MATING SIEGEL QUADRATIC POLYNOMIALS

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Abstract. Let F be a quadratic rational map of the sphere which has two fixed Siegel disks with bounded type rotation numbers θ and ν. Using a new degree 3 Blaschke product model for the dynamics of F and an adaptation of complex a priori bounds for renormalization of critical circle maps, we prove that F can be realized as the mating of two Siegel quadratic polynomials with the corresponding rotation numbers θ and ν.

Contents

1. Introduction

1.1. **Mating: Definitions and some history.** Mating quadratic polynomials is a topological construction suggested by Douady and Hubbard [**Do2**] to partially parametrize quadratic rational maps of the Riemann sphere by pairs of quadratic polynomials. Some results on matings of higher degree maps exist, but we will not discuss them in this paper. While there exist several, presumably equivalent, ways of describing the construction of mating, the following approach is perhaps the most standard. Consider two monic quadratic polynomials f_1 and f_2 whose filled Julia sets $K(f_i)$ are locally-connected. For each f_i, let Φ_i denote the conformal isomorphism between the basin of infinity $\overline{\mathbb{C}} \smallsetminus K(f_i)$ and $\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$, with $\Phi_i(\infty) = \infty$ and $\Phi_i'(\infty) = 1$.

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These *Böttcher maps* conjugate the polynomials to the squaring map:

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus K(f_i) & \xrightarrow{\Phi_i} & \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \\ \big\downarrow f_i & & \big\downarrow z \mapsto z^2 \\ \overline{\mathbb{C}} \setminus K(f_i) & \xrightarrow{\Phi_i} & \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{array}$$

By the Carathéodory's Theorem the inverse map Φ_i^{-1} has a continuous extension

$$\Phi_i^{-1} : \partial\mathbb{D} \to J(f_i),$$

where the Julia set $J(f_i) = \partial K(f_i)$ is the topological boundary of the filled Julia set. The induced parametrization

$$\gamma_i(t) \equiv \Phi_i^{-1}(e^{2\pi i t}) : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to J(f_i)$$

is commonly referred to as the *Carathéodory loop* of $J(f_i)$. Note that by the above commutative diagram, $\gamma_i(2t) = f_i(\gamma_i(t))$. Consider the topological space

$$X = (K(f_1) \sqcup K(f_2))/(\gamma_1(t) \sim \gamma_2(-t))$$

obtained by gluing the two filled Julia sets along their Carathéodory loops in reverse directions.

Definition I. Assume that the space X as defined above is homeomorphic to the 2-sphere S^2. Then the pair of polynomials (f_1, f_2) is called *topologically mateable*. The induced map of S^2

$$f_1 \sqcup_{\mathcal{T}} f_2 = (f_1|_{K_1} \sqcup f_2|_{K_2})/(\gamma_1(t) \sim \gamma_2(-t))$$

is the *topological mating* of f_1 and f_2.

It may seem surprising at this point that topologically mateable quadratics even exist, however, we shall see below that such examples are abundant. For any mateable pair (f_1, f_2), their topological mating is a degree 2 branched covering of the sphere, and it is natural to ask whether it possesses an invariant conformal structure.

Definition II. A quadratic rational map $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called a *conformal mating*, or simply a *mating*, of f_1 and f_2,

$$F = f_1 \sqcup f_2,$$

if it is conjugate to the topological mating $f_1 \sqcup_{\mathcal{T}} f_2$ by a homeomorphism which is conformal in the interiors of $K(f_1)$ and $K(f_2)$ in case there is an interior. If such F is unique up to conjugation by a Möbius transformation, we refer to it as *the* mating of f_1 and f_2.

Before proceeding to formulate the known existence results, let us describe another equivalent method of defining a mating. Let ⓒ denote the complex plane $\mathbb{C}$ compactified by adjoining a circle of directions at infinity, $\{\infty \cdot e^{2\pi i t} | t \in \mathbb{T}\}$ with the natural topology. Each f_i extends continuously to a copy of $\copyright_i$, acting as the squaring map $z \mapsto z^2$ on the circle at infinity. Gluing the disks $\copyright_i$ together via the equivalence relation $\sim_\infty$ identifying the point $\infty \cdot e^{2\pi it} \in \copyright_1$ with $\infty \cdot e^{-2\pi it} \in \copyright_2$, we obtain a 2-sphere $(\copyright_1 \sqcup \copyright_2)/\sim_\infty$. The well-defined map $f_1 \sqcup_{\mathcal{F}} f_2$ on this sphere given by f_i on $\copyright_i$ is a degree 2 branched covering of the sphere with an invariant equator. We shall refer to this map as the *formal mating* of f_1, f_2.

Recall that the *external ray of* f_i *at angle* t is the preimage

$$R_i(t) = \Phi_i^{-1}(\{re^{2\pi it} | r > 1\})$$

for $t \in \mathbb{T}$. Let $\hat{R}_i(t)$ denote the closure of $R_i(t)$ in $\copyright_i$. The *ray equivalence relation* $\sim_r$ on $(\copyright_1 \sqcup \copyright_2)/\sim_\infty$ is defined as follows. The points z and w are equivalent, $z \sim_r w$ if and only if there exists a collection of closed rays $\hat{R}_j = \hat{R}_i(t_j)$, $i \in \{1, 2\}$ and $j = 1, \ldots, n$, such that $z \in \hat{R}_1$, $w \in \hat{R}_n$ and $\hat{R}_j \cap \hat{R}_{j+1} \neq \emptyset$ for $j = 1, \ldots, n-1$. It follows immediately from the definition that if f_1 and f_2 are topologically mateable, then the quotient of $(\copyright_1 \sqcup \copyright_2)/\sim_\infty$ modulo $\sim_r$ is again a 2-sphere, and

$$(f_1 \sqcup_{\mathcal{F}} f_2)/\sim_r \simeq f_1 \sqcup_{\mathcal{T}} f_2.$$

Finally, let us formulate another definition of conformal mating, equivalent to the previously given, but more convenient for further application:

Definition IIa. Let f_1 and f_2 be quadratic polynomials with locally-connected Julia sets. A quadratic rational map F of the Riemann sphere is called a *conformal mating* of f_1 and f_2 if there exist continuous semiconjugacies

$$\varphi_i : K(f_i) \to \overline{\mathbb{C}}, \text{ with } \varphi_i \circ f_i = F \circ \varphi_i,$$

conformal in the interiors of the filled Julia sets in case there is an interior, such that $\varphi_1(K(f_1)) \cup \varphi_2(K(f_2)) = \overline{\mathbb{C}}$ and for $i, j = 1, 2$, $\varphi_i(z) = \varphi_j(w)$ if and only if $z \sim_r w$.

We are now prepared to give an account of known results. The simplest example of a non-mateable pair is given by quadratic polynomials $f_{c_1}(z) = z^2 + c_1$ and $f_{c_2}(z) = z^2 + c_2$ with locally-connected Julia sets whose parameter values c_1 and c_2 belong to the conjugate limbs of the Mandelbrot set. In this case the rays $\{R_1(t_j)\}$ and $\{R_2(t_j)\}$ landing at the dividing fixed points α_1, α_2 of the two polynomials have opposite angles (see e.g. [**Mi3**]). This implies that $\alpha_1 \sim_r \alpha_2$, and it is not hard to check that the quotient of $(\copyright_1 \sqcup \copyright_2)/\sim_\infty$ modulo $\sim_r$ is not homeomorphic to the 2-sphere.

Recall that two branched coverings F and G of S^2 with finite postcritical sets P_F and P_G are equivalent *combinatorially* or *in the sense of Thurston* if there exist two orientation preserving homeomorphisms $\phi, \psi : S^2 \to S^2$, such that $\phi \circ F = G \circ \psi$, and ψ is isotopic to ϕ rel P_F. Using Thurston's characterization of critically finite rational maps as branched coverings of the sphere (see [**DH**]), Tan Lei [**Tan**] and Rees [**Re1**] established the following:

Theorem. *Let c_1 and c_2 be two parameter values not in conjugate limbs of the Mandelbrot set such that f_{c_1} and f_{c_2} are postcritically finite. Then the map F is combinatorially equivalent to a quadratic rational map, where F is either the formal mating $f_{c_1} \sqcup_{\mathcal{F}} f_{c_2}$ or a certain degenerate form of it.*

Taking this line of investigation further, Rees [**Re2**] and Shishikura [**Sh**] demonstrated:

Theorem. *Under the assumptions of the previous theorem, f_{c_1} and f_{c_2} are topologically mateable. Moreover, their conformal mating $f_{c_1} \sqcup f_{c_2}$ exists.*

The case where the critical points of f_{c_i} are periodic was considered by Rees, the complementary case was done by Shishikura. Note, in particular, that when none of the critical points is periodic, the Julia sets are dendrites with no interior, which makes the result particularly striking. An example of this phenomenon is analyzed in detail in Milnor's recent paper [**Mi4**] in which he considers the self-mating $F = f_{c_{1/4}} \sqcup f_{c_{1/4}}$, where the quadratic polynomial $f_{c_{1/4}}$ is the landing point of the 1/4- external ray of the Mandelbrot set. It is not hard to deduce that F is a Lattès map, its Julia set $J(F) = \overline{\mathbb{C}}$ is obtained by pasting together two copies of the dendrite $J(f_{c_{1/4}})$.

The issue of topological mateability is usually settled using the following result of R. L. Moore [**Mo**]. Recall that an equivalence relation $\sim$ on S^2 is *closed* if $x_n \to x$, $y_n \to y$ and $x_n \sim y_n$ implies $x \sim y$.

Theorem (Moore). *Suppose that $\sim$ is a closed equivalence relation on the 2-sphere S^2 such that every equivalence class is a compact connected non-separating proper subset of S^2. Then the quotient space $S^2/\sim$ is again homeomorphic to S^2.*

For the application at hand, the theorem is replaced by the following corollary (see for example Proposition 4.4. of [**ST**]):

Corollary. *Let f_1 and f_2 be two quadratic polynomials with locally-connected Julia sets, such that every class of the ray equivalence relation $\sim_r$ is non-separating and contains at most N external rays for a fixed $N > 0$. Then f_1 and f_2 are topologically mateable.*

By means of a standard quasiconformal surgery, the theorem of Rees and Shishikura can be extended to any pair f_{c_1}, f_{c_2} where c_i belong to hyperbolic components H_1, H_2 of the Mandelbrot set which do not belong to conjugate limbs. Mating thus yields an isomorphism between the product $H_1 \times H_2$ and a hyperbolic component in the parameter space of quadratic rational maps. This isomorphism, however, does not necessarily extend as a continuous maps to the product of closures $\overline{H}_1 \times \overline{H}_2$, as was recently shown by A. Epstein [**Ep**].

So far no example of conformal matings without using Thurston's theorem (that is going beyond postcritically finite/hyperbolic case) has appeared in the literature.

However, Jiaqi Luo in his dissertation [**Luo**] has outlined a proof of the existence of conformal matings of Yoccoz polynomials with star-like polynomials (centers of hyperbolic components attached to the main cardioid of the Mandelbrot set). His approach consists of locating a candidate rational map for the mating, and then using *Yoccoz puzzle partitions* and complex bounds of Yoccoz to prove that this candidate rational map *is* a mating. A somewhat similar philosophy plays a role in this paper.

The question of constructing matings of polynomials with connected but non locally-connected Julia sets has been completely untouched. While there are definitions of mating which would carry over to non locally-connected case (such as approximate matings discussed in [**Mi2**], p. 54) no examples of such matings are known.

1.2. **Statement of the results.** Consider an irrational number $0 < \theta < 1$ and the quadratic polynomial $z \mapsto e^{2\pi i\theta} z + z^2$ which has an indifferent fixed point with multiplier $e^{2\pi i\theta}$ at the origin. To make this polynomial monic, we conjugate it by an affine map of $\mathbb{C}$ to put it in the normal form

$$f_\theta : z \mapsto z^2 + c_\theta, \text{ with } c_\theta = \frac{e^{2\pi i\theta}}{2}\left(1 - \frac{e^{2\pi i\theta}}{2}\right). \tag{1.1}$$

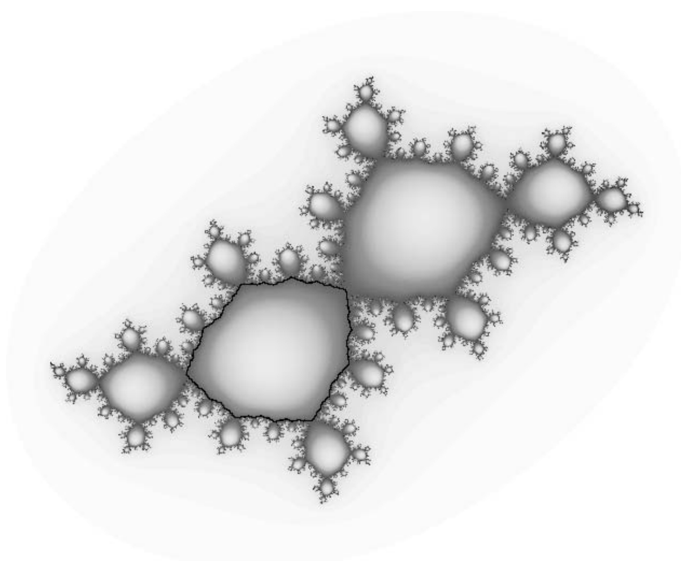

FIGURE 1. *Filled Julia set $K(f_\theta)$ for $\theta = (\sqrt{5}-1)/2$.*

The corresponding indifferent fixed point of f_θ is denoted by α. Assuming θ is irrational of bounded type, a classical result of Siegel [**CG**] implies that f_θ is linearizable near α, i.e., there exists an open neighborhood U of α and a conformal isomorphism $\phi : U \xrightarrow{\simeq} \mathbb{D}$ which conjugates f_θ on U to the rigid rotation $\varrho_\theta : z \mapsto e^{2\pi i\theta} z$:

$$\phi \circ f_\theta \circ \phi^{-1} = \varrho_\theta.$$

The maximal such linearization domain is a simply-connected neighborhood of α called the *Siegel disk* of f_θ. The following result has recently been proved by Petersen [**Pe**]:

Theorem (Petersen). *Let $0 < \theta < 1$ be an irrational of bounded type. Then the Julia set of the quadratic polynomial f_θ is locally-connected and has Lebesgue measure zero.*

Fig. 1 shows the filled Julia set of the quadratic polynomial f_θ for the golden mean $\theta = (\sqrt{5} - 1)/2$.

In proving his theorem, Petersen does not work directly with the Julia set of f_θ, but instead considers a certain Blaschke product , which is related to f_θ via a quasi-conformal surgery procedure. A simplified version of his argument, based on complex *a priori* bounds for renormalization of critical circle maps was presented by one of the authors in [**Ya**]. Since the Julia set of f_θ is locally-connected, we may pose mateability questions for these polynomials. Our main result is the following theorem:

Main Theorem. *Let $0 < \theta, \nu < 1$ be two irrationals of bounded type and $\theta \neq 1 - \nu$. Then the polynomials f_θ and f_ν are topologically mateable. Moreover, there exists a quadratic rational map F such that*

$$F = f_\theta \sqcup f_\nu.$$

Any two such rational maps are conjugate by a Möbius transformation.

In other words, one can paste any two filled Julia sets of the type shown in Fig. 1 along their boundaries to obtain a 2-sphere, and the actions of the polynomials on their filled Julia sets match up to give an action on the sphere which is conjugate to a quadratic rational map with two fixed Siegel disks. Fig. 2 shows the result of this pasting in the case $\theta = \nu = (\sqrt{5} - 1)/2$. In this picture we normalize the quadratic rational map $f_\theta \sqcup f_\theta$ to put the centers of the Siegel disks at zero and infinity. The black and gray regions are the images of the copies of the corresponding filled Julia sets in Fig. 1. There are, however, some prominent differences between these regions and the original filled Julia sets. First, there are infinitely many "pinch points" in the "ends" of the black and gray regions that are not present in the original filled Julia sets. An explicit combinatorial description of these pinch points will be presented in §8. Also, as J. Milnor pointed out to us, an infinite chain of preimages of the Siegel disk in the filled Julia set in Fig. 1 which lands at an endpoint in $J(f_\theta)$ maps to a chain in Fig. 2 which appears very stretched out near the end. This indicates that the continuous semiconjugacies between the filled Julia sets and their corresponding regions, although conformal in the interior of the sets, have a great amount of distortion near the boundary.

In the case $\theta = 1 - \nu$ the existence of a mating is ruled out for algebraic reasons. In fact, the polynomials are not even topologically mateable. Under the assumptions

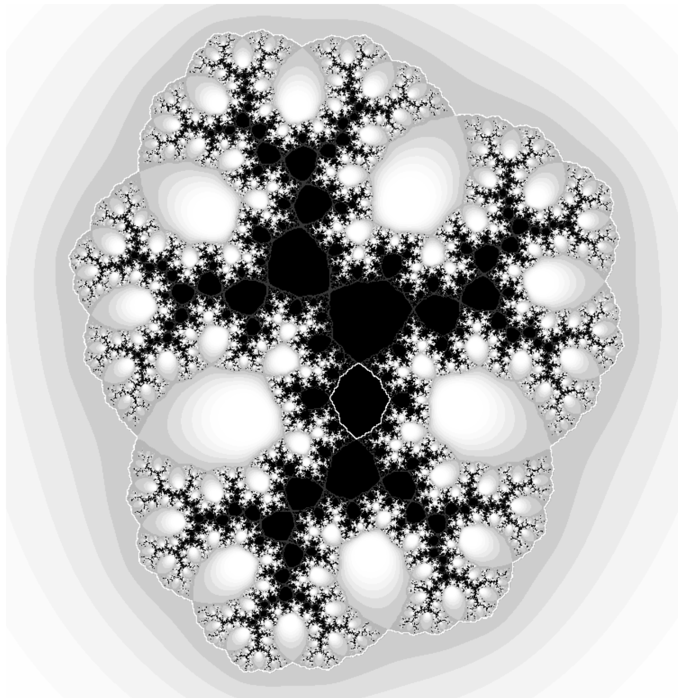

FIGURE 2. *The Julia set of the mating $f_\theta \sqcup f_\theta$ for $\theta = (\sqrt{5}-1)/2$.*

of the theorem, the candidate rational map F can be specified algebraically, and the main difficulty lies in establishing that F is indeed a mating. To fix the ideas we may assume that the candidate F has a Siegel disk Δ^0 with rotation number θ centered at 0, and another one Δ^∞ with rotation number ν centered at ∞. There is an unambiguous way to construct the semiconjugacies of Definition IIa in the interiors of the filled Julia sets, by mapping the preimages of the Siegel disk of f_θ to the corresponding preimages of Δ^0 and similarly the preimages of the Siegel disk of f_ν to the corresponding preimages of Δ^∞. To guarantee that these semiconjugacies extend continuously to the filled Julia sets we need to demonstrate that the boundaries $\partial\Delta^0$ and $\partial\Delta^\infty$ are Jordan curves each containing a critical point of F and that the Euclidean diameter of the n-th preimages of Δ^0 and Δ^∞ goes to zero uniformly in n. Proving these properties of the map F directly seems to be quite out of reach. We establish the first property by using a new Blaschke product model for the dynamics of F that was discovered by one of the authors when he was working on dynamics of cubic Siegel polynomials [**Za2**]. We then adapt the complex bounds from [**Ya**] to this model to prove the second property. Further properties of the semiconjugacies of Definition IIa are demonstrated by a combinatorial argument using spines and itineraries.

The symmetry of the construction in the case of a self-mating (i.e., when $\theta = \nu$) has a nice corollary. In this case the mating $F = f_\theta \sqcup f_\theta$ given by the Main Theorem commutes with the Möbius involution $\mathcal{I}$ which interchanges the centers of the two Siegel disks and fixes the third fixed point of F. Hence one can pass to the quotient Riemann surface $\overline{\mathbb{C}}/\mathcal{I} \simeq \overline{\mathbb{C}}$ to obtain a new quadratic rational map G. It is not hard to see that G is the mating of f_θ with the Chebyshev quadratic polynomial $f_{\text{cheb}} : z \mapsto z^2 - 2$ whose filled Julia set is the interval $[-2, 2]$:

Theorem. *Let $0 < \theta < 1$ be any irrational of bounded type. Then there exists a quadratic rational map G such that*

$$G = f_\theta \sqcup f_{\text{cheb}}.$$

Moreover, G is unique up to conjugation with a Möbius transformation.

Acknowledgements. We would like to express our gratitude to John Milnor for posing the problem and encouraging the dynamics group at Stony Brook to look at it. His picture of the "presumed mating of golden ratio Siegel disk with itself" (Fig. 2 in this paper) posted in the IMS at Stony Brook was the inspiration for this work. Adam Epstein, who also was enthusiastic about this problem and had learned about our similar ideas, brought the two of us together. We are indebted to him because this joint paper would have never existed without his persistence. Finally, we gratefully acknowledge the important role that Carsten Petersen's ideas in [**Pe**] play in our work.

2. Background Material

2.1. **Notations and terminology.** The unit disk in the complex plane will be denoted by $\mathbb{D}$, its boundary is the unit circle $\mathbb{T}$. For a set X in the plane, we use $\overline{X}$ and $\overset{\circ}{X}$ for the closure and the interior of X respectively. We use $|J|$ for the length of an interval J, dist and diam for the Euclidean distance and diameter in $\mathbb{C}$. We write $[a, b]$ for the closed interval with endpoints a and b in $\mathbb{R}$ without specifying their order. For a hyperbolic Riemann surface X, dist_X will denote the distance in the hyperbolic metric in X.

We call two real numbers a and b *K-commensurable* or simply *commensurable* if $K^{-1} \leq |a|/|b| \leq K$ for some $K > 1$ independent of a, b. Two sets X and Y in $\mathbb{C}$ are K-commensurable, if their diameters are. A configuration of points $x_1, \ldots x_n$ is called *K-bounded* if any two intervals $[x_i, x_j]$, and $[x_k, x_l]$ are K-commensurable. For a pair of intervals $I \subset J$ we say that I is *well inside of* J if there exists a universal constant $K > 0$, such that for each component L of $J \setminus I$ we have $|L| \geq K|I|$.

For two points a, b on the circle which are not diagonally opposite $[a, b]$ will denote, unless otherwise specified, the shorter of the two closed arcs connecting them. When working with a homeomorphism f of the unit circle, which extends beyond the circle, we will reserve the notation $f^{-i}(z)$ for the i-th preimage of $z \in \mathbb{T}$ contained in the circle $\mathbb{T}$.

2.2. **Quadratic rational maps.** The reader may find a detailed discussion of the dynamics of quadratic rational maps in Milnor's paper [**Mi2**]. Below we give a brief summary of some relevant facts. A quadratic rational map of the Riemann sphere $\overline{\mathbb{C}}$ may be expressed as a ratio

$$F(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

with one of the coefficients a_0, b_0 different from 0. The six-tuple $(a_0 : a_1 : a_2 : b_0 : b_1 : b_2)$ may be viewed as a point in the complex projective space $\mathbb{CP}^5$. The space of all quadratic rational maps $\mathbf{Rat}_2$ is identified in this way with a Zariski open subset of $\mathbb{CP}^5$ (see [**Mi2**] for a description of the topology of this set). From the point of view of complex dynamics the quadratic rational maps which are conjugate by a conformal isomorphism of the Riemann sphere are identified. That is, we consider the quotient space of $\mathbf{Rat}_2$ by the action of the group $\mathbf{M\ddot{o}b} \simeq PSL_2(\mathbb{C})$ of Möbius transformations. This *moduli space* of quadratic rational maps will be denoted $\mathcal{M}_2$. The action of $\mathbf{M\ddot{o}b}$ on $\mathbf{Rat}_2$ is locally free, and the quotient space has the structure of a 2-dimensional complex orbifold branched over a set $\mathcal{S} \subset \mathcal{M}_2$. This *symmetry locus* $\mathcal{S}$ consists of maps possessing a nontrivial automorphism group.

