

# AN INFORMAL INTRODUCTION TO MULTIPLIER IDEALS

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## 1. INTRODUCTION

Given a smooth complex variety  $X$  and an ideal (or ideal sheaf)  $\mathfrak{a}$  on  $X$ , one can attach to  $\mathfrak{a}$  a collection of *multiplier ideals*  $\mathcal{J}(\mathfrak{a}^c)$  depending on a rational weighting parameter  $c > 0$ . These ideals, and the vanishing theorems they satisfy, have found many applications in recent years. In the global setting they have been used to study pluricanonical and other linear series on a projective variety ([Dem93], [AS95], [Siu98], [EL97], [EL99], [Dem99]). More recently they have led to the discovery of some surprising uniform results in local algebra ([ELS01], [ELS], [ELSV]). The purpose of these lectures is to give an easy-going and gentle introduction to the algebraically-oriented local side of the theory.

Multiplier ideals can be approached (and historically emerged) from three different viewpoints. In commutative algebra they were introduced and studied by Lipman [Lip93] in connection with the Briançon-Skoda theorem.<sup>1</sup> On the analytic side of the field, Nadel [Nad90] attached a multiplier ideal to any plurisubharmonic function, and proved a Kodaira-type vanishing theorem for them.<sup>2</sup> This machine was developed and applied with great success by Demailly, Siu and others. Algebra-geometrically, the foundations were laid in passing by Esnault and Viehweg in connection with their work involving the Kawamata-Viehweg vanishing theorem. More systematic developments of the geometric theory were subsequently undertaken by Ein, Kawamata and the second author. We will take the geometric approach here.

The present notes follow closely a short course on multiplier ideals given by the second author at the Introductory Workshop for the Commutative Algebra Program at the MSRI in September 2002<sup>3</sup>. The three main lectures were supplemented with a presentation by the first author on multiplier ideals associated to monomial ideals (which appears here in §3). We have tried to preserve in this write-up the informal tone of these talks: thus we emphasize simplicity over generality in statements of results, and we present

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Research of the second author partially supported by NSF Grant DMS 0139713.

<sup>1</sup>Lipman used the term “adjoint ideals”, but this has come to refer to a different construction.

<sup>2</sup>In fact, the “multiplier” in the name refers to their analytic construction (see §2.4)

<sup>3</sup>Handwritten notes and the lectures on streaming video are available at <http://www.msri.org/publications/video/index05.html>

very few proofs. Our primary hope is to give the reader a feeling for what multiplier ideals are and how they are used. For a detailed development of the theory from an algebro-geometric perspective we refer to Part Three of the forthcoming book [Laz]. The analytic picture is covered in Demailly's lectures [Dem01].

We conclude this Introduction by fixing the set-up in which we work and giving a brief preview of what is to come. Throughout these notes,  $X$  denotes a smooth affine variety over an algebraically closed field  $k$  of characteristic zero and  $R = k[X]$  is the coordinate ring of  $X$ , so that  $X = \text{Spec } R$ . We consider a non-zero ideal  $\mathfrak{a} \subseteq k[X]$  (or equivalently a sheaf of ideals  $\mathfrak{a} \subseteq \mathcal{O}_X$ ). Given a rational number  $c \geq 0$  our plan is to define and study the multiplier ideal

$$\mathcal{J}(c \cdot \mathfrak{a}) = \mathcal{J}(\mathfrak{a}^c) \subseteq k[X].$$

As we proceed, there are two ideas to keep in mind. The first is that  $\mathcal{J}(\mathfrak{a}^c)$  measures in a somewhat subtle manner the singularities of the divisor of a typical function  $f$  in  $\mathfrak{a}$ : for fixed  $c$ , “nastier” singularities are reflected by “deeper” multiplier ideals. Secondly,  $\mathcal{J}(\mathfrak{a}^c)$  enjoys remarkable formal properties arising from the Kawamata-Viehweg-Nadel Vanishing theorem. One can view the power of multiplier ideals as arising from the confluence of these facts.

The theory of multiplier ideals described here has striking parallels with the theory of tight closure developed by Hochster and Huneke in positive characteristic. Many of the uniform local results that can be established geometrically via multiplier ideals can also be proven (in more general algebraic settings) via tight closure. For some time the actual connections between the two theories were not well understood. However very recent work [HY], [Taka] of Hara-Yoshida and Takagi has generalized tight closure theory to define a so called test ideal  $\tau(\mathfrak{a})$ , which corresponds to the multiplier ideal  $\mathcal{J}(\mathfrak{a})$  under reduction to positive characteristic. This provides a first big step towards identifying concretely the links between these theories.

Concerning the organization of these notes, we start in §2 by giving the basic definition and several examples. Multiplier ideals of monomial ideals are discussed in detail in §3. Invariants arising from multiplier ideals, with some applications to uniform Artin-Rees numbers, are taken up in §4. Section 5 is devoted to a discussion of some basic results about multiplier ideals, notably Skoda's theorem and the restriction and subadditivity theorems. We consider asymptotic constructions in §6, with applications to uniform bounds for symbolic powers following [ELS01].

We are grateful to Karen Smith for suggestions concerning these notes.

## 2. DEFINITION AND EXAMPLES

As just stated,  $X$  is a smooth affine variety of dimension  $n$  over an algebraically closed field of characteristic zero, and we fix an ideal  $\mathfrak{a} \subseteq k[X]$  in the coordinate ring of  $X$ . Very little is lost by focusing on the case  $X = \mathbb{C}^n$

of affine  $n$ -space over the complex numbers  $\mathbb{C}$ , so that  $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$  is an ideal in the polynomial ring in  $n$  variables.

**2.1. Log resolution of an ideal.** The starting point is to realize the ideal  $\mathfrak{a}$  geometrically.

**Definition 2.1.** A *log resolution* of an ideal sheaf  $\mathfrak{a} \subseteq \mathcal{O}_X$  is a proper, birational map  $\mu : Y \rightarrow X$  whose exceptional locus is a divisor  $E = \text{Exceptional}(\mu)$  such that

- (1)  $Y$  is non-singular.
- (2)  $\mathfrak{a} \cdot \mathcal{O}_Y = \mu^{-1}\mathfrak{a} = \mathcal{O}_Y(-F)$  with  $F = \sum r_i E_i$  an effective divisor.
- (3)  $F + E$  has simple normal crossing support.

Recall that a (Weil) divisor  $D = \sum \alpha_i D_i$  has simple normal crossing support if each of its irreducible components  $D_i$  is smooth, and if locally analytically one has coordinates  $x_1, \dots, x_n$  of  $Y$  such that  $\text{Supp } D = \sum D_i$  is defined by  $x_1 \cdot \dots \cdot x_a$  for some  $a$  between 1 and  $n$ . In other words, all the irreducible components of  $D$  are smooth and intersect transversally. The existence of a log resolution for any sheaf of ideals in any variety over a field of characteristic zero is essentially Hironaka’s celebrated result on resolution of singularities [Hir64]. Nowadays there are more elementary constructions of such resolutions, for instance [BM97], [EV00] or [Par99].

**Example 2.2.** Let  $X = \mathbb{A}^2 = \text{Spec } k[x, y]$  and  $\mathfrak{a} = (x^2, y^2)$ . Blowing up the origin in  $\mathbb{A}^2$  yields

$$Y = \text{Bl}_0(\mathbb{A}^2) \xrightarrow{\mu} \mathbb{A}^2 = X.$$

Clearly,  $Y$  is nonsingular. Computing on the chart for which the blowup  $\mu$  is a map from  $\mathbb{A}^2 \rightarrow \mathbb{A}^2$  given by  $(u, v) \mapsto (u, uv)$  shows that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-2E)$ . On the described chart we have  $\mathfrak{a} \cdot \mathcal{O}_Y = (u^2, u^2v^2) = (u^2)$  and  $(u = 0)$  is the equation of the exceptional divisor. This resolution is illustrated in Figure 1, where we have drawn schematically the curves in  $\mathbb{A}^2$  defined by typical  $k$ -linear combinations of generators of  $\mathfrak{a}$ , and the proper transforms of these curves on  $Y$ . Note that these proper transforms do not meet: this reflects the fact that  $\mathfrak{a}$  has become principal on  $Y$ .

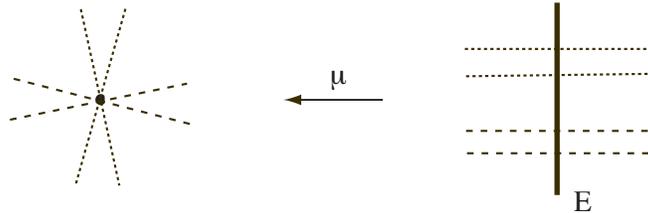


FIGURE 1. Log resolution of  $(x^2, y^2)$

**Example 2.3.** Now let  $\mathfrak{a} = (x^3, y^2)$ . In this case a log resolution is constructed by the familiar sequence of three blowings-up used to resolve a cuspidal curve (Figure 2). Here we have  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-2E_1 - 3E_2 - 6E_3)$  where  $E_i$  is the exceptional divisor of the  $i$ th blowup.

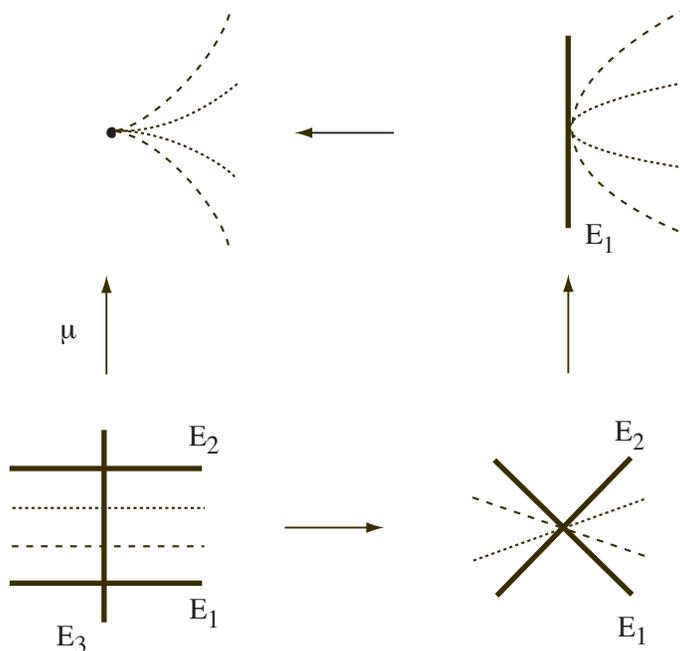


FIGURE 2. Log resolution of  $(x^3, y^2)$ .

These examples illustrate the principle that a log resolution of an ideal  $\mathfrak{a}$  is very close to being the same as a resolution of singularities of a divisor of a general function in  $\mathfrak{a}$ .

