THE GONALITY CONJECTURE ON SYZYGIES OF ALGEBRAIC CURVES OF LARGE DEGREE

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Introduction

The purpose of this note is to show that a small variant of the methods used by Voisin in [13, 14] leads to a surprisingly quick proof of the gonality conjecture of [9], asserting that one can read off the gonality of an algebraic curve C from its syzygies in the embedding defined by any one line bundle of sufficiently large degree. More generally, we establish a necessary and sufficient condition for the asymptotic vanishing of the weight one syzygies of the module associated to an arbitrary line bundle on C.

Let C be a smooth complex projective curve of genus $g \geq 2$, and let L be a very ample line bundle of degree d on C defining an embedding

$$\mathrm{C} \subseteq \mathbf{P}\mathrm{H}^0(\mathrm{C}, \mathrm{L}) = \mathbf{P}^r.$$

Starting with the work of Green in [7, 8] there has been a great deal of interest in understanding connections between the geometry of C and L and their syzygies. More precisely, write $\mathrm{S} = \mathrm{Sym}\, \mathrm{H}^0(\mathrm{C}, \mathrm{L})$ for the homogeneous coordinate ring of $\mathbf{P}^r$, and denote by

$$\mathrm{R} = \mathrm{R}(\mathrm{L}) = \bigoplus_m \mathrm{H}^0(\mathrm{C}, m\mathrm{L})$$

the graded S-module associated to L. Consider next the minimal graded free resolution $\mathrm{E}_\bullet = \mathrm{E}_\bullet(\mathrm{L})$ of R over S:

$$0 \longrightarrow \mathrm{E}_{r-1} \longrightarrow \cdots \longrightarrow \mathrm{E}_2 \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E}_0 \longrightarrow \mathrm{R} \longrightarrow 0,$$

where $\mathrm{E}_p \cong \oplus \mathrm{S}(-a_{p,j})$. Note that if L is normally generated then $\mathrm{E}_0 = \mathrm{S}$, in which case $\mathrm{E}_\bullet$ gives rise to a minimal resolution of the homogeneous ideal $\mathrm{I} = \mathrm{I}_{\mathrm{C}/\mathbf{P}^r}$ of C in $\mathbf{P}^r$. As customary, we denote by $\mathrm{K}_{p,q}(\mathrm{C}; \mathrm{L})$ the vector space of minimal generators of E_p in degree $p+q$, so that

$$\mathrm{E}_p = \bigoplus_q \mathrm{K}_{p,q}(\mathrm{C}; \mathrm{L}) \otimes_{\mathbf{C}} \mathrm{S}(-p-q).$$

We will be concerned here with investigating the grading of $\mathrm{E}_\bullet(\mathrm{L})$—i.e. determining which of the $\mathrm{K}_{p,q}$ are non-vanishing—when L has very large degree.

Research of the first author partially supported by NSF grant DMS-1001336.
Research of the second author partially supported by NSF grant DMS-1439285.

DOI 10.1007/s10240-015-0072-2

Published online: 10 March 2015

It is elementary that if $H^1(C, L) = 0$ then $K_{p,q}(C; L) = 0$ for $q \geq 3$. Moreover, work of Green [7] and others shows that if $d = \deg(L) \gg 0$, so that in particular $r = d - g$, then:

$$K_{p,0}(C; L) \neq 0 \iff p = 0;$$
$$K_{p,2}(C; L) \neq 0 \iff r - g \leq p \leq r - 1.$$

It follows from this that

$$K_{p,1}(C; L) \neq 0 \quad \text{for } 1 \leq p \leq r - 1 - g,$$

but these results leave open the question of when $K_{p,1}(C; L) \neq 0$ for $p \in [r - g, r - 1]$. Our first main result is that this is determined by the gonality $\mathrm{gon}(C)$ of C, i.e. the least degree of a branched covering $C \to \mathbf{P}^1$.

