POSITIVITY AND EXCESS INTERSECTION

by

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## §0. Introduction.

Consider a variety $M$, and a projective local complete intersection

$$X \subseteq M$$

of pure codimension $e$. Then for any subvariety $Y \subseteq M$ of dimension $k \geq e$, the intersection class

$$X \cdot Y \in A_{k-e}(X)$$

is defined up to rational equivalence on $X$. One of the most basic facts of intersection theory is that if $Y$ meets $X$, and does so properly, then $X \cdot Y$ is non-zero and in fact has positive degree with respect to any projective embedding of $X$. On the other hand, if the intersection of $X$ and $Y$ is improper, then $X \cdot Y$ may be zero or of negative degree. Our purpose here is to give some conditions on $X$ to guarantee the non-negativity or positivity of the intersection class in the case of possibly excess intersection. These conditions take the form of hypotheses on the normal bundle $N_{X/M}$ to $X$ in $M$, the theme being that positivity of the vector bundle $N_{X/M}$ forces the positivity of $X \cdot Y$ provided only that $Y$ meets $X$. We give several simple applications and related results, including a lower bound for the multiplicity of a proper intersection, generalizing a classical result for curves on a surface.



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## §1. Excess intersections with positive normal bundle.

We deal with a variety $M$ - not necessarily smooth or complete - and a local complete intersection $X \subseteq M$ of pure codimension $e$, which we assume to be projective. Denote by $N$ the normal bundle to $X$ in $M$, and let $L$ be a fixed ample line bundle on $X$. We are interested in intersecting $X$ with a subvariety $Y \subseteq M$ of dimension $k \geq e$.

Theorem 1. (A). *If* $S^m(N)$ *is generated by its global sections for some* $m > 0$, *then*

$$\deg_L(X \cdot Y) \geq 0.$$

(B). *If* $S^m(N) \otimes \check{L}$ *is generated by its global sections for some* $m > 0$, *then*

$$\deg_L(X \cdot Y) \geq \frac{1}{m^{\dim(X)}} \cdot \deg_L(X \cap Y).$$

(For an $\ell$-dimensional cycle or cycle class $\alpha$ on $X$, $\deg_L(\alpha)$ denotes the degree of the zero-dimensional class $c_1(L)^{\ell} \cap \alpha$. In (B), $\deg_L(X \cap Y)$ is the sum of the L-degrees of the irreducible components of $X \cap Y$, taken with their reduced structures.) The hypothesis in (B) is equivalent to the assumption that the normal bundle $N$ is *ample* in the sense of Hartshorne [H1], and the proof will show that in fact a somewhat better inequality holds.

Before proceeding, we record several simple applications:

Corollary 1. *In the situation of the theorem, if* $S^m(N)$ *is generated by its global sections for some* $m > 0$, *then* $X$ *is numerically effective in the sense that*

$$(*) \qquad \deg(X \cdot Y) \geq 0$$

*for any subvariety* $Y \subseteq M$ *of pure dimension* $e = \operatorname{codim}(X)$. *If moreover* $N$ *is ample, then strict inequality holds in* (*) *provided that* $Y$ *is numerically equivalent to an effective cycle whose support meets* $X$. ■

Corollary 2. *Let* $V_1, \ldots, V_r \subseteq \mathbf{P}^n$ *be subvarieties of degrees* $d_1, \ldots, d_r$. *Then*

$$\deg(V_1 \cap \ldots \cap V_r) \leq d_1 \cdot \ldots \cdot d_r .$$

This was originally proved by R. MacPherson and the first author. As before, the left-hand side denotes the sum of the degrees of the irreducible components of $V_1 \cap \ldots \cap V_r$ with their reduced structures.

Proof. By passing to a larger projective space, and to cones over the $V_i$, we may assume that $\Sigma \dim(V_i) \geq (r-1)n$. Let $M = \mathbf{P}^n \times \ldots \times \mathbf{P}^n$ (r times), and let $X = \mathbf{P}^n \subseteq M$ be the diagonal. The hypotheses of statement (B) of the theorem are satisfied with $L = \mathcal{O}_{\mathbf{P}^n}(1)$ and $m = 1$. Taking $Y = V_1 \times \ldots \times V_r$, the corollary follows. ■

Exercise. Assuming $r = 2$, show that if equality holds in Corollary 2 then $V_1$ and $V_2$ lie in a linear subspace of $\mathbf{P}^n$ in which they meet properly.