**2.2. Definition of multiplier ideals.** Besides a log resolution of  $\mu : Y \rightarrow X$  of the ideal  $\mathfrak{a}$ , the other ingredient for defining the multiplier ideal is the relative canonical divisor

$$K_{Y/X} = K_Y - \mu^* K_X = \text{div}(\det(\text{Jac } \mu)).$$

It is unique as a divisor (and not just as a divisor class) if one requires its support to be contained in the exceptional locus of  $\mu$ . Alternatively,  $K_{Y/X}$  is the effective divisor defined by the vanishing of the determinant of the Jacobian of  $\mu$ . The canonical divisor  $K_X$  is just the class corresponding to the canonical line bundle  $\omega_X$ . If  $X$  is smooth,  $\omega_X$  is just the sheaf of top differential forms  $\Omega_X^n$  on  $X$ .

Extremely useful for basic computations of multiplier ideals is the following proposition, see [Har77], Exercise II.8.5.

**Proposition 2.4.** *Let  $Y = \text{Bl}_Z X$  where  $Z$  is a smooth subvariety of the smooth variety  $X$  of codimension  $c$ . Then the relative canonical divisor  $K_{Y/X}$  is  $(c-1)E$ ,  $E$  being the exceptional divisor of the blowup.*

Now we can give a provisional definition of the multiplier ideal of an ideal  $\mathfrak{a}$ : it coincides in our setting with Lipman's construction in [Lip93].

**Definition 2.5.** Let  $\mathfrak{a} \subseteq k[X]$  be an ideal. Fix a log resolution  $\mu : Y \rightarrow X$  of  $\mathfrak{a}$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ , where  $F = \sum r_i E_i$ , and  $K_{Y/X} = \sum b_i E_i$ . The *multiplier ideal* of  $\mathfrak{a}$  is

$$\begin{aligned} \mathcal{J}(\mathfrak{a}) &= \mu_* \mathcal{O}_Y(K_{Y/X} - F) \\ &= \{ h \in k[X] \mid \text{div}(\mu^* h) + K_{Y/X} - F \geq 0 \} \\ &= \{ h \in k[X] \mid \text{ord}_{E_i}(\mu^* h) \geq r_i - b_i \text{ for all } i \}. \end{aligned}$$

(We will observe later that this is independent of the choice of resolution.)

The definition may seem at first blush a little mysterious. One way to motivate it is to note that  $\mathcal{J}(\mathfrak{a})$  is the push-forward of a bundle which is very natural from the viewpoint of vanishing theorems. In fact, the bundle  $\mathcal{O}_Y(-F)$  appearing above is (close to being) ample for the map  $\mu$ . Therefore  $K_{Y/X} - F$  has the shape to which Kodaira-type vanishing results will apply. In any event, the definition will justify itself before long through the properties of the ideals so defined.

**Exercise 2.6.** Use the fact that  $\mu_* \omega_Y = \omega_X$  to show that  $\mathcal{J}(\mathfrak{a})$  is indeed an ideal in  $k[X]$ .

**Exercise 2.7.** Show that the integral closure  $\bar{\mathfrak{a}}$  of  $\mathfrak{a}$  is equal to  $\mu_* \mathcal{O}_Y(-F)$ . Use this to conclude that  $\mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq \mathcal{J}(\mathfrak{a}) = \overline{\mathcal{J}(\mathfrak{a})}$ . (Recall that the integral closure of an ideal  $\mathfrak{a}$  consists of all elements  $f$  such that  $v(f) \geq v(a)$  for all valuations  $v$  of  $\mathcal{O}_X$ .)

**Exercise 2.8.** Verify that for ideals  $\mathfrak{a} \subseteq \mathfrak{b}$  one has  $\mathcal{J}(\mathfrak{a}) \subseteq \mathcal{J}(\mathfrak{b})$ . Use this and the previous exercise to show that  $\mathcal{J}(\mathfrak{a}) = \mathcal{J}(\bar{\mathfrak{a}})$ .

The above definition of the multiplier ideal is not general enough for the most interesting applications. As it turns out, allowing an additional rational (or real) parameter  $c$  considerably increases the power of the theory.

Note that a log resolution of an ideal  $\mathfrak{a}$  is at the same time a log resolution of any integer power  $\mathfrak{a}^n$  of that ideal. Thus we extend the last definition, using the same log resolution for every  $c \geq 0$ :

**Definition 2.9.** For every rational number  $c \geq 0$ , the *multiplier ideal* of the ideal  $\mathfrak{a}$  with exponent (or coefficient)  $c$  is

$$(1) \quad \begin{aligned} \mathcal{J}(\mathfrak{a}^c) &= \mathcal{J}(c \cdot \mathfrak{a}) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor) \\ &= \{ h \in k[X] \mid \text{ord}_{E_i}(\mu^* h) \geq \lfloor cr_i \rfloor - b_i \text{ for all } i \} \end{aligned}$$

where  $\mu : Y \rightarrow X$  is a log resolution of  $\mathfrak{a}$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ .

Note that we do not assign any meaning to  $\mathfrak{a}^c$  itself, only to  $\mathcal{J}(\mathfrak{a}^c)$ .<sup>4</sup> The round-down operation  $\lfloor \cdot \rfloor$  applied to a  $\mathbb{Q}$ -divisor  $D = \sum a_i D_i$  for distinct prime divisors  $D_i$  is just rounding down the coefficients. That is,  $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$ . The round up  $\lceil D \rceil = -\lfloor -D \rfloor$  is defined analogously.

**Exercise 2.10** (Caution with rounding). Show that rounding does not in general commute with restriction or pullback.

**Exercise 2.11.** Let  $\mathfrak{m}$  be the maximal ideal of a point  $x \in X$ . Show that

$$\mathcal{J}(\mathfrak{m}^c) = \begin{cases} \mathfrak{m}^{\lfloor c \rfloor + 1 - n} & \text{for } c \geq n = \dim X. \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

**Example 2.12.** Let  $\mathfrak{a} = (x^2, y^2) \subseteq k[x, y]$ . For the log resolution of  $\mathfrak{a}$  as calculated above we have  $K_{Y/X} = E$ . Therefore,

$$\mathcal{J}(\mathfrak{a}^c) = \mu_*(\mathcal{O}_Y(E - \lfloor 2c \rfloor E)) = (x, y)^{\lfloor 2c \rfloor - 1}$$

(In view of Exercise 2.8, this is a special case of Exercise 2.11.)

**Example 2.13.** Let  $\mathfrak{a} = (x^2, y^3)$ . In this case we computed a log resolution with  $F = 2E_1 + 3E_2 + 6E_3$ . Using the basic formula for the relative canonical divisor of a blowup along a smooth center, one computes  $K_{Y/X} = E_1 + 2E_2 + 4E_3$ . Therefore,

$$\begin{aligned} \mathcal{J}(\mathfrak{a}^c) &= \mu_*(\mathcal{O}_Y(E_1 + 2E_2 + 4E_3 - \lfloor c(2E_1 + 3E_2 + 6E_3) \rfloor)) \\ &= \mu_*(\mathcal{O}_Y((1 - \lfloor 2c \rfloor)E_1 + (2 - \lfloor 3c \rfloor)E_2 + (4 - \lfloor 6c \rfloor)E_3)). \end{aligned}$$

This computation shows that for  $c < 5/6$  the multiplier ideal is trivial, *i.e.*,  $\mathcal{J}(\mathfrak{a}^c) = \mathcal{O}_X$ . Furthermore,  $\mathcal{J}(\mathfrak{a}^{5/6}) = (x, y)$ . The next coefficient for which the multiplier ideal changes is  $c = 1$ . This behavior of multiplier ideals to be piecewise constant with discrete jumps is true in general and will be discussed in more detail later.

**Exercise 2.14** (Smooth ideals). Suppose that  $\mathfrak{q} \subseteq k[X]$  is the ideal of a smooth subvariety  $Z \subseteq X$  of pure codimension  $e$ . Then

$$\mathcal{J}(\mathfrak{q}^\ell) = \mathfrak{q}^{\ell + 1 - e}.$$

(Blowing up  $X$  along  $Z$  yields a log resolution of  $\mathfrak{q}$ .) The case of fractional exponents is similar.

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<sup>4</sup>There is a way to define the integral closure of an ideal  $\mathfrak{a}^c$ , for  $c \geq 0$  rational, such that it is consistent with the definition of the multiplier ideal. For  $c = p/q$  with positive integers  $p$  and  $q$ , set  $f \in \mathfrak{a}^{p/q}$  if and only if  $f^q \in \overline{\mathfrak{a}^p}$ , where the bar denotes the integral closure.

**2.3. Two basic properties.** The definitions of the previous subsection are justified by the fact that they lead to two very basic results.

The first point is that the ideal  $\mathcal{J}(\mathfrak{a}^c)$  constructed in Definition 2.9 is actually independent of the choice of resolution.

**Theorem 2.15.** *If  $X_1 \xrightarrow{\mu_1} X$  and  $X_2 \xrightarrow{\mu_2} X$  are two log resolutions of the ideal  $\mathfrak{a} \subseteq \mathcal{O}_X$  such that  $\mathfrak{a}\mathcal{O}_{X_i} = \mathcal{O}_{X_i}(-F_i)$ , then*

$$\mu_{1*}(\mathcal{O}_{X_1}(K_{X_1/X} - \lfloor c \cdot F_1 \rfloor)) = \mu_{2*}(\mathcal{O}_{X_2}(K_{X_2/X} - \lfloor c \cdot F_2 \rfloor)).$$

As one would expect, the proof involves dominating  $\mu_1$  and  $\mu_2$  by a third resolution. It is in the course of this argument that it becomes important to know that  $F_1$  and  $F_2$  have normal crossing support, see [Laz, Chapter 9].

**Exercise 2.16.** By contrast, give an example to show that if  $c$  is non-integral, then the ideal  $\mu_*(-\lfloor cF \rfloor)$  may indeed depend on the log resolution  $\mu$ .

The second fundamental fact is a vanishing theorem for the sheaves computing multiplier ideals.

**Theorem 2.17** (Local Vanishing Theorem). *Consider an ideal  $\mathfrak{a} \subseteq k[X]$  as above, and let  $\mu : Y \rightarrow X$  be a log resolution of  $\mathfrak{a}$  with  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ . Then*

$$R^i \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor cF \rfloor) = 0$$

for all  $i > 0$  and  $c > 0$ .

This leads one to expect that the multiplier ideal, being the zeroth derived image of  $\mathcal{O}_Y(K_{Y/X} - \lfloor cF \rfloor)$  under  $\mu_*$ , will display particularly good cohomological properties.

Theorem 2.17 is a special case of the Kawamata-Viehweg vanishing theorem for a mapping, see [Laz, Chapter 9]. It is the essential fact underlying all the applications of multiplier ideals appearing in these notes. When  $c$  is a natural number, the result can be seen as a slight generalization of the classical Grauert-Riemenschneider Vanishing Theorem. However as we shall see it is precisely the possibility of working with non-integral  $c$  that opens the door to applications of a non-classical nature.

