it suffices to prove that $X \times_{\mathbf{P}^2} \tilde{D}$ is irreducible. As any connected locally irreducible curve is irreducible, it suffices in turn to show that $X \times_{\mathbf{P}^2} \tilde{D}$ is connected. But this follows from Theorem 4.1(A).

For (B), let $V(\tilde{D})$ be the normal bundle to the C^∞ immersion $\eta : \tilde{D} \to \mathbf{P}^2$. Extend η to a C^∞ immersion ϕ from an ε-neighborhood $V_\varepsilon(\tilde{D})$ of the zero-section to an ε-neighborhood of D in $\mathbf{P}^2$. With proper choice of metric - so that near a node the two branches become perpendicular two-planes in Euclidean four-space - one sees that $V_\varepsilon(\tilde{D}) - \phi^{-1}(C)$ is a bundle over $\tilde{D} - \eta^{-1}(S) = D - S$ with fibre a punctured two-disk; as above, S denotes the set of singular points of C. In particular, the tubular neighborhood V_D is a deformation retract of $V_\varepsilon(\tilde{D}) - \phi^{-1}(C)$.

Now consider the product mapping $F = i \times \eta$, where i is the inclusion of $\mathbb{P}^2 - C$ in $\mathbb{P}^2$:

$$F : (\mathbb{P}^2 - C) \times D \to \mathbb{P}^2 \times \mathbb{P}^2 \quad .$$

By the connectedness theorem (3.1(B)), the fundamental group of the inverse image $F^{-1}(\Delta_\varepsilon)$ of an ε-neighborhood of the diagonal surjects onto the fundamental group of $(\mathbb{P}^2 - C) \times \tilde{D}$. One verifies (see [11]) that $F^{-1}(\Delta_\varepsilon)$ contains $V_\varepsilon(\tilde{D}) - \phi^{-1}(C)$, and therefore V_D , as a deformation retract. Thus $\pi_1(V_D) \twoheadrightarrow \pi_1(\mathbb{P}^2 - C)$, and therefore $X_o \times_{\mathbb{P}^2 - C} V_D$ is connected, as asserted. ∎

The above method yields the following corollary (cf. [54]).

<u>COROLLARY 8.4.</u> (B) <u>Let</u> C <u>be a node curve meeting an arbitrary non-empty curve</u> C' <u>transversely. Then there is a central extension</u>

$$1 \to A \to \pi_1(\mathbb{P}^2 - C \cup C') \to \pi_1(\mathbb{P}^2 - C') \to 1$$

<u>where</u> A <u>is a free abelian group on</u> r <u>generators,</u> r <u>being the number of irreducible components of</u> C .

<u>COROLLARY 8.5.</u> <u>Let</u> C <u>be a node curve defined by an irreducible homogenous polynomial</u> $F(X,Y,Z)$ <u>of degree</u> d . <u>(In characteristic</u> p , <u>assume that</u> $p \nmid d$.) Let $V \subseteq \mathbb{A}^3$ <u>be the non-singular affine surface with equation</u> $F(X,Y,Z) = 1$. <u>Then</u> V <u>is (tamely) simply connected.</u>

<u>Proof</u>. The canonical map $V \to \mathbb{P}^2 - C$ is a d-sheeted unramified covering. The corollary is therefore equivalent to the assertion that $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/d\mathbb{Z}$. ∎

The results of this section extend to higher dimensions. If H is a hypersurface in $\mathbb{P}^n$, $n \geqslant 2$, whose only singularities in codimension one are normal crossings, then $\pi_1(\mathbb{P}^n - H)$ is abelian, and is calculated as in Corollary 8.2. One may deduce this from the two-dimensional case by taking a generic plane section and applying the

Bertini theorem (1.1).

NOTES. (1) Zariski [66] began the study of $\pi_1(\mathbf{P}^2 - C)$ in the course of his investigations of surfaces as branched coverings of $\mathbf{P}^2$. He stated the theorem of this section, but the proof he proposed depends on the still unproved assertion of Severi that a node curve of degree d can be degenerated to d lines in general position.

(2) The proofs here follow the outline of Abhyankar [1]. His argument worked completely when every component D of C is non-singular, for then Bertini's theorem is enough to show that $f^{-1}(D)$ is irreducible for a finite covering f branched along C (see [59]). The stronger connectedness theorem of §3 allows the case of a general node curve to be handled in the same fashion. This was done in the algebraic case in [16], and in the topological setting in [10, 11]. The reader will see that the use of the connectedness theorem in Lemma 8.3(A) was motivated by the work [20] on branched coverings described in §6.