Theorem **A.** — *If* $\deg(L) \gg 0$, *then*

$$K_{p,1}(C; L) \neq 0 \iff 1 \leq p \leq r - \mathrm{gon}(C).$$

Thus one can read off the gonality of a curve from the resolution of the ideal of C in the embedding defined by any one line bundle of sufficiently large degree. The cases $p = r - 1, p = r - 2$ were established by Green [7], and the general statement was conjectured in [9], where it was observed that if $1 \leq p \leq r - \mathrm{gon}(C)$, then $K_{p,1}(C; L) \neq 0$.[1] Using Voisin's results [13, 14] on syzygies of general canonical curves, Aprodu and Voisin [1, 3] proved the statement of the theorem for a general curve of each gonality. We show (Remark 2.2) that the conclusion of the theorem holds for instance once $\deg(L) \geq g^3$, but we suspect that it should be enough to assume a lower bound on d that is linear in g.[2]

Theorem A follows from a more general result concerning the weight one asymptotic syzygies associated to an arbitrary divisor B. Specifically, fix a line bundle B on C, and with L as above consider the $S = \mathrm{Sym}\, H^0(L)$ module

$$R = R(B; L) = \bigoplus_m H^0(C, B + mL).$$

One can again form the graded minimal free resolution $E_\bullet(B; L)$ of $R(B; L)$ over S, giving rise to Koszul cohomology groups $K_{p,q}(C, B; L)$. As in the case $B = \mathcal{O}_C$ discussed in the previous paragraphs, the $K_{p,0}$ and the $K_{p,2}$ are completely controlled when $\deg L \gg 0$, and so the issue is to understand the weight one groups $K_{p,1}(C, B; L)$ when L has large degree.

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[1] In fact, suppose that $p : C \to \mathbf{P}^1$ is a branched covering of degree k. Then when $\deg(L) \gg 0$ the linear spaces spanned by the fibres of p sweep out a k-dimensional scroll $S \subset \mathbf{P}^r$ containing C. But the resolution of $I_{S/\mathbf{P}^r}$ has a linear strand of length $r - k$, which forces $K_{p,1}(C; L) \neq 0$ for $1 \leq p \leq r - k$. Thus the essential content of the theorem is that if $K_{r-k,1}(C; L) \neq 0$ and $\deg L \gg 0$, then C carries a pencil of degree $\leq k$.

[2] Rathmann has recently established such a result: see Remark 2.5.

Recall that B is said to be *p-very ample* if every effective divisor ξ of degree $(p+1)$ on C imposes independent conditions on the sections of B, i.e. if the natural map

$$H^0(C, B) \longrightarrow H^0(C, B \otimes \mathcal{O}_\xi)$$

is surjective for every $\xi \in C_{p+1} =_{\mathrm{def}} \mathrm{Sym}^{p+1} C$. Our second main result is:

Theorem **B.** — *Fix* B *and* $p \geq 0$. *Then*

$$K_{p,1}(C, B; L) = 0 \quad \textit{for all } L \textit{ with } \deg L \gg 0$$

if and only if B *is p-very ample.*

Serre duality implies that the vector spaces

$$K_{p,q}(C, B; L) \quad \text{and} \quad K_{r-1-p,2-q}(C, K_C - B; L)$$

are naturally dual, K_C being the canonical divisor of C, and one then finds that Theorem A is equivalent to the case $B = K_C$ of Theorem B. While this is arguably the most interesting instance of the result, it will become clear that decoupling B and L is helpful in guiding the argument.

When B fails to be p-very ample, it is natural to introduce the invariant

$$\gamma_p(B) = \dim\big\{\xi \in C_{p+1} \bigm| H^0(B) \longrightarrow H^0(B \otimes \mathcal{O}_\xi) \text{ not surjective}\big\}.$$

Theorem **C.** — *Let* $L_d = dA + E$, *where* A *is an ample line bundle on* C *and* E *is arbitrary. Fix* B *and* p, *and assume that* B *is not p-very ample. Then there is a polynomial* $P(d)$ *of degree* $\gamma_p(B)$ *in d such that*

$$\dim K_{p,1}(C, B; L_d) = P(d) \quad \textit{for } d \gg 0.$$

In some cases, we are also able to compute the leading coefficient of $P(d)$. We note that Yang [15] has recently proven (by somewhat related arguments) that the dimensions of the vector spaces $K_{p,0}$ and $K_{p,1}$ grow polynomially on an arbitrary variety.