**2.4. Analytic construction of multiplier ideals.** We sketch briefly the analytic construction of multiplier ideals. Let  $X$  be a smooth complex affine variety, and  $\mathfrak{a} \subseteq \mathbb{C}[X]$  an ideal. Choose generators  $g_1, \dots, g_p \in \mathfrak{a}$ . Then

$$\mathcal{J}(\mathfrak{a}^c)^{\text{an}} =_{\text{locally}} \left\{ h \text{ holomorphic} \mid \frac{|h|^2}{\left(\sum |g_i|^2\right)^c} \text{ is locally integrable} \right\}.$$

In other words, the analytic ideal associated to  $\mathcal{J}(\mathfrak{a}^c)$  arises as a sheaf of “multipliers”. See [Dem99, (5.9)] or [Laz, Chapter 9.3.D] for the proof. In brief the idea is to show that both the algebraic and the analytic definitions lead to ideals that transform the same way under birational maps. This reduces one to the situation where  $\mathfrak{a}$  is the principal ideal generated by

a single monomial in local coordinates. Here the stated equality can be checked by an explicit calculation.

**2.5. Multiplier ideals via tight closure.** As already hinted at in the introduction there is an intriguing parallel between effective results in local algebra obtained via multiplier ideals on the one hand and tight closure methods on the other. Almost all the results we will discuss in these notes are of this kind: there are tight closure versions of the Briançon-Skoda theorem, the uniform Artin-Rees lemma and even of the result on symbolic powers we present as an application of the asymptotic multiplier ideals in Section 6.4.<sup>5</sup> There is little understanding for why such different techniques (characteristic zero, analytic in origin vs. positive characteristic) seem to be tailor made to prove the same results.

Recently, Hara-Yoshida and Takagi strengthened this parallel by constructing in [HY] and [Taka, TiW, HT, Takb] multiplier-like ideals using techniques modelled after tight closure theory. Their construction builds on earlier work of Smith [Smi00] and Hara [Har01], who had established a connection between the multiplier ideal associated to the unit ideal (1) on certain singular varieties with the so-called test ideal in tight closure. The setting of the work of Hara and Yoshida is a regular<sup>6</sup> local ring  $R$  of positive characteristic  $p$ . For simplicity one might again assume  $R$  is the local ring of a point in  $\mathbb{A}^n$ . Just as with multiplier ideals, one assigns to an ideal  $\mathfrak{a} \subseteq R$  and a rational parameter  $c \geq 0$ , the *test ideal*

$$\tau(\mathfrak{a}^c) = \{ h \in R \mid hI^{*\mathfrak{a}^c} \subseteq I \text{ for all ideals } I \}.$$

Here  $I^{*\mathfrak{a}^c}$  denotes the  $\mathfrak{a}^c$ -tight closure of an ideal, specifically introduced for the purpose of constructing these test ideals  $\tau(\mathfrak{a}^c)$ .<sup>7</sup> The properties the test ideals enjoy are strikingly similar to those of the multiplier ideal in characteristic zero: For example the Restriction Theorem (Theorem 5.8) and Subadditivity (Theorem 5.10) hold. What makes the test ideal a true analog of the multiplier ideal is that under the process of reduction to positive characteristic the multiplier ideal  $\mathcal{J}(\mathfrak{a}^c)$  corresponds to the test ideal  $\tau(\mathfrak{a}^c)$ , or more precisely to the test ideal of the reduction mod  $p$  of  $\mathfrak{a}^c$ .

### 3. THE MULTIPLIER IDEAL OF MONOMIAL IDEALS

Even though multiplier ideals enjoy extremely good formal properties, they are very hard to compute in general. An important exception is the class of monomial ideals, whose multiplier ideals are described by a simple

<sup>5</sup>The tight closure analogues of these result can be found in [HH90], [Hun92] and [HH02], respectively.

<sup>6</sup>One feature of their theory is that there is no reference to resolutions of singularities. As a consequence no restriction on the singularity of  $R$  arises, whereas for multiplier ideals at least some sort of  $\mathbb{Q}$ -Gorenstein assumption is needed.

<sup>7</sup>Similarly as for tight closure,  $x \in I^{*\mathfrak{a}^c}$  if there is a  $h \neq 0$  such that for all  $q = p^e$  one has  $hx^q \mathfrak{a}^{\lceil qc \rceil} \subseteq I^{\lceil q \rceil}$ . Note that  $I^{\lceil q \rceil}$  denotes the ideal generated by all  $q$ th powers of the elements of  $I$ , whereas  $\mathfrak{a}^{\lceil qc \rceil}$  is the usual  $\lceil qc \rceil$ th power of  $\mathfrak{a}$ .

combinatorial formula, established by Howald [How01]. By way of illustration we discuss this result in detail.

To state the result let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be a monomial ideal, that is an ideal generated by monomials of the form  $x^m = x_1^{m_1} \cdot \dots \cdot x_n^{m_n}$  for  $m \in \mathbb{Z}^n \subseteq \mathbb{R}^n$ . In this way we can identify a monomial ideal  $\mathfrak{a}$  of  $k[x_1, \dots, x_n]$  with the set of exponents (contained in  $\mathbb{Z}^n$ ) of the monomials in  $\mathfrak{a}$ . The convex hull of this set in  $\mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R}$  is called the *Newton polytope* of  $\mathfrak{a}$  and it is denoted by  $\text{Newt}(\mathfrak{a})$ . Now Howald's result states:

**Theorem 3.1.** *Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be a monomial ideal. Then for every  $c > 0$ ,*

$$\mathcal{J}(\mathfrak{a}^c) = \langle x^m \mid m + (1, \dots, 1) \in \text{interior of } c \cdot \text{Newt}(\mathfrak{a}) \rangle$$

For example, the picture of the Newton polytope of the monomial ideal  $\mathfrak{a} = (x^4, xy^2, y^4)$  in Figure 3 shows, using Howald's result, that  $\mathcal{J}(\mathfrak{a}) = (x^2, xy, y^2)$ . Note that even though  $(0, 1) + (1, 1)$  lies in the Newton polytope  $\text{Newt}(\mathfrak{a})$  it does not lie in the interior. Therefore, the monomial  $y$  corresponding to  $(0, 1)$  does *not* lie in the multiplier ideal  $\mathcal{J}(\mathfrak{a})$ . But for all  $c < 1$ , clearly  $y \in \mathcal{J}(\mathfrak{a}^c)$ .

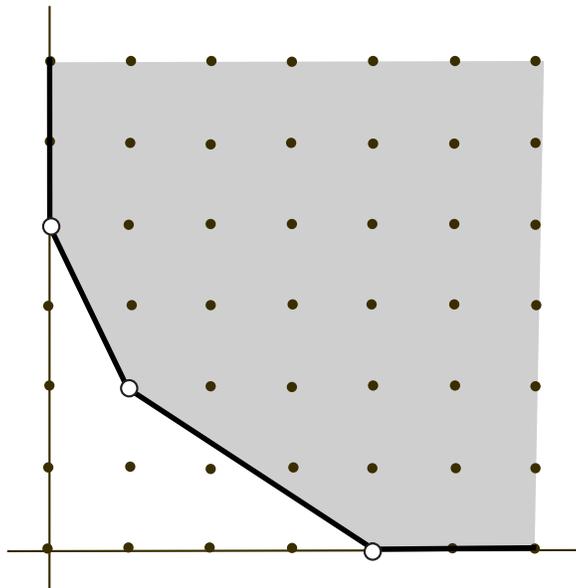


FIGURE 3. Newton polytope of  $(x^4, xy^2, y^4)$

To pave the way for clean proofs we need to formalize our setup slightly and recall some results from toric geometry.

**3.1. Basic notions from toric geometry.** Note that  $k[X] = k[x_1, \dots, x_n]$  carries a natural  $\mathbb{Z}^n$ -grading by giving a monomial  $x^m = x_1^{m_1} \cdot \dots \cdot x_n^{m_n}$  degree

$m \in \mathbb{Z}^n$ . Equivalently we note that the  $n$ -dimensional torus

$$T^n = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cong (k^*)^n$$

acts on  $k[X]$  via  $\lambda \cdot x^m = \lambda^m x^m$  for  $\lambda \in (k^*)^n$ . In terms of the varieties this means that  $X = \mathbb{A}^n$  contains the torus  $T^n$  as a dense open subset, and the action of  $T^n$  on itself naturally extends to an action of  $T^n$  on all of  $X$ . Under this action, the torus fixed ( $= \mathbb{Z}^n$ -graded) ideals are precisely the monomial ideals. We denote the lattice  $\mathbb{Z}^n$  in which the grading takes place by  $M$ . It is just the lattice of the exponents of the Laurent monomials of  $k[T^n]$ .

As indicated above, the Newton polytope  $\operatorname{Newt}(\mathfrak{a})$  of a monomial ideal  $\mathfrak{a}$  is just the convex hull in  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  of the set  $\{m \in M \mid x^m \in \mathfrak{a}\}$ . The Newton polytope of a principal ideal  $(x^v)$  is just the positive orthant in  $M_{\mathbb{R}}$  shifted by  $v$ . In general, the Newton polytope of any ideal is an unbounded region contained in the first orthant. With every point  $v$  the Newton polytope also contains the first orthant shifted by  $v$ .

**Exercise 3.2.** Let  $\mathfrak{a}$  be a monomial ideal in  $k[x_1, \dots, x_n]$ . Then the lattice points (viewed as exponents) in the Newton polytope  $\operatorname{Newt}(\mathfrak{a})$  of  $\mathfrak{a}$  define an ideal  $\bar{\mathfrak{a}} \supseteq \mathfrak{a}$ . Show that  $\bar{\mathfrak{a}}$  is the integral closure of  $\mathfrak{a}$  (see [Ful93]).

The property of  $X = \mathbb{A}^n$  to contain the torus  $T^n$  as a dense open set such that the action of  $T^n$  on itself extends to an action on  $X$  as just described is the definition of a *toric variety*. The language of toric varieties is the most natural to phrase, prove (and generalize, see [Bli]) Howald's result. To set this up completely would take us somewhat afield, so we choose to take a more direct approach using only a bare minimum of toric geometry.

A first fact we have to take without proof from the theory of toric varieties is that log resolutions of torus fixed ideals of  $k[X]$  exist in the category of toric varieties.<sup>8</sup>

**Theorem 3.3.** *Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be a monomial ideal. Then there is a log resolution  $\mu : Y \rightarrow X$  of  $\mathfrak{a}$  such that  $\mu$  is a map of toric varieties and consequently  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$  is such that  $F$  is fixed by the torus action on  $Y$ .*

*Indication of proof.* This follows from the theory of toric varieties. First one takes the normalized blowup of  $\mathfrak{a}$ , which is a (possibly singular) toric variety since  $\mathfrak{a}$  was a torus invariant ideal. Then one torically resolves the singularities of the resulting variety as described in [Ful93]. Note that this is a much easier task than resolution of singularities in general. It comes down to a purely combinatorial procedure.

An alternative proof could use Encinas and Villamayor's [EV00] equivariant resolution of singularities. They give an algorithmic procedure of constructing a log resolution of  $\mathfrak{a}$  such that the torus action is preserved — that is by only blowing up along torus fixed centers.  $\square$

<sup>8</sup>To be precise, a toric variety comes with the datum of the torus embedding  $T^n \subseteq X$ . Maps of toric varieties are such that they preserve the torus action.