(3) For historical discussions of previous work on these problems, we recommend Chapter VIII of [68], with its appendices by Abhyankar and Mumford, and the introduction to [69] by Artin and Mazur. We record here only a few of the main previous results. Zariski, Popp, and Alibert and Maltsiniotis have proved Severi's assertion on the degeneration of node curves if the number of nodes is large. On the other hand, Abhyankar and Prill proved that $\pi_1(\mathbf{P}^2 - C)$ is abelian when the number of nodes is small. For an irreducible node curve of degree d with δ nodes, the theorem was known then for $\delta > d^2/2 - 9d/4 + 1$ [2], and for $\delta < d^2/4$ [55].

Edmonds and Geyer (see [22]) had shown that any finite solvable quotient of $\pi_1(\mathbf{P}^2 - C)$ must be abelian. Geyer and Oka had reduced the problem to the case where C is irreducible [54]. Oka, Sakamoto and

Randell had related the problem to properties of the Milnor fibration associated to the defining equation of the curve. For example, Randell [56] had proved that the kernel of the map from $\pi_1(\mathbb{P}^2 - C)$ to $H_1(\mathbb{P}^2 - C)$ is a perfect group if C is a node curve.

(4) The techniques of some of these authors extend to simply connected surfaces other than $\mathbb{P}^2$. Lê and Saito have recently generalized the results of [16] and [11] to complete intersections. For singularities other than nodes only a few first steps have been made in calculating the fundamental group of the complement. Prill has shown that the fundamental group remains abelian if the number of cusps and nodes is small. Zariski's example [66] of two sextic curves with six cusps, where the fundamental group is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ when the cusps are on a conic, but abelian when they are not, indicates the subtleties of these problems. Dolgachev and Libgober have recently carried forward this work of Zariski, and have found other interesting examples. Moishezon has studied the case in which the curve is the branch curve of a projection to $\mathbb{P}^2$, and has related the fundamental group to the algebra of associated braid groups. (Progress on these last two projects was reported at this conference.)

§9. Higher Homotopy

Deligne has shown how a theorem of Goresky and MacPherson can be used to bring higher homotopy groups into the setting of the connectedness theorem. We give here an account of the results so obtained. We deal in this section with complex algebraic varieties. A local complete intersection is a connected but possibly reducible variety (or scheme) which is locally a complete intersection in some smooth variety. For simplicity of notation, we supress base-points of homotopy groups.

The following deep theorem of Goresky and MacPherson, conjectured by Deligne for smooth varieties in [10] and [11], is the basic fact upon which the results of this section depend. An announcement with indications of proofs appears in [23, §4].

THEOREM 9.1. ([23]). *Let* X *be a local complete intersection of pure dimension* n . *Let*

$$f : X \to \mathbf{P}^m$$

be a quasi-finite (i.e. finite-to-one) morphism, and let $L \subseteq \mathbf{P}^m$ *be a linear space of codimension* d . *Denote by* L_ε *an* ε*-neighborhood of* L *with respect to some Riemannian metric on* $\mathbf{P}^m$. *Then for sufficiently small* ε, *one has*

$$\pi_i(X , f^{-1}(L_\varepsilon)) = 0 \quad \text{for} \quad i \leqslant n - d \ .$$

This theorem may be viewed as a non-compact strengthening of the Lefschetz hyperplane theorem (cf. [47, §7]). When $i = 1$, and X is irreducible, one recovers a form of the Bertini theorem (2.1(B)). For non-singular X , Goresky and MacPherson also treat the case in which f has positive dimensional fibres ([23, Thm. 4.1]).

The first consequence of Theorem 9.1 is Deligne's generalization of the connectedness theorem:

THEOREM 9.2. ([12]) *Let* X *be a compact local complete intersection of pure dimension* n , *and let*

$$f : X \to \mathbf{P}^m \times \mathbf{P}^m$$

be a finite morphism. Denote by Δ *the diagonal in* $\mathbf{P}^m \times \mathbf{P}^m$.

a) *If* $n - m \geqslant 1$, *then* $\pi_1(X , f^{-1}(\Delta))$ *is trivial.*

b) <u>If</u> $n - m \geqslant 2$, <u>one has an exact sequence</u>

$$\pi_2(f^{-1}(\Delta)) \to \pi_2(X) \to \mathbb{Z} \to \pi_1(f^{-1}(\Delta)) \to \pi_1(X) \to 1 \quad .$$

c) <u>If</u> $2 < i \leqslant n - m$, <u>then</u> $\pi_i(X, f^{-1}(\Delta)) = 0$.

The map from $\pi_2(X)$ to $\mathbb{Z}$ occuring in (b) can be identified with the difference of the two homomorphisms .

$$(pr_1 \circ f)_* \ , \ (pr_2 \circ f)_* : \pi_2(X) \to \pi_2(\mathbf{P}^m) = \mathbb{Z}$$

where pr_i $(i = 1,2)$ are the projections of $\mathbf{P}^m \times \mathbf{P}^m$ onto its factors. Observe that statement (a) implies that $f^{-1}(\Delta)$ is connected, and that $\pi_1(f^{-1}(\Delta)) \twoheadrightarrow \pi_1(X)$. (Compare Theorem 3.1 and Corollary 3.3.)