A more useful parametrization of the moduli space $\mathcal{M}_2$ comes from the following considerations. Every map $F \in \mathbf{Rat}_2$ has three not necessarily distinct fixed points. Let μ_1, μ_2, μ_3 denote the multipliers of the fixed points. (By definition, the multiplier of F at a fixed point p is simply the derivative $F'(p)$ with appropriate modification if $p = \infty$.) Let

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3,\ \sigma_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3,\ \sigma_3 = \mu_1\mu_2\mu_3$$

be the elementary symmetric functions of these multipliers.

Proposition ([**Mi2**], Lemma 3.1). *The numbers σ_1, σ_2, σ_3 determine F up to a Möbius conjugacy, and are subject only to the restriction that*

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2$ is canonically isomorphic to $\mathbb{C}^2$, with coordinates σ_1 and σ_2.

Note that for any choice of μ_1, μ_2 with $\mu_1\mu_2 \neq 1$ there exists a quadratic rational map F, unique up to a Möbius conjugacy, which has distinct fixed points with these multipliers. The third multiplier can be computed as $\mu_3 = (2 - \mu_1 - \mu_2)/(1 - \mu_1\mu_2)$.

As a special case, let F be a quadratic rational map which has two Siegel disks centered at two fixed points of multipliers $e^{2\pi i\theta}$ and $e^{2\pi i\nu}$, where $0 < \theta, \nu < 1$. Note that we necessarily have $\theta \neq 1 - \nu$. By conjugating F with a Möbius transformation which sends the two centers to 0 and ∞ and the third fixed point to 1, we obtain a quadratic rational map which fixes $0, 1, \infty$ and has multipliers $e^{2\pi i\theta}$ at 0 and $e^{2\pi i\nu}$ at ∞. It is easy to see that these conditions determine the map uniquely. In fact, we obtain the normal form

$$F_{\theta,\nu} : z \mapsto z\,\frac{(1-e^{2\pi i\theta})z + e^{2\pi i\theta}(1-e^{2\pi i\nu})}{(1-e^{2\pi i\theta})e^{2\pi i\nu}z + (1-e^{2\pi i\nu})}. \tag{2.1}$$

2.3. **Critical circle maps.** Throughout this paper, we shall identify the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the affine manifold $\mathbb{R}/\mathbb{Z}$ using the canonical projection from the real line given by $x \mapsto e^{2\pi i x}$. By definition, a *critical circle map* is an orientation-preserving homeomorphism of the circle $\mathbb{T}$ of class C^3 with a single critical point c. We further assume that the critical point is of cubic type. This means that for a lift $\hat{f} : \mathbb{R} \to \mathbb{R}$ of f with critical points at integer translates of $\hat{c}$,

$$\hat{f}(x) - \hat{f}(\hat{c}) = (x-\hat{c})^3(\text{const} + O(x-\hat{c})).$$

The standard examples of analytic critical circle maps are provided by the projections to $\mathbb{T}$ of homeomorphisms in the *Arnold family*:

$$A^t : x \mapsto x + t - \frac{1}{2\pi}\sin 2\pi x.$$

Another group of examples, more relevant for our considerations, is given by the family of degree 3 Blaschke products

$$Q^t : z \mapsto e^{2\pi i t} z^2 \left(\frac{z-3}{1-3z}\right).$$

The restriction of Q^t to the unit circle $\mathbb{T}$ is a real-analytic homeomorphism. Every Q^t has a critical point of cubic type at $1 \in \mathbb{T}$ and no other critical points in $\mathbb{T}$, thus $Q^t|_{\mathbb{T}}$ is a critical circle map.

The quantity

$$\rho(f) = \lim_{n\to\infty} \frac{\hat{f}^{\circ n}(x)}{n} \pmod 1$$

is independent both of the choice of $x \in \mathbb{R}$ and the lift $\hat{f}$ of a critical circle map f, and is referred to as the *rotation number* of f. The rotation number is rational of the form $\rho(f) = p/q$ if and only if f has an orbit of period q. To further illustrate the connection between the number-theoretic properties of $\rho(f)$ and the dynamics of f, let us introduce the notion of a closest return of the critical point c. The iterate $f^{\circ n}(c)$ is a *closest return*, or equivalently, n is a *closest return moment*, if the interior of the arc $[f^{\circ n}(c), c]$ contains no iterates $f^{\circ j}(c)$ with $j < n$. Consider the representation of

$\rho(f)$ as a (possibly finite) continued fraction

$$\rho(f) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

with the a_i being positive integers. For convenience we will write $\rho(f) = [a_1, a_2, a_3, \dots]$. The *n-th convergent* of the continued fraction of $\rho(f)$ is the rational number

$$\frac{p_n}{q_n} = [a_1, a_2, \dots, a_n]$$

written in the reduced form. We set $p_0 = 0$, $q_0 = 1$. One easily verifies the recursive relations

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2},$$

for $n \geq 2$. In this notation, the iterates $\{f^{\circ q_n}(c)\}$ are the consecutive closest returns of the critical point c (see for example [**dMvS**]).

The rotation number $\rho(f)$ is said to be of *bounded type* if $\sup a_i < \infty$. We will make use of two linearization theorems for critical circle maps. Let us denote by ϱ_θ the rigid rotation $x \mapsto x + \theta \pmod{\mathbb{Z}}$. Yoccoz [**Yo1**] has shown:

Theorem. *Let f be a critical circle map with irrational rotation number θ. Then there exists a homeomorphic change of coordinates $h : \mathbb{T} \to \mathbb{T}$ such that*

$$h \circ f \circ h^{-1} = \varrho_\theta.$$

In general the homeomorphism h may not be regular at all, even if the map f is real-analytic. However, some regularity for h may be gained at the expense of extra assumptions on the rotation number $\rho(f)$. The following theorem of Herman [**He**] provides us with a sharp result which will be useful further in performing a quasiconformal surgery. Recall that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called K-*quasisymmetric* if

$$0 < K^{-1} \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq K < +\infty$$

for all x and all $t > 0$. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is K-quasisymmetric if its lift to $\mathbb{R}$ is such a homeomorphism. We simply call h quasisymmetric if it is K-quasisymmetric for some K.

Theorem. *A critical circle map f is conjugate to a rigid rotation by a quasisymmetric homeomorphism h if and only if the rotation number $\rho(f)$ is irrational of bounded type.*

The above result is based on the following a priori estimates called the *Świątek-Herman real a priori bounds* (see [**Sw**],[**dFdM**]):

Theorem. *Let f be a critical circle map with irrational rotation number. Let I_n denote the n-th closest return interval $[c, f^{\circ q_n}(c)]$. Then there exists $N = N(f) > 0$ such that*

$$K^{-1}|I_n| \leq |I_{n+1}| \leq K|I_n|$$

for $n \geq N$ and a universal constant $K > 1$. Moreover, let $\alpha_n : \mathbb{R} \to \mathbb{R}$ denote the affine map which restricts to a map $I_{n-1} \to [0,1]$ sending c to 0, and set $q(z) = z^3$. Then, there exists a C^2-compact family $\mathcal{F}$ of C^3 diffeomorphisms of the interval $[0,1]$ into $\mathbb{R}$ such that for $n > N$,

$$\alpha_n \circ f^{\circ q_n} \circ \alpha_n^{-1}|_{[0,1]} = H_n \circ q \circ h_n,$$

where $H_n \in \mathcal{F}$ and h_n is a C^3 diffeomorphism of $[0,1]$ with $h_n \to \mathrm{id}$ in C^2-topology.

We conclude this section with a useful observation on the combinatorics of closest returns. Let the continued fraction expansion $[a_1, a_2, \dots]$ of the rotation number $\rho(f)$ of a critical circle map f contain at least $n+1$ terms. Then (see [**dMvS**]) for any $i \leq n$, the consecutive closest returns $f^{\circ q_i}(c)$ and $f^{\circ q_{i+1}}(c)$ occur on different sides of the critical point c, that is $[f^{\circ q_i}(c), f^{\circ q_{i+1}}(c)] \ni c$. Let us list some of the points in the forward orbit of c in the order they are encountered when going from $f^{\circ q_{i-1}}(c)$ to $f^{q_i}(c)$:

$$f^{\circ q_{i-1}}(c), f^{\circ q_{i-1}+q_i}(c), f^{\circ q_{i-1}+2q_i}(c), \dots, f^{\circ q_{i-1}+a_{i+1}q_i}(c) = f^{\circ q_{i+1}}(c), c, f^{-q_{i+1}}(c), f^{\circ q_i}(c).$$

When $\rho(f)$ is irrational, Świątek-Herman real a priori bounds imply that for every $N > 0$ there exists a universal constant K_N such that the following holds. For all sufficiently large i, the arcs $[f^{\circ q_{i-1}+(j-1)q_i}(c), f^{\circ q_{i-1}+jq_i}(c)]$, $[f^{-(j-1)q_i}(c), f^{-jq_i}(c)]$ and $[c, f^{\circ q_{i-1}}(c)]$ are K_N-commensurable, for $1 \leq j \leq a_{i+1} - 1$ with $\min(j, a_{i+1} - j) < N$.

3. The Blaschke Model For Petersen's Theorem

As a motivation for further discussion, we present with slight modifications the construction of a model Blaschke product for a Siegel quadratic polynomial used by Petersen in [**Pe**]. Much of the tools developed in this section will carry over to the Blaschke product model for mating introduced in §4. It is somewhat easier, however, to discuss them in this context. Let us define

$$Q^t : z \mapsto e^{2\pi i t} z^2 \left(\frac{z-3}{1-3z} \right). \tag{3.1}$$

As we have seen in the previous section, the restriction $Q^t|_{\mathbb{T}}$ is a critical circle map with critical value $t \in \mathbb{T}$. The standard monotonicity considerations imply that for each irrational number $0 < \theta < 1$ there exists a unique value $t(\theta)$ for which the rotation number $\rho(Q^{t(\theta)}|_{\mathbb{T}}) = \theta$. Let us set $Q_\theta = Q^{t(\theta)}$.

3.1. **Elementary properties.** For the moment, let us work with a fixed irrational θ and abbreviate $Q = Q_\theta$. As seen from (3.1), Q has superattracting fixed points at 0 and ∞ and a double critical point at $z = 1$. The immediate basin of attraction of infinity, which we denote by $A(\infty)$, is a simply-connected region on which Q acts as a degree 2 branched covering. Q commutes with the reflection $\mathcal{T} : z \mapsto 1/\overline{z}$ through $\mathbb{T}$, so we have a similar description for $A(0) = \mathcal{T}(A(\infty))$, the immediate basin of attraction of the origin.

Just as in the polynomial case, there exists a unique conformal isomorphism $\varphi : A(\infty) \xrightarrow{\simeq} \overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ with $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$, which conjugates φ on $A(\infty)$ to the squaring map $z \mapsto z^2$ on $\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. We may use it to define the *external rays* $R^e(t) = \varphi^{-1}\{re^{2\pi i t} : r > 1\}$ for $t \in \mathbb{T}$, and the *equipotentials* $E_r = \varphi^{-1}\{re^{2\pi i t} : t \in \mathbb{T}\}$ for $r > 1$. The ray $R^e(t)$ *lands* at p if $\lim_{r\to 1} \varphi^{-1}(re^{2\pi i t}) = p$.

Proposition 3.1. $A(\infty) = \overline{\mathbb{C}} \smallsetminus \overline{\bigcup_{n\geq 0} Q^{-n}(\mathbb{D})}$.

Proof. Let us put $U = \overline{\mathbb{C}} \smallsetminus \overline{\bigcup_{n\geq 0} Q^{-n}(\mathbb{D})}$. Clearly $A(\infty) \subset U$ and $f(U) \subset U$. Since $\overline{\bigcup_{n\geq 0} Q^{-n}(\mathbb{T})} = J(Q)$, U is a subset of the Fatou set of Q. Assume by way of contradiction that $A(\infty) \neq U$. Then there must be a connected component of U other than $A(\infty)$ which eventually maps to a periodic Fatou component V by Sullivan's No Wandering Theorem. We have $V \neq A(\infty)$, since otherwise Q would have to have a pole $\neq \infty$ in U. According to Fatou-Sullivan, V is either the attracting basin of an attracting or parabolic periodic point, or a Siegel disk or a Herman ring. In the first two cases, there must be a critical point in V which converges to the periodic orbit. But $V \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ and there is no critical point of Q in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$. In the last two cases, some critical point in $J(Q)$ must accumulate on the boundary of the Siegel disk or Herman ring. The only critical point in $J(Q)$ is $z = 1$ whose forward orbit is dense on the unit circle $\mathbb{T}$. It follows that $\mathbb{T}$ must be the boundary of the Siegel disk or a component of the boundary of the Herman ring. Evidently this is impossible since $\mathbb{T}$ is accumulated from both sides by points in $J(Q)$ near the critical point $z = 1$. □

By the theorem of Yoccoz (see subsection 2.3), there exists a unique homeomorphism $h : \mathbb{T} \to \mathbb{T}$ with $h(1) = 1$ such that $h \circ Q|_{\mathbb{T}} = \varrho_\theta \circ h$, where $\varrho_\theta : z \mapsto e^{2\pi i\theta} z$ is the rigid rotation by angle θ. Let $H : \mathbb{D} \to \mathbb{D}$ be a homeomorphic extension of h to the unit disk. To have a canonical homeomorphism at hand, we assume that H is given by the Douady-Earle extension of circle homeomorphisms [**DE**]. Define a modified Blaschke product

$$\tilde{Q}(z) = \tilde{Q}_\theta(z) = \begin{cases} Q(z) & |z| \geq 1 \\ (H^{-1} \circ \varrho_\theta \circ H)(z) & |z| \leq 1 \end{cases} \tag{3.2}$$

where the two definitions match along the boundary of $\mathbb{D}$. Evidently, $\tilde{Q}$ is a degree 2 branched covering of the sphere which is holomorphic outside of the unit disk and is topologically conjugate to a rigid rotation on the unit disk. Imitating the polynomial

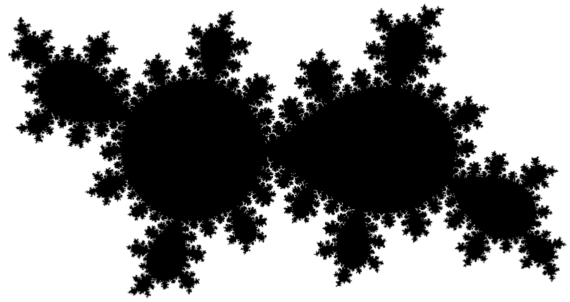

FIGURE 3. *"Filled Julia set" $K(\tilde{Q}_\theta)$ for $\theta = (\sqrt{5}-1)/2$.*

case, we define the "filled Julia set" of $\tilde{Q}$ by

$$K(\tilde{Q}) = \{z \in \mathbb{C} : \text{The orbit } \{\tilde{Q}^{\circ n}(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the "Julia set" of $\tilde{Q}$ as the topological boundary of $K(\tilde{Q})$:

$$J(\tilde{Q}) = \partial K(\tilde{Q}).$$

By Proposition 3.1, we have

$$K(\tilde{Q}) = \overline{\mathbb{C}} \smallsetminus A(\infty), \quad J(\tilde{Q}) = \partial A(\infty).$$

In particular, $K(\tilde{Q})$ is full. Fig. 3 shows the set $K(\tilde{Q})$ for the golden mean $\theta = (\sqrt{5}-1)/2$; In this case, $t(\theta) = 0.613648\ldots$.

3.2. **Drops and their addresses.** In what follows we collect basic facts about the "drops" associated with $\tilde{Q}$ and their addresses (see [**Pe**], and compare [**Za2**] for a more general notion of a drop in a similar family of degree 5 Blaschke products). By definition, the unit disk $\mathbb{D}$ is called the 0-*drop* of $\tilde{Q}$. For $n \geq 1$, any component U of $\tilde{Q}^{-n}(\mathbb{D}) \smallsetminus \mathbb{D}$ is a Jordan domain called an n-*drop*, with n being the *depth* of U. The map $\tilde{Q}^{\circ n} = Q^{\circ n} : U \to \mathbb{D}$ is a conformal isomorphism. The unique point $z = z(U) \in U$ with the property $\tilde{Q}^{\circ n}(z) = H^{-1}(0)$ is called the *center* of U. This is the point in U which eventually maps to the fixed point of the topological rotation on $\tilde{Q} : \mathbb{D} \to \mathbb{D}$. The unique point $\tilde{Q}^{-n}(1) \cap \partial U$ is called the *root* of U and is denoted by $x(U)$. The boundary ∂U is a real-analytic Jordan curve except at the root where it has a definite angle $\pi/3$. We simply refer to U as a drop when the depth is not important for us. Note that there is a unique 1-drop U_1 which is the large Jordan domain attached to the unit disk at its root $x = 1$ (see Fig. 3).

Let U and V be two drops of depths m and n respectively. Then either $\overline{U} \cap \overline{V} = \emptyset$, or else $\overline{U}$ and $\overline{V}$ intersect at a unique point, in which case we necessarily have $m \neq n$. If we assume for example that $m < n$, then it is easy to check that $\overline{U} \cap \overline{V} = x(V)$.

When this is the case, we call U the *parent* of V, or V a *child* of U. It is not hard to check that every n-drop with $n \geq 1$ has a unique parent which is an m-drop with $0 \leq m < n$. In particular the root of this n-drop belongs to the boundary of its parent.

By definition, $\mathbb{D}$ is said to be of *generation* 0. Any child of $\mathbb{D}$ is of generation 1. In general, a drop is of generation k if and only if its parent is of generation $k-1$.

Lemma 3.2 (Roots determine children). *Given a point $p \in \bigcup_{n \geq 0} \tilde{Q}^{-n}(1) \smallsetminus \mathbb{D}$, there exists a unique drop U with $x(U) = p$. In particular, two distinct children of a parent have distinct roots.*

Proof. It suffices to show that U_1 is the only child of $\mathbb{D}$ whose root is $z = 1$. Suppose that $U \neq U_1$ is an n-drop with $x(U) = 1$. Then $\tilde{Q}^{\circ n-1}(U) = U_1$ implies $\tilde{Q}^{\circ n-1}(x(U)) = x(U_1)$, or $\tilde{Q}^{\circ n-1}(1) = 1$. Since $n > 1$ by the assumption, this contradicts the fact that the rotation number of $\tilde{Q}|_{\mathbb{T}} = Q|_{\mathbb{T}}$ is irrational. □

We give a symbolic description of various drops by assigning an address to every drop. This is a slightly modified version of Petersen's approach, based on a suggestion of J. Milnor. Set $U_0 = \mathbb{D}$. For $n \geq 1$, let $x_n = \tilde{Q}^{-n+1}(1) \cap \mathbb{T}$ and U_n be the n-drop with root x_n, which is well-defined by Lemma 3.2. Now let $\iota = \iota_1 \iota_2 \cdots \iota_k$ be any multi-index of length k, where each ι_j is a positive integer. We inductively define the $(\iota_1 + \iota_2 + \cdots + \iota_k)$-drop $U_{\iota_1 \iota_2 \cdots \iota_k}$ of generation k with root

$$x(U_{\iota_1 \iota_2 \cdots \iota_k}) = x_{\iota_1 \iota_2 \cdots \iota_k} \tag{3.3}$$

as follows. We have already defined these for $k = 1$. For the induction step, suppose that we have defined $x_{\iota_1 \iota_2 \cdots \iota_{k-1}}$ for all multi-indices $\iota_1 \iota_2 \cdots \iota_{k-1}$ of length $k-1$. Then, we define

$$x_{\iota_1 \iota_2 \cdots \iota_k} = \begin{cases} \tilde{Q}^{-1}(x_{(\iota_1 - 1)\iota_2 \cdots \iota_k}) \cap \partial U_{\iota_1 \iota_2 \cdots \iota_{k-1}} & \text{if } \iota_1 > 1 \\ \tilde{Q}^{-1}(x_{\iota_2 \cdots \iota_k}) \cap \partial U_{\iota_1 \iota_2 \cdots \iota_{k-1}} & \text{if } \iota_1 = 1 \end{cases} \tag{3.4}$$

The drop $U_{\iota_1 \iota_2 \cdots \iota_k}$ will then be determined by (3.3) and Lemma 3.2 (see Fig. 4).

By the way these drops are given addresses, we have

$$\tilde{Q}(U_{\iota_1 \iota_2 \cdots \iota_k}) = \begin{cases} U_{(\iota_1 - 1)\iota_2 \cdots \iota_k} & \text{if } \iota_1 > 1 \\ U_{\iota_2 \cdots \iota_k} & \text{if } \iota_1 = 1 \end{cases} \tag{3.5}$$

3.3. **Limbs and wakes.** Let us fix a drop $U_{\iota_1 \cdots \iota_k}$. By definition, the *limb* $L_{\iota_1 \cdots \iota_k}$ is the closure of the union of this drop and all its descendants (i.e., children and grand children etc.):

$$L_{\iota_1 \cdots \iota_k} = \overline{\bigcup U_{\iota_1 \cdots \iota_k \cdots}} \, .$$

Note that $L_0 = K(\tilde{Q})$. If $\iota_1 \cdots \iota_k \neq 0$, we call $x_{\iota_1 \cdots \iota_k}$ the root of $L_{\iota_1 \cdots \iota_k}$.

It is not immediatley clear from this definition that limbs provide a useful partition of the filled Julia set $K(\tilde{Q})$. Indeed, it may happen *a priori* that the boundary of a

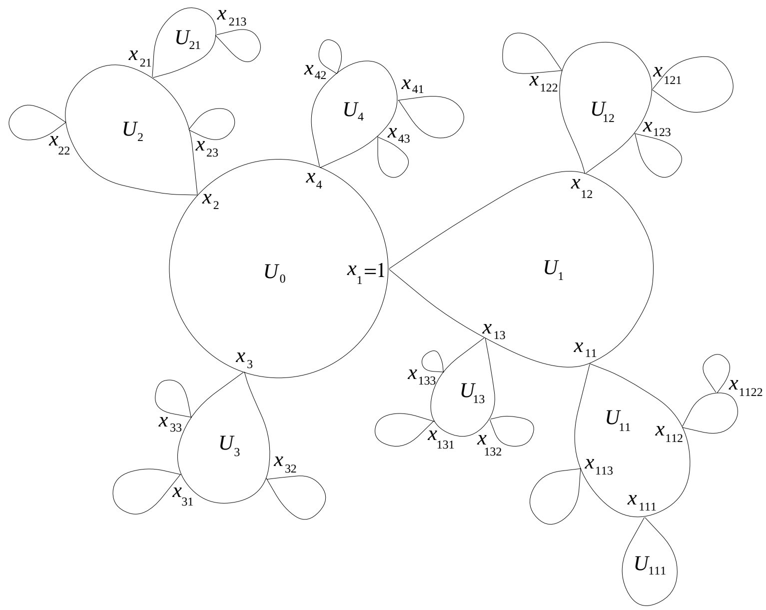

FIGURE 4. *Examples of some drops and their addresses.*

limb$\neq L_0$ is the whole $J(\tilde{Q})$. This is ruled out by the following key lemma of Petersen [**Pe**]:

Lemma 3.3 (Only two rays). *Suppose that $0 < \theta < 1$ is an irrational number. Then the critical point $z = 1$ of Q_θ is the landing point of two and only two external rays $R^e(t)$ and $R^e(s)$ in $A(\infty)$.*

Let W_1 denote the connected component of $\mathbb{C} \smallsetminus (R^e(t) \cup R^e(s) \cup \{1\})$ containing the drop U_1. We call W_1 the *wake* with root x_1. Given an arbitrary multi-index $\iota_1 \cdots \iota_k$, we define the wake $W_{\iota_1 \cdots \iota_k}$ as the appropriate pull-back of W_1. More precisely, consider the two external rays landing at $x_{\iota_1 \cdots \iota_k}$ which map to $R^e(t)$ and $R^e(t)$ under $\tilde{Q}^{\circ n}$, where $n = \iota_1 + \cdots + \iota_k$. These rays separate the plane into two simply-connected regions. The wake $W_{\iota_1 \cdots \iota_k}$ will then be the region containing the drop $U_{\iota_1 \cdots \iota_k}$. It is immediately clear that

$$L_{\iota_1 \cdots \iota_k} = \overline{W}_{\iota_1 \cdots \iota_k} \cap K(\tilde{Q})$$

(see Fig. 5). The integers n and k are respectively called the depth and generation of $W_{\iota_1 \cdots \iota_k}$ as well as $L_{\iota_1 \cdots \iota_k}$.