Corollary 3. *In the setting of the theorem, suppose that $M$ is acted on transitively by a connected algebraic group. If the normal bundle $N$ to $X$ in $M$ is ample, then $X$ meets any subvariety $Y \subseteq M$ of dimension $\geq \operatorname{codim}(X)$.*

Proof. The homogeneity of $M$ implies that $Y$ is algebraically equivalent to a subvariety $Z$ which meets $X$, and $X \cdot Z \neq 0$ by the theorem. ■

This simplifies and extends somewhat a result of Lübke [L].

Remark. Corollary 3 is closely related to two conjectures of Hartshorne ([H2] III.4.4, III.4.5) concerning smooth subvarieties of a non-singular variety $M$ in characteristic zero.

Conjecture A. If $X \subseteq M$ has an ample normal bundle, then some multiple of $X$ moves (as a cycle) in a large algebraic family.

Conjecture B. If both $X \subseteq M$ and $Y \subseteq M$ have ample normal bundles, and if $\dim(X) + \dim(Y) \geq \dim(M)$, then $X$ meets $Y$.

Observe that if one knew that some multiple of $[X]$ moved in an algebraic family large enough to cover $M$, then Conjecture B would follow from Theorem 1 as in the proof of Corollary 3. However, one has the following:

Counter-example to Conjecture A. Gieseker [Gi] has constructed an ample vector bundle E on $\mathbb{P}^2$ arising as a quotient of the form

$$(*) \qquad 0 \to \mathcal{O}_{\mathbb{P}^2}(-n)^2 \to \mathcal{O}_{\mathbb{P}^2}(-1)^4 \to E \to 0.$$

for suitable $n >> 0$. Take M to be the total space of E, and $X \subseteq M$ to be the zero-section. We claim that there are no projective surfaces $Y \subseteq M$ other than X itself, from which it follows that no multiple of X moves in any non-trivial algebraic family. Indeed, an embedding $Y \subseteq M$ distinct from the zero-section would give rise to a non-zero section of $f^*E$ on the normalization $\tilde{Y}$ of Y, where f is the composition of the natural maps $\tilde{Y} \to Y \to X$. But $H^1(\tilde{Y}, f^*\mathcal{O}_{\mathbb{P}^2}(-n)) = 0$ by the Mumford-Ramanujam vanishing theorem, and hence $H^0(\tilde{Y}, f^*E) = 0$.

Conjecture B remains open, although it seems to us plausible that a counter-example may exist. The general picture that appears to emerge from ([Han], [Fa], [Go], [L], [FL1]) is that ampleness of normal bundles has global consequences for subvarieties of homogeneous spaces, but not necessarily in general.

There are two inputs to the proof of Theorem 1. The first, which is the essential feature of the intersection theory developed by R. MacPherson and the first author ([FM1], [FM2], [Fu]), consists in reducing to an infinitesimal problem. Specifically, in the situation of the theorem, consider the fibre square

$$\begin{array}{ccc} X \cap Y & \subseteq & Y \\ \cap\!\shortmid & & \cap\!\shortmid \\ X & \subseteq & M, \end{array}$$

and denote by C the normal cone of $X \cap Y$ in Y. Then C has pure dimension $k = \dim Y$, and sits naturally as a subscheme in the total space of the normal bundle $N = N_{X/M}$. One can intersect C with the zero section of N to obtain a well-defined rational equivalence class

$$z(C,N) \in A_{k-e}(X) .$$

($z(C,N)$ is actually defined in $A_{k-e}(X \cap Y)$.) We call this the cone class determined by C in N. The basic fact then is that

$$X \cdot Y = z(C,N)$$

(cf. [FM1], [Fu]). Theorem 1 now follows from a general positivity statement for the classes determined by cones in vector bundles satisfying hypotheses (A) or (B) of the theorem:

Theorem 2 (cf. [FL2]). *Let $N$ be a vector bundle of rank $e$ on a projective variety $X$, and let $C \subseteq N$ be an irreducible cone of dimension $k \geq e$. Let $L$ be an ample line bundle on $X$.*

(A) *If $S^m(N)$ is generated by its global sections for some $m > 0$, then*

$$\deg_L(z(C,N)) \geq 0.$$

(B) *If $S^m(N) \otimes \check{L}$ is generated by its global sections for some $m > 0$, then*

$$\deg_L(z(C,N)) \geq \frac{1}{m^{\dim(\mathrm{Supp}\, C) + e-k}}\, s(C) \deg_L(\mathrm{Supp}\, C)\,,$$

*where $s(C)$ is the multiplicity of $C$ along its zero-section.*

We will prove the theorem under the stronger hypotheses:

(A') $N$ is generated by its global sections.