3.1.1. *Toric Divisors.* A toric variety  $X$  has a finite set of torus fixed prime (Weil) divisors. Indeed, since an arbitrary torus fixed prime divisor cannot meet the torus ( $T^n$  acts transitively on itself and is dense in  $X$ ) it has to lie in the boundary  $Y - T^n$ , which is a variety of dimension  $\leq n - 1$  and thus can only contain finitely many components of dimension  $n - 1$ . Furthermore, these torus fixed prime divisors  $E_1, \dots, E_r$  generate the lattice of all torus fixed divisors which we shall denote by  $L^X$ . We denote the sum of all torus invariant prime divisors  $E_1 + \dots + E_r$  by  $1_X$ .

The torus invariant rational functions of a toric variety are just the Laurent monomials  $x_1^{m_1} \cdot \dots \cdot x_n^{m_n} \in k[T^n]$ . For the toric variety  $X = \mathbb{A}^n$  one clearly has the identification of  $M$ , the lattice of exponents, with  $L^X$  by sending  $m$  to  $\operatorname{div} x^m$ . In general this map will not be surjective and its image is precisely the set of torus invariant Cartier divisors. We note the following easy lemma which will nevertheless play an important role in our proof of Theorem 3.1. It makes precise the idea that a log resolution of a monomial ideal  $\mathfrak{a}$  corresponds to turning its Newton polytope  $\operatorname{Newt}(\mathfrak{a}) \subseteq M_{\mathbb{R}}$  into a translate of the first orthant in  $L_{\mathbb{R}}^X$ .

**Lemma 3.4.** *Let  $\mu : Y \rightarrow X = \operatorname{Spec} k[x_1, \dots, x_n]$  be a toric resolution of the monomial ideal  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ . Then, for  $m \in M$  we have*

$$c \cdot m \in c' \operatorname{Newt}(\mathfrak{a}) \iff c \cdot \mu^* \operatorname{div} x^m \geq c' \cdot F$$

for all rational  $c, c' > 0$ .

*Proof.* We first show the case  $c = c' = 1$ . Assume that  $m \in \operatorname{Newt}(\mathfrak{a})$ . By Exercise 3.2, this is equivalent to  $x^m \in \bar{\mathfrak{a}}$ , the integral closure of  $\mathfrak{a}$ . Since, by Exercise 2.7,  $\bar{\mathfrak{a}} = \mu_* \mathcal{O}_Y(-F)$  it follows that  $x^m \in \bar{\mathfrak{a}}$  if and only if  $\mu^* x^m \in \mathcal{O}_Y(-F)$ . This, finally, is equivalent to  $\mu^*(\operatorname{div} x^m) \geq F$ .

For the general case express  $c$  and  $c'$  as integer fractions. Clearing denominators and noticing that for an integer  $a$  one has  $a \operatorname{Newt}(\mathfrak{a}) = \operatorname{Newt}(\mathfrak{a}^a)$  one reduces to the previous case.  $\square$

3.1.2. *Canonical divisor.* As the final ingredient for computing the multiplier ideal we need an understanding of the canonical divisor (class) of a toric variety.

**Lemma 3.5.** *Let  $X$  be a (smooth) toric variety and let  $E_1, \dots, E_r$  denote the collection of all torus invariant prime Weil divisors. Then the canonical divisor is  $K_Y = -\sum E_i = -1_X$ .*

We leave the proof as an exercise or alternatively refer to [Ful93] or [Dan78] for this basic result. We verify it for  $X = \mathbb{A}^n$ . Then  $E_i = (x_i = 0)$  for  $i = 1, \dots, n$  are the torus invariant divisors and  $K_X$  is represented by the divisor of the  $T^n$ -invariant rational  $n$ -form  $\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$ , which is  $-(E_1 + \dots + E_n)$ . As a consequence of the last lemma we get the following lemma.

**Lemma 3.6.** *Let  $\mu : Y \rightarrow X = \mathbb{A}^n$  be a birational map of (smooth) toric varieties. Then  $K_{Y/X} = \mu^*1_X - 1_Y$  and the support of  $\mu^*1_X$  is equal to the support of  $1_Y$ .*

*Proof.* As the strict transform of a torus invariant divisor on  $X$  is a torus invariant divisor on  $Y$  it follows that  $\mu^*1_X - 1_Y$  is supported on the exceptional locus of  $\mu$ . Since  $-1_X$  represents the canonical class  $K_X$  and respectively for  $Y$ , the first assertion follows from the definition of  $K_{Y/X}$ . Since  $\mu^*1_X$  is torus invariant clearly its support is included in  $1_Y$ . Since  $\mu$  is an isomorphism over the torus  $T^n \subseteq X$  it follows that  $\mu^{-1}(1_X) \supseteq 1_Y$  which implies the second assertion.  $\square$

**Exercise 3.7.** This exercise shows how to avoid taking Lemma 3.5 on faith but instead using a result of Russel Goward [Gow02] which states that a log resolution of a monomial ideal can be obtained by a sequence of monomial blowups.

A *monomial blowup*  $Y = Bl_Z(Y)$  of  $\mathbb{A}^n$  is the blowing up of  $\mathbb{A}^n$  at the intersection  $Z$  of some of the coordinate hyperplanes  $E_i = (x_i = 0)$  of  $\mathbb{A}^n$ .

For such monomial blowup  $\mu : Y = Bl_Z(X) \rightarrow X \cong \mathbb{A}^n$  show that  $Y$  is a smooth toric variety which is canonically covered by  $\text{codim}(Z, X)$  many  $\mathbb{A}^n$  patches. Show that  $1_Y = E_1 + \dots + E_n + E$  where  $E$  is the exceptional divisor of  $\mu$ . Via a direct calculation verify the assertions of the last two lemmata for  $Y$ .

Since a monomial blowup is canonically covered by affine spaces one can repeat the process and obtains the notion of a *sequence of monomial blowups*. Using Goward's result show directly that a monomial ideal has a toric log resolution  $\mu : Y \rightarrow \mathbb{A}^n$  with the properties as in Lemma 3.6.

**3.2. Proof of Theorem 3.1.** By the existence of a toric (or equivariant) log resolution of a monomial ideal  $\mathfrak{a}$  it follows immediately that the multiplier ideal  $\mathcal{J}(\mathfrak{a}^c)$  is also generated by monomials. Thus, in order to determine  $\mathcal{J}(\mathfrak{a}^c)$  it is enough to decide which monomials  $x^m$  lie in  $\mathcal{J}(\mathfrak{a}^c)$ . With our preparations this now an easy task.

*Proof of Theorem 3.1.* As usual we denote  $\text{Spec } k[x_1, \dots, x_n]$  by  $X$  and let  $\mu : Y \rightarrow X$  be a toric log resolution of  $\mathfrak{a}$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ .

Abusing notation by identifying  $\text{div}(x_1 \dots x_n) = 1_X \in L^X$  with  $(1, \dots, 1) \in M$ , the condition of the theorem that  $m + 1_X$  is in the interior of the Newton polytope  $c \cdot \text{Newt}(\mathfrak{a})$  is equivalent to

$$m + 1_X - \varepsilon 1_X \in c \cdot \text{Newt}(\mathfrak{a})$$

for small enough rational  $\varepsilon > 0$ . By Lemma 3.4 this holds if and only if

$$\mu^* \text{div } g + \mu^* 1_X - \varepsilon \mu^* 1_X \geq c F.$$

Using the formula  $K_{Y/X} = \mu^* 1_X - 1_Y$  from Lemma 3.5 this is furthermore equivalent to

$$\mu^* \text{div } g + K_{Y/X} + [1_Y - \varepsilon \mu^* 1_X - c F] \geq 0$$

for sufficiently small  $\varepsilon > 0$ . Since by Lemma 3.6,  $\mu^*1_X$  is effective with the same support as  $1_Y$  it follows that all coefficients appearing in  $1_Y - \varepsilon\mu^*1_X$  are very close to but strictly smaller than 1 for small  $\varepsilon > 0$ . Therefore,  $\lfloor 1_Y - \varepsilon\mu^*1_X - cF \rfloor = \lfloor -cF \rfloor = -\lfloor cF \rfloor$ . Thus we can finish our chain of equivalences with

$$\mu^* \operatorname{div} g \geq -K_{Y/X} + \lfloor cF \rfloor$$

which says nothing but that  $g \in \mathcal{J}(\mathfrak{a}^c)$ .  $\square$

This formula for the multiplier ideal of a monomial ideal will be applied in the next section to concretely compute certain invariants arising from multiplier ideals.

#### 4. INVARIANTS ARISING FROM MULTIPLIER IDEALS AND APPLICATIONS

We keep the notation of a smooth affine variety  $X$  over an algebraically closed field of characteristic zero, and an ideal  $\mathfrak{a} \subseteq k[X]$ . In this section we use multiplier ideals to attach some invariants to  $\mathfrak{a}$ , and we study their influence on some algebraic questions.

**4.1. The log canonical threshold.** If  $c > 0$  is very small, then  $\mathcal{J}(\mathfrak{a}^c) = k[X]$ . For large  $c$ , on the other hand, the multiplier ideal  $\mathcal{J}(\mathfrak{a}^c)$  is clearly nontrivial. This leads one to define:

**Definition 4.1.** The *log canonical threshold* of  $\mathfrak{a}$  is the number

$$\operatorname{lct}(\mathfrak{a}) = \operatorname{lct}(X, \mathfrak{a}) = \inf\{c > 0 \mid \mathcal{J}(\mathfrak{a}^c) \neq \mathcal{O}_X\}.$$

The following exercise shows that  $\operatorname{lct}(\mathfrak{a})$  is a rational number, and that the infimum appearing in the definition is actually a minimum. Consequently, the log canonical threshold is just the smallest  $c > 0$  such that  $\mathcal{J}(\mathfrak{a}^c)$  is nontrivial.

**Exercise 4.2.** As usual, fixing notation of a log resolution  $\mu : Y \rightarrow X$  with  $\mathfrak{a} \cdot \mathcal{O}_Y = \sum r_i E_i$  and  $K_{Y/X} = \sum b_i E_i$ , show that  $\operatorname{lct}(X, \mathfrak{a}) = \min\{\frac{b_i+1}{r_i}\}$ .

Recall the notions from singularity theory [Kol97] in which a pair  $(X, \mathfrak{a}^c)$  is called *log terminal* if and only if  $b_i - cr_i + 1 > 0$  for all  $i$ . It is called *log canonical* if and only if  $b_i - cr_i + 1 \geq 0$  for all  $i$ . The last exercise also shows that  $(X, \mathfrak{a}^c)$  is log terminal if and only if the multiplier ideal  $\mathcal{J}(\mathfrak{a}^c)$  is trivial.

**Example 4.3.** Continuing previous examples we observe that  $\operatorname{lct}((x^2, y^2)) = 1$  and  $\operatorname{lct}((x^2, y^3)) = \frac{5}{6}$ .