Theorems B and C follow in a surprisingly simple manner from a small variant of the Hilbert scheme computations pioneered by Voisin in her proof [13, 14] of Green's conjecture for general canonical curves. It is well-known that $K_{p,1}(C, B; L)$ can be computed as the cohomology of the Koszul-type complex

$$\begin{aligned}\Lambda^{p+1}H^0(L) \otimes H^0(B) &\longrightarrow \Lambda^{p}H^0(L) \otimes H^0(B + L)\\ &\longrightarrow \Lambda^{p-1}H^0(L) \otimes H^0(B + 2L),\end{aligned}$$

and the basic strategy is to realize this complex geometrically. In brief, a line bundle B on C determines a vector bundle $E_B = E_{p+1,B}$ of rank $p+1$ on the symmetric product

C_{p+1} whose fibre at a point $\xi \in C_{p+1}$ is the vector space $H^0(C, B \otimes \mathcal{O}_\xi)$. The natural map $H^0(B) \longrightarrow H^0(B \otimes \mathcal{O}_\xi)$ globalizes to a homomorphism of vector bundles

$$(*) \qquad \mathrm{ev}_B = \mathrm{ev}_{p+1,B} : H^0(C, B) \otimes_{\mathbf{C}} \mathcal{O}_{C_{p+1}} \longrightarrow E_B,$$

and evidently ev_B is surjective as a map of vector bundles if and only if B is p-very ample. On the other hand, if $N_L = \det E_L$, then it is well-known that $H^0(N_L) = \Lambda^{p+1} H^0(C, L)$, and twisting (*) by N_L gives rise to a vector bundle map

$$(**) \qquad H^0(C, B) \otimes N_L \longrightarrow E_B \otimes N_L.$$

Computations of Voisin identify $H^0(C_{p+1}, E_B \otimes N_L)$ with the space $Z_{p,1}(C, B; L)$ of Koszul cycles, and hence $K_{p,1}(C, B; L) = 0$ if and only if the homomorphism

$$H^0(C, B) \otimes H^0(C_{p+1}, N_L) \longrightarrow H^0(C_{p+1}, E_B \otimes N_L)$$

determined by (**) is surjective. But assuming that B is p-very ample, so that (**) is surjective as a map of bundles, this follows for $\deg L \gg 0$ simply by applying Serre–Fujita vanishing to the kernel of (**). We note that the main difference from Voisin's set-up—apart from separating B and L, which clarifies the issue—is that we push down to the symmetric product rather than working on the universal family over it. Some related computations had earlier appeared in the paper [10], where it was shown that one could see the syzygies of canonical curves in cohomology related to the cotangent bundle E_{Ω_C} of the symmetric product, but it has to be admitted that nothing came of these.

We are grateful to Marian Aprodu, Gabi Farkas, B. Purnaprajna, Frank Schreyer, David Stapleton, Bernd Sturmfels, Brooke Ullery and Claire Voisin for valuable discussions and encouragement.

1. Proofs

This section is devoted to the proofs of Theorems A, B and C from the Introduction. We keep the notation introduced there.[3] Thus C is a smooth projective curve of genus g, and L is a very ample line bundle of degree d on C defining an embedding

$$C \subseteq \mathbf{P}H^0(L) = \mathbf{P}^r.$$

We fix an arbitrary line bundle on B on C, and we are interested in the Koszul cohomology groups

$$K_{p,q}(B; L) = K_{p,q}(C, B; L)$$

[3] In addition, we continue to allow ourselves to be a little sloppy in confounding additive and multiplicative notation for divisors and line bundles.

arising as the cohomology of the Koszul-type complex:

$$\Lambda^{p+1}H^0(L) \otimes H^0\big(B + (q-1)L\big) \longrightarrow \Lambda^{p}H^0(L) \otimes H^0(B + qL) \longrightarrow \Lambda^{p-1}H^0(L) \otimes H^0\big(B + (q+1)L\big).$$

We recall that results of Green and others imply that if $d = \deg(L) \gg 0$, then $K_{p,q}(B; L) = 0$ for all $q \geq 3$, and:

$$K_{p,0}(B; L) \neq 0 \iff p \in \big[0, h^0(B) - 1\big]$$
$$K_{p,2}(B; L) \neq 0 \iff p \in \big[r - h^1(B), r - 1\big]$$

(cf. [5, Proposition 5.1, Corollary 5.2]).[4] So the issue is to understand which of the groups $K_{p,1}(B; L)$ vanish when $\deg L \gg 0$.