<u>Proof.</u> We use the construction introduced in the proof of Theorem 3.1:

$$\begin{tikzcd}
 & X^* \arrow[r, "q"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\mathbf{P}^{2m+1} \supset & V \arrow[r, "p"'] & \mathbf{P}^m \times \mathbf{P}^m \\
L \arrow[ur, hook] \arrow[rr, "\simeq", "p|L"'] & & \Delta \arrow[u, hook]
\end{tikzcd}$$

Recall that p and q are $\mathbf{C}^*$-bundles, V is an open subset of $\mathbf{P}^{2m+1}$, and $L \subseteq V$ is an m-dimensional linear space mapping isomorphically to Δ . The hypotheses on X imply that X^* is a (non-compact) local complete intersection of pure dimension $n + 1$. Hence by Theorem 9.1, $\pi_i(X, f^{*-1}(L_\varepsilon)) = 0$ for sufficiently small ε when $i \leqslant \dim X^* - \operatorname{codim} L = n - m$. But since f^* is proper, and L is a closed analytic submanifold of V , $f^{*-1}(L)$ is a deformation retract of $f^{*-1}(L_\varepsilon)$ provided that ε is small enough* Thus

$$(*) \qquad \pi_i(X^*, f^{*-1}(L)) = 0 \quad \text{for} \quad i \leqslant n - m \quad .$$

Consider now the commutative square

$$\begin{tikzcd} \pi_i(f^{*-1}(L)) \arrow[r] \arrow[d] & \pi_i(f^{-1}(\Delta)) \arrow[d] \\ \pi_i(X^*) \arrow[r] & \pi_i(X) \end{tikzcd} .$$

Since q gives rise to an isomorphism $f^{*-1}(L) \to f^{-1}(\Delta)$, the top horizontal map is an isomorphism for all i . On the other hand, by (*), the vertical map on the left is bijective for $i < n - m$, and surjective if $i = n - m$. The three assertions of the theorem then follow from the long exact sequence of the $\mathbb{C}^*$-bundle $q : X^* \to X$, which yields isomomphisms $\pi_i(X^*) \to \pi_i(X)$ when $i = 0$ and $i > 2$, and an exact sequence

$$0 \to \pi_2(X^*) \to \pi_2(X) \to \mathbb{Z} \to \pi_1(X^*) \to \pi_1(X) \to 1 \quad .$$

Finally, to check that the map from $\pi_2(X)$ to $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ in (b) is as described, it suffices to consider the case $X = \mathbb{P}^m \times \mathbb{P}^m$ and f = identity, where the verification is routine. ∎

<u>REMARK 9.3.</u> If X is not compact, the theorem remains valid for quasi-finite f provided that the diagonal Δ is replaced by a small ε-neighborhood Δ_ε with respect to some Riemannian metric on $\mathbb{P}^m \times \mathbb{P}^m$. In fact, the C^*-bundle $p : V \to \mathbb{P}^m \times \mathbb{P}^m$ is trivial over Δ_ε when ε is sufficiently small. For an appropriate Riemannian metric on $\mathbb{P}^{2m+1}$ (which may be taken to be a product metric near L)

* One can avoid invoking this fact by arguing as in the proof of Corollary 3.3.

the restriction of p to an ε-neighborhood L_ε of L is a trivial bundle $L_\varepsilon \to \Delta_\varepsilon$ with fibre an open disk. The same is therefore true of the pull-back $f^{*-1}(L_\varepsilon) \to f^{-1}(\Delta_\varepsilon)$. The argument then proceeds as before.

The exact sequence in statement (b) of the theorem proves slightly awkward to use in practice. The applications we have in mind follow most easily from a variant of Deligne's result for mappings of the form $X \times Y \to \mathbf{P}^m \times \mathbf{P}^m$. We adopt the following <u>notation</u>: $\hat{\mathbf{P}}^m$ denotes the natural $\mathbf{C}^*$-bundle $\hat{\mathbf{P}}^m = \mathbf{C}^{m+1} - \{0\} \to \mathbf{P}^m$. If $f : X \to \mathbf{P}^m$ is a morphism, we let $\hat{X} = X \times_{\mathbf{P}^m} \hat{\mathbf{P}}^m \to X$ be the pull-back bundle, and $\hat{f} : \hat{X} \to \hat{\mathbf{P}}^m$ the induced map.