(B') $N \otimes \check{L}$ is generated by its global sections.

The general case is treated by combining the proof below with the arguments in §2 of [FL2].

Proof. We may assume that $\mathrm{Supp}(C) = X$. If $N$ is generated by its global sections, then a general section of $N$ meets $C$ properly or not at all. Therefore, $z(C,N)$ is represented by an effective (or zero) cycle, and this proves (A).

Turning to (B), after possibly replacing $C \subseteq N$ by $C \oplus L \subseteq N \oplus L$ --which leaves the cone class unchanged ([FL2§1])--we may assume that $C$ maps to its support with fibre dimension $\geq 1$. In this situation, one has the formula

$$z(C,N) = \pi_*(c_{e-1}(Q_{\mathbf{P}(N)}) \cap [\mathbf{P}(C)]),$$

where $Q_{\mathbf{P}(N)} = \pi^* N/\mathcal{O}_{\mathbf{P}(N)}(-1)$ is the rank $e-1$ universal quotient bundle on the projectivization $\pi : \mathbf{P}(N) \to X$. Thus

$$\deg_L(z(C,N)) = \int_{\mathbf{P}(C)} c_1(\pi^*L)^{k-e}\, c_{e-1}(Q_{\mathbf{P}(N)})$$

$$= \sum_{i=0}^{e-1} \int_{\mathbf{P}(C)} c_1(\pi^*L)^{k-e+i}\, c_{e-1-i}(Q_{\mathbf{P}(N)} \otimes \pi^*L^\vee) .$$

By the hypothesis (B'), $Q_{\mathbf{P}(N)} \otimes \pi^*L^\vee$ is generated by its global sections, and it follows that its Chern classes are represented by effective (or zero) cycles. Thus all the terms in the sum above are non-negative. Therefore, letting $n = \dim X = \dim \mathrm{Supp}(C)$, one has

$$\deg_L(z(C,E)) \geq \int_{\mathbf{P}(C)} c_1(\pi^*L)^n\, c_{k-1-n}(Q_{\mathbf{P}(N)} \otimes \pi^*L^\vee)$$

$$= \int_{\mathbf{P}(C)} c_1(\pi^*L)^n\, c_{k-1-n}(Q_{\mathbf{P}(N)})$$

$$= \deg\{c_1(L)^n \cap s_n(C,N)\} ,$$

where $s_n(C,N) \in A_n(X)$ is the n-dimensional Segre class of C. But $s_n(C,N) = s(C) \cdot [X]$, and the theorem follows. ■

Remark. As a special case of statement (B) of Theorem 2, one finds that if N is ample, and if $C \subseteq N$ is a cone of pure dimension $e = \mathrm{rk}(N)$, then the cone class $z(C,N)$ has strictly positive degree. This was proved in [FL2], where it was used to determine all numerically positive polynomials in the Chern classes of an ample vector bundle.

## §2. Intersection multiplicities.

If C and D are curves on a surface M, and $P \in C \cap D$ is an isolated point of intersection, then a classical formula of Max Noether expresses the intersection multiplicity $m_P(C \cdot D)$ of the given curves at P in terms of their proper transforms on the blow-up $\tilde{M}$ of M at P. Specifically, Noether's formula states that

$$m_P(C \cdot D) = e_P(C) \cdot e_P(D) + \sum_{Q \in E} m_Q(\tilde{C} \cdot \tilde{D}),$$

$\tilde{C}$ and $\tilde{D}$ being the proper transforms of C and D, where the sum on the right is taken over all points on the exceptional divisor $E \subseteq \tilde{M}$. In particular, since this sum is non-negative, one obtains the familiar lower bound for the intersection multiplicity. In this section we discuss a generalization of Noether's formula to higher dimensions. Unlike the situation for curves on a surface, it can happen in general that the proper transforms of the given varieties no longer meet properly in a neighborhood of the exceptional divisor. In this case, positivity results come into play in order to bound from below the contribution of this intersection.

Let $V_1,\dots,V_r$ be subvarieties of a smooth variety $M$, with $\Sigma\, \mathrm{codim}(V_i,M) = \dim(M)$. Assume that $V_1,\dots,V_r$ intersect properly at the point $P \in M$. As we are interested in local questions, we will suppose that $P$ is the only point at which the $V_i$ meet. Denote by $\tilde{M}$ the blow-up of $M$ at $P$, so that the exceptional divisor $E$ is a projective space, with

$$L = \mathcal{O}(-E)\,|_E$$

the (ample) hyperplane bundle. If $\tilde{V}_i \subseteq \tilde{M}$ is the blow-up of $V_i$ at $P$, then $\bigcap \tilde{V}_i$ is contained in $E$, and hence $\tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r$ is a well-defined rational equivalence class of dimension zero on $E$.