Write C_k for the kth symmetric product of C, viewed as parameterizing all effective divisors on C of degree k. We consider the commutative diagram:

(1.1)
$$\begin{array}{ccccc} & & C & & \\ & \overset{pr_1}{\nearrow} & & \overset{pr_1}{\nwarrow} & \\ C \times C_p & & \overset{j_{p+1}}{\hookrightarrow} & & C \times C_{p+1} \\ & \underset{\sigma_{p+1}}{\searrow} & & \underset{pr_2}{\swarrow} & \\ & & C_{p+1} & & \end{array}$$

where σ_{p+1} and j_{p+1} are the maps defined by

$$\sigma_{p+1}(x, \xi) = x + \xi, \qquad j_{p+1}(x, \xi) = (x, x + \xi).$$

Note that σ_{p+1} realizes $C \times C_p$ as the universal family of degree $p+1$ divisors over C_{p+1}.

The proofs revolve around two well-studied tautological sheaves on C_{p+1}. First given a line bundle B on C, define

$$E_B = E_{p+1,B} =_{\text{def}} \sigma_{p+1,*}pr_1^*(B).$$

Thus E_B is a vector bundle of rank $p+1$ on C_{p+1} whose fibre at $\xi \in C_{p+1}$ is identified with the vector space $H^0(C, B \otimes \mathcal{O}_\xi)$. It follows from the construction that $H^0(C_{p+1}, E_B) = H^0(C, B)$, which gives rise to a homomorphism:

(1.2)
$$ev_B = ev_{p+1,B} : H^0(C, B) \otimes_{\mathbf{C}} \mathcal{O}_{C_{p+1}} \longrightarrow E_B$$

[4] In particular, if $H^0(B) = 0$ then $K_{p,0}(B; L) = 0$ for all p, and if $H^1(B) = 0$, then $K_{p,2}(B; L) = 0$ for all p provided that $\deg L \gg 0$.

of vector bundles on C_{p+1}. Evidently ev_B is surjective if and only if B is p-very ample. Next, given a line bundle L on C, put

$$N_L = N_{p+1,L} = \det E_L.$$

Alternatively, N_L characterized by the fact that its pullback to the Cartesian product is isomorphic to the $(p+1)$-fold box product of L with itself twisted by the ideal of the sum of the pairwise-diagonals. Note that $\Lambda^{p+1} ev_L$ determines a map

$$\Lambda^{p+1} H^0(C, L) \longrightarrow H^0(C_{p+1}, N_L),$$

and it was established e.g. in [6, 13] that this is an isomorphism. Twisting ev_B by N_L, one arrives at the vector bundle map

$$H^0(C, B) \otimes_{\mathbf{C}} N_L \longrightarrow E_B \otimes N_L \tag{1.3}$$

that lies at the heart of the proof.

Our main results follow immediately from two lemmas whose proofs appear at the end of this section. The first, which is effectively due to Voisin, states that $K_{p,1}(B; L) = 0$ if and only if (**1.3**) is surjective on global sections. The second asserts that as L gets very positive on C, the corresponding line bundles N_L become sufficiently positive on C_{p+1} to satisfy a Serre-type vanishing theorem.

Lemma **1.1** *(Voisin). — The global sections of* $E_B \otimes N_L$ *are identified with the space*

$$Z_{p,1}(B; L) = \ker\big(\Lambda^p H^0(L) \otimes H^0(B + L) \longrightarrow \Lambda^{p-1} H^0(L) \otimes H^0(B + 2L)\big)$$

of Koszul cycles, and the homomorphism

$$\begin{aligned} &H^0(C, B) \otimes H^0(C_{p+1}, N_L) \\ &\quad = H^0(C, B) \otimes \Lambda^{p+1} H^0(C, L) \longrightarrow H^0(C_{p+1}, E_B \otimes N_L) \end{aligned}$$

arising from (**1.3**) *is identified with the Koszul differential. In particular,*

$$K_{p,1}(C, B; L) = 0$$

if and only if the bundle map (**1.3**) *determines a surjection on global sections.*