**Example 4.4** (The log canonical threshold of a monomial ideal). The formula for the multiplier ideal of a monomial ideal  $\mathfrak{a}$  on  $X = \operatorname{Spec} k[x_1, \dots, x_n]$  shows that  $\mathcal{J}(\mathfrak{a}^c)$  is trivial if and only if  $1_X = (1, \dots, 1)$  is in the interior of the Newton polytope  $c \operatorname{Newt}(\mathfrak{a})$ . This allows to compute the log canonical threshold of  $\mathfrak{a}$ :  $\operatorname{lct}(\mathfrak{a})$  is the largest  $t > 0$  such that  $1_X \in t \cdot \operatorname{Newt}(\mathfrak{a})$ .

**Example 4.5.** As a special case of the previous example, take

$$\mathbf{a} = (x_1^{a_1}, \dots, x_n^{a_n}).$$

Then the Newton polytope is the subset of the first orthant consisting of points  $(v_1, \dots, v_n)$  satisfying  $\sum \frac{v_i}{a_i} \geq 1$ . Therefore  $1_X \in t \cdot \text{Newt}(\mathbf{a})$  if and only if  $\sum \frac{1}{a_i} \geq t$ . In particular,  $\text{lct}(\mathbf{a}) = \sum \frac{1}{a_i}$ .

**4.2. Jumping numbers.** The log-canonical threshold measures the triviality or non-triviality of a multiplier ideal. By using the full algebraic structure of these ideals, it is natural to see this threshold as merely the first of a sequence of invariants. These so-called jumping numbers were first considered (at least implicitly) in work of Libgober, Loeser and Vaquié ([Lib83], [LV90]). They are studied more systematically in the paper [ELSV].

We start with a lemma:

**Lemma 4.6.** *For  $\mathbf{a} \subseteq \mathcal{O}_X$ , there is an increasing discrete sequence of rational numbers*

$$0 = \xi_0 < \xi_1 < \xi_2 < \dots$$

*such that  $\mathcal{J}(\mathbf{a}^c)$  is constant for  $\xi_i \leq c < \xi_{i+1}$  and  $\mathcal{J}(\mathbf{a}^{\xi_i}) \not\supseteq \mathcal{J}(\mathbf{a}^{\xi_{i+1}})$ .*

We leave the (easy) proof to the reader.

The  $\xi_i = \xi_i(\mathbf{a})$  are called the *jumping numbers* or *jumping coefficients* of  $\mathbf{a}$ . Referring to the log resolution  $\mu$  appearing in Example 4.2, note that the only candidates for jumping numbers are those  $c$  such that  $cr_i$  is an integer for some  $i$ . Clearly the first jumping number  $\xi_1(\mathbf{a})$  is the log canonical threshold  $\text{lct}(\mathbf{a})$ .

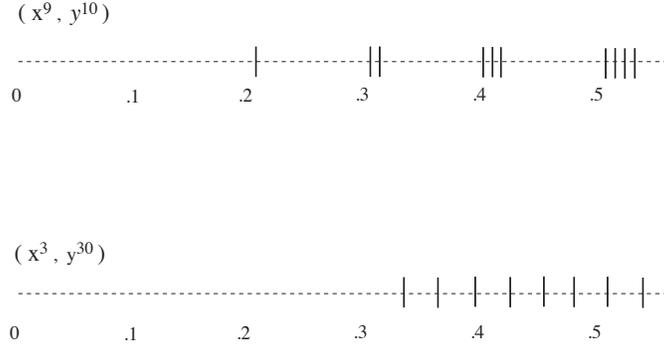
**Example 4.7** (Jumping numbers of monomial ideals). Let  $\mathbf{a} \subseteq k[x_1, \dots, x_n]$  be a monomial ideal. For the multiplier ideal  $\mathcal{J}(\mathbf{a}^c)$  to jump at  $c = \xi$ , it is equivalent that some monomial, say  $x^v$ , is in  $\mathcal{J}(\mathbf{a}^\xi)$  but not in  $\mathcal{J}(\mathbf{a}^{\xi-\varepsilon})$  for all  $\varepsilon > 0$ . Thus, the largest  $\xi > 0$  such that  $v + (1, \dots, 1) \in \xi \text{Newt}(\mathbf{a})$  is a jumping number. Doing this construction for all  $v \in \mathbb{N}^n$  one obtains all jumping numbers of  $\mathbf{a}$  (this uses the fact that the multiplier ideal of a monomial ideal is a monomial ideal).

**Exercise 4.8.** Consider again  $\mathbf{a} = (x_1^{a_1}, \dots, x_n^{a_n})$ . Then the jumping numbers of  $\mathbf{a}$  are precisely the rational numbers of the form

$$\frac{v_1+1}{a_1} + \dots + \frac{v_n+1}{a_n}$$

where  $(v_1, \dots, v_n)$  ranges over  $\mathbb{N}^n$ . Note however that different vectors  $(v_1, \dots, v_n)$  may give the same jumping number.

It is instructive to picture the jumping numbers of an ideal graphically. Figure 4, taken from [ELSV], shows the jumping numbers of the two ideals  $(x^9, y^{10})$  and  $(x^3, y^{30})$ : the exponents are chosen so that the two ideals have the same Samuel multiplicity, and so that the pictured jumping coefficients occur “with multiplicity one” (in a sense whose meaning we leave to the reader).

FIGURE 4. Jumping numbers of  $(x^9, y^{10})$  and  $(x^3, y^{30})$ 

**4.3. Jumping length.** Jumping numbers give rise to an additional invariant in the case of principal ideals.

**Lemma 4.9.** *Let  $f \in k[X]$  be a non-zero function. Then  $\mathcal{J}(f) = (f)$  but for  $c < 1$  one has  $(f) \subsetneq \mathcal{J}(f^c)$ . In other words,  $\xi = 1$  is a jumping number of the principal ideal  $(f)$ .*

Deferring the proof for a moment, we note that the Lemma means that  $\xi_\ell(f) = 1$  for some index  $\ell$ . We define  $\ell = \ell(f)$  to be the *jumping length* of  $f$ . Thus  $\ell(f)$  counts the number of jumping coefficients of  $(f)$  that are  $\leq 1$ .

**Example 4.10.** Let  $f = x^4 + y^3 \in \mathbb{C}[x, y]$ . One can show that  $f$  is sufficiently generic so that  $\mathcal{J}(f^c) = \mathcal{J}((x^4, y^3)^c)$  provided that  $c < 1$ . Therefore the first few jumping numbers of  $f$  are

$$0 < \text{lct}(f) = \frac{1}{4} + \frac{1}{3} < \frac{2}{4} + \frac{1}{3} < \frac{1}{4} + \frac{2}{3} < 1,$$

and  $\ell(f) = 4$ .

*Proof of Lemma 4.9.* Let  $\mu : Y \rightarrow X$  be a log resolution of  $(f)$  and denote the integral divisor  $(f = 0)$  by  $D = \sum a_i D_i$ . Clearly,  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\mu^* D)$  and  $\mu^* D$  is also an integral divisor. Thus

$$\begin{aligned} \mathcal{J}(f) &= \mu_* \mathcal{O}_Y(K_{Y/X} - \mu^* D) \\ &= \mu_* (\mathcal{O}_Y(K_{Y/X}) \otimes \mu^* \mathcal{O}_X(-D)) \\ &= \mathcal{O}_X \otimes \mathcal{O}_X(-D) \\ &= (f). \end{aligned}$$

On the other hand, choose a general point  $x \in D_i$  on any of the components of  $D = \text{div}(f) = \sum a_i D_i$ . Then  $\mu$  is an isomorphism over  $x$  and consequently

$$\text{ord}_{D_i}(\mathcal{J}(f^c)) < a_i \text{ for } 0 < c < 1.$$

Therefore  $\mathcal{J}((f)^c) \subsetneq (f)$  whenever  $c < 1$ .  $\square$

Finally, we note that the jumping length can be related to other invariants of the singularities of  $f$ :

**Proposition 4.11** ([ELSV]). *Assume the hypersurface defined by the vanishing of  $f$  has at worst an isolated singularity at  $x \in X$ . Then*

$$\ell(f) \leq \tau(f, x) + 1,$$

where  $\tau(f, x)$  is the Tjurina number of  $f$  at  $x$ , defined as the colength in  $\mathcal{O}_{x, X}$  of  $(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$  for  $z_1, \dots, z_n$  parameters around  $x$ .

**4.4. Application to uniform Artin-Rees numbers.** We next discuss a result relating jumping lengths to uniform Artin-Rees numbers of a principal ideal.

To set the stage, recall the statement of the Artin-Rees lemma in a simple setting:

**Theorem** (Artin-Rees). *Let  $\mathfrak{b}$  be an ideal and  $f$  an element of  $k[X]$ . There exists an integer  $k = k(f, \mathfrak{b})$  such that*

$$\mathfrak{b}^m \cap (f) \subseteq \mathfrak{b}^{m-k} \cdot (f)$$

for all  $m \geq k$ . In other words, if  $fg \in \mathfrak{b}^m$  then  $g \in \mathfrak{b}^{m-k}$ .

Classically,  $k$  is allowed to depend both on  $\mathfrak{b}$  and  $f$ . However in his paper [Hun92], Huneke showed that in fact there is a single integer  $k = k(f)$  which works simultaneously for all ideals  $\mathfrak{b}$ .<sup>9</sup> Any such  $k$  is called a *uniform Artin-Rees number* of  $f$ .

The next result shows that the jumping length gives an effective estimate (of moderate size!) for uniform Artin-Rees numbers.

**Theorem 4.12** ([ELSV]). *As above, write  $\ell(f)$  for the jumping length of  $f$ . Then the integer  $k = \ell(f) \cdot \dim X$  is a uniform Artin-Rees number of  $f$ .*

If  $f$  defines a smooth hypersurface, its jumping length is 1 and it follows that  $n = \dim X$  is a uniform Artin-Rees number in this case. (In fact, Huneke showed that  $n - 1$  also works in this case.)

If  $f$  defines a hypersurface with only an isolated singular point  $x \in X$ , it follows from Proposition 4.11 and the Theorem that  $k = n \cdot (\tau(f, x) + 1)$  is a uniform Artin-Rees number. (One can show using the next Lemma and some observations of Huneke that in fact  $k = \tau(f, x) + n$  also works: see [ELSV, §3].)

The essential input to Theorem 4.12 is a statement involving consecutive jumping coefficients:

**Lemma 4.13.** *Consider two consecutive jumping numbers*

$$\xi = \xi_i(f) < \xi_{i+1}(f) = \xi'$$

of  $f$ , and let  $\mathfrak{b} \subseteq k[X]$  be any ideal. Then given a natural number  $m > n = \dim X$ , one has

$$\mathfrak{b}^m \cdot \mathcal{J}(f^\xi) \cap \mathcal{J}(f^{\xi'}) \subseteq \mathfrak{b}^{m-n} \cdot \mathcal{J}(f^{\xi'}).$$

---

<sup>9</sup>We stress that both the classical Artin-Rees Lemma and Huneke's theorem are valid in a much more general setting.

