Preface:

This preprint consists of two articles concerning the dynamics of subgroup actions on homogeneous spaces. The first, "On Actions of Epimorphic Subgroups on Homogeneous Spaces" (abbreviated **AES**), was written by Nimish A. Shah and myself in 1997, and the second, "Unique Ergodicity on Compact Homogeneous Spaces" (**UEC**), was written by myself in 1999. Since the results of the second paper complement those of the first, I thought it would be a good idea for them to appear as one preprint. For the benefit of the readers who are not experts in the theory of subgroup actions on homogeneous spaces I have prefaced the papers with some general remarks explaining and motivating our results, and the connection between them.

The remarks are organized as a comparison between facts which had been previously known about the action of the geodesic and horocycle flow on finite-volume Riemann surfaces – the simplest nontrivial example that falls into our framework – and our results on subgroup actions on homogeneous spaces. This might seem a somewhat strange way to present things: it fails to mention numerous intermediate results which were obtained by various authors (many of which are indispensable for our proofs), the results described are not stated in complete generality nor with perfect precision, and no references are given other than to our two papers. Many readers will thus prefer to proceed directly to the papers themselves. Nevertheless I hope that some readers, who might otherwise regard our results as dauntingly technical, will find them easier to read once I have sketched the features shared by our results and the example.

Let X be the unit tangent bundle to a Riemann surface of finite area, and consider the geodesic flow $T = \{g_t : t \in \mathbb{R}\}$ and the horocycle flow $U = \{u_t : t \in \mathbb{R}\}$ on X. These flows satisfy the commutation relation

$$g_t u_s g_{-t} = u_{e^{2t}s}. \tag{1}$$

The following are well-known, through the work of many authors:

1. The algebraic setup: The flows T and U admit algebraic descriptions as *subgroup actions on a homogeneous space*. This means that the space X can be identified with L/Λ for $L = \mathrm{SL}(2, \mathbb{R})$ and Λ a lattice in L (a discrete subgroup such that there is a fundamental domain for the action of Λ on L with finite haar measure). L and any of its subgroups now act on L/Λ by

$$g \cdot (\ell\Lambda) = (g\ell)\Lambda.$$

The geodesic (respectively horocycle) flow is realized by the action of the one-parameter subgroup of L consisting of diagonal matrices (resp. upper triangular unipotent matrices). We continue to denote these subgroups by T and U. The relation (1) is reflected in the fact that T normalizes U.

Let $F = T \cdot U$. From the representation theory of L it follows that for any linear representation (of real algebraic groups) $\rho : L \to \mathrm{GL}(V)$ and any $v \in V$,

$$\rho(F)v = v \Longrightarrow \rho(L)v = v;$$

thus F is an *epimorphic* subgroup of L.

Stony Brook IMS Preprint #1999/7a
May 1999

2. Unique ergodicity and minimality for F: F acts both uniquely ergodically (Liouville measure is the only finite F-invariant Borel measure) and minimally (all F-orbits are dense in X).

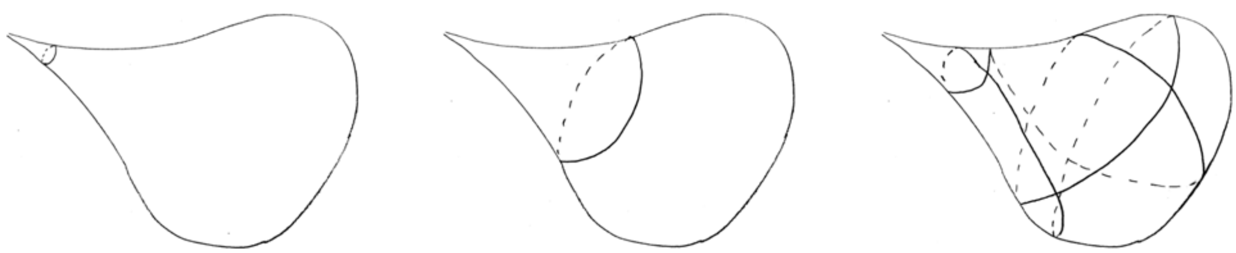
3. More in the cocompact case: If X is compact then U already acts minimally and uniquely ergodically.