The next proposition follows directly from the above definitions:

Proposition 3.4 (Properties of limbs and wakes). *Consider $\tilde{Q}_\theta$ for an irrational number $0 < \theta < 1$. Then*

(i) *If a drop U is contained in a limb L, then any child of U is also contained in L.*
(ii) *Any two limbs and any two wakes are either disjoint or nested.*

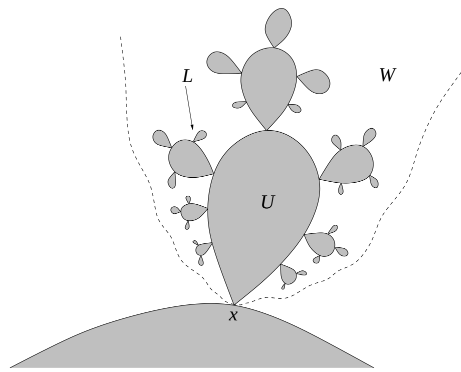

FIGURE 5. *A drop U with root x, and the associated limb L and wake W.*

(iii) *For any limb $L_{\iota_1\cdots\iota_k}$, we have*

$$\tilde{Q}_\theta(L_{\iota_1\cdots\iota_k}) = \begin{cases} L_{(\iota_1-1)\iota_2\cdots\iota_k} & \text{if } \iota_1 > 1 \\ L_{\iota_2\cdots\iota_k} & \text{if } \iota_1 = 1 \end{cases}$$

In particular, every limb eventually maps to L_1 and then to the whole filled Julia set $K(\tilde{Q}_\theta)$. The same relation holds for wakes.

The following theorem is a central result of [**Pe**].

Theorem 3.5 (Local-connectivity). *Suppose that $0 < \theta < 1$ is an irrational number. Then as the depth of a limb L of $\tilde{Q}_\theta$ goes to infinity,* $\mathrm{diam}(L) \to 0$. *This implies that the Julia set $J(Q_\theta)$, hence $J(\tilde{Q}_\theta)$, is locally-connected.*

In particular, it follows that the diameter of a drop goes to zero as the depth goes to infinity, simply because every drop is a subset of the limb with the same root.

One important implication of this result is the lack of the so-called "ghost limbs":

Corollary 3.6 (No ghost limbs). *Suppose that $0 < \theta < 1$ is an irrational number. Then the filled Julia set $K(\tilde{Q}_\theta)$ is the union of $\overline{\mathbb{D}}$ and all the limbs of generation 1:*

$$K(\tilde{Q}_\theta) = \overline{\mathbb{D}} \cup \bigcup_{n \geq 1} L_n.$$

This follows from the fact that distinct L_n's are separated by their wakes and $\mathrm{diam}(L_n) \to 0$ as $n \to \infty$.

3.4. **Drop-chains.**

Definition 3.7. Consider a sequence of drops $\{U_0 = \mathbb{D}, U_{\iota_1}, U_{\iota_1\iota_2}, U_{\iota_1\iota_2\iota_3}, \cdots\}$ where each $U_{\iota_1\cdots\iota_k}$ is the parent of $U_{\iota_1\cdots\iota_{k+1}}$. The closure of the union

$$\mathcal{C} = \overline{\bigcup_k U_{\iota_1\cdots\iota_k}}$$

is called a *drop-chain.*

Since in a drop-chain $\mathcal{C}$ each parent touches the child at its root and the diameter of the subsequent children goes to zero by Theorem 3.5, the tail of $\mathcal{C}$ must converge to a well-defined point in the Julia set of $\tilde{Q}$. In other words, there exists a unique point $p = p(\mathcal{C})$ such that in the Hausdorff topology, $\lim_{k\to\infty} \overline{U}_{\iota_1\cdots\iota_k} = \{p\}$. It follows that

$$\mathcal{C} = \bigcup_k \overline{U}_{\iota_1\cdots\iota_k} \cup \{p\}.$$

In particular, $\mathcal{C}$ is compact, connected and locally-connected.

Another way to characterize $p(\mathcal{C})$ is as follows: Consider the corresponding limbs

$$K(\tilde{Q}) = L_0 \supset L_{\iota_1} \supset L_{\iota_1\iota_2} \supset L_{\iota_1\iota_2\iota_3} \supset \cdots$$

which are nested by Proposition 3.4. Since $\operatorname{diam}(L_{\iota_1\cdots\iota_k}) \to 0$ as $k \to \infty$ by Theorem 3.5, the intersection of these limbs must be a unique point, namely $p(\mathcal{C})$:

$$p(\mathcal{C}) = \bigcap_k L_{\iota_1\cdots\iota_k}.$$

By a *ray* in a drop U we mean a hyperbolic geodesic which connects some boundary point $p \in \partial U$ to the center $z(U)$. This ray is denoted by $[\![p, c(U)]\!]$. For two distinct points $p, q \in \partial U$, we use the notation $[\![p, q]\!]$ for the union of the rays $[\![p, c(U)]\!] \cup [\![c(U), q]\!]$.

Given any drop-chain $\mathcal{C}$, there exists a unique "most efficient" path $R = R(\mathcal{C})$ in $\mathcal{C}$ which connects 0 to $p(\mathcal{C})$. In fact, if $\mathcal{C}$ is of the form $\overline{\bigcup_k U_{\iota_1\cdots\iota_k}}$, we define

$$R(\mathcal{C}) = [\![0, x_{\iota_1}]\!] \cup \bigcup_{k\geq 2} [\![x_{\iota_1\cdots\iota_k}, x_{\iota_1\cdots\iota_{k+1}}]\!] \cup \{p(\mathcal{C})\}.$$

(see Fig. 6). It is easy to see that $R(\mathcal{C})$ is a piecewise analytic embedded arc in the plane. We call $R(\mathcal{C})$ the *drop-ray* associated with $\mathcal{C}$. We often say that $R(\mathcal{C})$, or $\mathcal{C}$, *lands* at $p(\mathcal{C})$.

Proposition 3.8. *Every point in the filled Julia set $K(\tilde{Q}_\theta)$ either belongs to the closure of a drop or is the landing point of a unique drop-chain.*

Proof. Let $p \in K(\tilde{Q}_\theta)$ and assume that p does not belong to the closure of any drop. Then by Corollary 3.6, p belongs to some limb L_{ι_1}, and inductively, it follows that it belongs to the intersection of a decreasing sequence of limbs $L_{\iota_1} \supset L_{\iota_1\iota_2} \supset L_{\iota_1\iota_2\iota_3} \supset \cdots$. Hence p is the landing point of the corresponding drop-chain $\mathcal{C} = \overline{\bigcup_k U_{\iota_1\cdots\iota_k}}$. Uniqueness of this drop-chain follows from Proposition 3.9 below. $\square$

It follows from the next proposition that the union of drop-rays associated with all drop-chains has the structure of an infinite topological tree (a "dendrite") in the plane.

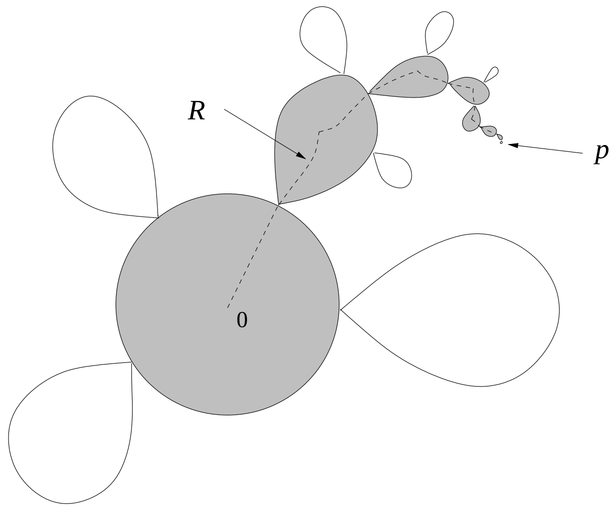

FIGURE 6. *A drop-chain and the drop-ray associated with it.*

Proposition 3.9. *The assignment $\mathcal{C} \mapsto p(\mathcal{C})$ is one-to-one. In other words, different drop-rays land at distinct points.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct drop-chains. Let $U_{\iota_1 \cdots \iota_k} \subset \mathcal{C}_1$ be the drop of smallest generation k which is disjoint from $\mathcal{C}_2$, and similarly define $U_{\iota'_1 \cdots \iota'_k} \subset \mathcal{C}_2$. The limbs $L_{\iota_1 \cdots \iota_k}$ and $L_{\iota'_1 \cdots \iota'_k}$ are disjoint by Proposition 3.4. Since $p(\mathcal{C}_1) \in L_{\iota_1 \cdots \iota_k}$ and $p(\mathcal{C}_2) \in L_{\iota'_1 \cdots \iota'_k}$, we will have $p(\mathcal{C}_1) \neq p(\mathcal{C}_2)$. $\square$

3.5. **Surgery.** The modified Blaschke product $\tilde{Q} = \tilde{Q}_\theta$ as defined in (3.2) is a degree 2 branched covering of the sphere. When the rotation number θ is irrational of bounded type, the action of $\tilde{Q}_\theta$ is in fact conjugate to that of a quadratic polynomial. This follows from a *quasiconformal surgery* construction due to Douady, Ghys, Herman, and Shishikura [**Do3**].

Let us fix an irrational number $0 < \theta < 1$ of bounded type. By Herman's Theorem (see subsection 2.3) the unique homeomorphism $h : \mathbb{T} \to \mathbb{T}$ with $h(1) = 1$ which conjugates $Q|_{\mathbb{T}}$ to ϱ_θ is quasisymmetric. In this case, the Douady-Earle extension $H : \mathbb{D} \to \mathbb{D}$ of h is a quasiconformal homeomorphism whose dilatation only depends on the dilatation of h [**DE**]. The modified Blaschke product $\tilde{Q}_\theta$ of (3.2) is then a quasiregular branched covering of the sphere. We define a $\tilde{Q}_\theta$-invariant conformal structure σ_θ on the plane as follows: On $\mathbb{D}$, let σ_θ be the pull-back $H^*\sigma_0$ of the standard conformal structure σ_0. Since ϱ_θ preserves σ_0, $\tilde{Q}_\theta$ will preserve σ_θ on $\mathbb{D}$. For every $n \geq 1$, pull $\sigma_\theta|_{\mathbb{D}}$ back by $\tilde{Q}_\theta^{\circ n} = Q_\theta^{\circ n}$ on $\tilde{Q}_\theta^{-n}(\mathbb{D}) \smallsetminus \mathbb{D}$, which consists of all drops of Q_θ of depth n. Since $Q_\theta^{\circ n}$ is holomorphic, this does not increase the dilatation of σ_θ. Finally, let $\sigma_\theta = \sigma_0$ on the rest of the plane. By construction, σ_θ has bounded dilatation and is invariant under $\tilde{Q}_\theta$. Therefore, by the Measurable Riemann

Mapping Theorem (see for example [**AB**]), we can find a unique quasiconformal homeomorphism $\psi_\theta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, normalized by $\psi_\theta(\infty) = \infty$, $\psi_\theta(H^{-1}(0)) = e^{2\pi i\theta}/2$ and $\psi_\theta(1) = 0$, such that $\psi_\theta^* \sigma_0 = \sigma_\theta$. Set

$$f_\theta = \psi_\theta \circ \tilde{Q}_\theta \circ \psi_\theta^{-1}. \tag{3.6}$$

Then f_θ is a quasiregular self-map of the sphere which preserves σ_0, hence it is holomorphic. Also $f_\theta : \mathbb{C} \to \mathbb{C}$ is a proper map of degree 2 since $\tilde{Q}_\theta$ has the same properties. Therefore f_θ is a quadratic polynomial.

Since the action of f_θ on $\psi_\theta(\mathbb{D})$ is quasiconformally conjugate to a rigid rotation, $\psi_\theta(\mathbb{D})$ is contained in a Siegel disk for f_θ with rotation number θ. As $\psi_\theta(1) = 0$ is a critical point for f_θ, it follows that the entire orbit $\{f_\theta^{\circ n}(0)\}_{n \geq 0}$ lies on the boundary of this Siegel disk. But $\{f_\theta^{\circ n}(0)\}_{n \geq 0}$ is dense on $\psi_\theta(\mathbb{T})$, so $\psi_\theta(\mathbb{T})$ is exactly the boundary of this Siegel disk, which is a quasicircle passing through the critical point 0 of f_θ. Up to affine conjugacy there is only one quadratic polynomial with a fixed Siegel disk of the given rotation number θ. By the way we normalized ψ_θ, we must have $f_\theta : z \mapsto z^2 + c_\theta$ as in (1.1).

We summarize the above as follows:

Theorem 3.10 (Douady, Ghys, Herman, Shishikura)**.** *Let f be a quadratic polynomial which has a fixed Siegel disk Δ of rotation number θ. If θ is of bounded type, then f is quasiconformally conjugate to $\tilde{Q}_\theta$ in* (3.2)*. In particular, $\partial\Delta$ is a quasicircle passing through the critical point of f.*

In particular, this surgery procedure allows us to define drops, limbs, wakes, drop-chains and drop-rays for the quadratic polynomial f_θ.

4. A Blaschke Model For Mating

The object of this section is to construct, for a pair of numbers $0 < \theta, \nu < 1$ with $\theta \neq 1 - \nu$, a Blaschke product $B_{\theta,\nu}$. When θ and ν are irrationals of bounded type, $B_{\theta,\nu}$ plays the role of a model for the quadratic rational map $F_{\theta,\nu}$ of (2.1) in the same way as Q_θ does for the quadratic polynomial f_θ. After showing the existence of such $B_{\theta,\nu}$, we will define drops, limbs, drop-chains and drop-rays for the "modified" $\tilde{B}_{\theta,\nu}$ in an analogous way.

4.1. **Existence.** We would like to prove the following result:

Theorem 4.1 (Existence of Blaschke models for mating)**.** *Let $0 \leq \theta < 1$, $0 \leq \nu < 1$ and $\theta \neq 1 - \nu$. Then there exists a degree 3 Blaschke product*

$$B = B_{\theta,\nu} : z \mapsto \frac{e^{-2\pi i\nu}}{ab}\ z\ \left(\frac{z-a}{1-\overline{a}z}\right)\left(\frac{z-b}{1-\overline{b}z}\right) \tag{4.1}$$

with the following properties:

(i) *$0 < |a| < 1$ and $|b| = |a|^{-1} > 1$, with $a\overline{b} \neq 1$,*
(ii) *B has a double critical point at $z = 1$, and*
(iii) *The restriction $B|_{\mathbb{T}}$ is a critical circle map with rotation number θ.*

The proof of this theorem will be given in the rest of this subsection. In (i) the condition $a\overline{b} \neq 1$ is necessary simply because when $a\overline{b} = 1$, B reduces to the linear map $z \mapsto e^{-2\pi i\nu} z$.

For simplicity, let us set

$$\begin{array}{l} \kappa = ab, \text{ where } |\kappa| = 1 \text{ by (i)} \\ \zeta = a + b \end{array} \tag{4.2}$$

Using the equation (4.1), the condition $B'(z) = 0$ may be written in the form

$$A_1 z^4 + A_2 z^3 + A_3 z^2 + \overline{A}_2 z + \overline{A}_1 = 0,$$

where

$$\begin{array}{l} A_1 = \overline{a}\,\overline{b} = \overline{\kappa}, \\ A_2 = -2(\overline{a} + \overline{b}) = -2\overline{\zeta}, \\ A_3 = 2 + |a+b|^2 = 2 + |\zeta|^2. \end{array} \tag{4.3}$$

A brief computation shows that the condition of $z = 1$ being a double critical point of B translates into

$$\begin{cases} 4A_1 + 3A_2 + 2A_3 = -\overline{A}_2 \\ 3A_1 + 2A_2 + A_3 = \overline{A}_1 \end{cases}$$

or by (4.3)

$$\begin{cases} 2\kappa - 3\zeta + 2 + |\zeta|^2 = \overline{\zeta} \\ 3\kappa - 4\zeta + 2 + |\zeta|^2 = \overline{\kappa} \end{cases} \tag{4.4}$$

Subtracting the second equation in (4.4) from the first equation, we find that

$$\zeta - \kappa = \overline{\zeta} - \overline{\kappa} \Longrightarrow \zeta - \kappa \in \mathbb{R}.$$

Set $\kappa = x + iy$ and $\zeta = u + iy$ and substitute them into the first equation in (4.4) to obtain

$$u^2 - 4u + (2x + y^2 + 2) = 0,$$

which, by $x^2 + y^2 = 1$, has solutions $u = x + 1$ and $u = -x + 3$. These correspond to $\zeta = \kappa + 1$ and $\zeta = -\overline{\kappa} + 3$. By (4.2), the choice of $\zeta = \kappa + 1$ leads to $a = \kappa$ or $a = 1$, which is not appropriate since we want $|a| < 1$. Therefore, we are left with the only possibility

$$\zeta = -\overline{\kappa} + 3. \tag{4.5}$$

Let $\kappa = e^{2\pi i t}$ with $t \in \mathbb{R}$. From (4.2) and (4.5) it follows that a and b are the solutions of the quadratic equation

$$z^2 + (\overline{\kappa} - 3)z + \kappa = 0. \tag{4.6}$$

Lemma 4.2. *As $\kappa = e^{2\pi it}$ goes around the unit circle, the two solutions of the quadratic equation (4.6) define two closed curves $t \mapsto a(t)$ and $t \mapsto b(t)$ in the complex plane with the following properties (see Fig. 7):*

(i) *$a(t+1) = a(t)$ and $b(t+1) = b(t)$,*
(ii) *$0 < |a(t)| \leq 1$ and hence $|b(t)| = |a(t)|^{-1} \geq 1$,*
(iii) *$|a(t)| = 1$ if and only if $t \in \mathbb{Z}$, or equivalently $\kappa = 1$, in which case $a(t) = b(t) = 1$,*
(iv) *$a(t)\overline{b(t)} \neq 1$ unless $t \in \mathbb{Z}$ so that $a(t) = b(t) = 1$.*

Proof. Let us first note that the solutions z_1, z_2 of (4.6) lie on the unit circle $\mathbb{T}$ if and only if $\kappa = 1$ in which case there is a double root at $z_1 = z_2 = 1$. In fact, if $|z_1| = |z_2| = 1$, then

$$2 = 3 - |\overline{\kappa}| \leq |\overline{\kappa} - 3| = |z_1 + z_2| \leq |z_1| + |z_2| = 2.$$

Hence $|\overline{\kappa} - 3| = 2$, or equivalently, $\kappa = 1$.

Now let $\kappa = e^{2\pi it}$ go around $\mathbb{T}$. Then the double root at $z = 1$ splits into distinct roots $a = a(t)$ and $b = b(t)$ which by inspecting the explicit formula for a and b are real-analytic functions of t away from integer values and are labeled so that (ii) holds. Clearly a and b are $\mathbb{Z}$-periodic, so (i) holds trivially.

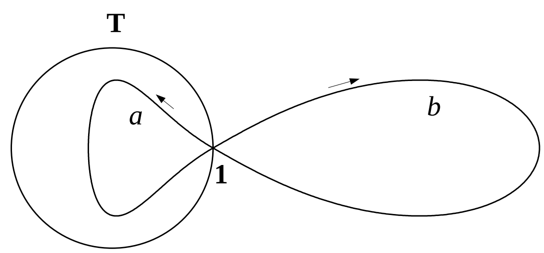

FIGURE 7

Finally, suppose that for some $t \in \mathbb{R}$, $a = a(t)$ and $b = b(t)$ satisfy $a\overline{b} = 1$. Then $a/\overline{a} = \kappa$, or $\overline{a} = a\,\overline{\kappa}$. Since a is a solution of (4.6), we have

$$\overline{a}^2 + (\kappa - 3)\overline{a} + \overline{\kappa} = 0 \Longrightarrow a^2\overline{\kappa}^2 + (\kappa - 3)a\overline{\kappa} + \overline{\kappa} = 0,$$

or, after multiplying by κ^2,

$$a^2 + \kappa(\kappa - 3)a + \kappa = 0. \tag{4.7}$$

Comparing (4.7) and (4.6) for $z = a$, we conclude that

$$\kappa(\kappa - 3) = \overline{\kappa} - 3 \Longrightarrow \kappa^2(\kappa - 3) = 1 - 3\kappa \Longrightarrow (\kappa - 1)^3 = 0$$

which shows $\kappa = 1$. $\square$

Lemma 4.3. *For any $z \in \mathbb{T}$, the closed curve $\Gamma_z : [0,1] \to \mathbb{T}$ defined by*

$$\Gamma_z(t) = \left(\frac{z - a(t)}{1 - \overline{a(t)}z} \right) \left(\frac{z - b(t)}{1 - \overline{b(t)}z} \right) \tag{4.8}$$

is null-homotopic.

Note that when $z = 1$, there is no ambiguity in the definition of Γ_z. In fact, by (4.2) and (4.5),

$$\Gamma_1 = \frac{1 - \zeta + \kappa}{1 - \overline{\zeta} + \overline{\kappa}} = \frac{-2 + \kappa + \overline{\kappa}}{-2 + \kappa + \overline{\kappa}} \equiv 1$$

so that Γ_1 is the constant loop 1.

Proof. Consider the two homotopies $(t,s) \mapsto a(t,s)$ and $(t,s) \mapsto b(t,s)$ rel $\{1\}$ defined by

$$a(t,s) = (1-s)a(t) + s, \qquad b(t,s) = (1-s)b(t) + s.$$

Note that $|a(t,s)| \leq 1$ and $|b(t,s)| \leq 1$, with the equality if and only if $a(t,s) = 1$ and $b(t,s) = 1$. Consider the map defined by

$$H(t,s) = \left(\frac{z - a(t,s)}{1 - \overline{a(t,s)}z} \right) \left(\frac{z - b(t,s)}{1 - \overline{b(t,s)}z} \right)$$

A brief computation shows that when $z = 1$, $H(t,s) \equiv 1$. Evidently H defines a homotopy between $H(\cdot, 0) = \Gamma_z$ and the constant loop $H(\cdot, 1) = 1$. □

Proof of Theorem 4.1. Start with the closed curves $t \mapsto a(t)$ and $t \mapsto b(t)$ of Lemma 4.2 and form the Blaschke product

$$B^t : z \mapsto e^{-2\pi i(\nu + t)}\ z\ \left(\frac{z - a(t)}{1 - \overline{a(t)}z} \right) \left(\frac{z - b(t)}{1 - \overline{b(t)}z} \right).$$

When t is not an integer, B^t has degree 3 by Lemma 4.2(iv) and satisfies conditions (i) and (ii) required by Theorem 4.1. Moreover, it maps the unit circle $\mathbb{T}$ to itself, and has no critical points in $\mathbb{T}$ other than 1, hence $B^t|_{\mathbb{T}}$ is a critical circle map. So to finish the proof, it suffices to show that for some $t \notin \mathbb{Z}$, the rotation number of the restriction of B^t to the circle $\mathbb{T}$ is equal to θ. To this end, consider the universal covering map $\mathbb{R} \to \mathbb{T}$ given by $z = z(w) = e^{2\pi i w}$. Since $B^0 : z \mapsto e^{-2\pi i \nu} z$, a lifting of B^0 to the real line will be the affine map $\hat{B}^0 : w \mapsto -\nu + w$. The loop $\{t \mapsto B^t\}_{0 \leq t \leq 1}$ can then be lifted to a path $\{t \mapsto \hat{B}^t\}_{0 \leq t \leq 1}$, with

$$\hat{B}^t : w \mapsto -\nu - t + w + \frac{1}{2\pi i} \log(\Gamma_{e^{2\pi i w}}(t)),$$

where Γ_z is the closed curve defined in (4.8). Let $\rho(t) = \lim_{n \to \infty} (\hat{B}^t)^{\circ n}(w)/n$. It is a standard fact that ρ is well-defined and independent of w and the map $t \mapsto$

$\rho(t)$ is continuous (see for example [**dMvS**]). The rotation number of B^t is then the fractional part of $\rho(t)$. Evidently $\rho(0) = -\nu$. Since Γ_z is null-homotopic by Lemma 4.3, we simply have $\hat{B}^1 : w \mapsto -\nu - 1 + w$, so that $\rho(1) = -\nu - 1$. It follows that for some t between 0 and 1, $\rho(t) \equiv \theta \pmod 1$. Hence the rotation number of the corresponding B^t is θ. $\square$

4.2. **Corollaries of the construction.** As we shall see below, the Blaschke product $B_{\theta,\nu}$ we constructed above and the Blaschke model Q_θ of §3 share many common properties. This will allow us to define drops, limbs, drop-chains etc. in a similar fashion for $B_{\theta,\nu}$. We will also describe a quasiconformal surgery transforming $B_{\theta,\nu}$ into the quadratic rational map $F_{\theta,\nu}$.