We will deduce this from Skoda's theorem in the next section. In the meantime, we observe that it leads immediately to the

*Proof of Theorem 4.12.* We apply the previous Lemma repetitively to successive jumping numbers in the chain of multiplier ideals

$$k[X] = \mathcal{J}(f^0) \supsetneq \mathcal{J}(f^{\xi_1}) \supsetneq \mathcal{J}(f^{\xi_2}) \supsetneq \dots \supsetneq \mathcal{J}(f^{\xi_\ell}) = \mathcal{J}(f) = (f).$$

After further intersection with  $(f)$  one finds:

$$\begin{aligned} \mathfrak{b}^m \cap (f) &\subseteq \mathfrak{b}^{m-n} \cdot \mathcal{J}(f^{\xi_1}) \cap (f) \\ &\subseteq \mathfrak{b}^{m-2n} \cdot \mathcal{J}(f^{\xi_2}) \cap (f) \\ &\dots \\ &\subseteq \mathfrak{b}^{m-\ell n}(f), \end{aligned}$$

as required.  $\square$

**Remark 4.14.** When  $\mathfrak{a} = (f)$  is a principal ideal, the jumping numbers of  $f$  are related to other invariants appearing in the literature. In particular, if  $f$  has an isolated singularity then (suitable translates of) the jumping coefficients appear in the Hodge-theoretically defined *spectrum* of  $f$ . See [ELSV, §5] for precise statements and references.

## 5. FURTHER LOCAL PROPERTIES OF MULTIPLIER IDEALS

In this section we discuss some results involving the local behavior of multiplier ideals. We start with Skoda's theorem and some variants. Then we discuss the restriction and subadditivity theorems, which will be used in the next section.

**5.1. Skoda's theorem.** An important (and early) example of a uniform result in local algebra was established by Skoda and Briançon [SB74] using analytic results of Skoda [Sko72]. In our language, Skoda's result is this:

**Theorem 5.1** (Skoda's Theorem, I). *Consider any ideal  $\mathfrak{b} \subseteq k[X]$  with  $X$  smooth of dimension  $n$ . Then for all  $m \geq n$*

$$\mathcal{J}(\mathfrak{b}^m) = \mathfrak{b} \cdot \mathcal{J}(\mathfrak{b}^{m-1}) = \dots = \mathfrak{b}^{m+1-n} \cdot \mathcal{J}(\mathfrak{b}^{n-1}).$$

**Remark 5.2.** As Hochster noted in his lectures, the statement in [Sko72] has a more analytic flavor. In fact, using the analytic interpretation of multiplier ideals (§2.4) one sees that (the analytic analogue of) Theorem 5.1 is essentially equivalent to the following statement.

Suppose that  $\mathfrak{b}$  is generated by  $(g_1, \dots, g_t)$ , and that  $f$  is a holomorphic function such that

$$\int \frac{|f|^2}{(\sum |g_i|^2)^m} < \infty$$

for some  $m \geq n = \dim X$ . Then locally there exist holomorphic functions  $h_i$  such that  $f = \sum h_i g_i$ , and moreover each of the  $h_i$  satisfies the local integrability condition  $\int \frac{|h_i|^2}{(\sum |g_i|^2)^{m-1}} < \infty$ .

(The hypothesis expresses the membership of  $f$  in  $\mathcal{J}(\mathfrak{b}^m)^{\text{an}}$  and the conclusion writes  $f$  as belonging to  $\mathfrak{b}^{\text{an}} \cdot \mathcal{J}(\mathfrak{b}^{m-1})^{\text{an}}$ .)

As a corollary of Skoda's theorem, one obtains the classical theorem of Briançon-Skoda.

**Corollary 5.3** (Briançon-Skoda). *With the notation as before,*

$$\overline{\mathfrak{b}^m} \subseteq \mathcal{J}(\mathfrak{b}^m) \subseteq \mathfrak{b}^{m+1-n}$$

where  $\overline{\phantom{x}}$  denotes the integral closure and  $n = \dim X$ .

*Sketch of proof of Theorem 5.1.* The argument follows ideas of Teissier and Lipman. We choose generators  $g_1, \dots, g_k$  for the ideal  $\mathfrak{b}$  and fix a log resolution  $\mu : Y \rightarrow X$  of  $\mathfrak{b}$  with  $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ . Write  $g'_i = \mu^*(g_i) \in \Gamma(Y, \mathcal{O}_Y(-F))$  to define the surjective map

$$(2) \quad \bigoplus_{i=0}^k \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-F)$$

by sending  $(x_1, \dots, x_k)$  to  $\sum x_i g'_i$ . Tensoring this map with  $\mathcal{O}_Y(K_{Y/X} - (m-1)F)$  yields the surjection

$$\bigoplus_{i=1}^k \mathcal{O}_Y(K_{Y/X} - (m-1)F) \xrightarrow{\varphi} \mathcal{O}_Y(K_{Y/X} - mF).$$

Further applying  $\mu_*$  we get the map  $\bigoplus_{i=0}^k \mathcal{J}(\mathfrak{b}^{m-1}) \xrightarrow{\mu_* \varphi} \mathcal{J}(\mathfrak{b}^m)$  which again sends a tuple  $(y_1, \dots, y_k)$  to  $\sum y_i g_i$ . Therefore, the image of  $\mu_*(\varphi)$  is

$$\text{Image}(\mu_* \varphi) = \mathfrak{b} \mathcal{J}(\mathfrak{b}^{m-1}) \subseteq \mathcal{J}(\mathfrak{b}^m).$$

What remains to show is that  $\mu_* \varphi$  is surjective. For this consider the Koszul complex on the  $g'_i$  on  $Y$  which resolves the map in (2).

$$\begin{aligned} 0 \rightarrow \mathcal{O}_Y((k-1)F) \rightarrow \bigoplus^k \mathcal{O}_Y((k-2)F) \rightarrow \dots \\ \dots \rightarrow \bigoplus^{\binom{k}{2}} \mathcal{O}_Y(F) \rightarrow \bigoplus^k \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-F) \rightarrow 0. \end{aligned}$$

As above, tensor through by  $\mathcal{O}_Y(K_{Y/X} - (m-1)F)$  to get a resolution of  $\varphi$ . Local vanishing (Theorem 2.17) applies to the  $m \geq n = \dim X$  terms on the right. Chasing through the sequence while taking direct images then gives the required surjectivity. See [Laz, Chapter 9] or [EL99] for details.  $\square$

It will be useful to have a variant involving several ideals and fractional coefficients. For this we extend slightly the definition of multiplier ideals.

5.1.1. *Mixed multiplier ideals.* Fix a sequence of ideals  $\mathfrak{a}_1, \dots, \mathfrak{a}_t$  and positive rational numbers  $c_1, \dots, c_t$ . Then we define the multiplier ideal

$$\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_t^{c_t})$$

starting with a log resolution  $\mu : Y \rightarrow X$  of the product  $\mathfrak{a}_1 \cdots \mathfrak{a}_t$ . Since this is at the same time also a log resolution of each  $\mathfrak{a}_i$  write  $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$  for simple normal crossing divisors  $F_i$ .

**Definition 5.4.** With the notation as indicated, the mixed multiplier ideal is

$$\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_t^{c_t}) = \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor c_1 F_1 + \dots + c_t F_t \rfloor)).$$

As before, this definition is independent of the chosen log resolution.

Note that once again we do not attempt to assign any meaning to the expression  $\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_t^{c_t}$  in the argument of  $\mathcal{J}$ . This expression is meaningful a priori whenever all  $c_i$  are positive integers and our definition is consistent with this prior meaning.

With this generalization of the concept of multiplier ideals we get the following variant of Skoda's theorem.

**Theorem 5.5** (Skoda's Theorem, II). *For every integer  $c \geq n = \dim X$  and any  $d > 0$  one has*

$$\mathcal{J}(\mathfrak{a}_1^c \cdot \mathfrak{a}_2^d) = \mathfrak{a}_1^{c-(n-1)} \mathcal{J}(\mathfrak{a}_1^{n-1} \cdot \mathfrak{a}_2^d).$$

The proof of this result is only a technical complication of the proof of the First Version we discussed above. We refer to [Laz, Chapter 9] for details.

We conclude by using Skoda's Theorem to prove (a slight generalization of) the Lemma 4.13 underlying the results on uniform Artin-Rees numbers in the previous section.

**Lemma 5.6.** *Let  $\mathfrak{a} \subseteq k[X]$  be an ideal and let  $\xi < \xi'$  be consecutive jumping numbers of  $\mathfrak{a}$ . Then for  $m > n$  we have*

$$\mathfrak{b}^m \cdot \mathcal{J}(\mathfrak{a}^\xi) \cap \mathcal{J}(\mathfrak{a}^{\xi'}) \subseteq \mathfrak{b}^{m-n} \cdot \mathcal{J}(\mathfrak{a}^{\xi'})$$

for all ideals  $\mathfrak{b} \subseteq k[X]$ .

*Proof.* We first claim that

$$\mathfrak{b}^m \mathcal{J}(\mathfrak{a}^\xi) \cap \mathcal{J}(\mathfrak{a}^{\xi'}) \subseteq \mathcal{J}(\mathfrak{b}^{m-1} \cdot \mathfrak{a}^{\xi'}).$$

This is shown via a simple computation. In fact, to begin with one can replace  $\xi$  by  $c \in [\xi, \xi')$  arbitrarily close to  $\xi'$  since this does not change the statement. Let  $\mu : Y \rightarrow X$  be a common log resolution of  $\mathfrak{a}$  and  $\mathfrak{b}$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$  and  $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B)$ . Let  $E$  be a prime divisor on  $Y$  and denote by  $a, b$  and  $e$  the coefficient of  $E$  in  $A, B$  and  $K_{Y/X}$ , respectively. Then  $f$  is in the left-hand side if and only if

$$\text{ord}_E f \geq \max(-e + mb + \lfloor ca \rfloor, -e + \lfloor \xi' a \rfloor).$$

If  $b = 0$  this implies that  $\text{ord}_E f \geq -e + (m-1)b + \lfloor \xi' a \rfloor$ . If  $b \neq 0$  then  $b$  is a positive integer  $\geq 1$ . Since  $c$  is arbitrarily close to  $\xi'$  we get  $\lfloor \xi' a \rfloor -$

$b \leq \lfloor \xi' a \rfloor - 1 \leq \lfloor ca \rfloor$ . Adding  $-e + mb$  it follows that also in this case  $\text{ord}_E f \geq -e + (m-1)b + \lfloor \xi' a \rfloor$ . Since this holds for all  $E$  it follows that  $f \in \mathcal{J}(\mathfrak{b}^{m-1} \cdot \mathfrak{a}^{\xi'})$ .

Now, using Theorem 5.5 we deduce

$$\mathcal{J}(\mathfrak{b}^{m-1} \cdot \mathfrak{a}^{\xi'}) \subseteq \mathfrak{b}^{m-n} \mathcal{J}(\mathfrak{b}^{n-1} \cdot \mathfrak{a}^{\xi'}) \subseteq \mathfrak{b}^{m-n} \mathcal{J}(\mathfrak{a}^{\xi'}).$$

Putting all the inclusions together, the Lemma follows.  $\square$

**Exercise 5.7.** Let  $\mathfrak{a} \subseteq k[X]$  be an ideal. Starting at  $\dim X - 1$ , the jumping numbers are periodic with period 1. That is,  $\xi \geq \dim X - 1$  is a jumping number if and only if  $\xi + 1$  is a jumping number.