PROPOSITION 9.4. *Let X and Y be compact local complete intersections of pure dimensions n and ℓ respectively, and let $f : X \to \mathbf{P}^m$ and $g : Y \to \mathbf{P}^m$ be finite morphisms. Then*

$$\pi_i(\hat{X} \times \hat{Y} , \hat{X} \times_{\hat{\mathbf{P}}^m} \hat{Y}) = 0 \quad \text{for} \quad i \le n + \ell - m \ .$$

Proof. Observe that there is a natural $\mathbf{C}^*$-bundle map $\hat{\mathbf{P}}^m \times \hat{\mathbf{P}}^m \to V$, where $V \subseteq \mathbf{P}^{2m+1}$ is the open set used in the previous proof, and that the diagonal $\hat{\mathbf{P}}^m = \hat{\Delta} \subseteq \hat{\mathbf{P}}^m \times \hat{\mathbf{P}}^m$ is the inverse image of the linear space $L \subseteq V$. This follows from the explicit descriptions of V and L given in the proof of Theorem 3.1. Thus starting from the finite map $F = f \times g : X \times Y \to \mathbf{P}^m \times \mathbf{P}^m$, one obtains the following commutative diagram of cartesian squares, in which W denotes the fibre product of $X \times Y$ and V over $\mathbf{P}^m \times \mathbf{P}^m$, with projection $H : W \to V$.

$$\begin{array}{ccccc}
\hat{X}\times\hat{Y} & \longrightarrow & W & \longrightarrow & X\times Y \\
{\scriptstyle \hat{F}\,=\,\hat{f}\times\hat{g}}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle f\times g\,=\,F} \\
\hat{\mathbf{P}}^m\times\hat{\mathbf{P}}^m & \longrightarrow & V & \xrightarrow[p]{} & \mathbf{P}^m\times\mathbf{P}^m \\
\uparrow & & \uparrow & & \\
\hat{\Delta} & \longrightarrow & L & &
\end{array}$$

The horizontal maps are $\mathbf{C}^*$-bundles, and the vertical maps are finite.

As in the proof of Theorem 9.2, the theorem of Goresky and MacPherson (9.1) implies that $\pi_i(W, H^{-1}(L)) = 0$ for $i \le n + \ell - m$. On the other hand, the pair $(\hat{X} \times \hat{Y}, \hat{F}^{-1}(\Delta))$ is the inverse image of the pair $(W, H^{-1}(L))$ under a bundle map, and consequently

$$\pi_i(\hat{X} \times \hat{Y}, \hat{F}^{-1}(\hat{\Delta})) \xrightarrow{\cong} \pi_i(W, H^{-1}(L))$$

for all i. Since $\hat{F}^{-1}(\hat{\Delta}) = \hat{X} \times_{\hat{\mathbf{P}}^m} \hat{Y}$, the proposition follows. ∎

REMARK 9.5. The proposition extends immediately to more than two factors. Specifically, if $X_1, \ldots, X_r$ are compact local complete intersections of pure dimensions $n_1, \ldots n_r$, and if $f_i : X_i \to \mathbf{P}^m$ $(1 \le i \le r)$ are finite morphisms, then

$$\pi_i(\hat{X}_1 \times \ldots \times \hat{X}_r, \hat{X}_1 \times_{\hat{\mathbf{P}}^m} \ldots \times_{\hat{\mathbf{P}}^m} \hat{X}_r) = 0$$

for $i \le n_1 + \ldots + n_r - (r-1)m$. The proof is the same as before, except that V becomes an open subset of $\mathbf{P}^{r(m+1)-1}$.

Proposition 9.4 has as a basic consequence the following

THEOREM 9.6. *Let X be a compact local complete intersection of pure dimension n, let $f : X \to \mathbf{P}^m$ be a finite map, and let $Y \subseteq \mathbf{P}^m$ be a closed local complete intersection of pure codimension d. Then the induced homomorphism*

$$f_* : \pi_i(X, f^{-1}(Y)) \to \pi_i(\mathbf{P}^m, Y)$$

<u>is bijective if</u> $i \le n - d$, <u>and surjective when</u> $i = n - d + 1$.
(Compare Corollary 4.3.)

<u>Proof</u>. Note to begin with that the theorem is equivalent to the assertion that

$$\hat{f}_* : \pi_i(\hat{X}, \hat{f}^{-1}(\hat{Y})) \to \pi_i(\hat{\mathbf{P}}^m, \hat{Y})$$

is an isomorphism for $i \le n - d$, and surjective if $i = n - d + 1$. In fact, one has the commutative square

$$\begin{array}{ccc} \pi_i(\hat{X}, \hat{f}^{-1}(\hat{Y})) & \longrightarrow & \pi_i(X, f^{-1}(Y)) \\ \hat{f}_* \big\downarrow & & \big\downarrow f_* \\ \pi_i(\hat{\mathbf{P}}^m, \hat{Y}) & \longrightarrow & \pi_i(\mathbf{P}^m, Y) \end{array} ,$$

in which the unlabeled homomorphisms are induced by the bundle maps $h_X : \hat{X} \to X$ and $h_{\mathbf{P}^m} : \hat{\mathbf{P}}^m \to \mathbf{P}^m$. Then since $\hat{Y} = h^{-1}_{\mathbf{P}^m}(Y)$ and $\hat{f}^{-1}(\hat{Y}) = h_X^{-1}(f^{-1}(Y))$, the horizontal homomorphisms are bijective for all i by a standard homotopy property of fibrations.