Let $0 < \theta < 1$ be irrational and $0 < \nu < 1$ be irrational of Brjuno type, and set $B = B_{\theta,\nu}$. By (4.1), $B(z) = e^{-2\pi i\nu} z + O(z^2)$ near $z = 0$, so by the theorem of Brjuno-Yoccoz [**Yo2**] the origin is the center of a Siegel disk U^0 for B. We have $U^0 \subset \mathbb{D}$ since the unit circle is a subset of the Julia set. Since B commutes with the reflection $\mathcal{T} : z \mapsto 1/\overline{z}$, there exists a Siegel disk $U^\infty = \mathcal{T}(U^0)$ centered at infinity. In the local coordinate $w = 1/z$ near infinity, the map $w \mapsto 1/B(1/w)$ has the form $w \mapsto e^{2\pi i\nu} w + O(w^2)$, so the rotation number of U^∞ is $\frac{1}{2\pi i} \log B'(\infty) = \nu$.

B has zeros at $\{0, a, b\}$ and poles at $\{\infty, 1/\overline{a}, 1/\overline{b}\}$. The preimage $B^{-1}(\mathbb{T})$ consists of $\mathbb{T}$ and an analytic closed curve homeomorphic to a figure eight with the double point at $z = 1$. This curve and the basic dynamics of B are shown in Fig. 8. By the

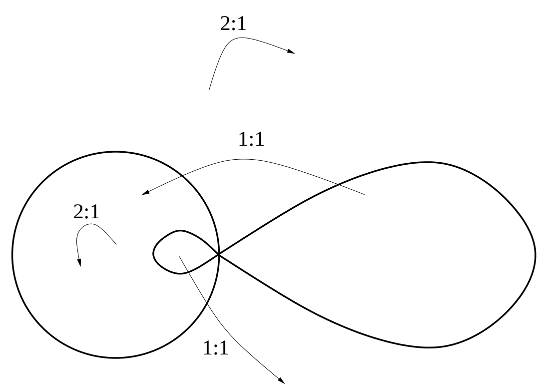

FIGURE 8. *The preimage $B^{-1}(\mathbb{T})$ and the basic dynamics of B.*

theorem of Yoccoz (see subsection 2.3), there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$, unique if we require that $h(1) = 1$, such that $h \circ B|_{\mathbb{T}} = \varrho_\theta \circ h$. Denoting by $H : \mathbb{D} \to \mathbb{D}$ the Douady-Earle extension of h, we define the modified map $\tilde{B}$ as

$$\tilde{B}(z) = \tilde{B}_{\theta,\nu}(z) = \begin{cases} B(z) & |z| \geq 1 \\ (H^{-1} \circ \varrho_\theta \circ H)(z) & |z| \leq 1 \end{cases} \tag{4.9}$$

The map $\tilde{B}$ is a degree 2 branched covering of the sphere, holomorphic outside of $\mathbb{D}$. It has a Siegel disk U^∞ centered at ∞ and a "topological Siegel disk," namely the unit disk $\mathbb{D}$, on which its action is topologically conjugate to an irrational rotation.

The definition of drops and their addresses for the map $\tilde{B}$ carries over word for word from subsection 3.2. In particular, the unit disk $\mathbb{D}$ is the 0-drop, and its immediate preimage $U_1 = \tilde{B}^{-1}(\mathbb{D}) \smallsetminus \mathbb{D}$ is the 1-drop of $\tilde{B}$. As before, the root of the drop $U_{\iota_1\iota_2\ldots\iota_k}$ is the point $x_{\iota_1\iota_2\ldots\iota_k} = \partial U_{\iota_1\iota_2\ldots\iota_{k-1}\iota_k} \cap \partial U_{\iota_1\iota_2\ldots\iota_{k-1}}$. As in subsection 3.4, for each sequence of drops $\{U_0 = \mathbb{D}, U_{\iota_1}, U_{\iota_1\iota_2}, \ldots\}$ where each $U_{\iota_1\ldots\iota_k}$ is the parent of $U_{\iota_1\ldots\iota_{k+1}}$, we define the drop-chain

$$\mathcal{C} = \overline{\bigcup_k U_{\iota_1\ldots\iota_k}}, \tag{4.10}$$

and the corresponding drop-ray $R(\mathcal{C}) \subset \mathcal{C}$. We can also define the limb $L_{\iota_1\ldots\iota_k}$ as the closure of the union of $U_{\iota_1\ldots\iota_k}$ and all its descendants:

$$L_{\iota_1\ldots\iota_k} = \overline{\bigcup U_{\iota_1\ldots\iota_k\ldots}}\ .$$

In anticipation of the analogue of Theorem 3.5, let us define the *accumulation set* of the drop-chain $\mathcal{C}$ in (4.10) as the intersection of the decreasing sequence of limbs $L_{\iota_1} \supset L_{\iota_1\iota_2} \supset L_{\iota_1\iota_2\iota_3} \supset \cdots$. In the case when this set is a single point $\{p\}$, we shall say that $R(\mathcal{C})$ or $\mathcal{C}$ *lands at* p.

As an analogue to the "filled Julia set" $K(\tilde{Q})$, we define

$$K(\tilde{B}) = K(\tilde{B}_{\theta,\nu}) = \{z \in \mathbb{C} : \text{The orbit } \{\tilde{B}^{\circ n}(z)\}_{n\geq 0} \text{ never intersects } U^\infty\}$$

and

$$J(\tilde{B}) = \partial K(\tilde{B}).$$

Both sets are nonempty and compact. However, $K(\tilde{B})$ is no longer full. The simply-connected basin of infinity for $\tilde{Q}$ is replaced by the Siegel disk U^∞ of $\tilde{B}$ and all its infinitely many preimages (compare Fig. 9).

Finally, if θ is of bounded type, we can perform the same kind of quasiconformal surgery as in subsection 3.5 to obtain a quadratic rational map from $\tilde{B}$. In this case by Herman's theorem (see subsection 2.3) the homeomorphism h which linearizes $B|_{\mathbb{T}}$ is quasisymmetric, therefore its Douady-Earl extension H is quasiconformal. The map $\tilde{B} = \tilde{B}_{\theta,\nu}$ is a quasiregular branched covering of the Riemann sphere. We define a $\tilde{B}_{\theta,\nu}$-invariant conformal structure $\sigma_{\theta,\nu}$ on the sphere by setting it equal to the standard structure σ_0 on $\mathbb{C} \smallsetminus K(\tilde{B}_{\theta,\nu})$, to $H^*\sigma_0$ on $\mathbb{D}$, and to $(\tilde{B}_{\theta,\nu}^{\circ n})^* H^*\sigma_0 = (B_{\theta,\nu}^{\circ n})^* H^*\sigma_0$ on every drop of depth n. The maximal dilatation of $\sigma_{\theta,\nu}$ is equal to the dilatation of H, and by the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\psi^*\sigma_0 = \sigma_{\theta,\nu}$. The conjugated map $F = \psi \circ B_{\theta,\nu} \circ \psi^{-1}$ is a degree 2 holomorphic branched covering of the sphere, that is a quadratic rational map. Let us normalize ψ by assuming $\psi(\infty) = \infty$, $\psi(H^{-1}(0)) = 0$

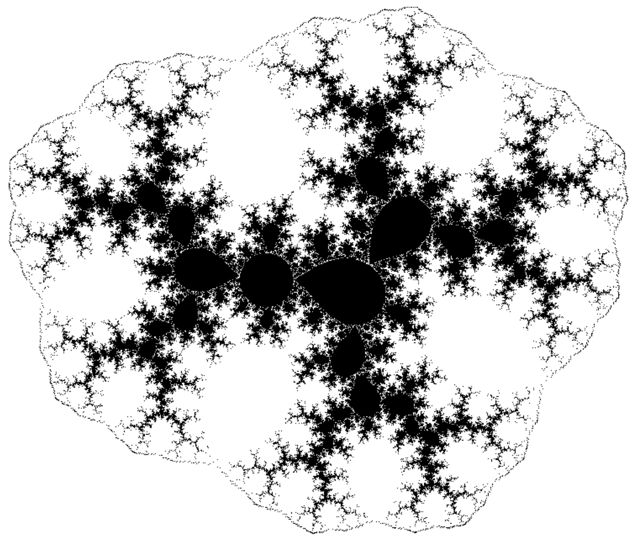

FIGURE 9. *Set* $K(\tilde{B}_{\theta,\nu})$ *for* $\theta = \nu = (\sqrt{5}-1)/2$. *Numerical experiment gives* $a = -0.019048 - 0.298116i$, $b = 3.280417 - 0.667122i$ *for these choices of* θ *and* ν. *There is a striking similarity with the corresponding picture for the quadratic rational map* F *of Fig. 2, up to a* $90°$ *rotation. The reason is the existence of a quasiconformal homeomorphism conjugating* $\tilde{B}_{\theta,\nu}$ *to* F *which is conformal in the white region.*

and $\psi(\beta) = 1$, where β denotes the fixed point of $B_{\theta,\nu}$ in $\mathbb{C} \smallsetminus (U^\infty \cup \mathbb{D})$. By inspection, we have $F = F_{\theta,\nu}$ in (2.1), so that

$$F_{\theta,\nu} = \psi \circ B_{\theta,\nu} \circ \psi^{-1}.$$

Recall that $F_{\theta,\nu}$ has two Siegel disks Δ^0 and Δ^∞ centered at 0 and ∞, which are the images $\Delta^0 = \psi(\mathbb{D})$ and $\Delta^\infty = \psi(U^\infty)$. As a first consequence we obtain

Theorem 4.1. *Let* $0 < \theta < 1$ *be an irrational of bounded type. Then the boundary of the Siegel disk* Δ^0 *of* $F_{\theta,\nu}$ *is a quasicircle passing through a single critical point of* $F_{\theta,\nu}$.

Observe that there is a natural symmetry

$$F_{\theta,\nu} = \mathcal{I} \circ F_{\nu,\theta} \circ \mathcal{I},$$

where $\mathcal{I}$ is the involution $z \mapsto 1/z$.

Corollary 4.4. *Suppose that both $0 < \theta < 1$ and $0 < \nu < 1$ are irrationals of bounded type. Then the boundaries of the Siegel disks Δ^0 and Δ^∞ of $F_{\theta,\nu}$ are disjoint quasicircles, each passing through a critical point of $F_{\theta,\nu}$.*

The involution $\mathcal{I}$ provides us with a quasiconformal conjugacy between $\tilde{B}_{\theta,\nu}$ and $\tilde{B}_{\nu,\theta}$. In particular, setting

$$K^\infty(\tilde{B}_{\theta,\nu}) = \overline{\mathbb{C} \setminus K(\tilde{B}_{\theta,\nu})},$$

we have

Corollary 4.5. *There exists a quasiconformal homeomorphism of the Riemann sphere mapping the set $K^\infty(\tilde{B}_{\theta,\nu})$ to $K(\tilde{B}_{\nu,\theta})$.*

Hence for the map $\tilde{B}_{\theta,\nu}$ we can naturally define the *drops growing from infinity* $U^\infty_{\iota_1\ldots\iota_k} \subset \mathbb{C} \setminus K(\tilde{B}_{\theta,\nu})$, with $U^\infty_0 = U^\infty$, limbs growing from infinity $L^\infty_{\iota_1\ldots\iota_k}$, etc.

We conclude with another immediate corollary of the above construction:

Corollary 4.6. *With the above notation, $\partial K(\tilde{B}_{\theta,\nu}) = \partial K^\infty(\tilde{B}_{\theta,\nu})$.*

Proof. Under the surgery construction, both sets $\partial K(\tilde{B}_{\theta,\nu})$ and $\partial K^\infty(\tilde{B}_{\theta,\nu})$ correspond to the Julia set $J(F_{\theta,\nu})$. ☐

5. Construction of Puzzle-Pieces

The goal of this section and the next one is to establish the following analogue of Theorem 3.5:

Theorem 5.1. *Let $0 < \theta, \nu < 1$ be irrationals of bounded type, with $\theta \neq 1 - \nu$, and consider the modified Blaschke product $\tilde{B}_{\theta,\nu}$ of (4.9). Then as the depth of a limb $L_{\iota_1\ldots\iota_k}$ goes to infinity,* $\mathrm{diam}(L_{\iota_1\ldots\iota_k})$ *goes to zero.*

It follows from Corollary 4.5 that $\mathrm{diam}(L^\infty_{\iota_1\ldots\iota_k}) \to 0$ as $\iota_1 + \ldots + \iota_k \to \infty$.

We start by constructing *puzzle-pieces*. Our construction closely parallels the one presented by Petersen in [**Pe**]. For simplicity, set $B = B_{\theta,\nu}$ and $\tilde{B} = \tilde{B}_{\theta,\nu}$. Denote by $\mathcal{C}$ the drop-chain

$$\mathcal{C} = \overline{U_0 \cup U_1 \cup U_{11} \cup U_{111} \cup \cdots}\,.$$

The following refinement of Douady-Hubbard-Sullivan Landing Theorem can be found in [**TY**]:

Lemma 5.2. *Let F be a rational map and let Λ denote the closure of the union of the postcritical set and possible rotation domains of F. Suppose that $\gamma : (-\infty, 0] \to \overline{\mathbb{C}} \setminus \Lambda$ is a curve with*

$$F^{\circ nk}(\gamma(-\infty, -k]) = \gamma(-\infty, 0]$$

for all positive integers k. Then $\lim_{t\to-\infty} \gamma(t)$ exists and is a repelling or parabolic periodic point of F whose period divides n.

We can apply the above lemma to the drop-chain $\mathcal{C}$, setting γ to be the drop-ray $R(\mathcal{C})$ parameterized so that the root of the $(k+1)$-st drop corresponds to $t=-k$. We conclude that $R(\mathcal{C})$ lands at the unique fixed point β of B in $\overline{\mathbb{C}} \smallsetminus (\mathbb{D} \cup U^\infty)$. Since β is necessarily repelling, the size of the drops in $\mathcal{C}$ decreases geometrically, and the drop-chain $\mathcal{C}$ lands at the point β. Repeating the argument, we see that the drop-ray $R(\mathcal{D})$ associated to the drop-chain

$$\mathcal{D} = \overline{U^\infty \cup U_1^\infty \cup U_{11}^\infty \cup U_{111}^\infty \cup \cdots}$$

lands at a fixed point as well, which is necessarily β. Let $\mathcal{C}'$ be the drop-chain $\overline{U_0 \cup U_2 \cup U_{21} \cup \cdots}$ mapped to $\mathcal{C}$ by $\tilde{B}$, and similarly define the drop-chain $\mathcal{D}' = \overline{U^\infty \cup U_2^\infty \cup U_{21}^\infty \cup \cdots}$. Then $\mathcal{C}'$ and $\mathcal{D}'$ have a common landing point $\beta' \neq \beta$, which is a preimage of β in $\overline{\mathbb{C}} \smallsetminus (\mathbb{D} \cup U^\infty)$.

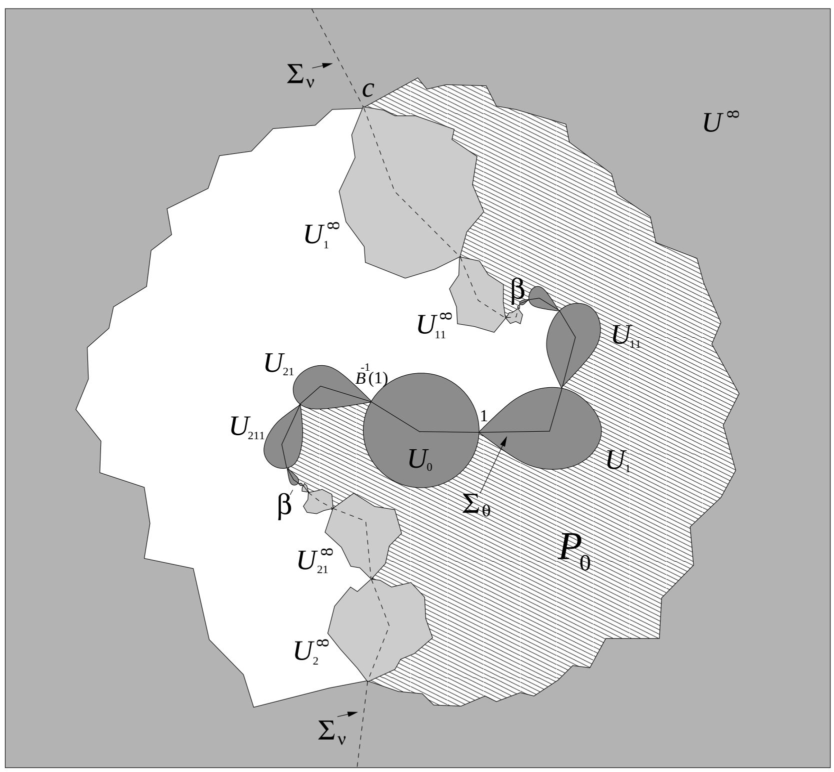

FIGURE 10. *The* 0*-th critical puzzle-piece* P_0 *and the "spines"* Σ_θ *and* Σ_ν *(see §7).*

As before, the moments of closest returns of the critical point $z=1$ are denoted by $\{q_n\}$. Recall that these numbers appear as the denominators of the convergents of the continued fraction of θ. We define the 0*-th critical puzzle-piece* P_0 as the closure of the connected component of

$$\overline{\mathbb{C}} \setminus (\mathbb{D} \cup U^\infty \cup \mathcal{C} \cup \mathcal{C}' \cup \mathcal{D} \cup \mathcal{D}')$$

which contains the arc $[1, B^{-1}(1)] \ni B^{\circ q_1}(1)$ in the boundary (see Fig. 10). We inductively define the n-th critical puzzle-piece $P_n \subset \overline{\mathbb{C}} \setminus \mathbb{D}$ as the closed set which is mapped homeomorphically onto P_{n-1} by $B^{\circ q_n}$ and which contains the arc $[1, B^{-q_n}(1)] \subset \mathbb{T}$ in the boundary. The following proposition summarizes some of the properties of critical puzzle-pieces:

Proposition 5.3 (Properties of puzzle-pieces)**.**

(i) *The puzzle-piece P_n intersects the unit circle $\mathbb{T}$ along the arc $[1, B^{-q_n}(1)]$.*
(ii) $B^{\circ q_n}(P_n \cap \partial U_1) = [B^{\circ q_n}(1), B^{-q_{n-1}}(1)]$.
(iii) $B^{\circ q_n + q_{n-1} + q_{n-2}}(P_n \cap \partial U_{q_n+1}) = [1, B^{\circ q_{n-1}+q_{n-2}}(1)]$.
(iv) *P_n contains the drop $U_{q_{n+2}+1}$.*

Proof. Observe that $B^{\circ q_n}$ is a homeomorphism $[B^{-q_n}(1), B^{-q_n - q_{n-1}}(1)] \xrightarrow{\simeq} [B^{-q_{n-1}}, 1]$ with one critical point at 1. Thus the univalent inverse branch B^{-q_n} sending P_{n-1} to P_n maps the arc $[B^{-q_{n-1}}, 1]$ onto the union of $[1, B^{-q_n}(1)]$ and a subarc of ∂U_1. The first three statements now follow by induction on n. As seen from the combinatorics of closest returns (see §2.3) $\partial U_{q_{n+2}+1} \cap \mathbb{T} = B^{-q_{n+2}}(1)$ is contained in the arc $[1, B^{-q_n}(1)]$. Evidently, the drop $U_{q_{n+2}+1}$ has no intersections with ∂P_n, thus $U_{q_{n+2}+1} \subset P_n$. □

The preimages of the puzzle-piece P_0 have the following nesting property:

Lemma 5.4. *Let A_1 and A_2 be two distinct univalent pull-backs of the puzzle-piece P_0 such that $\mathring{A}_1 \cap \mathring{A}_2 \neq \emptyset$. Then either $A_1 \subset A_2$ or $A_2 \subset A_1$.*

Proof. By construction, the boundary of the puzzle-piece P_0 consists of an open arc $\gamma \subset \mathcal{C} \cup \mathcal{C}'$ which is made up of the boundary arcs of various drops $U_{\iota_1 \ldots \iota_k}$, a similarly defined arc $\gamma^\infty \subset \mathcal{D} \cup \mathcal{D}'$ and points β, β' (see Fig. 10). Denote by γ_1, γ_1^∞, β_1, β_1' the corresponding parts of ∂A_1, and label the boundary of A_2 in the same way.

Evidently γ_1 does not intersect γ_2^∞ or the points β_2, β_2', so it can only intersect γ_2. Similarly, γ_1^∞ can only intersect γ_2^∞. Suppose that $y \in \{\beta_1, \beta_1'\} \cap \{\beta_2, \beta_2'\}$. Then $B^{-k}(\beta) = y$ for some choice of the inverse branch. Since β is not in the post-critical set of B, this branch of B^{-k} has a univalent extension to a neighborhood of β intersecting the boundary of P_0 along a non-empty open arc. Pulling back, it follows that for some neighborhood D of y, $\gamma_1 \cap D = \gamma_2 \cap D$ and $\gamma_1^\infty \cap D = \gamma_2^\infty \cap D$.