**5.2. Restriction theorem.** The next result deals with restrictions of multiplier ideals. Consider a smooth subvariety  $Y \subseteq X$  and an ideal  $\mathfrak{b} \subseteq k[X]$  which does not vanish on  $Y$ . There are then two ways to get an ideal on  $Y$ . First, one can compute the multiplier ideal  $\mathcal{J}(X, \mathfrak{b}^c)$  on  $X$  and then restrict it to  $Y$ . On the other hand, one can also restrict  $\mathfrak{b}$  to  $Y$  and then compute the multiplier ideal on  $Y$  of this restricted ideal. The Restriction Theorem – which is arguably the most important local property of multiplier ideals – states that there is always an inclusion among these ideals on  $Y$ .

**Theorem 5.8** (Restriction Theorem). *Let  $Y \subseteq X$  be a smooth subvariety of  $X$  and  $\mathfrak{b}$  an ideal of  $k[X]$  such that  $Y$  is not contained in the zero locus of  $\mathfrak{b}$ . Then*

$$\mathcal{J}(Y, (\mathfrak{b} \cdot k[Y])^c) \subseteq \mathcal{J}(X, \mathfrak{b}^c) \cdot k[Y].$$

One can think of the Theorem as reflecting the principle that singularities can only get worse under restriction.

In the present setting, the result is due to Esnault and Viehweg [EV92, Proposition 7.5] When  $Y$  is a hypersurface, the statement is proved using the local vanishing theorem 2.17. Since in any event a smooth subvariety is a local complete intersection, the general case then follows from this.

**Exercise 5.9.** Give an example where strict inclusion holds in the Theorem.

**5.3. Subadditivity theorem.** We conclude with a result due to Demailly, Ein and the second author [DEL00] concerning the multiplicative behavior of multiplier ideals. This subadditivity theorem will be used in the next section to obtain some uniform bounds on symbolic powers of ideals.

**Theorem 5.10** (Subadditivity). *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals in  $k[X]$ . Then for all  $c, d > 0$  one has*

$$\mathcal{J}(\mathfrak{a}^c \cdot \mathfrak{b}^d) \subseteq \mathcal{J}(\mathfrak{a}^c) \cdot \mathcal{J}(\mathfrak{b}^d).$$

*In particular, for every positive integer  $m$ ,  $\mathcal{J}(\mathfrak{a}^{cm}) \subseteq \mathcal{J}(\mathfrak{a}^c)^m$ .*

*Sketch of proof.* The idea of the proof is to pull back the data to the product  $X \times X$  and then to restrict to the diagonal  $\Delta$ . Specifically, assume for

simplicity that  $c = d = 1$ , and consider the product

$$\begin{array}{ccc} & X \times X & \\ & \swarrow \quad \searrow & \\ X & \xleftarrow{p_1} & X \end{array}$$

along with its projections as indicated. For log resolutions  $\mu_1$  and  $\mu_2$  of  $\mathfrak{a}$  and  $\mathfrak{b}$  respectively one can verify that  $\mu_1 \times \mu_2$  is a log resolution of the ideal  $p_1^{-1}(\mathfrak{a}) \cdot p_2^{-1}(\mathfrak{b})$  on  $X \times X$ . Using this one shows that

$$\mathcal{J}(X \times X, p_1^{-1}(\mathfrak{a}) \cdot p_2^{-1}(\mathfrak{b})) = p_1^{-1} \mathcal{J}(X, \mathfrak{a}) \cdot p_2^{-1} \mathcal{J}(X, \mathfrak{b}).$$

Now let  $\Delta \subseteq X \times X$  be the diagonal. Apply the Restriction Theorem 5.8 with  $Y = \Delta$  to conclude

$$\begin{aligned} \mathcal{J}(X, \mathfrak{a} \cdot \mathfrak{b}) &= \mathcal{J}(\Delta, p_1^{-1}(\mathfrak{a}) \cdot p_2^{-1}(\mathfrak{b}) \cdot \mathcal{O}_\Delta) \\ &\subseteq \mathcal{J}(X \times X, p_1^{-1}(\mathfrak{a}) \cdot p_2^{-1}(\mathfrak{b})) \cdot \mathcal{O}_\Delta \\ &= \mathcal{J}(X, \mathfrak{a}) \cdot \mathcal{J}(X, \mathfrak{b}), \end{aligned}$$

as required.  $\square$

## 6. ASYMPTOTIC CONSTRUCTIONS

There are many natural situations in geometry and algebra where one is forced to confront rings or algebras that fail to be finitely generated. For example, if  $D$  is a non-ample divisor on a projective variety  $V$ , then the section ring  $R(V, D) = \bigoplus \Gamma(V, \mathcal{O}_V(mD))$  is typically not finitely generated. Or likewise, if  $\mathfrak{q}$  is a radical ideal in some ring, the symbolic blow-up algebra  $\bigoplus \mathfrak{q}^{(m)}$  likewise fails to be finitely generated in general. It is nonetheless possible to extend the theory of multiplier ideals to such settings. It turns out that there is finiteness built into the resulting multiplier ideals that may not be present in the underlying geometry or algebra. This has led to some of the most interesting applications of the theory.

In the geometric setting, the asymptotic constructions have been known for some time, but it was only with Siu's work [Siu98] on deformation-invariance of plurigenera that their power became clear. Here we focus on an algebraic formulation of the theory from [ELS01]. As before, we work with a smooth affine variety  $X$  defined over an algebraically closed field  $k$  of characteristic zero.

**6.1. Graded systems of ideals.** We start by defining certain collections of ideals, to which we will later attach multiplier ideals.

**Definition 6.1.** A *graded system* or *graded family of ideals* is a family  $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$  of ideals in  $k[X]$  such that

$$\mathfrak{a}_\ell \cdot \mathfrak{a}_m \subseteq \mathfrak{a}_{\ell+m}$$

for all  $\ell, m \geq 1$ . To avoid trivialities, we also assume that  $\mathfrak{a}_k \neq (0)$  for  $k \gg 1$ .

The condition in the definition means that the direct sum

$$R(\mathfrak{a}_\bullet) \stackrel{\text{def}}{=} k[X] \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \dots$$

naturally carries a graded  $k[X]$ -algebra structure and  $R(\mathfrak{a}_\bullet)$  is called the *Rees algebra* of  $\mathfrak{a}_\bullet$ . In the interesting situations  $R(\mathfrak{a}_\bullet)$  is not finitely generated, and it is here that the constructions of the present section give something new. One can view graded systems as local objects displaying complexities similar to those that arise from linear series on a projective variety  $V$ .<sup>10</sup>

**Example 6.2.** We give several examples of graded systems.

- (i). Let  $\mathfrak{b} \subseteq k[X]$  be a fixed ideal, and set  $\mathfrak{a}_k = \mathfrak{b}^k$ . One should view the resulting as a trivial example.
- (ii). Let  $Z \subseteq X$  be a reduced subvariety defined by the radical ideal  $\mathfrak{q}$ . The symbolic powers

$$\mathfrak{q}^{(k)} \stackrel{\text{def}}{=} \{f \in k[X] \mid \text{ord}_z f \geq k, z \in Z \text{ generic}\}$$

form a graded system.<sup>11</sup>

- (iii). Let  $<$  be a term order on  $k[x_1, \dots, x_n]$  and  $\mathfrak{b}$  be an ideal. Then

$$\mathfrak{a}_k \stackrel{\text{def}}{=} \text{in}_{<}(\mathfrak{b}^k)$$

defines a graded system of monomial ideals, where  $\text{in}_{<}(\mathfrak{b}^k)$  denotes the initial ideal with respect to the given term order.

**Example 6.3** (Valuation ideals). Let  $\nu$  be a  $\mathbb{R}$ -valued valuation centered on  $k[X]$ . Then the valuation ideals

$$\mathfrak{a}_k \stackrel{\text{def}}{=} \{f \in k[X] \mid \nu(f) \geq k\}$$

form a graded family. Special cases of this construction are interesting even when  $X = \mathbb{A}_{\mathbb{C}}^2$ .

- (i). Let  $\eta : Y \rightarrow \mathbb{A}^2$  be a birational map with  $Y$  also smooth and let  $E \subseteq Y$  be a prime divisor. Define the valuation  $\nu(f) \stackrel{\text{def}}{=} \text{ord}_E(f)$ . Then

$$\mathfrak{a}_k \stackrel{\text{def}}{=} \mu_* \mathcal{O}_Y(-kE) = \{f \in \mathcal{O}_X \mid \nu(f) = \text{ord}_E(f) \geq k\}.$$

- (ii). In  $\mathbb{C}[x, y]$  put  $\nu(x) = 1$  and  $\nu(y) = \frac{1}{\sqrt{2}}$ . Then one gets a valuation by weighted degree. Here  $\mathfrak{a}_k$  is the monomial ideal generated by the monomials  $x^i y^j$  such that  $i + \frac{j}{\sqrt{2}} \geq k$ .
- (iii). Given  $f \in \mathbb{C}[x, y]$  define  $\nu(f) = \text{ord}_z(f(z, e^z - 1))$ . This yields a valuation giving rise to the graded system

$$\mathfrak{a}_k \stackrel{\text{def}}{=} (x^k, y - P_{k-1}(x))$$

<sup>10</sup>If  $D$  is an effective divisor on  $V$ , the base ideals  $\mathfrak{b}_k = \mathfrak{b}(|kD|) \subseteq \mathcal{O}_V$  form a graded family of ideal sheaves on  $V$ : this is the prototypical example.

<sup>11</sup>When  $Z$  is reducible, we ask that the condition hold at a general point of each component. The fact that this is equivalent to the usual algebraic definition is a theorem of Zariski and Nagata: see [Eis95, Chapter 3].

where  $P_{k-1}(x)$  is the  $(k-1)$ st Taylor polynomial of  $e^x - 1$ . Note that the general element in  $\mathfrak{a}_k$  defines a smooth curve in the plane.

**Remark 6.4.** Except for Example 6.2(i), all these constructions give graded families  $\mathfrak{a}_\bullet$  for which the corresponding Rees algebra need not be finitely generated.

**6.2. Asymptotic multiplier ideals.** We now attach multiplier ideals  $\mathcal{J}(\mathfrak{a}_\bullet^c)$  to a graded family  $\mathfrak{a}_\bullet$  of ideals. The starting point is:

**Lemma 6.5.** *Let  $\mathfrak{a}_\bullet$  be a graded system of ideals on  $X$ , and fix a rational number  $c > 0$ . Then for  $p \gg 0$  the multiplier ideals  $\mathcal{J}(\mathfrak{a}_p^{c/p})$  all coincide.*

**Definition 6.6.** Let  $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$  be a graded system of ideals on  $X$ . Given  $c > 0$  we define the *asymptotic multiplier ideal* of  $\mathfrak{a}_\bullet$  with exponent  $c$  to be the common ideal

$$\mathcal{J}(\mathfrak{a}_\bullet^c) \stackrel{\text{def}}{=} \mathcal{J}(\mathfrak{a}_p^{c/p})$$

for any sufficiently big  $p \gg 0$ .<sup>12</sup>

*Indication of Proof of Lemma 6.5.* We first claim that one has an inclusion of multiplier ideals  $\mathcal{J}(\mathfrak{a}_p^{c/p}) \subseteq \mathcal{J}(\mathfrak{a}_{pq}^{c/pq})$  for all  $p, q \geq 0$ . Granting this, it follows from the Noetherian condition that the collection of ideals  $\{\mathcal{J}(\mathfrak{a}_p^{c/p})\}_{p \geq 0}$  has a unique maximal element. This proves the lemma at least for sufficiently divisible  $p$ . (The statement for all  $p \gg 0$  requires a little more work; see [Laz, Chapter 11].)