Consider next the long exact sequences of the pairs $(\hat{X}, \hat{f}^{-1}(\hat{Y}))$ and $(\hat{\mathbf{P}}^m, \hat{Y})$:

$$(*) \qquad \begin{array}{ccccccccc} \cdots \to & \pi_i(\hat{X}, \hat{f}^{-1}(\hat{Y})) & \longrightarrow & \pi_{i-1}(\hat{f}^{-1}(\hat{Y})) & \xrightarrow{j_*} & \pi_{i-1}(\hat{X}) & \to \cdots \\ & \big\downarrow & & \big\downarrow \hat{f}_* & & \big\downarrow & \\ \cdots \to & \pi_i(\hat{\mathbf{P}}^m, \hat{Y}) & \longrightarrow & \pi_{i-1}(\hat{Y}) & \longrightarrow & \pi_{i-1}(\mathbf{P}^m) & \to \cdots , \end{array}$$

where j denotes the inclusion $\hat{f}^{-1}(\hat{Y}) \hookrightarrow \hat{X}$. Identifying $\hat{f}^{-1}(\hat{Y})$ in the natural way with $\hat{X} \times_{\hat{\mathbf{P}}^m} \hat{Y}$, Proposition 9.4 asserts that $j_* \times \hat{f}_* : \pi_{i-1}(\hat{f}^{-1}(\hat{Y})) \to \pi_{i-1}(\hat{X}) \times \pi_{i-1}(\hat{Y})$ is bijective for $i \le n - d$ and surjective if $i = n - d + 1$. Thus if $i \le n - d$, the top row in (*) forms a short exact sequence, and $\hat{f}_*$ restricts to an isomorphism $\ker(j_*) \xrightarrow{\cong} \pi_{i-1}(\hat{Y})$. Therefore the composition $\pi_i(\hat{X}, \hat{f}^{-1}(\hat{Y})) \to \pi_{i-1}(\hat{f}^{-1}(\hat{Y})) \to \pi_{i-1}(\hat{Y})$ is bijective when $i \le n - d$; one sees similarly that it is surjective for $i = n - d + 1$. The theorem now follows from the observation that $\pi_i(\hat{\mathbf{P}}^m, \hat{Y}) \xrightarrow{\cong} \pi_{i-1}(\hat{Y})$ when $i \le n - d + 1$, since in fact $\pi_k(\hat{\mathbf{P}}^m) = \pi_k(\mathbf{C}^{m+1} - \{0\}) = 0$ for $k \le 2m$. ∎

Theorem 9.6 contains as special cases strengthened forms of the Lefschetz hyperplane theorem, and the theorems of Barth et. al. on the topology of small codimensional subvarieties of projective space. To begin with, taking $Y = L \subseteq \mathbf{P}^m$ to be a linear space of codimension d, and noting that $\pi_i(\mathbf{P}^m, L) = 0$ for $i \le 2(m-d) + 1$, one finds that $\pi_i(X, f^{-1}(L)) = 0$ for $i \le n - d$. Thus we recover the Lefschetz-type theorem (9.1) of Goresky and MacPherson for compact X. On the other hand, applying Theorem 9.6 with $Y = X$, one obtains

<u>COROLLARY 9.7</u> If $X \subseteq \mathbf{P}^m$ <u>is a closed local complete intersection of pure dimension</u> n, <u>then</u>

$$\pi_i(\mathbf{P}^m, X) = 0 \quad \underline{\text{for}} \quad i \le 2n - m + 1 . \blacksquare$$

It follows for instance that X is simply connected if $2n > m$ (compare Corollary 5.3(B)). Note that by the Hurewicz theorem, Corollary 9.7 also implies that the natural maps $H_i(X, \mathbb{Z}) \to H_i(\mathbf{P}^m, \mathbb{Z})$ are isomorphisms for $i \le 2n - m$, and surjective when $i = 2n - m + 1$.

<u>COROLLARY 9.8.</u> <u>Let</u> $f : X \to \mathbf{P}^m$ <u>and</u> $Y \subseteq \mathbf{P}^m$ <u>be as in the statement of Theorem</u> 9.6. <u>Then</u>

$$\pi_i(X, f^{-1}(Y)) = 0 \quad \underline{\text{for}} \quad i \le \min(n - d, m - 2d + 1) .$$

<u>Proof.</u> When $i \le m - 2d + 1$, $\pi_i(\mathbf{P}^m, Y) = 0$ by the previous corollary. ∎

In particular, if $X, Y \subseteq \mathbf{P}^m$ are closed local complete intersections of pure dimensions n and ℓ respectively, then

$$\pi_i(X, X \cap Y) = 0 \quad \text{for} \quad i \leq \min(n + \ell - m\ ,\ 2\ell - m + 1)\quad .$$