Now assume that the claim is false. Let $A_1 = B^{-m}(P_0)$ and $A_2 = B^{-n}(P_0)$, with $m \leq n$. Then by the above observation, either γ_2 or γ_2^∞ intersects both $\mathring{A}_1$ and $\mathbb{C} \setminus A_1$. Therefore, either $B^{\circ m}(\gamma_2)$ or $B^{\circ m}(\gamma_2^\infty)$ intersects both $\mathring{P}_0$ and $\mathbb{C} \setminus P_0$. To fix the ideas, let us assume that $B^{\circ m}(\gamma_2)$ does. Note that $B^{\circ m}(\gamma_2) \cap \partial P_0 \subset \gamma$, hence $B^{\circ m}(\gamma_2)$ must intersect the union of the drop-rays $R(\mathcal{C}) \cup R(\mathcal{C}')$ *transversally* at a root x of a drop in $\mathcal{C} \cup \mathcal{C}'$. Now under $B^{\circ n-m}$ a small open subarc of $B^{\circ m}(\gamma_2)$ around

x maps homeomorphically to a subarc $\delta \subset \gamma$ around $B^{\circ n-m}(x)$. Since the orbit $x, B(x), \ldots, B^{\circ n-m}(x)$ does not contain the critical point 1, it follows that δ also intersects $R(\mathcal{C}) \cup R(\mathcal{C}')$ transversally at $B^{\circ n-m}(x)$, which is impossible. $\square$

Corollary 5.5. *For all $n \geq 0$ we have $P_{n+2} \subsetneqq P_n$.*

Proof. It is clear from the definition of critical puzzle-pieces that $\mathring{P}_{n+2} \cap \mathring{P}_n \neq \emptyset$. By Proposition 5.3(i), $P_{n+2} \cap \mathbb{T} \subsetneqq P_n \cap \mathbb{T}$. The claim now follows from Lemma 5.4. $\square$

Lemma 5.6. *Let U be a topological disk whose boundary is contained in a finite union of the boundary arcs of drops (resp. drops growing from infinity). Then U itself must be a drop (resp. drop growing from infinity).*

Proof. Let us consider the case of drops. The proof for the case of drops growing from infinity is similar. The modified Blaschke product $\tilde{B}$ is an open mapping, so it satisfies the Maximum Principle in $\mathbb{C} \smallsetminus U_1^\infty$. Since $\tilde{B}^{\circ n}(\partial U) \subset \mathbb{T}$ for a large n, we must have $\tilde{B}^{\circ n}(U) \subset \mathbb{D}$, which means U itself is a drop. $\square$

Lemma 5.7. *Let A be a univalent pull-back of the puzzle-piece P_0. Suppose that a drop at infinity $U^\infty_{\iota_1 \ldots \iota_k}$ is contained in A. Then A contains the whole limb $L^\infty_{\iota_1 \ldots \iota_k}$.*

Proof. Let us denote by $\gamma_A^\infty \subset \partial A$ the part of the boundary of A made up of the boundary arcs of drops at infinity. Assume by way of contradiction that there is a drop at infinity $U^\infty_{\iota_1 \ldots \iota_k \ldots \iota_{k+m}} \not\subset A$. Let $\mathcal{D}$ be a drop-chain containing $U^\infty_{\iota_1 \ldots \iota_k \ldots \iota_{k+m}}$. Let $\delta \subset \partial \mathcal{D}$ be an arc connecting the root of $U^\infty_{\iota_1}$ to a point in $\partial U^\infty_{\iota_1 \ldots \iota_k \ldots \iota_{k+m}} \smallsetminus \overline{A}$. Then δ goes in and out of A, but it only intersects ∂A at the points of γ_A^∞. Thus the curves δ and γ_A^∞ bound a topological disk $U \subset \mathring{A}$. By Lemma 5.6, U itself is a drop growing from infinity. Since U shares a non-trivial boundary arc with another drop growing from infinity, we arrive at a contradiction. $\square$

Lemma 5.8. *The puzzle-piece P_n contains a Euclidean disk D centered at a point in $J(B)$ with $\operatorname{diam}(D) > K|[1, B^{-q_n}(1)]|$ for some K independent of n.*

Proof. Note first that by Proposition 5.3(iv), $U_{q_{n+2}+1} \subset P_n$. Since $B^{\circ q_{n+2}}(1)$ is a closest return of the critical point $1 \in \mathbb{T}$, $B^{-q_{n+2}}|_{\mathbb{T}}$ maps the arc $(B^{-q_{n+2}}(1), B^{\circ q_{n+2}}(1))$ diffeomorphically onto $(B^{-2q_{n+2}}(1), 1)$. This inverse branch has a univalent extension to a neighborhood of 1, which we denote by ψ_n. By Świątek-Herman real a priori bounds (see the discussion in the end of §2.3), the segments $[B^{-2q_{n+2}}(1), B^{-q_{n+2}}(1)]$, $[B^{-q_{n+2}}(1), 1]$ and $[1, B^{\circ q_{n+2}}(1)]$ are K_1-commensurable. Here the constant K_1 becomes universal for sufficiently large n and therefore can be chosen independent of n. Moreover, $1/K_2 \leq |\psi_n'(1)| \leq K_2$ for some $K_2 > 1$ which is also independent of n. By Koebe Distortion Theorem we may choose a Euclidean disk D around the point 1 commensurable with $[1, B^{\circ q_{n+2}}(1)]$ such that ψ_n has bounded distortion in D. Now let us pull back a sub-disk $D' \subset D$ centered at a point in ∂U_1 to obtain a

Euclidean disk $D_1 \subset \mathbb{C} \setminus \mathbb{D}$ around a point in $\partial U_{q_{n+2}+1}$ such that both $\operatorname{diam} D_1$ and $\operatorname{dist}(D_1, B^{-q_{n+2}}(1))$ are K_3-commensurable with $[1, B^{\circ q_{n+2}}(1)]$ for some K_3 independent of n.

Denote by $D_1' \subset \mathbb{D}$ the disk symmetric to D_1 with respect to $\mathbb{T}$. Let $D_2 \subset U_1$ be given by $B(D_2) = D_1'$. It is clear that D_2 is again commensurable with $[1, B^{\circ q_{n+2}}(1)]$, and so is $\operatorname{dist}(D_2, 1)$. By Koebe Distortion Theorem, the image $\psi_n(D_2) \subset U_{q_{n+2}+1} \subset P_n$ contains a Euclidean disk with the desired properties. □

The last property of puzzle-pieces we need is the following:

Lemma 5.9. *There exists $N > 0$ such that for all $n \geq N$ the puzzle-piece P_n does not intersect ∂U^∞.*

Proof. Since the boundary of the Siegel disk U^∞ is forward-invariant, we only need to show the existence of one N such that $P_N \cap \partial U^\infty = \emptyset$. Assume this is false. Let us denote by l_n the boundary arc of P_n connecting 1 to ∂U^∞. By Lemma 5.4, the curves in the orbit

$$l_n, B(l_n), \ldots, B^{\circ q_n - 1}(l_n) \tag{5.1}$$

are disjoint. By the theorem of Yoccoz (see subsection 2.3) the maps $B|_{\mathbb{T}}$ and $B|_{\partial U^\infty}$ are topologically conjugate to rigid rotations. Since the inverse orbit of a point under an irrational rotation is dense on the circle, the maximum diameter of the pieces into which the curves (5.1) partition the boundaries of $\mathbb{D}$ and U^∞ goes to zero as $n \to \infty$. We may therefore construct an orientation-reversing topological conjugacy between the circle maps $B|_{\mathbb{T}}$ and $B|_{\partial U^\infty}$. This contradicts the fact that $\theta \neq 1 - \nu$. □

6. Complex Bounds

The proof of Petersen's Theorem presented in [**Ya**] is based on a version of estimates employed in the same paper for proving a renormalization convergence result. In renormalization theory it is customary to use the term *complex a priori bounds* for such estimates. Our goal in this section is to adapt these bounds to the Blaschke product model introduced in §4.

As before, let us fix irrationals $0 < \theta, \nu < 1$ of bounded type, with $\theta \neq 1 - \nu$, and set $B = B_{\theta,\nu}$, $\tilde{B} = \tilde{B}_{\theta,\nu}$. Recall that B is a Blaschke product of the form

$$B = z \mapsto \lambda\, z \, \left(\frac{z - a}{1 - \overline{a} z}\right) \left(\frac{z - b}{1 - \overline{b} z}\right),$$

where $|\lambda| = 1$, $0 < |a| < 1$ and $|b| = |a|^{-1}$. We set

$$B(1) = e^{2\pi i \tau} \quad \text{with } \; 0 < \tau < 1.$$

The convergents of the continued fraction $\theta = [a_1, a_2, a_3, \dots]$ will be denoted $\{p_n/q_n\}$. First note that $(B(z)-B(1))/(z-1)^3$ is a bounded holomorphic function in the domain $\mathbb{C} \smallsetminus \overline{(\mathbb{D} \cup U^\infty \cup U_1^\infty)}$. As a consequence,

$$C^{-1}|z-1|^3 < |B(z) - B(1)| < C|z-1|^3 \tag{6.1}$$

in this domain, for some positive constant C.

Let S be the translation-invariant infinite strip which is mapped onto the open topological annulus $\mathbb{C} \smallsetminus (\overline{U^0} \cup \overline{U^\infty})$ by the exponential map $z \mapsto e^{2\pi i z}$. Let us denote by S_J the domain obtained by removing from S the points of the real line that do not belong to the interval $J \subset \mathbb{R}$:

$$S_J = (S \smallsetminus \mathbb{R}) \cup J.$$

Let $\hat{B}(z)$ denote the (multi-valued) meromorphic function $\frac{1}{2\pi i} \log B(e^{2\pi i z})$ on S. On the real line $\hat{B}$ has singularities at the integer points, whose images lie at the integer translates of $0 < \tau < 1$. Its other singularities lie at the boundary curves of S at the points $\pm s + j,\ j \in \mathbb{Z}$, which are mapped by the exponential map to the critical points on the boundaries of the Siegel disks U^0 and U^∞ of B. By the Monodromy Theorem, in the domain $S_{(\tau+i,\tau+i+1)}$ with the critical values removed, we have well-defined branches $\phi_{i,m}$ of the inverse $\hat{B}^{-1}$, mapping the open interval $(\tau+i, \tau+i+1)$ homeomorphically onto the interval between two consecutive integers $(m, m+1)$ (see Fig. 11). The maps $\phi_{i,m}$ range over the simply-connected regions

$$S_{(m,m+1)} \smallsetminus \left[\left(\frac{\pm 1}{2\pi i} \log(\overline{U}_1^\infty)\right) \cup \left(\frac{\pm 1}{2\pi i} \log(\overline{U}_1)\right) \right]. \tag{6.2}$$

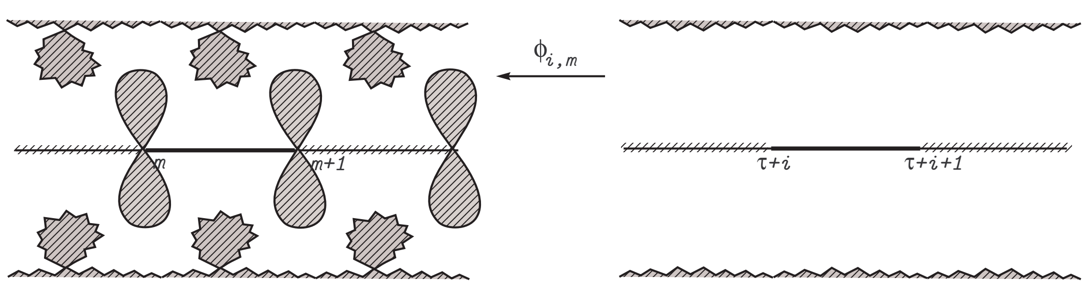

Figure 11

Denote by $\Upsilon : \mathbb{T} \smallsetminus \{B(1)\} \to I = (\tau - 1, \tau)$ the single-valued branch of $\frac{1}{2\pi i} \log(z)$ mapping 1 to 0. Define the (discontinuous) map $\phi : I \to I$ by

$$\phi(z) = \begin{cases} \phi_{-1,0}(z) & \text{for } z \in (\tau - 1, \Upsilon(B^{\circ 2}(1))], \\ \phi_{-1,-1}(z) & \text{for } z \in (\Upsilon(B^{\circ 2}(1)), \tau). \end{cases}$$

Let us fix an $n \geq 2$ and consider the inverse orbit

$$(1, B^{\circ q_n}(1)), (B^{-1}(1), B^{\circ q_n - 1}(1)), \ldots, (B^{-q_n}(1), 1). \tag{6.3}$$

Set $J_{-i} = \Upsilon((B^{-i}(1), B^{\circ q_n - i}(1)))$ and consider the ϕ-orbit

$$J_0, J_{-1}, J_{-2}, \ldots, J_{-q_n}. \tag{6.4}$$

By the combinatorics of closest returns (see subsection 2.3) the smallest value of $i > 0$ for which the arc $B^{-i}((B^{-q_n}(1), 1)) \subset \mathbb{T}$ contains the critical point 1 is q_{n+1}. Also, the smallest $j > 0$ for which $1 \in B^{\circ j}((B^{-q_n}(1), 1))$ is $q_{n+1} + q_n$. As $q_{n+1} \geq q_n + 2$, the interval $(B^{-k-2}(1), B^{\circ q_n - k - 2}(1))$ does not contain 1 for $0 \leq k \leq q_n - 1$. Hence, $B^{\circ 2}(1) \notin (B^{-k}(1), B^{\circ q_n - k}(1))$ for $0 \leq k \leq q_n - 1$. In other words, the intervals J_{-k} of the orbit (6.4) for $0 \leq k \leq q_n - 1$ do not contain the point of discontinuity of the map ϕ. By its definition, the map $\phi : J_{-k} \to J_{-k-1}$ for $0 \leq k \leq q_n - 1$ has a univalent extension to $S_{J_{-k}}$. As seen from (6.2) the range of this univalent map is a subset of $S_{J_{-k-1}}$, hence the composition $\phi^l : J_{-i} \to J_{-i-l}$ for $0 \leq i < i + l \leq q_n$ univalently extends to the entire $S_{J_{-i}}$.

Consider the univalent extensions of the iterates $\phi^k : J_0 \to J_{-k}$ to the strip S_{J_0} for $1 \leq k \leq q_n$. Applying these univalent branches to a point $z \in S_{J_0}$, we obtain the *inverse orbit, corresponding to the orbit (6.4)*

$$z = z_0, z_{-1}, z_{-2}, \ldots, z_{-q_n}, \text{ where } z_{-k} = \phi^k(z_0). \tag{6.5}$$

A corresponding inverse orbit of a subset of S_{J_0} is similarly defined.

Let $\mathbb{C}_J \supset S_J$ denote the slit plane $(\mathbb{C} \smallsetminus \mathbb{R}) \cup J$. One easily constructs a conformal mapping of this domain to the upper half-plane to verify that the hyperbolic neighborhood $\{z \in \mathbb{C}_J \,|\, \mathrm{dist}_{\mathbb{C}_J}(z, J) < r\}$ for $r > 0$ is the union $D_\theta(J)$ of two Euclidean disks of equal radii with common chord J intersecting the real axis at an outer angle $\theta = \theta(r)$ (see [**dMvS**]). An elementary computation yields in this case

$$r = \log \tan(\pi/2 - \theta/4).$$

The standard properties of conformal maps imply that the hyperbolic neighborhood $\{z \in S_J \,|\, \mathrm{dist}_{S_J}(z, J) < r\}$ also forms angles $\theta = \theta(r)$ with $\mathbb{R}$. We choose the notation $G_\theta(J)$ for this neighborhood. The Schwarz Lemma implies that $G_\theta(J) \subset D_\theta(J)$.

Let $\breve{S} \subset \mathbb{C}$ be a horizontal strip invariant under the unit translation, which is compactly contained in S. A specific choice of $\breve{S}$ will be made later in our arguments (see the remarks before Lemma 6.4). Let I be a bounded interval in $\mathbb{R}$. For a point $z \in S_I$ not belonging to $\mathbb{R}$, denote by $0 < \widehat{(z, I)} < \pi/2$ the least of the outer angles the segments joining z to the end-points of I form with the real line. The following adaptation of Lemma 2.1 of [**Ya**] will be used to control the expansion of inverse branches:

Lemma 6.1. *Let us fix n and consider the inverse orbit (6.5). Let $k \leq q_n - 1$. Assume that for some i between 0 and k, $z_{-i} \in \breve{S}$ and $\widehat{(z_{-i}, J_{-i})} > \epsilon > 0$. Then*

$$\frac{\operatorname{dist}(z_{-k}, J_{-k})}{|J_{-k}|} \leq C \ \frac{\operatorname{dist}(z_{-i}, J_{-i})}{|J_{-i}|}$$

for some constant $C = C(\epsilon, \breve{S}) > 0$.

Proof. First observe that $B^{-q_n}|_{\mathbb{T}}$ is a diffeomorphism on the arc $[B^{\circ 2q_n}(1), B^{-q_n}(1)] \subset \mathbb{T}$ which contains the arc $[B^{\circ q_n}(1), 1]$ in its interior. Moreover, by Świątek-Herman real a priori bounds (see subsection 2.3), the latter arc is contained well inside of the former. As seen from the combinatorics of closest returns, the iterates $B^{-j}([B^{\circ 2q_n}(1), B^{-q_n}(1)])$ do not contain $B^{\circ 2}(1)$ for $j \leq q_n - 1$. Setting $H = \Upsilon([B^{\circ 2q_n}(1), B^{-q_n}(1)])$, we see that J_0 is contained well inside of H, and $\phi^j : J_0 \to J_{-j}$ univalently extends to S_H for $1 \leq j \leq q_n - 1$. Set $T = \phi^i(H) \supset J_{-i}$. By Koebe Distortion Theorem, there exists $\rho > 0$ such that both components of $T \smallsetminus J_{-i}$ have length at least $2\rho|J_{-i}|$. Note that the iterate

$$\phi^{\circ k-i} : J_{-i} \to J_{-k}$$

has a univalent extension to S_T.

Let us normalize the situation by considering the orientation-preserving affine maps

$$\alpha_1 : J_{-i} \to [0,1] \text{ and } \alpha_2 : J_{-k} \to [0,1].$$

The composition $\alpha_2 \circ \phi^{\circ k-i} \circ \alpha_1^{-1}$ is defined in a straight horizontal strip

$$Y = \{z \in \mathbb{C}_{[-2\rho, 1+2\rho]} : |\operatorname{Im} z| < M\}$$

for some $M > 0$ independent of n. The space of normalized univalent maps of Y is compact by Koebe Theorem, thus the statement is true if $\operatorname{dist}(z, J_{-i})/|J_{-i}| < \rho$.

Now assume $\operatorname{dist}(z, J_{-i})/|J_{-i}| > \rho$. Consider the smallest closed hyperbolic neighborhood $\overline{G_\theta(J_{-i})}$ containing z_{-i}. Recall that z_{-i} is contained in a strip $\breve{S} \Subset S$. For a point $\zeta \in \mathbb{C}_I$ with $\operatorname{dist}(\zeta, I) > \rho|I|$ and $\widehat{(\zeta, I)} > \epsilon$, the smallest closed neighborhood $\overline{D_\theta(I)} \ni \zeta$ satisfies $\operatorname{diam} D_\theta(I) \leq C(\rho, \epsilon) \operatorname{dist}(\zeta, I)$ (see [**Ya**], Lemma 2.1). Therefore, we have $\operatorname{diam} G_\theta(J_{-i}) \leq C(\rho, \epsilon, \breve{S}) \operatorname{dist}(z, J_{-i})$ and by Koebe Theorem,

$$\frac{\operatorname{diam} G_\theta(J_{-i})}{|J_{-i}|} \sim \frac{\operatorname{diam} G_\theta(J_{-k})}{|J_{-k}|}.$$

By the Schwarz Lemma, $z_{-k} \in G_\theta(J_{-k})$ and the claim follows. □

Set $I_m = \Upsilon([1, B^{\circ q_m}(1)])$, and let G_m denote the hyperbolic neighborhood

$$G_\alpha(\Upsilon([B^{\circ q_{m+1}}(1), B^{q_m - q_{m+1}}(1)]))$$

where $0 < \alpha < \pi/2$ will be specified later. The following two lemmas are direct adaptations of Lemmas 4.2 and 4.4 of [**Ya**], for which the reader is referred for a detailed discussion supplemented with figures. In both lemmas we work with the orbit (6.5) for some fixed value of n.

Lemma 6.2. *Let J and J' be two consecutive returns of the orbit (6.4) of J_0 to I_m for $n > m > 1$ and let ζ, ζ' be the corresponding points of the inverse orbit (6.5). If $\zeta \in G_m$, then either $\zeta' \in G_m$ or $\widehat{(\zeta', J')} > \epsilon$ and $\operatorname{dist}(\zeta', J') < C|I_m|$ where the constants ϵ and C are independent of m.*

We remark that the constants ϵ and C will in general depend on the choice of the Blaschke product B. The argument is illustrated in Fig. 12.

Proof. Note that $J = J_{-i}$ and $J' = J_{-i-q_{m+1}}$ for some $i < q_n - q_{m+1}$. Recall that $G_m = G_\alpha(\Upsilon([B^{\circ q_{m+1}}(1), B^{q_m - q_{m+1}}(1)]))$. Let $\check{G}_m$ denote the pull-back of G_m along the inverse orbit $J, \ldots, J'$. Also let G'_m denote the pull-back of G_m along the piece of the orbit $J, \ldots, \phi^{\circ q_m - 1}(J)$, and let $G''_m = \phi(G'_m)$.

The combinatorics of closest returns (see subsection 2.3) implies that the restriction $B^{-q_{m-1}}|_{(B^{\circ q_{m+1}}(1), B^{q_m - q_{m+1}}(1))}$ is a diffeomorphism. Hence the pull-back of G_m along the orbit $J, \ldots, \phi^{\circ q_m - 1}(J)$ is univalent. By the Schwarz Lemma,

$$G'_m \subset G_\alpha(\Upsilon([B^{\circ q_{m+1} - q_m + 1}(1), B^{1 - q_{m+1}}(1)])).$$

By Świątek-Herman real a priori bounds, the critical value τ divides the interval $\Upsilon([B^{\circ q_{m+1} - q_m + 1}(1), B^{1 - q_{m+1}}(1)])$ into K_1-commensurable pieces, where K_1 becomes universal for large m, and can therefore be chosen simultaneously for all m. As the absolute value of the derivative of the exponential map is bounded away from 0 and ∞ on the strip S, the estimate (6.1) is still valid for the lifted map near the critical point. Together with elementary properties of the cube root map this implies that $G''_m \subset G_\beta([\Upsilon(B^{\circ q_{m+1} - q_m}(1), 1]))$ for some $\beta > 0$ independent of m. Let $V_0 \subset S$ be the union of the connected components of $\frac{\pm 1}{2\pi i} \log(\overline{U}_1)$ attached to 0 (see Fig. 12). Since the boundary of G''_m contains a segment of ∂V_0 which forms outer angles $\pi/3$ with $\mathbb{R}$ at 0, we have $G''_m \subset G_\gamma([\Upsilon(B^{\circ q_{m+1} - q_m}(1)), a_1]) \cup G_\sigma([a_2, 0])$ where the points $\Upsilon(B^{\circ q_{m+1} - q_m}(1)), a_1, a_2, 0$ form a K_2-bounded configuration with $K_2, \gamma > 0$ and $\sigma > \pi/2 > \alpha$ independent of m.

The pull-back of G''_m to $\check{G}_m$ is univalent. Applying the Schwarz Lemma we have $\check{G}_m \subset G_m \cup G_\gamma([0, \Upsilon(B^{-q_{m+1} + q_m}(a_1))])$ and the claim follows. □

Lemma 6.3. *Let J be the last return of the orbit (6.4) to the interval I_m preceding the first return to I_{m+1} for $n - 1 > m > 1$, and let J' and J'' be the first two returns to I_{m+1}. Let ζ, ζ' and ζ'' be the corresponding points in the inverse orbit (6.5), so that $\zeta' = \phi^{\circ q_m}(\zeta)$, $\zeta'' = \phi^{\circ q_{m+2}}(\zeta')$. Suppose that $\zeta \in G_m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon = \epsilon(B) > 0$ and $\operatorname{dist}(\zeta'', J'') < C(B)|I_{m+1}|$, or $\zeta'' \in G_{m+1}$.*

Proof. Note that $J \subset \Upsilon([B^{\circ q_{m+1} + q_m}(1), B^{\circ q_m}(1)])$. By the Schwarz Lemma,

$$\zeta' \in G_\beta(\Upsilon([B^{\circ q_{m+1} - q_m}(1), 1]))$$

for some $\beta > 0$ independent of m. Denote by $\hat{J}$ and $\check{J}$ the intervals of (6.4) such that $\phi^{\circ q_{m+1} - q_m}(J') = \hat{J}$ and $\phi^{\circ q_m}(\hat{J}) = \check{J}$, and let $\hat{\zeta}$, $\check{\zeta}$ be the corresponding points of (6.5).