To verify the claim let  $\mu : X' \rightarrow X$  be a common log resolution of  $\mathfrak{a}_p$  and  $\mathfrak{a}_{pq}$  with  $\mathfrak{a}_p \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_p)$  and  $\mathfrak{a}_{pq} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_{pq})$ . Since the  $\mathfrak{a}_k$  form a graded system one has  $\mathfrak{a}_p^q \subseteq \mathfrak{a}_{pq}$  and therefore  $-cqF_p \leq -cF_{pq}$ . Thus

$$\mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \frac{cq}{p} F_p \rfloor) \subseteq \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \frac{c}{pq} F_{pq} \rfloor)$$

as claimed.  $\square$

**Remark 6.7.** Lemma 6.5 shows that any information captured by the multiplier ideals  $\mathcal{J}(\mathfrak{a}_p^{c/p})$  is present already for any one sufficiently large index  $p$ . It is in this sense that multiplier ideals have some finiteness built in that may not be present in the underlying graded system  $\mathfrak{a}_\bullet$ .

**Exercise 6.8.** We return to the graded systems in Example 6.3 coming from valuations on  $\mathbb{A}^2$ .

- (ii). Here  $\mathfrak{a}_k$  is the monomial ideal generated by  $x^i y^j$  with  $i + \frac{j}{\sqrt{2}} \geq k$ , and  $\mathcal{J}(\mathfrak{a}_\bullet^c)$  is the monomial ideal generated by all  $x^i y^j$  with

$$(i+1) + \frac{(j+1)}{\sqrt{2}} > c.$$

(Compare with Theorem 3.1.)

<sup>12</sup>In [ELS01] and early versions of [Laz], one only dealt with the ideals  $\mathcal{J}(\mathfrak{a}_\bullet^\ell)$  for integral  $\ell$ , which were written  $\mathcal{J}(\|\mathfrak{a}_\ell\|)$ .

(iii). Now take the valuation  $\nu(f) = \text{ord}_z f(z, e^z - 1)$ . Then

$$\mathcal{J}(\mathfrak{a}_\bullet^c) = \mathbb{C}[x, y]$$

for all  $c > 0$ . (Use the fact that each  $\mathfrak{a}_k$  contains a smooth curve.)

**6.3. Growth of graded systems.** We now use the Subadditivity Theorem 5.10 to prove a result from [ELS01] concerning the multiplicative behavior of graded families of ideals:

**Theorem 6.9.** *Let  $\mathfrak{a}_\bullet$  be a graded system of ideals and fix any  $\ell \in \mathbb{N}$ . Then*

$$\mathcal{J}(\mathfrak{a}_\bullet^\ell) = \mathcal{J}(\mathfrak{a}_{\ell p}^{1/p}) \text{ for } p \gg 0.$$

Moreover for every  $m \in \mathbb{N}$  one has:

$$(3) \quad \mathfrak{a}_\ell^m \subseteq \mathfrak{a}_{\ell m} \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{\ell m}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\ell)^m.$$

In particular, if  $\mathcal{J}(\mathfrak{a}_\bullet^\ell) \subseteq \mathfrak{b}$  for some natural number  $\ell$  and ideal  $\mathfrak{b}$ , then  $\mathfrak{a}_{\ell m} \subseteq \mathfrak{b}^m$  for all  $m$ .

**Remark 6.10.** The crucial point here is the containment  $\mathcal{J}(\mathfrak{a}_\bullet^{\ell m}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\ell)^m$ : it shows that passing to multiplier ideals “reverses” the inclusion  $\mathfrak{a}_\ell^m \subseteq \mathfrak{a}_{\ell m}$ .

*Proof of Theorem 6.9.* For the first statement, observe that if  $p \gg 0$  then

$$\mathcal{J}(\mathfrak{a}_\bullet^\ell) = \mathcal{J}(\mathfrak{a}_p^{\ell/p}) = \mathcal{J}(\mathfrak{a}_{\ell p}^{\ell/\ell p}) = \mathcal{J}(\mathfrak{a}_{\ell p}^{1/p}),$$

where the second equality is obtained by taking  $\ell p$  in place of  $p$  as the large index in Lemma 6.5. For the containment  $\mathfrak{a}_{\ell m} \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{\ell m})$  it is then enough to prove that  $\mathfrak{a}_{\ell m} \subseteq \mathcal{J}(\mathfrak{a}_{\ell m p}^{1/p})$ . But we have  $\mathfrak{a}_{\ell m} \subseteq \mathcal{J}(\mathfrak{a}_{\ell m})$  thanks to Exercise 2.7, while the inclusion  $\mathcal{J}(\mathfrak{a}_{\ell m}) \subseteq \mathcal{J}(\mathfrak{a}_{\ell m p}^{1/p})$  was established during the proof of 6.5.

It remains only to prove that  $\mathcal{J}(\mathfrak{a}_\bullet^{\ell m}) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\ell)^m$ . To this end, fix  $p \gg 0$ . Then by the definition of asymptotic multiplier ideals and the Subadditivity Theorem one has

$$\begin{aligned} \mathcal{J}(\mathfrak{a}_\bullet^{\ell m}) &= \mathcal{J}(\mathfrak{a}_p^{\ell m/p}) \\ &\subseteq \mathcal{J}(\mathfrak{a}_p^{\ell/p})^m \\ &= \mathcal{J}(\mathfrak{a}_\bullet^\ell)^m, \end{aligned}$$

as required. □

**Example 6.11.** The Theorem gives another explanation of the fact that the multiplier ideals associated to the graded system  $\mathfrak{a}_\bullet$  from Example 6.3.(iii) are trivial. In fact, in this example the colength of  $\mathfrak{a}_k$  in  $\mathbb{C}[X]$  grows linearly in  $k$ . It follows from Theorem 6.9 that then  $\mathcal{J}(\mathfrak{a}_\bullet^\ell) = (1)$  for all  $\ell$ .

**Exercise 6.12.** Let  $\mathfrak{a}_k = \mathfrak{b}^k$  be the trivial graded family consisting of powers of a fixed ideal. Then  $\mathcal{J}(\mathfrak{a}_\bullet^c) = \mathcal{J}(\mathfrak{b}^c)$  for all  $c > 0$ . So we do not get anything new in this case.

**6.4. A comparison theorem for symbolic powers.** As a quick but surprising application of Theorem 6.9 we discuss a result due to Ein, Smith and the second author from [ELS01] concerning symbolic powers of radical ideals.

Consider a reduced subvariety  $Z \subseteq X$  defined by a radical ideal  $\mathfrak{q} \subseteq k[X]$ . Recall from Example 6.2(ii) that one can define the symbolic powers  $\mathfrak{q}^{(k)}$  of  $\mathfrak{q}$  to be

$$\mathfrak{q}^{(k)} \stackrel{\text{def}}{=} \{f \in \mathcal{O}_X \mid \text{ord}_z f \geq k, z \in Z\}.$$

Thus evidently  $\mathfrak{q}^k \subseteq \mathfrak{q}^{(k)}$ , and equality holds if  $Z$  is smooth. However if  $Z$  is singular then in general the inclusion is strict:

**Example 6.13.** Take  $Z \subseteq \mathbb{C}^3$  to be the union of the three coordinate axes, defined by the ideal

$$\mathfrak{q} = (xy, yz, xz) \subseteq \mathbb{C}[x, y, z].$$

Then  $xyz \in \mathfrak{q}^{(2)}$  since evidently the union of the three coordinate planes has multiplicity 2 at a general point of  $Z$ . But  $\mathfrak{q}^2$  is generated by monomials of degree 4, thus cannot contain  $xyz$ , which is of degree 3.

Swanson [Swa00] proved (in a much more general setting) that there exists an integer  $k = k(Z)$  such that

$$\mathfrak{q}^{(km)} \subseteq \mathfrak{q}^m$$

for all  $m \geq 0$ . At first glance, one might be tempted to suppose that for very singular  $Z$  the coefficient  $k(Z)$  will have to become quite large. The main result of [ELS01] shows that this isn't the case, and that in fact one can take  $k(Z) = \text{codim } Z$ :

**Theorem 6.14.** *Assume that every irreducible component of  $Z$  has codimension  $\leq e$  in  $X$ . Then*

$$\mathfrak{q}^{(em)} \subseteq \mathfrak{q}^m \text{ for all } m \geq 0.$$

*In particular,  $\mathfrak{q}^{(m \cdot \dim X)} \subseteq \mathfrak{q}^m$  for all radical ideals  $\mathfrak{q} \subseteq k[X]$  and all  $m \geq 0$ .*

**Example 6.15** (Points in the plane). Let  $T \subseteq \mathbb{P}^2$  be a finite set (considered as a reduced scheme), and let  $I \subseteq S = \mathbb{C}[x, y, z]$  be the homogeneous ideal of  $T$ . Suppose that  $f \in S$  is a homogeneous form which has multiplicity  $\geq 2m$  at each of the points of  $T$ . Then  $f \in I^m$ . (Apply Theorem 6.14 to the homogeneous ideal  $I$  of  $T$ .) In spite of the classical nature of this statement, we do not know a direct elementary proof.

*Proof of Theorem 6.14.* Applying Theorem 6.9 to the graded system  $\mathfrak{a}_k = \mathfrak{q}^{(k)}$ , it suffices to show that

$$(*) \quad \mathcal{J}(\mathfrak{a}_\bullet^e) \subseteq \mathfrak{q}.$$

Since  $\mathfrak{q}$  is radical, it suffices to test the inclusion  $(*)$  at a general point of  $Z$ . Therefore we can assume that  $Z$  is smooth, in which case  $\mathfrak{q}^{(k)} = \mathfrak{q}^k$ . Now Exercises 2.14 and 6.12 apply.  $\square$

**Remark 6.16.** Using their theory of tight closure, Hochster and Huneke [HH02] have extended Theorem 6.14 to arbitrary regular Noetherian rings containing a field.

**Remark 6.17.** Theorem 6.9 is applied in [ELS] to study the multiplicative behavior of Abhyankar valuations centered at a smooth point of a complex variety.

**Remark 6.18.** Working with the asymptotic multiplier ideals  $\mathcal{J}(\mathfrak{a}_\bullet^c)$  one can define the log-canonical threshold and jumping coefficients of a graded system  $\mathfrak{a}_\bullet$  much as in §4. However now these numbers need no longer be rational, the periodicity of jumping numbers (Exercise 5.7) may fail, and in fact the collection of jumping coefficients of  $\mathfrak{a}_\bullet$  can contain accumulation points. See [ELSV, §5].

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