4. Rays in T: If $\rho : G \to \mathrm{GL}(V)$ is a representation without $\rho(L)$-invariant vectors, and $v \in V$ is $\rho(U)$-invariant, then

$$\rho(t)v \to_{t \to -\infty} 0.$$

5. More in the non-cocompact case: If X is noncompact then there are periodic orbits for the horocycle flow. Let O be such an orbit, and let ν be the length measure on O. Then as t tends to $+\infty$, the sequence of measures $g_t\nu$ tends to the Liouville measure on X; this implies that for any open set W there is t_0 such that $g_tO \cap W \neq \emptyset$ for $t \geq t_0$.

In contrast, as t tends to $-\infty$ the sets g_tO leaves compact subsets of X and the sequence of measures $g_t\nu$ diverges (has no convergent subsequence) in the space of probability measures on X.



Let me indicate how our results generalize the results listed above.

1'. The algebraic setup: Our setup will be an action of a subgroup F on a homogeneous space L/Λ, where L is a real algebraic group, F is an algebraic subgroup, and Λ is a lattice in L. In the main case of interest F will have the form $F = T \cdot U$, where T normalizes U, T is commutative and diagonalizable and U is unipotent. Furthermore F will be epimorphic in a subgroup G of L. A typical example of our setup is obtained by setting $L = \mathrm{SL}(n, \mathbb{R})$, G a subgroup of L isomorphic to $\mathrm{SL}(k, \mathbb{R})$ for some $k < n$,

$$F = \left\{ \begin{pmatrix} * & 0 & \cdots & 0 & * \\ 0 & * & \cdots & 0 & * \\ & & \cdots & & \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{pmatrix} \right\} \cap G,$$

and letting T (resp. U) be the intersection of F with the diagonal (resp. upper triangular unipotent) matrices.

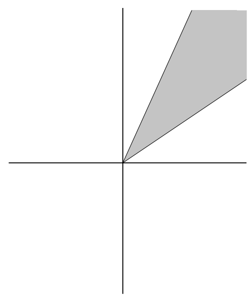
It is worth noting that there are subgroups T and U satisfying the hypotheses above which have lower dimensions then in this example, and in fact, any simple L contains such T and U where $\dim T = 1$ and $\dim U = 2$.

Note that the example fits our framework as part of the special case $L = G$.

2'. Orbit-closures and invariant measures for F: In this setup Shahar Mozes proved in 1992 that any F-invariant probability measure is G-invariant. We prove that for any $x \in L/\Lambda$, $\overline{Fx} = \overline{Gx}$. This is Corollary 1.3 of **AES**. The statements corresponding to these theorems in case $L = G$ are unique ergodicity and minimality of the action of F.

3'. More on the cocompact case: Theorems 1.2 and 1.4 of the paper **UEC** state that if $L = G$, or more generally, if the action of G on L/Λ is either minimal or uniquely ergodic, and Λ is cocompact, then the action of U is both minimal and uniquely ergodic.

4'. Cones in T: Our results depend essentially on a result we call the 'cone lemma' which asserts that for every representation $\rho : G \to \mathrm{GL}(V)$ without $\rho(G)$-invariant vectors, there is a cone T^- in T (a closed sub-semigroup of T with nonempty interior) such that for every $v \in V$ which is $\rho(U)$-invariant, $\rho(t)v \to 0$ as t leaves compact subsets of T^-. This lemma, which I proved in an earlier paper, is included as Lemma 1.5 of **UEC**. We let T^+ denote the cone opposite to T^-, i.e., $T^+ = \{t^{-1} : t \in T^-\}$.



5'. More Results: Our main result (Theorem 1.4 of **AES**) is reminiscent of what happens in the example when $t \to +\infty$. It asserts that if ν is any measure on L/Λ which is absolutely continuous with respect to the projection of the haar measure class along a U-orbit, then $t\nu$ tends to a G-invariant measure as t leaves compact subsets of T^+. A topological consequence of this is that for any $x \in L/\Lambda$, and any open set W intersecting $\overline{Gx}$, there is a compact subset $C \subset T^+$ such that for all $t \in T^+ \setminus C$, $W \cap Utx \neq \emptyset$.

The argument of the proof of Theorem 1.1 of **UEC** provides the following contrasting result: suppose L is semisimple and ν is a U-invariant ergodic measure which is not G-invariant. Then for any divergent sequence $\{t_n\}$ in T^-, the sequence of measures $\{t_n\nu\}$ also diverges. Moreover, for any $x \in L/\Lambda$, either Ux is dense in L/Λ or for any compact $K \subset L/\Lambda$ there exists a compact $C \subset T^-$ such that for all $t \in T^- \setminus C$, $Utx \cap K = \emptyset$.