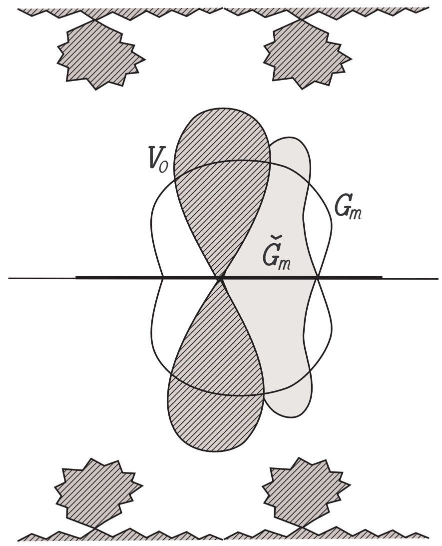

FIGURE 12

We have $\hat{J} \subset \Upsilon([B^{\circ q_m}(1), B^{\circ q_m - q_{m+1}}(1)])$ and $\hat{\zeta} \in G_\beta(\Upsilon([B^{\circ q_m}(1), B^{\circ q_m - q_{m+1}}(1)]))$. By the Schwarz Lemma and elementary properties of the map B (see (6.1)), there exist points b_1, b_2 in $\Upsilon([1, B^{-q_{m+1}}(1)])$ such that 0, b_1, b_2, $\Upsilon(B^{-q_{m+1}}(1))$ form a K-bounded configuration, and

$$\check{\zeta} \in G_\sigma([0, b_1]) \cup G_\gamma([b_2, \Upsilon(B^{-q_{m+1}}(1))])$$

for σ and γ independent of m and $\sigma > \pi/2$. The claim now follows from the Schwarz Lemma. $\square$

Let us now select a strip $\check{S} \Subset S$ used in Lemma 6.1. By Lemma 5.9 there exists $N > 0$ such that $P_n \cap \partial U^\infty = \emptyset$ for all $n \geq N$. Let E be an annulus around the unit circle, compactly contained in the domain $\mathbb{C} \smallsetminus (\overline{U}^\infty \cup \overline{U}^0)$ and such that $P_N \cup P_{N+1} \subset E$. We set $\check{S}$ to be the strip $\frac{1}{2\pi i}\log(E)$. Let $\hat{P}_n$ denote the component of $\frac{1}{2\pi i}\log(P_n)$ attached to $\Upsilon([1, B^{-q_n}(1)])$. Our argument culminates in the next lemma:

Lemma 6.4. *As before let P_n denote the n-th critical puzzle piece and N be as above. Then for all $n \geq N+3$ we have*

$$\operatorname{diam} P_n \leq C_1 \sqrt[3]{\frac{\operatorname{diam} P_{n-1}}{|[B^{\circ q_{n-1}}(1), 1]|}} \cdot |[1, B^{-q_n}(1)]| + C_2 \tag{6.6}$$

for positive constants C_1, C_2 independent of n. Moreover, for $z \in \hat{P}_n$, either $z \in G_{n-1}$ or $(\widehat{z, I_{n-1}}) > \epsilon > 0$, where ϵ is again independent of n.

Proof. Choose $\alpha > 0$ in the definition of G_n so that

$$\hat{P}_{N+2} \cup \hat{P}_{N+3} \subset G_\alpha(\Upsilon([B^{\circ q_{N+2}}(1), B^{q_{N+1}-q_{N+2}}(1)])) = G_{N+1}.$$

By Corollary 5.5, $P_{n+2} \subset P_n$ for all n, hence $\hat{P}_n \subset G_{N+1}$ for all $n \geq N+3$. Fix a value of $n > N+4$. Let

$$\Pi_0 = \hat{P}_{n-1}, \Pi_{-1}, \ldots, \Pi_{-q_n} = \hat{P}_n \tag{6.7}$$

be the inverse orbit corresponding to the orbit (6.4). We begin by establishing

$$\frac{\operatorname{diam} \Pi_{-(q_n-1)}}{|J_{-(q_n-1)}|} \leq K_1 \frac{\operatorname{diam} \hat{P}_{n-1}}{|J_0|} \tag{6.8}$$

for some constant K_1 which does not depend on n.

Let $z \in \partial \hat{P}_{n-1}$ and consider the inverse orbit (6.5). Let $m \leq n$ be the largest value for which $z \in G_m$. We will prove the estimate (6.8) using an induction on m. Let $T_{-1}, \ldots, T_{-k}$ be the consecutive returns of the orbit of J_0 as (6.4) to I_m until the first return to I_{m+1}, and let $\zeta_{-1}, \ldots, \zeta_{-k}$ be the corresponding points in (6.5). Note that by Świątek-Herman real a priori bounds, the intervals T_{-i} are all K-commensurable with J_0, for some K independent of n. It is easily seen from the combinatorics of the closest returns that the elements Π_{-k_i} of the inverse orbit (6.7) corresponding to the points ζ_{-i} intersect the real axis along a subset of $(\hat{P}_N \cup \hat{P}_{N+1}) \cap \mathbb{R}$. By Lemma 5.4, $\Pi_{-k_i} \subset \hat{P}_N \cup \hat{P}_{N+1}$, so $\zeta_{-i} \in \breve{S}$. By Lemma 6.2, either there exists a moment i between 0 and k such that

$$(\widehat{\zeta_{-i}, I_m}) > \epsilon \text{ and } \operatorname{dist}(\zeta_{-i}, T_{-i}) < C|I_m|,$$

or $\zeta_{-k} \in G_m$. In the former case we derive (6.8) from Lemma 6.1. In the latter case, consider the point ζ'' which corresponds to the second return of (6.4) to I_{m+1}. By Lemma 6.3, either $(\widehat{\zeta'', I_{m+1}}) > \epsilon$ and $\operatorname{dist}(\zeta'', I_{m+1}) < C|I_{m+1}|$, or $\zeta'' \in G_{m+1}$.

In the first case we are done again by Lemma 6.1. In the second case the proof of (6.8) is completed by induction on m. The same argument implies that either $(\widehat{z_{-q_n}, J_{-q_n}}) > \epsilon$, or $z_{-q_n} \in G_{n-1}$. The estimate (6.6) follows from (6.8) and (6.1). $\square$

The estimate (6.6) implies that if $\dfrac{\operatorname{diam} P_{n-1}}{|[B^{\circ q_{n-1}}(1), 1]|} > K$ for a large $K > 0$, then

$$1 < \frac{\operatorname{diam} P_n}{|[1, B^{-q_n}(1)]|} < \frac{1}{2} \cdot \frac{\operatorname{diam} P_{n-1}}{|[B^{\circ q_{n-1}}(1), 1]|}.$$

This implies that for large n the puzzle-piece P_n is commensurable with its base arc $[1, B^{-q_n}(1)]$. In combination with the previous lemma, this shows that $P_n \subset G_\sigma(\Upsilon(I_{n-1}))$ for some fixed $\sigma > 0$. Applying the Schwarz Lemma to the inverse orbit

$$P_n, B^{\circ q_{n+1}-q_n}(P_{n+1}), B^{\circ q_{n+1}-2q_n}(P_{n+1}), \ldots, B^{\circ q_{n-1}}(P_{n+1}), P_{n+1},$$

we see that

Corollary 6.5. *There exists an angle $\gamma > 0$ such that for large values of n,*
$$\hat{P}_{n+1} \subset G_\gamma(\Upsilon([1, B^{-q_{n+1}}(1)])).$$

Let us summarize the consequences. We first prove the following:

Lemma 6.6 (Only two drop-chains)**.** *There are exactly two drop-chains of the form $\mathcal{D}_1 = \overline{\bigcup_k U^\infty_{\iota_1\ldots\iota_k}}$ and $\mathcal{D}_2 = \overline{\bigcup_k U^\infty_{\iota'_1\ldots\iota'_k}}$ accumulating at the critical point 1. Moreover, both of these drop-chains land at 1, and they separate U_1 from $\mathbb{D}$, in the sense that U_1 and $\mathbb{D}$ belong to different components of $\overline{\mathbb{C}} \smallsetminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.*

Proof. Let $\mathcal{D} = \overline{\bigcup_k U^\infty_{\iota_1\ldots\iota_k}}$ be any drop-chain accumulating at 1. This implies that for an arbitrarily large n there is a drop $U^\infty_{\iota_1\ldots\iota_k} \subset \mathcal{D}$ which intersects the critical puzzle-piece P_n. Since $U^\infty_{\iota_1\ldots\iota_k}$ cannot intersect ∂P_n, $U^\infty_{\iota_1\ldots\iota_k} \subset P_n$. By Lemma 5.7, the whole limb $L^\infty_{\iota_1\ldots\iota_k}$ is contained in P_n. By Corollary 6.5, $\operatorname{diam} P_n \to 0$, hence the drop-chain $\mathcal{D}$ lands at 1.

By Lemma 4.6 every puzzle-piece P_n contains a drop at infinity $U^\infty_{\iota_1\ldots\iota_k}$. Since $P_{n+2} \subset P_n$ (Corollary 5.5) and $\mathring{P}_n \cap \mathring{P}_{n+1} = \emptyset$, there exist at least two distinct drop-chains landing at 1 (passing through P_n's with even and odd n's respectively). Clearly these drop-chains separate U_1 from $\mathbb{D}$.

Assume that there is a third drop-chain landing at 1. This implies that there are two distinct drop-chains landing at the critical value $B(1)$. Then the complement of the union of these drop-chains has a component O which does not contain any of the drops U_i. This implies that $O \subset \bigcup B^{-n}(U^\infty)$, which is a contradiction. $\square$

The above lemma implies that for every $i \geq 1$ there are exactly two drop-chains $\mathcal{D}^i_1$, $\mathcal{D}^i_2$ accumulating at the point $x_i = B^{-i+1}(1) \in \mathbb{T}$. These drop-chains land at x_i and separate U_i from $\mathbb{D}$. We may now define, as in subsection 3.3, the *wake with root* x_i to be the the connected component W_i of $\overline{\mathbb{C}} \smallsetminus (\mathcal{D}^i_1 \cup \mathcal{D}^i_2)$ containing U_i. For the corresponding limb we clearly have $L_i \subset \overline{W}_i$. Due to the symmetry of the surgery (Corollary 4.5), all the objects we have defined have their symmetric counterparts. That is there is a sequence of critical puzzle-pieces P^∞_n converging to the critical point $c \in \partial U^\infty$, wakes $W^\infty_i \supset U^\infty_i$ with $L^\infty_i \subset \overline{W}^\infty_i$, etc.

We now proceed to give the proof of Theorem 5.1, which will occupy the rest of the section.

Proof of Theorem 5.1. Let $\mathcal{D} = \overline{\bigcup_k U^\infty_{\iota_1\ldots\iota_k}}$ be a drop-chain accumulating at a point $z \in J(\tilde{B})$. We would like to show that $\operatorname{diam} L^\infty_{\iota_1\ldots\iota_k} \to 0$, which in turn will imply that $\mathcal{D}$ lands at z. By symmetry of the surgery (Corollary 4.5) this will prove the desired statement. Denote by z_i the forward iterate $B^{\circ i}(z)$. Let us consider the two possibilities:

- *Case 1.* There exist n and m such that for $i > m$, $z_i \notin P_n \cup P_{n+1} \cup P^\infty_n \cup P^\infty_{n+1}$. Let ζ be a limit point of the sequence $\{z_i\}$. Since the rotation numbers θ, ν are irrational, our assumption implies that $\zeta \notin \mathbb{T} \cup \partial U^\infty$. Clearly, the point ζ must be contained in a wake at infinity, which we call W_j^∞. Denote by i_k the moments $z_{i_k} \in L_j^\infty$, and by Ω_k the univalent pull-back of W_j^∞ along the orbit $z, z_1, \ldots, z_{i_k}$. We refer to the following lemma to show that $\operatorname{diam}(\Omega_k) \to 0$ as $k \to \infty$ (see for example [**Lyu**], Prop. 1.10):

Lemma 6.7 (Shrinking Lemma). *Let F be a rational map. Let $\{F_i^{-m}\}$ be a family of univalent branches of the inverse maps in a domain U. If $U \cap J(F) \neq \emptyset$, then for any V such that $\overline{V} \subset U$, we have* $\operatorname{diam}(F_i^{-m}V) \to 0$ *as $m \to \infty$.*

Applying this lemma to our situation, we conclude that $\operatorname{diam}\Omega_k \to 0$. A drop $U^\infty_{\iota_1\ldots\iota_k}$ does not intersect the boundary of Ω_k. Moreover, by the same argument as in Lemma 5.7, if a drop $U^\infty_{\iota_1\ldots\iota_k}$ is contained in Ω_k, then $L^\infty_{\iota_1\ldots\iota_k} \subset \Omega_k$. Thus the diameters of the limbs $L^\infty_{\iota_1\ldots\iota_k}$ shrink to zero, and hence the drop-chain $\mathcal{D}$ lands at z.

• *Case 2.* To fix the ideas, let us assume that the critical point 1 is a limit point of the sequence $\{z_i\}$. Let z_{i_n} be the first point in the orbit $\{z_i\}$ contained in the puzzle-piece P_n. Denote by

$$Y_0 = P_n, Y_{-1}, \ldots, Y_{-i_n} \tag{6.9}$$

the univalent preimages of P_n along the inverse orbit $z_{i_n}, \ldots, z$.

Lemma 6.8. *There exist at most one moment i between 1 and i_n such that element Y_{-i} of the inverse orbit (6.9) hits the critical point 1. Moreover, the pull-back (6.9) decomposes into two maps with bounded distortion and, possibly, one iterate of B^{-1} near the critical value.*

Proof. Let us prove the first statement. To be definite let us assume that P_n is above the critical point 1. Note that if $Y_{-i} \cap \mathbb{T} = \emptyset$ for some $i \leq q_{n+1}$, then the inverse orbit (6.9) never hits the critical point for $1 < i \leq i_n$. Otherwise denote by A and B the "above" and "below" $B^{\circ q_{n+1}}$-preimages of P_n. One verifies directly, using the observations made in Lemma 5.3 that $A \cap (\mathbb{T}) \subsetneq P_n \cap (\mathbb{T})$ (compare [**Ya**], Lemma 6.11). By Lemma 5.4, $A \subset P_n$, and thus $z_{q_{n+1}} \notin A$. The next possible moment when (6.9) hits 1 is $i = q_{n+1} + q_n$. However, if $Y_{-q_{n+1}-q_n} \cap \mathbb{T} \neq \emptyset$, then we may verify again that $Y_{-q_{n+1}-q_n} \subset P_n$, which is not possible by our assumption.

Now let $k \leq i_n$ be the last moment when $Y_{-k} \cap \mathbb{T} \neq \emptyset$. As seen from the above argument, in combination with Świątek-Herman real a priori bounds and Corollary 6.5, the pull-back $Y_0 \to \cdots \to Y_{-k}$ decomposes into two maps with bounded distortion and, possibly, one branch of B^{-1} near the critical value. The combinatorics of closest returns and real a priori bounds also imply that $\operatorname{dist}(Y_{-k}, B(1))$ is greater than $K_1 \operatorname{diam} Y_{-k}$ for some constant $K_1 > 0$. Hence the distance from Y_{-k-1} to $\mathbb{T} \cup \partial U^\infty$ is greater than $K_2 \operatorname{diam} Y_{-k-1}$ for $K_2 > 0$, and the rest of the pull-back $Y_{-k} \to \cdots \to Y_{-i_n}$ has bounded distortion by the Koebe Theorem. □

By Lemma 5.8 and Corollary 6.5 the puzzle-piece P_n contains a Euclidean disk, whose diameter is commensurable with $\operatorname{diam} P_n$, centered at a point in $J(B)$. Therefore, by

Lemma 6.8, the domain $Y_{-i_n} \ni z$ contains a Euclidean disk centered at a point of $J(B)$ whose diameter is commensurable with $\operatorname{diam} Y_{-i_n}$. This implies that $\operatorname{diam} Y_{-i_n} \to 0$. By Lemma 5.7, if $U^\infty_{\iota_1\ldots\iota_k} \subset Y_{-i_n}$, then $L^\infty_{\iota_1\ldots\iota_k} \subset Y_{-i_n}$. So the diameters of limbs $L^\infty_{\iota_1\ldots\iota_k}$ shrink to zero, and the drop-chain $\mathcal{D}$ lands at z. □

7. The Proof

Throughout this section we fix a pair of irrationals θ and ν of bounded type, with $\theta \neq 1-\nu$. In what follows we prove the Main Theorem, that is we show that the quadratic rational map $F_{\theta,\nu}$ of (2.1) is in fact *the* mating of the quadratic polynomials f_θ and f_ν in the sense we described in the introduction.

7.1. **Spines and itineraries.** Let $\tilde{Q}_\theta$ be the modified Blaschke product $\tilde{Q}_\theta$ of (3.2). Consider the two drop-chains

$$\mathcal{C} = \overline{U_0 \cup U_1 \cup U_{11} \cup \cdots}, \quad \mathcal{C}' = \overline{U_0 \cup U_2 \cup U_{21} \cup \cdots}$$

with $\tilde{Q}_\theta(\mathcal{C}') = \mathcal{C}$. Applying Lemma 5.2 again, we see that $\mathcal{C}$ and $\mathcal{C}'$ land respectively at the repelling fixed point β and its preimage β'. By the *spine* of $\tilde{Q}_\theta$ we mean the union of the drop-rays

$$S_\theta = R(\mathcal{C}) \cup R(\mathcal{C}')$$

(compare Fig. 13, where the image of the spine of $\tilde{Q}_\theta$ is shown in the filled Julia set of the quadratic polynomial f_θ for $\theta = (\sqrt{5}-1)/2$). Every point on the spine which is not in the interior of $K(\tilde{Q}_\theta)$ is either one of the endpoints β, β', or a preimage of the critical point $z=1$.

By Petersen's Theorem 3.5 the Julia set $J(\tilde{Q}_\theta)$ is locally-connected. Thus the Böttcher map extends continuously from the basin of infinity of $\tilde{Q}_\theta$ to its boundary. As a consequence, there exists a Carathéodory loop $\eta_\theta : \mathbb{R}/\mathbb{Z} \to J(\tilde{Q}_\theta)$ which conjugates the angle-doubling map to $\tilde{Q}_\theta$. A point $z \in J(\tilde{Q}_\theta)$ is the landing point of an external ray $R^e(t)$ if and only if $\eta_\theta(t) = z$. It is easy to see that $\eta_\theta(0) = \beta$ and $\eta_\theta(1/2) = \beta'$.

By Lemma 3.3 the critical point $z=1$, hence every preimage of it, is *biaccessible*, that is it is the landing point of exactly two external rays. For the quadratic polynomial f_θ the converse statement is true for an arbitrary θ of Brjuno type: Every biaccessible point in the Julia set $J(f_\theta)$ eventually hits the critical point [**Za1**]. The two external rays landing at the critical point of $\tilde{Q}_\theta$ are both mapped to the external ray landing at the critical value $\tilde{Q}_\theta(1)$. This means that they have angles of the form $\omega/2$ and $(\omega+1)/2$, where $\omega = \omega(\theta)$ is a well-defined irrational number in the interval $(0,1)$. It can be shown that the function $\theta \mapsto \omega(\theta)$ is effectively computable (see [**BS**] and compare with subsection 8.2).

Consider the two connected subsets of the Julia set

$$\begin{aligned} J^0_\theta = &\ \{z \in J(\tilde{Q}_\theta) : z = \eta_\theta(t) \text{ for some } 0 \le t < 1/2\}, \\ J^1_\theta = &\ \{z \in J(\tilde{Q}_\theta) : z = \eta_\theta(t) \text{ for some } 1/2 \le t < 1\}. \end{aligned} \tag{7.1}$$

By local-connectivity of $J(\tilde{Q}_\theta)$ (Theorem 3.5), $J^0_\theta \cup J^1_\theta = J(\tilde{Q}_\theta)$, and evidently $J^0_\theta \cap J^1_\theta = (\bigcup_{n=0}^{\infty} \tilde{Q}_\theta^{-n}(1)) \cap S_\theta = \{1 = x_1, x_{11}, x_{111}, \dots\} \cup \{x_2, x_{21}, x_{211}, \dots\}$.

We proceed to define the *itinerary* of a point $z \in J(\tilde{Q}_\theta)$ with respect to S_θ. This will be a dynamically-defined infinite sequence of 0's and 1's which gives the binary expansion of the angle of an external ray landing at z (see [**Do1**] for a general dicscussion on how one computes angles in similar situations). In the case where z is biaccessible, we define two different itineraries corresponding to the angles of the two rays landing at z. Set $z_0 = z,\ z_k = \tilde{Q}_\theta(z_{k-1})$. We consider three distinct cases:

- *Case 1.* The orbit of z never hits the spine. Then z is not biaccessible and hence there exists a unique angle t with $z = \eta_\theta(t)$. Define the itinerary of z to be the sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$, where $\varepsilon_i \in \{0, 1\}$ is determined by the condition

$$z_i \in J^{\varepsilon_i}_\theta, \quad i = 0, 1, \dots$$

Then it is easy to see that the angle t has binary expansion $0.\varepsilon_0\varepsilon_1\varepsilon_2\cdots$.

- *Case 2.* The orbit of z eventually hits the β-fixed point, i.e., there exists the smallest integer $n \geq 0$ such that $z_n = \beta$. In this case, the angle t with $z = \eta_\theta(t)$ is still unique. The itinerary of z is defined as $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, 0, 0, 0, \dots)$, where

$$z_i \in J^{\varepsilon_i}_\theta, \quad i = 0, 1, 2, \dots, n.$$

The binary digits of the angle t are then given by the itinerary of z.

- *Case 3.* The orbit of z eventually hits the critical point at 1. In this case there are exactly two angles $0 < t < s < 1$ with $\eta_\theta(t) = \eta_\theta(s) = z$. Let us assume that the angles corresponding to the critical point have binary expansions $\omega/2 = 0.0\omega_1\omega_2\dots$ and $(\omega + 1)/2 = 0.1\omega_1\omega_2\dots$. Then the critical value $v = \tilde{Q}_\theta(1)$ has a unique ray landing on it with angle $\omega = 0.\omega_1\omega_2\dots$. Since v can never hit the spine, by *Case 1* above, the binary digits of ω are uniquely determined by the condition

$$\tilde{Q}^{\circ i}_\theta(1) \in J^{\omega_i}_\theta, \quad i = 1, 2, \dots$$

We are going to define two itineraries for z. Let $n \geq 0$ be the smallest integer such that $z_n \in S_\theta \smallsetminus \{\beta, \beta'\}$. Define the *initial segment* $(\varepsilon_0, \dots, \varepsilon_{n-1})$ of both itineraries of z by the condition

$$z_i \in J^{\varepsilon_i}_\theta, \quad i = 0, 1, \dots, n-1.$$

(If $n = 0$, we define the initial segment to be empty.) Let $m \geq 1$ be defined by the condition $z_{n+m} = v = \tilde{Q}_\theta(1)$. Since $z_n, \dots, z_{n+m-1}$ all belong to the intersection $J^0_\theta \cap J^1_\theta$, there is an ambiguity in assigning digits to the points of this part of the orbit of z. So consider z_n and replace it by two points $a_n \in J^0_\theta$ and $b_n \in J^1_\theta$, both sufficiently close to z_n. It is easy to see that the points of the orbits $a_n, \dots, a_{n+m-1}$ and $b_n, \dots, b_{n+m-1}$ have well-defined itineraries $(\varepsilon_n, \dots, \varepsilon_{n+m-1})$ and $(\varepsilon'_n, \dots, \varepsilon'_{n+m-1})$ determined by the conditions

$$a_i \in J^{\varepsilon_i}_\theta, \quad i = n, n+1, \dots, n+m-1,$$
$$b_i \in J^{\varepsilon'_i}_\theta, \quad i = n, n+1, \dots, n+m-1.$$

We call these two segments *ambiguous*. Note that $\varepsilon_i + \varepsilon'_i = 1$ for $n \leq i \leq n+m-1$. Finally, follow these two by the well-defined itinerary of the critical value. We thus obtain two itineraries for z:

$$\varepsilon = (\underbrace{\varepsilon_0, \ldots, \varepsilon_{n-1}}_{\text{initial segment}}, \underbrace{\varepsilon_n, \ldots, \varepsilon_{n+m-1}}_{\text{ambiguous segment}}, \underbrace{\omega_1, \omega_2, \ldots}_{\text{itinerary of } v}),$$

$$\varepsilon' = (\underbrace{\varepsilon_0, \ldots, \varepsilon_{n-1}}_{\text{initial segment}}, \underbrace{\varepsilon'_n, \ldots, \varepsilon'_{n+m-1}}_{\text{ambiguous segment}}, \underbrace{\omega_1, \omega_2, \ldots}_{\text{itinerary of } v}).$$

These two itineraries give the binary digits of the two angles t and s.

Since $\tilde{Q}_\theta$ and f_θ are quasiconformally conjugate for θ of bounded type, with the conjugacy being conformal in the basin of infinity, we have a completely similar description for the spine and itineraries of points in the quadratic Julia set $J(f_\theta)$. Fig. 13 shows the spine and selected rays for f_θ with $\theta = (\sqrt{5}-1)/2$.

We summarize the above discussion in the following proposition:

Proposition 7.1.