Lemma **1.2.** *— Let* $\mathcal{F}$ *be any coherent sheaf on* C_{p+1}*. There exists an integer* $d_0 = d_0(\mathcal{F})$ *having the property that if* $d = \deg(L) \geq d_0(\mathcal{F})$*, then*

$$H^i(C_{p+1}, \mathcal{F} \otimes N_L) = 0 \quad \textit{for } i > 0.$$

Granting the lemmas for now, we prove the main results.

Proof of Theorem B. — Assume that B is p-very ample, so that ev_B in (**1.2**) is surjective. Denote by $M_B = M_{p+1,B}$ its kernel:

$$0 \longrightarrow M_B \longrightarrow H^0(C, B) \otimes \mathcal{O}_{C_{p+1}} \longrightarrow E_B \longrightarrow 0. \tag{1.4}$$

To show that $K_{p,1}(B; L) = 0$ when $\deg L \gg 0$, it suffices by Lemma 1.1 to prove that

$$H^1(C_{p+1}, M_B \otimes N_L) = 0 \tag{1.5}$$

for very positive L. But this follows from Lemma 1.2. Conversely, if ev_B is not surjective, then it is elementary—and we will see momentarily in the proof of Theorem C—that $K_{p,1}(B; L) \neq 0$ for every sufficiently positive L. □

Remark **1.3.** — Proposition 2.1 below gives an effective lower bound on deg(L) that is sufficient to guarantee the vanishing (**1.5**). □

Proof of Theorem C. — Denote by $M_B = M_{p+1,B}$ and $\mathcal{F}_B$ respectively the kernel and cokernel of ev_B:

$$0 \longrightarrow M_B \longrightarrow H^0(B) \otimes \mathcal{O}_{C_{p+1}} \longrightarrow E_B \longrightarrow \mathcal{F}_B \longrightarrow 0. \tag{1.6}$$

Taking $L_d = dA + E$ as in the statement of the theorem, put $N_d = N_{L_d}$. We will see in the proof of Lemma 1.2 below that

$$N_d = N_E + dS_A,$$

where S_A is an ample divisor on C_{p+1}. On the other hand, it follows from the two lemmas that for $d \gg 0$

$$K_{p,1}(C, B; L_d) = H^0(C_{p+1}, \mathcal{F}_B \otimes N_d).$$

Therefore $\dim K_{p,1}(C, B; L_d)$ is given for $d \gg 0$ by the Hilbert polynomial of $\mathcal{F}_B \otimes N_E$ with respect to S_A. But $\gamma_p(B) = \dim \operatorname{Supp} \mathcal{F}_B$, and the result follows. □

Remark **1.4.** — This argument shows that $K_{p,0}(C, B; L_d) = H^0(C_{p+1}, M_B \otimes N_d)$ provided that d is large. Hence (assuming that $p \leq r(B)$) the dimension of this Koszul group always grows as a polynomial of degree $(r(B) - p)$ in d when $d \gg 0$.[5] In other words, it is the growth of the $K_{p,1}$ groups that exhibit interesting dependence on geometry. □

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[5] The arguments of [15] show that analogously on a variety of dimension n, $\dim K_{p,0}$ grows as a polynomial of degree $n(r(B) - p)$.

We next recall the well-known argument that the case $B = K_C$ of Theorem B implies the Gonality Conjecture.

Proof of Theorem A. — Fix $p \leq g$. We need to show that if $\deg(L) \gg 0$, and if

$$(*) \qquad K_{r(L)-p,1}(C; L) \neq 0,$$

then C carries a pencil of degree $\leq p$. By duality, (*) implies that

$$K_{p-1,1}(C, K_C; L) \neq 0,$$

and hence by Theorem B there exists an effective divisor $\xi \in C_p$ of degree p that fails to impose independent conditions on $|K_C|$. But then ξ moves in a non-trivial linear series thanks to geometric Riemann–Roch. □

We conclude this section by proving the two lemmas stated above.