<u>REMARK 9.9.</u> Using Remark 9.5, one may generalize Theorem 9.6 to more than one subvariety $Y \subset \mathbf{P}^m$. In fact, let $f : X^n \to \mathbf{P}^m$ be as in the theorem, and let $Y_1, \ldots, Y_r \subseteq \mathbf{P}^m$ be closed local complete intersections of pure codimensions $d_1, \ldots, d_r$. Set $d = d_1 + \ldots + d_r$. Then the <u>natural homorphism</u>

$$\pi_i(X\ ,\ f^{-1}(\cap Y_j)) \to \prod_{j=1}^{r} \pi_i(\mathbf{P}^m\ ,\ Y_j)$$

<u>is bijective for</u> $i \leq n - d$, <u>and surjective when</u> $i = n - d + 1$.

This is already interesting when $X = \mathbf{P}^m$. For example, suppose that $Z \subseteq \mathbf{P}^m$ can be expressed as the intersection of $r \geq 2$ local complete intersections $Y_j \subseteq \mathbf{P}^m$ of pure codimension c. Then $\pi_i(\mathbf{P}^m, Y_j) = 0$ for $i \leq m - 2c + 1$ (Corollary 9.7), and one finds that $\pi_i(\mathbf{P}^m, Z) = 0$ for $i \leq m - rc$. When the intersection $Z = \cap Y_j$ is proper, this says that $\pi_i(\mathbf{P}^m, Z) = 0$ for $i \leq \dim Z$; by contrast, Corollary 9.7 only applies here in the range $i \leq \dim Z - \operatorname{codim} Z + 1$.

<u>NOTES</u>. (1) One of the most classical results concerning the topology of algebraic varieties is Lefschetz's theorem on hyperplane sections. In its original form, this theorem asserted that if $X \subseteq \mathbf{P}^m$ is a smooth projective variety of dimension n, and if $L \subseteq \mathbf{P}^m$ is a hyperplane meeting X transversely, then the maps $H_i(X \cap L) \to H_i(X)$ induced by inclusion are isomorphisms for $i < n - 1$, and surjective if $i = n - 1$. Contemporary proofs, using Morse theory, give a stronger assertion: if $f : X \to \mathbf{P}^m$ is a finite map, then $\pi_i(X, f^{-1}(L)) = 0$ when $i \leq n - 1$ for any hyperplane L (cf.[47,§7]).

The extension of this result to non-compact varieties was initiated by Zariski [67], who showed that if X is the complement of a hypersurface in $\mathbb{P}^m$, and if L is a sufficiently general hyperplane, then $\pi_1(X \cap L) \xrightarrow{\cong} \pi_1(X)$ for $m \geq 3$, and $\pi_1(X \cap L) \twoheadrightarrow \pi_1(X)$ when $m = 2$. Modern proofs have been given by Cheniot [9] and Hamm and Lê [31]. Hamm and Lê show that in fact $\pi_i(X, X \cap L) = 0$ for $i \leq m - 1$. The same statement holds if X is an arbitrary Zariski-open subset of $\mathbb{P}^m$ (cf. [11]). The theorem of Goresky and MacPherson was conjectured by Deligne in [10] and [11]. In [12] he showed how the conjecture (as it was at the time) leads to Theorem 9.2.

(2) Hamm [29] has studied the topology of local complete intersections. He shows that if X is a complex analytic subset of some neighborhood of the origin in $\mathbb{C}^m$ which is locally defined by r equations, and if $X_\varepsilon = \{z \in X \mid \|z\| \leq \varepsilon\}$, then $X_\varepsilon - \{0\}$ is $(m - r - 2)$ - connected for small ε. This result is used by Goresky and MacPherson in the proof of Theorem 9.1. Hamm had also proved that the Lefschetz hyperplane theorem holds for compact local complete intersections.

(3) Barth proved in [5] that if $X, Y \subseteq \mathbb{P}^m$ are smooth projective varieties of dimensions n and ℓ respectively, then

$$(*) \qquad H^i(Y, Y \cap X ; \mathbb{Q}) = 0 \quad \text{for} \quad i \leq \min(n+\ell-m \,,\, 2n-m+1) ,$$

at least if X and Y meet properly. Taking $Y = \mathbb{P}^m$, he deduced

$$(**) \qquad H^i(\mathbb{P}^m, X ; \mathbb{Q}) = 0 \quad \text{for} \quad i \leq 2n - m + 1 .$$

These are the prototypes for Corollaries 9.7 and 9.8, and they aroused considerable interest when they appeared. Hartshorne [34] subsequently showed that in fact (**) is a simple consequence of the Hard Lefschetz theorem. An algebraic proof of (**), valid for local complete intersections, was given by Ogus [52]. In [53], Ogus proved (*) for local complete intersections, without the hypothesis that X and Y meet properly, under the assumption that $n > \ell$.