(i) *Let $z \in J(\tilde{Q}_\theta)$. Then the angle(s) of the external ray(s) landing at z is(are) determined by the itinerary(ies) of z, that is by the answer to the purely topological question of whether a point in the forward orbit of z belongs to J^0_θ, J^1_θ, or to which point of the spine. In particular, two points in the Julia set having the same itinerary must coincide.*

(ii) *Every infinite sequence of 0's and 1's can be realized as the itinerary of a unique point in $J(\tilde{Q}_\theta)$.*

7.2. **Main reduction.** A key ingredient in the proof of the main theorem is the following reduction step:

Theorem 7.2. *Let $0 < \theta, \nu < 1$ be irrationals of bounded type and $\theta \neq 1 - \nu$. Then there exist continuous maps $\zeta_\theta : K(\tilde{Q}_\theta) \to \overline{\mathbb{C}}$ and $\zeta_\nu : K(\tilde{Q}_\nu) \to \overline{\mathbb{C}}$ such that*

$$\begin{aligned} \zeta_\theta \circ \tilde{Q}_\theta &= \tilde{B}_{\theta,\nu} \circ \zeta_\theta \quad \text{on } K(\tilde{Q}_\theta) \\ \zeta_\nu \circ \tilde{Q}_\nu &= \tilde{B}_{\theta,\nu} \circ \zeta_\nu \quad \text{on } K(\tilde{Q}_\nu). \end{aligned} \tag{7.2}$$

ζ_θ and ζ_ν can be chosen to be quasiconformal homeomorphisms in the interiors of $K(\tilde{Q}_\theta)$ and $K(\tilde{Q}_\nu)$ respectively. Moreover, $\zeta_\theta(K(\tilde{Q}_\theta)) \cup \zeta_\nu(K(\tilde{Q}_\nu)) = \overline{\mathbb{C}}$ and $\zeta_\theta(z) = \zeta_\nu(w)$ if and only if there exists an angle $t \in \mathbb{R}/\mathbb{Z}$ such that $z = \eta_\theta(t)$ and $w = \eta_\nu(-t)$.

Before starting the proof, we fix some notation. For simplicity, we set $K(\tilde{Q}_\theta) = K_\theta$, $K(\tilde{Q}_\nu) = K_\nu$. We also recall the definition of the compact set $K(\tilde{B}_{\theta,\nu}) = K_{\theta,\nu}$ as the set of all points whose forward orbits under the iteration of $\tilde{B}_{\theta,\nu}$ never hit the Siegel disk U^∞. Similarly, $K^\infty_{\theta,\nu} = \overline{\mathbb{C} \smallsetminus K_{\theta,\nu}}$ is the set of points whose forward orbits never hit the "Siegel disk" $U_0 = \mathbb{D}$.

Proof of Theorem 7.2. We begin by constructing ζ_θ. The map ζ_ν can be constructed

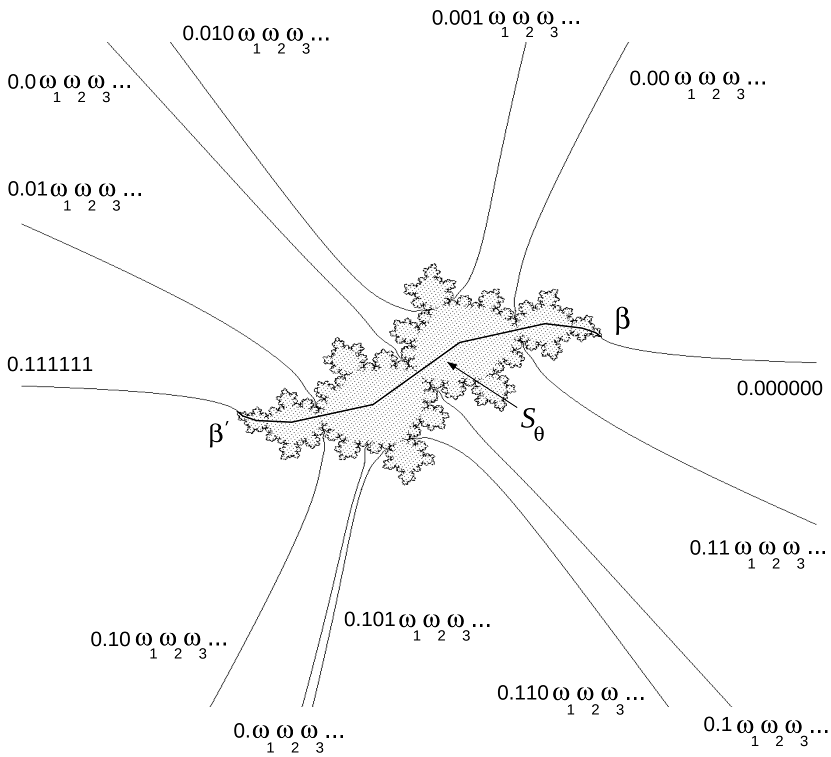

FIGURE 13. *This picture shows the filled Julia set of the quadratic polynomial f_θ, for $\theta = (\sqrt{5}-1)/2$. The spine is shown by a thick path connecting the repelling fixed point β to its preimage β'. Selected rays and angles in base 2 are shown. Here $\omega = 0.\omega_1\omega_2\omega_3\ldots$ is the unique angle corresponding to the ray which lands at the critical value. For this value of θ, ω is given by the continued fraction $[1,2,2,2^2,2^3,2^5,\ldots]$, where the powers of 2 form the Fibonacci sequence. Hence $\omega_1 = 1$, $\omega_2 = 0$, $\omega_3 = 1$, etc.*

in a similar fashion. Consider the modified Blaschke products $\tilde{Q}_\theta$ of (3.2) and $\tilde{B}_{\theta,\nu}$ of (4.9). Since both of these are quasiconformally conjugate to the rigid rotation $z \mapsto e^{2\pi i\theta}z$ on the unit disk, one can define a quasiconformal conjugacy $\zeta_\theta : \mathbb{D} \to \mathbb{D}$ between them, which extends homeomorphically to a conjugacy $\zeta_\theta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. This ζ_θ can be extended to the union of the closures of all drops of $\tilde{Q}_\theta$ by pulling back. To this end, let $U_{\iota_1\ldots\iota_k}$ be any drop of $\tilde{Q}_\theta$ of generation k and consider the corresponding drop $U'_{\iota_1\ldots\iota_k}$ of $\tilde{B}_{\theta,\nu}$ *with the same address.* Let $n = \iota_1 + \cdots + \iota_k$ and define $\zeta_\theta : \overline{U}_{\iota_1\ldots\iota_k} \xrightarrow{\simeq} \overline{U'}_{\iota_1\ldots\iota_k}$ by

$$\zeta_\theta = \tilde{B}^{-n}_{\theta,\nu} \circ \zeta_\theta \circ \tilde{Q}^{\circ n}_\theta .$$

An easy induction on n shows that ζ_θ defined this way is a conjugacy between $\tilde{Q}_\theta$ and $\tilde{B}_{\theta,\nu}$ on $\bigcup_k \bigcup_{\iota_1,\dots,\iota_k} \overline{U}_{\iota_1\dots\iota_k}$ which is quasiconformal on the union $\bigcup_k \bigcup_{\iota_1,\dots,\iota_k} U_{\iota_1\dots\iota_k} = \mathrm{int}(K_\theta)$.

We would like to extend ζ_θ to a continuous semiconjugacy $K_\theta \to K_{\theta,\nu}$. By Proposition 3.8, every point in K_θ is either in the closure of a drop or is the landing point of a unique drop-chain. Since ζ_θ is already defined on $\bigcup_k \bigcup_{\iota_1,\dots,\iota_k} \overline{U}_{\iota_1\dots\iota_k}$, it suffices to define it at the landing points of drop-chains of $\tilde{Q}_\theta$. Take a drop-chain $\mathcal{C} = \overline{\bigcup_k U_{\iota_1\dots\iota_k}}$ which lands at p and consider the corresponding drop-chain of $\tilde{B}_{\theta,\nu}$, $\mathcal{C}' = \overline{\bigcup_k U'_{\iota_1\dots\iota_k}}$, whose drops have *the same* addresses. By Theorem 5.1, the diameter of $U'_{\iota_1\dots\iota_k}$ goes to zero as $k \to \infty$, hence $\mathcal{C}'$ lands at a well-defined point $p' \in K_{\theta,\nu}$. Define $\zeta_\theta(p) = p'$.

Evidently ζ_θ defined this way has the property that for any limb $L_{\iota_1\dots\iota_k}$ of $\tilde{Q}_\theta$, the image $\zeta_\theta(L_{\iota_1\dots\iota_k})$ is precisely the limb $L'_{\iota_1\dots\iota_k}$ of $\tilde{B}_{\theta,\nu}$ with the same address. We would like to show that ζ_θ is continuous as a map from K_θ into $\overline{\mathbb{C}}$. Take a point $p \in K_\theta$ and a sequence $p_n \in K_\theta$ converging to p. When p belongs to the interior of K_θ continuity is trivial. So let us assume that $p \in \partial K_\theta$. By Proposition 3.8, we have two possibilities:

- *Case 1.* p is the landing point of a drop-chain $\mathcal{C} = \overline{\bigcup_k U_{\iota_1\dots\iota_k}}$. Fix a multi-index $\iota_1 \dots \iota_k$ and observe that p belongs to the wake $W_{\iota_1\dots\iota_k}$. Therefore, for n large enough, $p_n \in L_{\iota_1\dots\iota_k}$, which implies $\zeta_\theta(p_n) \in L'_{\iota_1\dots\iota_k}$. It follows that $\mathrm{dist}(\zeta_\theta(p), \zeta_\theta(p_n)) < \mathrm{diam}(L'_{\iota_1\dots\iota_k})$. Since $\mathrm{diam}(L'_{\iota_1\dots\iota_k}) \to 0$ as $k \to \infty$ by Theorem 5.1, we have $\zeta_\theta(p_n) \to \zeta_\theta(p)$ as $n \to \infty$.
- *Case 2.* p belongs to the boundary of a drop $U_{\iota_1\dots\iota_k}$ of $\tilde{Q}_\theta$ of *smallest* possible generation. It might be the case that p is the root of a child $U_{\iota_1\dots\iota_k\iota_{k+1}}$ in which case $p = \partial U_{\iota_1\dots\iota_k} \cap \partial U_{\iota_1\dots\iota_k\iota_{k+1}}$. If for all sufficiently large n, p_n belongs to $\overline{U}_{\iota_1\dots\iota_k}$ (or to $\overline{U}_{\iota_1\dots\iota_k} \cup \overline{U}_{\iota_1\dots\iota_k\iota_{k+1}}$ if p is the root of $U_{\iota_1\dots\iota_k\iota_{k+1}}$), then $\zeta_\theta(p_n) \to \zeta_\theta(p)$ is immediate. Hence it suffices to prove the convergence in the case $p_n \notin \overline{U}_{\iota_1\dots\iota_k}$ (or $p_n \notin \overline{U}_{\iota_1\dots\iota_k} \cup \overline{U}_{\iota_1\dots\iota_k\iota_{k+1}}$ if p is the root of $U_{\iota_1\dots\iota_k\iota_{k+1}}$). Under this assumption, it follows from $p_n \to p$ that p_n belongs to a limb $L(n)$ with root $x(n) \in \partial U_{\iota_1\dots\iota_k}$ (or $x(n) \in \partial U_{\iota_1\dots\iota_k} \cup \partial U_{\iota_1\dots\iota_k\iota_{k+1}}$ if p is the root of $U_{\iota_1\dots\iota_k\iota_{k+1}}$) such that $x(n) \to p$ as $n \to \infty$. Then $\zeta_\theta(p_n)$ belongs to the limb $L'(n)$ of $\tilde{B}_{\theta,\nu}$ with the same address as $L(n)$ whose root $x'(n) = \zeta_\theta(x(n))$ converges to $\zeta_\theta(p)$ as $n \to \infty$. Since $\mathrm{diam}(L'(n)) \to 0$ by Theorem 5.1, we must have $\zeta_\theta(p_n) \to \zeta_\theta(p)$ as $n \to \infty$ as well. This finishes the proof of continuity.

We can define ζ_ν and prove its continuity in a similar way. It is clear from the above construction that the semiconjugacy relations (7.2) hold and $\zeta_\theta(K_\theta) = K_{\theta,\nu}$ and similarly $\zeta_\nu(K_\nu) = K^\infty_{\theta,\nu}$.

It remains to prove the last property of ζ_θ and ζ_ν. Consider the spines S_θ and S_ν for $\tilde{Q}_\theta$ and $\tilde{Q}_\nu$ as in subsection 7.1 and map them to get simple arcs $\Sigma_\theta = \zeta_\theta(S_\theta)$ and $\Sigma_\nu = \zeta_\theta(S_\nu)$ (compare Fig. 10). Set

$$\Sigma = \Sigma_\theta \cup \Sigma_\nu.$$

Lemma 7.3. *Two simple curves Σ_θ and Σ_ν do not intersect except at the two end-points β and β'. Hence Σ is a Jordan curve on the Riemann sphere.*

Proof. Clearly the intersection $\Sigma_\theta \cap \Sigma_\nu$ is a subset of $\partial K_{\theta,\nu} \cap \Sigma$. Every point in the latter intersection is either β or β', or is a preimage of 1 or c, where c is the critical point of $B_{\theta,\nu}$ on the boundary of U^∞. Since 1 and c have disjoint forward orbits, the conclusion follows. $\square$

Now consider the four connected sets

$$\Lambda_\theta^i = \zeta_\theta(J_\theta^i),\ \Lambda_\nu^i = \zeta_\nu(J_\nu^i) \quad i = 0, 1,$$

where J_θ^i and J_ν^i are the subsets of the Julia sets $J(\tilde{Q}_\theta)$ and $J(\tilde{Q}_\nu)$ we defined in (7.1). Let

$$X = \{1 = x_1, x_{11}, x_{111}, \dots, x_2, x_{21}, x_{211}, \dots\}$$

and

$$Y = \{c = x_1^\infty, x_{11}^\infty, x_{111}^\infty, \dots, x_2^\infty, x_{21}^\infty, x_{211}^\infty, \dots\}$$

be the preimages of the critical points 1 and c along Σ. It is clear from the definition that

$$X \subset \Lambda_\theta^0 \cap \Lambda_\theta^1 \subset X \cup Y,$$

$$Y \subset \Lambda_\nu^0 \cap \Lambda_\nu^1 \subset X \cup Y.$$

But in fact we have the following much sharper statement:

Lemma 7.4. *With the above notation, we have*

$$\Lambda_\theta^0 \cap \Lambda_\theta^1 = \Lambda_\nu^0 \cap \Lambda_\nu^1 = X \cup Y.$$

Proof. Take a point $y \in Y$ and assume that $\tilde{B}_{\theta,\nu}^{\circ n}(y) = c$. By Lemma 6.6, there are exactly two drop-chains which land at the critical point c from different sides of Σ_ν. Then the pull-backs of these drop-chains along the orbit $y, \tilde{B}_{\theta,\nu}(y), \dots, \tilde{B}_{\theta,\nu}^{\circ n}(y) = c$ give two drop-chains which land at y from different sides of Σ_ν. These drop-chains are clearly subsets of the compact set $K_{\theta,\nu}$. The fact that they land at y from different sides of Σ_ν implies $y \in \Lambda_\theta^0 \cap \Lambda_\theta^1$. The proof of the other equality is similar. $\square$

Corollary 7.5. *With the above notation, we have*

$$\Lambda_\theta^0 = \Lambda_\nu^1 \quad \textit{and} \quad \Lambda_\theta^1 = \Lambda_\nu^0.$$

Proof. Let $\overline{\mathbb{C}} \smallsetminus \Sigma = O_1 \cup O_2$, where O_i are disjoint topological disks with $\Lambda_\theta^0 \subset \overline{O}_1$ and $\Lambda_\theta^1 \subset \overline{O}_2$. Taking the orientations on the sphere into account, we have $\Lambda_\nu^1 \subset \overline{O}_1$ and $\Lambda_\nu^0 \subset \overline{O}_2$. Since $\Lambda_\theta^0 \cup \Lambda_\theta^1 = \Lambda_\nu^0 \cup \Lambda_\nu^1 = \partial K_{\theta,\nu}$ and $\Lambda_\theta^0 \cap \Lambda_\theta^1 = \Lambda_\nu^0 \cap \Lambda_\nu^1$ by Lemma 7.4, it follows that $\Lambda_\theta^0 = \Lambda_\nu^1$ and $\Lambda_\theta^1 = \Lambda_\nu^0$. $\square$

We can now define the itinerary of a point $p \in \partial K_{\theta,\nu}$ with respect to Σ_θ by looking at the points in the forward orbit of p and deciding whether they belong to Λ^0_θ, Λ^1_θ, or to $\Lambda^0_\theta \cap \Lambda^1_\theta$. As in the discussion of itineraries for the points in the Julia set $J(\tilde{Q}_\theta)$ (see subsection 7.1), we may face an ambiguity in defining the digits when some forward iterate of p, say p_n, belongs to $\Lambda^0_\theta \cap \Lambda^1_\theta$. In this case, we perturb p_n to obtain a pair of nearby points in Λ^0_θ and Λ^1_θ and keep iterating the two points to decide to which piece of the Julia set they belong. After a finite number of iterations, we are off the spine Σ and the rest of the itinerary can be defined in an unambiguous way. Since $\partial K_{\theta,\nu} = \partial K^\infty_{\theta,\nu}$ by Corollary 4.6, a similar procedure can be used to define the itinerary or two itineraries of p with respect to Σ_ν. In short,

Proposition 7.6 (Two or four itineraries). *Let $p \in \partial K_{\theta,\nu}$. Then, either p is not a preimage of 1 or c in which case it has unique itineraries ε_θ with respect to Σ_θ and ε_ν with respect to Σ_ν, or p is a preimage of 1 or c in which case it has two different itineraries $\varepsilon_\theta, \varepsilon'_\theta$ with respect to Σ_θ and $\varepsilon_\nu, \varepsilon'_\nu$ with respect to Σ_ν.*

Since the i-th digit of the itinerary or itineraries of a point p with respect to Σ_θ is determined by the condition $\tilde{B}^{\circ i}_{\theta,\nu}(p) \in \Lambda^0_\theta$ or Λ^1_θ, and similarly for the itineraries with respect to Σ_ν, we have the following consequence of Corollary 7.5:

Proposition 7.7 (Σ_θ- and Σ_ν-itineraries have opposite digits). *Let $p \in \partial K_{\theta,\nu}$ have itinerary $\varepsilon_\theta(p) = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ with respect to Σ_θ. Then the itinerary of p with respect to Σ_ν is obtained by converting all 0's to 1's and all 1's to 0's in ε_θ. In other words, $\varepsilon_\nu(p) = \overline{\varepsilon}_\theta(p) = (\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots)$, where $\overline{\varepsilon}_i = 1 - \varepsilon_i$. In the case where p has two itineraries, we have $\varepsilon_\nu(p) = \overline{\varepsilon}_\theta(p)$ and $\varepsilon'_\nu(p) = \overline{\varepsilon'}_\theta(p)$.*

The following lemma is a straightforward consequence of the above construction:

Lemma 7.8 (Itineraries match). *Let $z \in K_\theta$ and $p = \zeta_\theta(z) \in K_{\theta,\nu}$.*

(i) *Suppose that z is not a preimage of the critical point 1. Then the unique itinerary of z with respect to S_θ coincides with $\varepsilon_\theta(p)$ when p is not a preimage of c, and it coincides with one of the two itineraries $\varepsilon_\theta(p)$ or $\varepsilon'_\theta(p)$ when p is a preimage of c.*

(ii) *Suppose that z is a preimage of 1. Then the two itineraries of z with respect to S_θ coincide with the two itineraries $\varepsilon_\theta(p)$ and $\varepsilon'_\theta(p)$.*

Corollary 7.9 (Itineraries determine points). *Two points in $\partial K_{\theta,\nu}$ with the same itinerary with respect to Σ_θ or Σ_ν must coincide.*

Proof. Let $p, q \in \partial K_{\theta,\nu}$ and assume for example that $\varepsilon_\theta(p) = \varepsilon_\theta(q)$. When p (hence q) is a preimage of 1 or c, it is easy to see that identical Σ_θ-itineraries implies $p = q$. So let us assume that p and q are not preimages of 1 or c. Since $\zeta_\theta : K_\theta \to K_{\theta,\nu}$ is surjective, we have $p = \zeta_\theta(u)$ and $q = \zeta_\theta(v)$ for some $u, v \in \partial K_\theta = J(\tilde{Q}_\theta)$. By the above lemma, u and v have the same itineraries with respect to S_θ. By Proposition 7.1(i), $u = v$. Hence $p = q$. □

Now consider two points $z \in K_\theta$ and $w \in K_\nu$ such that $z = \eta_\theta(t)$ and $w = \eta_\nu(-t)$ for some $t \in \mathbb{T}$. Set $p = \zeta_\theta(z)$ and $q = \zeta_\nu(w)$. The binary digits $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ of the angle t form an itinerary of z with respect to S_θ. Since $t = 0.\varepsilon_0\varepsilon_1\varepsilon_2\dots$ in base 2, $-t$ has the binary expansion $0.\overline{\varepsilon}_0\overline{\varepsilon}_1\overline{\varepsilon}_2\dots$. Hence $(\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots)$ is an itinerary of w with respect to S_ν. Thus by Lemma 7.8, $\varepsilon_\theta(p) = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ and $\varepsilon_\nu(q) = (\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots)$. By Proposition 7.7, $\varepsilon_\theta(q) = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$, which means p and q have the same itinerary with respect to Σ_θ. This, by Corollary 7.9, implies $p = q$.

Conversley, assume that $\zeta_\theta(z) = \zeta_\nu(w) = p$. We consider two cases: First assume that p is not a preimage of 1 or c. Then it follows from Proposition 7.7 that $\varepsilon_\theta(p) = \overline{\varepsilon}_\nu(p) = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ and these itineraries are unique. By Lemma 7.8, $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ is the S_θ-itinerary of z and $(\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots)$ is the S_ν-itinerary of w. Setting $t = 0.\varepsilon_0\varepsilon_1\varepsilon_2\dots$ in base 2, we have $z = \eta_\theta(t)$ and $w = \eta_\nu(-t)$ and we are done. Next, assume that p is a preimage of, say, 1. Then, as 1 and c have disjoint orbits under $\tilde{B}_{\theta,\nu}$, p cannot be a preimage of c. This implies that z is a preimage of the critical point 1 of $\tilde{Q}_\theta$ and therefore has two itineraries, and w is not a preimage of the critical point 1 of $\tilde{Q}_\nu$ and so has a unique itinerary. Let $w = \eta_\nu(-t)$, where the unique $t \in \mathbb{T}$ has binary expansion $t = 0.\varepsilon_0\varepsilon_1\varepsilon_2\dots$. By Lemma 7.8, $\varepsilon_\nu(p) = (\overline{\varepsilon}_0, \overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots)$ is one of the Σ_ν-itineraries of p. Hence by Proposition 7.7, $\varepsilon_\theta(p) = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ is one of the Σ_θ-itineraries of p. Therefore, by another application of Lemma 7.8, $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ is one of the two S_θ-itineraries of z, implying $z = \eta_\theta(t)$.