Proof of Lemma 1.1. — It follows from the projection formula and the constructions that

$$\begin{aligned} H^0(C_{p+1}, E_B \otimes N_L) &= H^0\big(C \times C_p, pr_1^*B \otimes \sigma_{p+1}^* N_L\big) \\ &= H^0\big(C \times C_p, (j_{p+1})^*\big(pr_1^*B \otimes pr_2^* N_L\big)\big). \end{aligned}$$

Moreover the map induced by (**1.3**) on global sections is identified with the restriction

$$H^0(C \times C_{p+1}, B \boxtimes N_L) \longrightarrow H^0\big(C \times C_p, (B \boxtimes N_L)|(C \times C_p)\big).$$

But this is exactly Voisin's Hilbert-schematic interpretation of Koszul cohomology, and from this point one can argue just as in [2, Lemma 5.4]. In brief, one observes that on $C \times C_p$ one has an isomorphism

$$j_{p+1}^*(N_{p+1,L}) = (L \boxtimes N_{p,L})(-D),$$

where $D \subseteq C \times C_p$ is the image of $j_p : C \times C_{p-1} \hookrightarrow C \times C_p$. Therefore

$$H^0\big(C \times C_p, (j_{p+1})^*(B \boxtimes N_{p+1,L})\big)$$

is identified with

$$\begin{aligned} &\ker\big(H^0\big(C \times C_p, \mathcal{O}_C(B + L) \boxtimes N_{p,L}\big) \\ &\quad \longrightarrow H^0\big(C \times C_{p-1}, \mathcal{O}_C(B + 2L) \boxtimes N_{p-1,L}\big)\big), \end{aligned}$$

and the assertion follows. □

Proof of Lemma 1.2. — Given a divisor A on C, the divisor $\mathrm{T_A} =_{\mathrm{def}} \sum pr_i^*(\mathrm{A})$ on the Cartesian product C^{p+1} descends to a divisor $\mathrm{S_A} = \mathrm{S}_{p+1,\mathrm{A}}$ on C_{p+1}. For example, if $\mathrm{A} = x_1 + \cdots + x_d$, then

$$\mathrm{S_A} = \mathrm{C}_{p,x_1} + \cdots + \mathrm{C}_{p,x_d} \in \mathrm{Div}(\mathrm{C}_{p+1}),$$

where $\mathrm{C}_{p,x}$ denotes the image of the map $\mathrm{C}_p \hookrightarrow \mathrm{C}_{p+1}$ given by $\xi \mapsto \xi + x$. One has $\mathrm{S}_{\mathrm{A}_1+\mathrm{A}_2} = \mathrm{S}_{\mathrm{A}_1} + \mathrm{S}_{\mathrm{A}_2}$, and $\mathrm{S_A}$ is ample on C_{p+1} if and only if A is ample on C. Observe next that if L is line bundle on C, then $\mathrm{N_{L+A}} = \mathrm{N_L} + \mathrm{S_A}$ on C_{p+1}. This is well-known, but it can be checked directly from the definitions by observing that if $x \in \mathrm{C}$ is a point then there is an exact sequence

$$0 \longrightarrow \mathrm{E_L} \longrightarrow \mathrm{E}_{\mathrm{L}(x)} \longrightarrow \mathcal{O}_{\mathrm{C}_{p,x}} \longrightarrow 0$$

of sheaves on C_{p+1}.

Now fix an ample divisor A of degree a on C and a coherent sheaf $\mathcal{F}$ on C_{p+1}. By Fujita–Serre vanishing, there exists an integer $m_0 = m_0(\mathcal{F})$ such that if P is any nef divisor on C_{p+1}, then

$$(*) \qquad \mathrm{H}^i\big(\mathrm{C}_{p+1}, \mathcal{F}(m\mathrm{S_A} + \mathrm{P})\big) = 0 \quad \text{for } i > 0$$

whenever $m \geq m_0$. Put

$$d_0 = d_0(\mathcal{F}) = (2g + p) + m_0 a,$$

and suppose that $\deg(\mathrm{L}) \geq d_0$. Then $\mathrm{L} = \mathrm{L}_0 + m_0\mathrm{A}$ where L_0 is p-very ample, and in particular $\mathrm{N}_{\mathrm{L}_0}$ is globally generated. Therefore

$$\mathrm{N_L} = m_0\mathrm{S_A} + (\mathrm{nef}),$$

and so $(*)$ gives the required vanishing. Alternatively, one could prove the lemma by pulling back to the symmetric product. □

2. Complements

This section is devoted to some additional results, and a conjecture about what one might hope for in higher dimensions.[6]

We start by establishing an effective version of Theorem B. Since the statement is presumably far from optimal we only sketch the proof.