Generalizations of (*) and (**) to homotopy were obtained by Larsen and Sommese. Specifically, Larsen [44] proved Corollary 9.7 for smooth X. Barth asked in [6] whether Larsen's result remains true for local complete intersections. The homotopy analogue of (*) when X is smooth is due to Sommese [62]. Sommese in fact proves similar theorems for homogeneous spaces other than $\mathbf{P}^m$.

Hamm [30] has given a local generalization of Larsen's result. He shows that if X is an irreducible n-dimensional complex analytic subset of a neighborhood of 0 in $\mathbf{C}^m$, with $X - \{0\}$ non-singular, then $X_\varepsilon - \{0\}$ is $(2n - m - 1)$ - connected for sufficiently small ε. (As above, $X_\varepsilon = \{z \in X \mid \|z\| \leq \varepsilon\}$.) By taking X to be the cone over a smooth subvariety of projective space, he recovers Larsen's theorem as an immediate consequence.

(4) There is a Barth-type theorem for branched coverings of projective space which extends Corollary 6.3: if X is a non-singular projective variety of dimension n, and if $f : X \to \mathbf{P}^n$ is a branched covering of degree d, then the homomorphisms $f : \pi_i(X) \to \pi_i(\mathbf{P}^n)$ are bijective for $i \leq n + 1 - d$, and surjective if $i = n + 2 - d$ ([45], [46]). For the proof, one shows first that f factors canonically through an embedding of X into the total space of a certain vector bundle $E \to \mathbf{P}^n$ of rank $d - 1$. This vector bundle enjoys a strong positivity property:

(*) $E(-1)$ is generated by its global sections.

Then one uses Theorems 9.1 and 9.2 to prove that if $E \to \mathbf{P}^n$ is a vector bundle of rank e satisfying (*), and if $X \subseteq E$ is a compact local complete intersection of pure dimension n, then $\pi_i(E, X) = 0$ for

$i \le n - e + 1$. This yields the stated theorem for coverings, and by taking E to be the direct sum of $e = m - n$ copies of the hyper - plane line bundle on $\mathbf{P}^n$, one also recovers Corollary 9.7. The proof that the vector bundle associated to a branched covering $f : X \to \mathbf{P}^n$ satisfies (*) uses the Kodaira vanishing theorem on X , which accounts for the non-singularity hypothesis.

(5) The results discussed in Remark 9.9 have a number of interesting antecedents and generalizations. At least in the non-singular case, it is classical that if $Z \subseteq \mathbf{P}^m$ is a complete intersection of r hypersurfaces, then $\pi_i(\mathbf{P}^m, Z) = 0$ for $i \le \dim Z$. Kato and Oka (see [41, Lemma 6.1]) observed that if Z is the common zero-locus of <u>any</u> r homogeneous polynomials on $\mathbf{P}^m$, then $\pi_i(\mathbf{P}^m, Z) = 0$ for $i \le m - r$. Newstead [50] has recently extended this to the statement that if $X \subseteq \mathbf{P}^m$ is a smooth projective variety of dimension n , and if $Z \subseteq X$ is the intersection of X with any r hypersurfaces in $\mathbf{P}^m$, then $\pi_i(X, Z) = 0$ for $i \le n - r$. These results follow from (9.9), except that Newstead in fact allows X to be arbitrarily singular along Z .

More generally, one can consider a vector bundle E of rank r on a smooth n-dimensional projective variety X , and the zero-locus $Z(s)$ of a section s of E . For example, if E is a direct sum of positive line bundles on $\mathbf{P}^m$, then $Z(s)$ is the intersection of r hypersurfaces. When E is suitably <u>positive</u>, or <u>ample</u>, one expects to have Lefschetz-type results. Griffiths [24, p. 205], working with a differential-geometric definition of positivity, shows that $\pi_i(X, Z(s)) = 0$ for $i \le n - r$ provided that the section s vanishes transversely (although he only states the result for homology.) Using a weaker cohomological notion of ampleness, Sommese [60] proves that $H^i(X, Z(s); \mathbb{Z}) = 0$ when $i \le n - r$ for transversely vanishing s. In fact, his ingenious argument works for the zero locus of an arbitrary section s. Another result along these lines is proved in [46] using the theorem (9.1) of Goresky and MacPherson. Here X is a local complete intersection, and E is a vector bundle satisfying the (very strong) requirement that it be a quotient of a direct sum of copies of an ample line bundle L, where L is generated by its global sections. The conclusion is that $\pi_i(X, Z(s)) = 0$ when $i \leq n - r$, for any section s.

§10. Open Questions

We list in this section some open questions, and topics for further investigation.