Theorem 3. *With the preceding notation,*

(A) $m_P(V_1 \cdot \ldots \cdot V_r) = \prod_{i=1}^{r} e_P(V_i) + \deg(\tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r)$, *where* $e_P(V_i)$ *is the multiplicity of* $V_i$ *at* $P$.

(B) $\deg(\tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r) \geq \deg_L(\tilde{V}_1 \cap \ldots \cap \tilde{V}_r)$.

We will prove (A) and (B) when each $V_i$ is a divisor on $M$. The proof of (A) in general uses the deformation to the normal bundle, as in [FM1]. For (B), one cannot apply Theorem 1 to the diagonal imbedding of $\tilde{M}$ in $\tilde{M} \times \ldots \times \tilde{M}$, since the normal bundle to this imbedding is not ample. Instead, one imbeds $\tilde{M}$ in the blow-up of $\tilde{M} \times \ldots \times \tilde{M}$ along $E \times \ldots \times E$, where the normal bundle *is* ample. We refer to [Fu] §12.4 for details.

*Proof*. Let $\pi : \tilde{M} \to M$ and $\eta : E \to P$ be the canonical maps. From the projection formula and the identification of $\mathcal{O}(E)\,|_E$ with $\mathcal{O}(-1)$, one has:

(i) $\eta_*(\pi^* V_1 \cdot \ldots \cdot \pi^* V_r) = V_1 \cdot \ldots \cdot V_r$

(ii) $\eta_*(\pi^* V_1 \cdot \ldots \cdot \pi^* V_j \cdot E^{r-j}) = 0 \qquad (1 \leq j < r)$

(iii) $\eta_*(E^r) = (-1)^{r-1}[P]$ .

Let $m_i = e_P(V_i)$. Equivalently, one has an equation of divisors on $\tilde{M}$:

(iv) $\pi^* V_i = m_i E + \tilde{V}_i \qquad (1 \leq i \leq r)$.

By (i) - (iv) and bilinearity of intersection products,

$$\eta_*(\tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r) = \eta_*((\pi^* V_1 - m_1 E) \cdot \ldots \cdot (\pi^* V_r - m_r E))$$
$$= V_1 \cdot \ldots \cdot V_r - m_1 \cdot \ldots \cdot m_r [P]$$

which proves (A). Shrinking $M$, we may assume each $V_i$ is a principal Cartier divisor on $M$, so that, by (iv), $\mathcal{O}(\tilde{V}_i) = \mathcal{O}(-m_i E)$. The intersection class $\tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r$ may be constructed from the fibre square

$$\begin{array}{ccc} \bigcap \tilde{V}_i & \subseteq & \tilde{M} \\ \cap\!| & & \cap\!| \\ \tilde{V}_1 \times \ldots \times \tilde{V}_r & \subseteq & \tilde{M} \times \ldots \times \tilde{M} . \end{array}$$

The normal bundle to $\tilde{V}_1 \times \ldots \times \tilde{V}_r$ in $\tilde{M} \times \ldots \times \tilde{M}$ restricts to $\bigoplus_{i=1}^{r} L^{\otimes m_i}$ on $\bigcap \tilde{V}_i \subseteq E$. Theorem 2(B) then applies, as in the proof of Theorem 1, to show that

$$\deg \tilde{V}_1 \cdot \ldots \cdot \tilde{V}_r \geq \deg_L(\tilde{V}_1 \cap \ldots \cap \tilde{V}_r) . \quad \blacksquare$$

<u>Remark</u>. When the $V_i$ are Cartier divisors, the theorem holds even if $M$ is singular. In fact, the preceding proof shows that if $m_i$ are any positive integers such that

$$\pi^* V_i = m_i E + W_i$$

for some effective Cartier divisors $W_i$ on $\tilde{M}$, then

$$m_P(V_1 \cdot \ldots \cdot V_r) = \prod_{i=1}^{r} m_i \cdot e_P(M) + \deg(W_1 \cdot \ldots \cdot W_r)$$

with $\deg(W_1 \cdot \ldots \cdot W_r) \geq \deg_L(W_1 \cap \ldots \cap W_r)$. In place of (iii) one uses the equation $\eta_*(E^r) = (-1)^{r-1} e_P(M) \cdot [P]$.

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