[6] Some subsequent developments are stated in Remark 2.5 appearing at the end of the section. These involve a substantial improvement of Proposition 2.1 due to Rathmann, and the fact that Conjecture 2.4 has been established by Yang and the authors.

Proposition **2.1.** — *Assume that* B *is* p*-very ample. Then* $K_{p,1}(C, B; L) = 0$ *for every line bundle* L *with*

$$\textbf{(2.1)} \qquad \deg(L) > \big(p^2 + p + 2\big)(g-1) + (p+1)\deg(B).$$

Sketch of proof. — Keeping notation as in the proof of Theorem B, one needs to prove that $H^1(C_{p+1}, M_B \otimes N_L) = 0$ when $\deg(L)$ satisfies the stated bound. If $h^0(C, B) > 2(p+1)$, we replace $H^0(C, B)$ in (**1.4**) by a general subspace of dimension $2p+2$ to define a vector bundle M_B' of rank $p+1$ sitting in an exact sequence

$$0 \longrightarrow M_B' \longrightarrow M_B \longrightarrow \oplus \mathcal{O}_{C_{p+1}} \longrightarrow 0,$$

and one is reduced to proving that $H^1(C_{p+1}, M_B' \otimes N_L) = 0$. Note that $M_B' \otimes N_B$—being a quotient of $\Lambda^p(M_B^*)$—is globally generated, and that $\det M_B' = -N_B$.

We assert that if L satisfies (**2.1**), then

$$(*) \qquad N_L - (p+1)N_B - K_{C_{p+1}} \quad \text{is ample.}$$

Granting this, we see that if (**2.1**) holds, then

$$M_B' \otimes N_L = \big(M_B' \otimes N_B\big) \otimes \det\big(M_B' \otimes N_B\big) \otimes K_{C_{p+1}} \otimes A$$

where is A is ample, so the Griffiths vanishing theorem [11, 7.3.2] applies. For (*), it is equivalent to check the statement after pulling back by the quotient $\pi : C^{p+1} \to C_{p+1}$. One has $\pi^* N_L = T_L - \Delta$, where $T_L = \sum pr_i^* L$ is the symmetrization of L and $\Delta \in \mathrm{Div}(C^{p+1})$ is the union of the pairwise diagonals. Since $K_{C_{p+1}} = N_{K_C}$, the claim (*) reduces with some computation to the fact that if D is a divisor on C, then $T_D + \Delta$ is nef on C^{p+1} if and only if $\deg D \geq p(g-1)$. □

Remark **2.2.** — The Proposition guarantees that we can detect whether K_C is p-very ample (or equivalently, whether $\mathrm{gon}(C) \geq p+2$) by the vanishing of $K_{p,1}(C, K_C; L)$ for any L with

$$\deg(L) > \big(p^2 + 3p + 3\big)(g-1).$$

But in any event $\mathrm{gon}(C) \leq \frac{g+3}{2}$, and it follows (with some computation) that the gonality of C is determined by the weight one syzygies of C with respect to any line bundle of degree $\geq g^3$. However we expect that such cubic bounds are far from optimal: one hopes that it is enough that the degree of L grows linearly in g, as suggested by the results of [1, 3].[7] □

[7] See Remark 2.5.