1) The work of Faltings [15] described in the notes to §3 generalizes the connectedness theorem (3.1(A)) to homogeneous spaces G/P, at least in characteristic zero. It is natural to ask whether the analogue of (3.1(B)) holds as well. It would also be interesting to know whether Sommese's generalization of the Barth-Larsen theorem to homogeneous spaces [61, 62, 63] can be extended to the framework of §9. For example, if $\mathbf{P}^m$ is replaced in Theorem 9.6 by a Grassmannian G of linear spaces of $\mathbf{P}^m$, is the conclusion of the theorem valid when $n - d$ is replaced by $n - d - k$, where $k = \dim G - m$? (This is the "ampleness" of the tangent bundle to G in the sense of Sommese [60]; see also [32].) There might be interesting geometric applications of connectedness results for abelian varieties. Theorems in this direction have been obtained by Barth [4] and Sommese [60, 61, 62, 63].

2) Bearing in mind the counter-examples to the conjecture described in Note 2 of §4, are there any useful conditions on an embedding $Y \hookrightarrow Z$ of projective varieties that guarantee the connectedness

of $f^{-1}(Y)$ when $f : X \to Z$ is a proper morphism with $\dim f(X) > \operatorname{codim}(Y, Z)$? Perhaps one should assume in addition that $\dim Y \geqslant \operatorname{codim} Y$ (cf. [33, Problem 4.8]).

3) In applications, the connectedness theorem has been used exclusively for mappings of the form $f \times g : X \times Y \to \mathbf{P}^m \times \mathbf{P}^m$. Does it yield interesting results in more imaginative settings?

4) There is still something to be learned about how singularities affect the results in the topological setting. For example in π_1 statements, as we have seen, one generally needs the assumption of local irreducibility. So it is somewhat surprising that this hypothesis is not required in Corollary 5.3. In the case of higher homotopy, Goresky and MacPherson [23, Prop. 4.2] give a variant of (9.1) for compact X which takes into account arbitrary singularities. Other Lefschetz-type results for singular varieties have been proved by Gerstner and Kaup [21], Kato [41], Karchyauskas [40], and Newstead [51]. Extensions of the Barth theorems to the singular case were obtained by Ogus [52, 53]. The range of applicability of these results depends on the local structure of the singularities. It would be interesting to find a unified statement, and to prove analogues of the results of §9.

5) The results of §§4 - 7 suggest a number of related questions. Suppose that $X \subseteq \mathbf{P}^m$ is a smooth non-degenerate projective variety of dimension n . If $\pi : X \to \mathbf{P}^k$ is a generic projection from $\mathbf{P}^m$, which of the Thom-Boardman singularities that might occur will necessarily exist? Along slightly different lines, if $\gamma : X \to \operatorname{Grass}(\mathbf{P}^n, \mathbf{P}^m)$ is the Gauss mapping, what Schubert cycles $\Sigma \subseteq \operatorname{Grass}(\mathbf{P}^n, \mathbf{P}^m)$ will meet the image $\gamma(X)$ of X ? More generally, can one say anything about the connectivity of the pair $(X, \gamma^{-1}(\Sigma))$? For example if $L \subseteq \mathbf{P}^m$ is a linear space of codimension

$n+1$, it follows from Zak's result (7.2) on the finiteness of γ that the set

$$S = \{x \in X \mid T_x \text{ meets } L\}$$

is an <u>ample</u> divisor on X, and so $\pi_i(X, S) = 0$ for $i \leq n-1$. (This fact was noticed by A. Landman and A. Sommese). See also question (7), below.

6) Can any of Zak's techniques be extended to make further progress on Hartshorne's conjecture [34] on complete intersections? The most obvious question is whether one could prove the projective normality of every smooth subvariety of sufficiently small codimension in projective space.

7) A number of interesting problems in algebraic geometry can be formulated in terms of the degeneracy loci of a map of vector bundles (cf. the discussion of Brill-Noether theory in [3].) Specifically, let

$$\sigma : F \to E$$

be a homomorphism of vector bundles of ranks f and e on a smooth projective variety X, and set

$$D_k(\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \leqslant k\} .$$

If non-empty, $D_k(\sigma)$ has codimension $\leqslant (f-k)(e-k)$ in X. Under suitable positivity hypotheses on the bundles involved, is there a Lefschetz-type theorem for the pair $(X, D_k(\sigma))$ which reduces to the results discussed in Note 5 to §9 when F is a trivial line bundle, and $k = 0$ (so that $D_k(\sigma)$ is the zero-locus of a section of E)?

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NOTE: Since the manuscript was typed, it has come to our attention that Roberts' notes [57] have been subsumed by a joint paper with T. Fujita entitled "Varieties with small secant varieties: the extremal case". Besides an exposition of Zak's work, this paper contains a partial classification of varieties $X^n \subseteq \mathbb{P}^r$, with $3n = 2(r - 2)$, such that $\dim \operatorname{Sec} X = r - 1$.

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