This covers all the cases and completes the proof of Theorem 7.2. $\square$

We conclude with the following:

Corollary 7.10 (At most three points). *Let $p \in \partial K_{\theta,\nu}$. Then $\zeta_\theta^{-1}(p) \cup \zeta_\nu^{-1}(p)$ contains at most 3 points.*

Proof. Since p has at most two itineraries with respect to Σ_θ and two itineraries with respect to Σ_ν, Lemma 7.8 and Proposition 7.1 imply that $\zeta_\theta^{-1}(p)$ and $\zeta_\nu^{-1}(p)$ each contain at most two points. So to prove the corollary, we assume by way of contradiction that there are four distinct point $z_1, z_2 \in K_\theta$ and $z_3, z_4 \in K_\nu$ such that $\zeta_\theta(z_1) = \zeta_\theta(z_2) = \zeta_\nu(z_3) = \zeta_\nu(z_4) = p$. By Theorem 7.2, all four points have to be biaccessible. Pick, for example, z_1 and z_3 and note that they eventually map to the critical points of $\tilde{Q}_\nu$ and $\tilde{Q}_\theta$ [**Za1**]. Hence $p = \zeta_\theta(z_1)$ eventually maps to the critical point 1 of $\tilde{B}_{\theta,\nu}$ and $p = \zeta_\nu(z_3)$ also maps to the critical point c of $\tilde{B}_{\theta,\nu}$. This is clearly impossible since c and 1 have disjoint orbits. $\square$

7.3. **End of the proof.** We can now prove the main theorem of this paper:

Theorem 7.11 (Bounded type Siegel quadratics are mateable). *Let $0 < \theta, \nu < 1$ be two irrationals of bounded type and $\theta \neq 1 - \nu$. Then the quadratic polynomials f_θ and f_ν are topologically mateable. Moreover, there exists a quadratic rational map F such that $F = f_\theta \sqcup f_\nu$. Any two such rational maps are conjugate by a Möbius transformation.*

Proof. The last assertion is immediate since every quadratic rational map with two fixed Siegel disks of rotation numbers θ and ν is holomorphically conjugate to the normalized map $F_{\theta,\nu}$ defined in (2.1). By Definition IIa of the introduction, it suffices to construct continuous maps $\varphi_\theta : K(f_\theta) \to \overline{\mathbb{C}}$ and $\varphi_\nu : K(f_\nu) \to \overline{\mathbb{C}}$ with the following properties:

(a) $\varphi_\theta \circ f_\theta = F_{\theta,\nu} \circ \varphi_\theta$ and $\varphi_\nu \circ f_\nu = F_{\theta,\nu} \circ \varphi_\nu$.
(b) $\varphi_\theta(K(f_\theta)) \cup \varphi_\nu(K(f_\nu)) = \overline{\mathbb{C}}$.
(c) φ_θ and φ_ν are conformal in the interiors of $K(f_\theta)$ and $K(f_\nu)$.
(d) $\varphi_\theta(z) = \varphi_\nu(w)$ if and only if z and w are ray equivalent.

It is clear from the preceding discussion what these maps should be. By the surgery construction of subsections 3.5 and 4.2, there exist quasiconformal homeomorphisms $\psi_\theta, \psi_\nu, \psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\begin{aligned} \psi_\theta \circ \tilde{Q}_\theta &= f_\theta \circ \psi_\theta, \\ \psi_\nu \circ \tilde{Q}_\nu &= f_\nu \circ \psi_\nu, \\ \psi \circ \tilde{B}_{\theta,\nu} &= F_{\theta,\nu} \circ \psi. \end{aligned} \tag{7.3}$$

Consider the semiconjugacies ζ_θ and ζ_ν of Theorem 7.2 and define

$$\varphi_\theta = \psi \circ \zeta_\theta \circ \psi_\theta^{-1},$$

$$\varphi_\nu = \psi \circ \zeta_\nu \circ \psi_\nu^{-1}.$$

Properties (a) and (b) above are immediate consequences of the corresponding properties of ζ_θ and ζ_ν stated in Theorem 7.2. So to finish the proof, we must show (c) and (d).

To show (c), recall the surgery construction of subsection 3.5. Consider the Douady-Earle extension H_θ used in defining the modified Blaschke product $\tilde{Q}_\theta$ in (3.2). The invariant conformal structure σ_θ on the unit disk $\mathbb{D}$ is given by the pull-back of the standard conformal structure σ_0 under H_θ. Similarly, we have the Douady-Earle extension $H_{\theta,\nu}$ for the linearizing homeomorphism of $B_{\theta,\nu} : \mathbb{T} \to \mathbb{T}$ used in defining the modified Blaschke product $\tilde{B}_{\theta,\nu}$ in (4.9), and the invariant conformal structure $\sigma_{\theta,\nu}$ on $\mathbb{D}$ as the pull-back of σ_0 under $H_{\theta,\nu}$. Both H_θ and $H_{\theta,\nu}$ conjugate $\tilde{Q}_\theta$ and $\tilde{B}_{\theta,\nu}$ to the rigid rotation $z \mapsto e^{2\pi i\theta} z$. By definition of ζ_θ, we have $\zeta_\theta = H_{\theta,\nu}^{-1} \circ H_\theta$ on $\mathbb{D}$. This means that ζ_θ pulls $\sigma_{\theta,\nu}$ back to σ_θ on the unit disk. It follows that the composition $\varphi_\theta = \psi \circ \zeta_\theta \circ \psi_\theta^{-1}$ on $\mathbb{D}$ pulls σ_0 back to σ_0, hence it is conformal there. Then (a) and the fact that f_θ and $F_{\theta,\nu}$ are holomorphic show that ζ_θ is conformal in the interior of $K(f_\theta)$. A similar argument applies to ζ_ν.

To show (d), we note that the quasiconformal conjugacies ψ_θ and ψ_ν are conformal outside the filled Julia sets, so they preserve the external angles. Therefore $\gamma_\theta = \psi_\theta \circ \eta_\theta$ and $\gamma_\nu = \psi_\nu \circ \eta_\nu$, where γ_θ and γ_ν are the Carathéodory loops of $J(f_\theta)$ and $J(f_\nu)$. By Theorem 7.2, $\varphi_\theta(z) = \varphi_\nu(w)$ implies that $z = \gamma_\theta(t)$ and $w = \gamma_\nu(-t)$ for some $t \in \mathbb{T}$, which means z and w are ray equivalent. The converse statement is almost immediate because if $z \in K(f_\theta)$ is ray equivalent to $w \in K(f_\nu)$, the same is true for $\psi_\theta^{-1}(z)$ and $\psi_\nu^{-1}(w)$. Since every pair of ray equivalent points of the form $(\eta_\theta(t), \eta_\nu(-t))$ is mapped to the same point under $(\zeta_\theta, \zeta_\nu)$, the same must be true for arbitrary pairs of ray equivalent points. Hence $\zeta_\theta(\psi_\theta^{-1}(z)) = \zeta_\nu(\psi_\nu^{-1}(w))$, or $\varphi_\theta(z) = \varphi_\nu(w)$. This proves (d), and finishes the proof of the Main Theorem 7.11. $\square$

8. Concluding Remarks

In this section, we discuss some corollaries of Theorem 7.11. In particular, we describe the nature of the pinch points already observed in Fig. 2. Then we prove a number-theoretic corollary of the topological mateability part of Theorem 7.11 which is related to the rotation sets of the angle-doubling map on the circle. Finally, we conclude with a discussion of the special case of a self-mating $f_\theta \sqcup f_\theta$ and mating f_θ with the Chebyshev polynomial $z \mapsto z^2 - 2$.

8.1. **Ray equivalence classes and pinch points.** Consider two irrationals θ and ν of bounded type, with $\theta \neq 1 - \nu$, and the quadratic polynomials f_θ and f_ν and the rational map $F_{\theta,\nu}$. Let

$$K(F_{\theta,\nu}) = \{z \in \mathbb{C} : \text{The orbit } \{F_{\theta,\nu}^{\circ n}(z)\}_{n \geq 0} \text{ never intersects } \Delta^\infty\},$$

and similarly

$$K^\infty(F_{\theta,\nu}) = \{z \in \mathbb{C} : \text{The orbit } \{F_{\theta,\nu}^{\circ n}(z)\}_{n \geq 0} \text{ never intersects } \Delta^0\}.$$

(In Fig. 2 these two sets are the compact sets in black and gray respectively.) As we have already noted in the introduction, $K(F_{\theta,\nu})$ is not a full set. In fact, it is evident from Fig. 2 that there are infinitely many identifications between pairs of landing points of drop-chains in $K(F_{\theta,\nu})$ which correspond to the pinch points of $K^\infty(F_{\theta,\nu})$, that is the preimages of the critical point $c \in \partial\Delta^\infty$. Similar fact holds for drop-chains of $K^\infty(F_{\theta,\nu})$ and the pinch points of $K(F_{\theta,\nu})$. We gave a precise version of this statement in Lemma 6.6. It follows that every precritical point in the Julia set of f_θ (resp. f_ν) is identified with the landing points of two distinct drop-chains of f_ν (resp. f_θ). Theorem 7.11 allows us to determine exactly which two drop-chains correspond to the given pinch point. Throughout the following discussion we continue using notations from §7.

Recall that the quasiconformal conjugacies ψ_θ (between $\tilde{Q}_\theta$ and f_θ) and ψ_ν (between $\tilde{Q}_\nu$ and f_ν) in (7.3) are conformal in the basins of infinity, so they preserve the ray equivalence classes. From this fact and Corollary 7.10, it follows that for the formal mating of f_θ and f_ν, every ray equivalence class intersects $K(f_\theta) \cup K(f_\nu)$ in at most three points. Let E denote the intersection of a ray equivalence class with the union $K(f_\theta) \cup K(f_\nu)$. We only have three possibilities for E:

- *Case 1.* $E = \{z, w\}$, where $z \in K(f_\theta)$ and $w \in K(f_\nu)$ are both the landing points of unique rays, hence $z = \gamma_\theta(t)$ and $w = \gamma_\nu(-t)$ for a unique $t \in \mathbb{T}$.

• *Case 2.* $E = \{z, z', w\}$, where $z, z' \in K(f_\theta)$ are both the landing points of unique rays and $w \in K(f_\nu)$ is biaccessible, hence a preimage of the critical point of f_ν. In this case, there exist $s, t \in \mathbb{T}$ such that $z = \gamma_\theta(s)$, $z' = \gamma_\theta(t)$, and $w = \gamma_\nu(-s) = \gamma_\nu(-t)$.

• *Case 3.* $E = \{z, w, w'\}$, where $z \in K(f_\theta)$ is biaccessible, and $w, w' \in K(f_\nu)$ are both the landing points of unique rays. In this case, there exist $s, t \in \mathbb{T}$ such that $z = \gamma_\theta(s) = \gamma_\theta(t)$, $w = \gamma_\nu(-t)$, $w' = \gamma_\nu(-s)$.

Corollary 8.1 (Pinch points in $K(F_{\theta,\nu})$). *The compact set $K(F_{\theta,\nu})$ is homeomorphic to the quotient of the filled Julia set $K(f_\theta)$ by an equivalence relation $\sim$ defined as follows. Two points $z \neq z'$ in $K(f_\theta)$ satisfy $z \sim z'$ if and only if they are the landing points of unique rays at angles $s, t \in \mathbb{T}$, $z = \gamma_\theta(s)$, $z' = \gamma_\theta(t)$, such that $\gamma_\nu(-s) = \gamma_\nu(-t)$. Every non-trivial equivalence class of $\sim$ contains exactly two points which are necessarily the landing points of two distinct drop-chains of f_θ.*

Proof. Since $\varphi_\theta : K(f_\theta) \to K(F_{\theta,\nu})$ is a surjective map, $K(F_{\theta,\nu})$ is homeomorphic to $K(f_\theta)/\sim$, where $z \sim z'$ if and only if z and z' belong to the same fiber of φ_θ. By the above discussion (*Case 2*), for distinct points $z \neq z'$, we have $\varphi_\theta(z) = \varphi_\theta(z')$ if and only if there exist $w \in K(f_\nu)$ and distinct angles $s, t \in \mathbb{T}$ such that $z = \gamma_\theta(s)$, $z' = \gamma_\theta(t)$, and $w = \gamma_\nu(-s) = \gamma_\nu(-t)$. In this case w is a preimage of the critical point of f_ν. Both z and z' are landing points of distinct drop-chains of f_θ, for otherwise z or z' would belong to the closure of a drop (Proposition 3.8), hence $\varphi_\theta(z) = \varphi_\theta(z')$ would eventually map to the boundary of the Siegel disk Δ^0 of $F_{\theta,\nu}$. On the other hand, $\varphi_\theta(z) = \varphi_\nu(w)$ eventually maps to the critical point of $F_{\theta,\nu}$ on the boundary of Δ^∞. This would contradict $\partial\Delta^0 \cap \partial\Delta^\infty = \emptyset$. $\square$

This completely describes which identifications are made in $K(f_\theta)$ in order to obtain $K(F_{\theta,\nu})$: Take any precritical point in the Julia set of f_ν and calculate the angles s, t of the two external rays landing on it. Then find the landing points of the external rays at angles $-s$ and $-t$ for f_θ, which are ends of distinct drop-chains, and identify them in $K(f_\theta)$. This creates a "pinch point." After all such possible identifications are made, we obtain a homeomorphic copy of $K(F_{\theta,\nu})$. Note that not all the landing points of drop-chains of f_θ undergo this identification, simply because there are uncountably many drop-chains and only countably many pinch points.

8.2. **Rotation sets of the doubling map.** The angle $\omega = \omega(\theta)$ of the external ray landing at the critical value of the quadratic polynomial f_θ may be described in terms of the rotation sets of the angle-doubling map on $\mathbb{T}$ defined by $m_2 : x \mapsto 2x \pmod 1$. A subset $E \subset \mathbb{T}$ is called a *rotation set* if the restriction of m_2 to E is order-preserving, with $m_2(E) \subset E$. It is easy to see that in this case E must be contained in a closed semicircle. Hence the restriction $m_2|_E$ can be extended to a degree 1 monotone map of the circle, which has a well-defined rotation number, denoted by $\rho(E) \in [0, 1)$. The following theorem can be found in [**BS**]:

Theorem 8.2 (Rotation sets of the doubling map)**.**

(i) *For any* $0 \leq \theta < 1$ *there exists a unique compact rotation set* $E_\theta \subset \mathbb{T}$ *with* $\rho(E_\theta) = \theta$. *When* θ *is rational* E_θ *is a single periodic orbit of* m_2. *On the other hand, when* θ *is irrational,* E_θ *is a Cantor set contained in a well-defined semicircle* $[\omega/2, (\omega+1)/2]$, *with* $\{\omega/2, (\omega+1)/2\} \subset E_\theta$, *and the action of* m_2 *on* E_θ *is minimal. In this case the angle* ω *can be computed in terms of* θ *as*

$$\omega = \sum_{0<p/q<\theta} 2^{-q}, \tag{8.1}$$

where the sum is taken over all (not necessarily reduced) fractions p/q.

(ii) *For every* $0 < \omega < 1$, *the semicircle* $[\omega/2, (\omega+1)/2]$ *contains a unique compact minimal rotation set* E^ω. *The graph of* $\omega \mapsto \rho(E^\omega)$ *is a devil's staircase.*

The mapping $\omega \mapsto \rho(E^\omega)$ is intimately connected with the parameter rays defining the limbs of the Mandelbrot set [**BS**].

Now consider the quadratic polynomial f_θ for an irrational θ of bounded type. Then the Julia set $J(f_\theta)$ is locally-connected, and the boundary of the Siegel disk Δ of f_θ is a quasicircle passing through the critical point 0 (compare Theorem 3.5 and Theorem 3.10). We know that 0 is the landing point of exactly two external rays at angles $\omega/2$ and $(\omega+1)/2$, where $0 < \omega < 1$. Define

$$E = \{t \in \mathbb{T} : \gamma_\theta(t) \in \partial\Delta\}.$$

It is easy to see that E is compact and contained in the semicircle $[\omega/2, (\omega+1)/2]$, hence by the above theorem, $E = E_\omega$. On the other hand, the order of the points in the orbit $\{f_\theta^{\circ n}(0)\}_{n\geq 0}$ on the boundary $\partial\Delta$ determines the rotation number θ uniquely [**dMvS**]. At the same time this order coincides with the order of the orbit of ω under m_2 on the circle. It follows that $\rho(E_\omega) = \theta$.

Corollary 8.3. *When* $0 < \theta < 1$ *is an irrational of bounded type, the angle* $0 < \omega(\theta) < 1$ *of the external ray landing at the critical value of the quadratic polynomial* f_θ *is given by (8.1).*

It is interesting to investigate number-theoretic properties of the numbers $\omega(\theta)$ when θ is irrational. For example, it follows from the above discussion that for irrational $0 < \theta < 1$, $\omega(\theta)$ is also irrational. When θ is of bounded type, we have the much sharper statement that $\omega(\theta)$ is not $(2 + (\sqrt{5}-1)/2 - \delta)$-Diophantine for any $\delta > 0$ [**BS**]. In particular, by Roth's theorem, $\omega(\theta)$ is transcendental over $\mathbb{Q}$. The topological mateability part of Theorem 7.11 allows us to draw a further conclusion:

Theorem 8.4. *Suppose that* $0 < \theta, \nu < 1$ *are irrationals of bounded type, with* $\theta \neq 1 - \nu$, *and consider the angles* $\omega(\theta)$ *and* $\omega(\nu)$. *Then the equation*

$$2^n \omega(\theta) + 2^m \omega(\nu) \equiv 0 \pmod 1 \tag{8.2}$$

does not have any solution in non-negative integers n, m.

Note that the condition $\theta \neq 1-\nu$ is necessary because $\omega(\theta)+\omega(1-\theta)=1$. Also, when $\theta=\nu$ this theorem is only saying that $\omega(\theta)$ is irrational, a fact that is clear from Theorem 8.2.

Proof. Suppose that (8.2) holds for some n, m. Set $t=\omega(\theta)/2^m$, so that $-2^{n+m}t \equiv 2^m\omega(\nu) \pmod 1$. Let $z=\gamma_\theta(t)\in J(f_\theta)$ and $w=\gamma_\nu(-t)\in J(f_\nu)$. Then $f_\theta^{\circ m}(z)=c_\theta$ is the critical value of f_θ and $f_\nu^{\circ n+m}(w)=f_\nu^{\circ m}(c_\nu)$ belongs to the forward orbit of the critical point of f_ν. By Theorem 7.11, $F_{\theta,\nu}=f_\theta \sqcup f_\nu$, so $\varphi_\theta(z)\in J(F_{\theta,\nu})$ and $\varphi_\nu(w)\in J(F_{\theta,\nu})$ eventually hit $\partial\Delta^0$ and $\partial\Delta^\infty$ respectively. But z and w are ray equivalent, so $\varphi_\theta(z)=\varphi_\nu(w)$ by Theorem 7.11. This contradicts $\partial\Delta^0\cap\partial\Delta^\infty=\emptyset$. $\square$

8.3. **Mating with Chebyshev quadratic polynomial.** When $\theta=\nu$, the self-mating $F=F_{\theta,\theta}=f_\theta\sqcup f_\theta$ given by Theorem 7.11 has a natural symmetry, i.e., it commutes with the involution $\mathcal{I}: z\mapsto 1/z$ of the sphere. As was apparently first observed by C. Petersen, if we destroy this symmetry by passing to the quotient space, we can create new examples of mating.

Consider the quotient of the Riemann sphere by the action of $\mathcal{I}$. The resulting space is again a Riemann surface conformally isomorphic to the sphere $\overline{\mathbb{C}}$. Since $F\circ\mathcal{I}=\mathcal{I}\circ F$, there is a well-defined rational map G which makes the following diagram commute:

$$\begin{tikzcd} \overline{\mathbb{C}} \arrow[r, "F"] \arrow[d, "\pi"] & \overline{\mathbb{C}} \arrow[d, "\pi"] \\ \overline{\mathbb{C}} \arrow[r, "G"] & \overline{\mathbb{C}} \end{tikzcd}$$

Here $\pi:\overline{\mathbb{C}}\to\overline{\mathbb{C}}/\mathcal{I}\simeq\overline{\mathbb{C}}$ is the degree 2 natural projection. Chasing around this diagram shows that G is a quadratic rational map which clearly has one Siegel disk of rotation number θ. Therefore this way of collapsing the sphere identifies the two critical points of F but creates a new critical point of its own. It is not hard to check that G is Möbius conjugate to the map

$$z\mapsto \frac{4z}{((1+z)+e^{2\pi i\theta}(1-z))^2}, \tag{8.3}$$

with a fixed Siegel disk centered at 1. The critical point $c_1=(e^{2\pi i\theta}+1)/(e^{2\pi i\theta}-1)$ of this map has the finite orbit $c_1\mapsto\infty\mapsto 0$. The second critical point $c_2=-c_1$ belongs to the boundary of the Siegel disk (compare Fig. 14).

Recall that the *Chebyshev* quadratic polynomial is $f_{\text{cheb}}: z\mapsto z^2-2$. It is easy to see that the filled Julia set $K(f_{\text{cheb}})=J(f_{\text{cheb}})$ is the closed interval $[-2,2]$. Its Carathéodory loop $\gamma_{\text{cheb}}:\mathbb{T}\to J(f_{\text{cheb}})$ is simply given by $\gamma_{\text{cheb}}(t)=2\cos t$, hence $\gamma_{\text{cheb}}(t)=\gamma_{\text{cheb}}(s)$ if and only if $t=-s$.

We would like to show that G is the mating of f_θ with f_{cheb}. Recall that γ_θ is the Carathéodory loop of $J(f_\theta)$ and $\varphi_\theta: K(f_\theta)\to\overline{\mathbb{C}}$ is the semiconjugacy between f_θ and F given by Theorem 7.11. Denote by φ_1 the composition $\pi\circ\varphi_\theta: K(f_\theta)\to\overline{\mathbb{C}}$, which

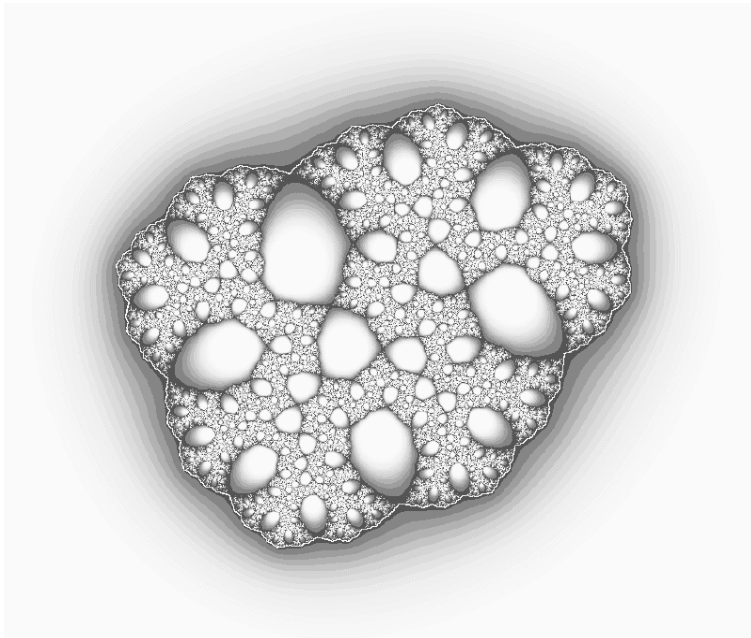

FIGURE 14. *The Julia set of the mating* $f_\theta \sqcup f_{\text{cheb}}$, *where* $\theta = (\sqrt{5}-1)/2$. *To get a better picture we have conjugated the map in (8.3) by* $w = 1/(z-1)$ *so as to put the center of the Siegel disk at infinity and the finite critical orbit at* $(e^{2\pi i\theta}+1)/2 \mapsto 0 \mapsto -1$.

conjugates f_θ to the quadratic rational map G. It is clear from the symmetry of the construction that

$$\varphi_\theta(\gamma_\theta(-t)) = \mathcal{I}(\varphi_\theta(\gamma_\theta(t)))$$

for all $t \in \mathbb{T}$. It follows that the composition $\varphi_\theta \circ \gamma_\theta$ conjugates the map $t \mapsto -t$ on $\mathbb{T}$ to the involution $\mathcal{I}$. Hence it descends to a map $\varphi_2 : K(f_{\text{cheb}}) \to \overline{\mathbb{C}}$ which conjugates f_{cheb} to G. It is easy to check that the pair (φ_1, φ_2) satisfies the conditions of Definition IIa of the introduction. Hence,

Theorem 8.5 (Mating with the Chebyshev map)**.** *Let* $0 < \theta < 1$ *be any irrational of bounded type. Then there exists a quadratic rational map* G *such that*

$$G = f_\theta \sqcup f_{\text{cheb}}.$$

Moreover, G *is unique up to conjugation with a Möbius transformation.*

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