As suggested by Schreyer, we observe next that in some cases one can use the proof of Theorem C to get more information about the polynomial $P(d)$ appearing there. We focus on the most interesting case $B = K_C$, and content ourselves with illustrating the method in a simple instance. Specifically, suppose that C carries finitely many pencils

$$\alpha_1, \ldots, \alpha_s \in W^1_{p+1}(C)$$

of degree $p+1$, while no other divisors of degree $p+1$ on C move in non-trivial linear series. We assume also that each α_i is (scheme-theoretically) an isolated point in $W^1_{p+1}(C)$ in the sense that the multiplication maps

$$(*) \qquad H^0(\alpha_i) \otimes H^0(K_C - \alpha_i) \longrightarrow H^0(K)$$

are surjective for each i.[8]

Proposition **2.3.** — *Under the hypotheses just stated, take* $L_d = d \cdot x$ *for some point* $x \in C$. *Then for* $d \gg 0$,

$$\dim K_{p,1}(C, K_C; L_d) = s \cdot d + \textit{(constant)}.$$

We note that Yang has made some interesting computations of the dimensions of the groups $K_{p,0}(C, K_C; L_d)$ on a general curve, including determining the leading coefficient of the resulting polynomial.

Sketch of proof of Proposition 2.3. — Note that each α_i determines a copy of $\mathbf{P}^1 = |\alpha_i|$ sitting in the symmetric product C_{p+1}, and these are precisely the positive-dimensional fibres of the Abel–Jacobi map

$$u = u_{p+1} : C_{p+1} \longrightarrow \mathrm{Jac}^{p+1}(C).$$

Now when $B = K_C$, the evaluation (**1.2**) is identified with the coderivative du of u, and by a well-known computation [4, Chap. IV.4], the condition (*) implies that

$$\operatorname{coker} du = \bigoplus_{i=1}^{s} \Omega^1_{|\alpha_i|}.$$

In particular, the sheaf $\mathcal{F}_{K_C}$ appearing in (**1.6**) has rank one along each $\mathbf{P}^1 = |\alpha_i|$. On the other hand, if $L_d = d \cdot x$ then the divisor N_d has degree $d +$ (constant) along $|\alpha_i|$, so each of these copies of $\mathbf{P}^1$ contributes a term of the same shape to the Hilbert polynomial of $\mathcal{F}_{K_C}$. □

[8] Recall that the Gieseker–Petri theorem asserts that the hypothesis holds automatically for a general curve of genus $g = 2p$, in which case s is given by a certain Catalan number.

Finally, we make some remarks about what one might expect in higher dimensions. Let X be a smooth projective variety of dimension n, and let $L_d = dA + E$ where A is an ample and E an arbitrary divisor on X. Given a line bundle B on X, one would like to give geometric conditions on B in order that

$$K_{p,1}(X, B; L_d) = 0 \quad \text{for all } d \gg 0 : \tag{2.2}$$

as explained above and in [5, Problem 7.2] this is the most interesting group from an asymptotic viewpoint. It is conceivable that it suffices to assume that B is p-very ample in the sense that $H^0(B)$ imposes independent conditions on every subscheme $\xi \subseteq X$ of length $p+1$, but this seems out of reach. On the other hand, recall that B is said to be *p-jet very ample* if for every effective zero-cycle

$$z = a_1x_1 + \cdots + a_sx_s$$

of degree $p+1$ on X, the natural mapping

$$H^0(X, B) \longrightarrow H^0\big(X, B \otimes \mathcal{O}_X/\mathfrak{m}_1^{a_1} \cdot \cdots \cdot \mathfrak{m}_s^{a_s}\big)$$

is surjective, where $\mathfrak{m}_i \subseteq \mathcal{O}_X$ is the ideal sheaf of x_i. When $\dim X = 1$ this is the same as p-very ample, but in higher dimensions the condition on jets is stronger.

Conjecture **2.4.** — *If* B *is p-jet very ample, then* (**2.2**) *holds.*

It is very possible that the ideas of [15] will be helpful for this.

Remark **2.5** *(Added in February, 2015)*. — We mention two developments that have occurred since the paper was written, but that have not yet appeared. First, Rathmann [12] has shown that if B is p-very ample, then $K_{p,1}(C, B; L) = 0$ as soon as

$$H^1(C, L - B) = H^1(L) = 0.$$

In particular the gonality conjecture holds for any line bundle L with $\deg L \geq 4g - 3$. Secondly, the authors and Yang have established Conjecture 2.4. The proof will appear elsewhere.

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Manuscrit reçu le 16 juillet 2014
Manuscrit accepté le 24 février 2015