BIACCESSIBILITY IN QUADRATIC JULIA SETS I: THE LOCALLY-CONNECTED CASE

SAEED ZAKERI

ABSTRACT. Let $f : z \mapsto z^2 + c$ be a quadratic polynomial whose Julia set J is locally-connected. We prove that the Brolin measure of the set of biaccessible points in J is zero except when $f(z) = z^2 - 2$ is the Chebyshev quadratic polynomial for which the corresponding measure is one.

§1. Introduction. Let $f : z \mapsto z^2 + c$ be a quadratic polynomial with connected filled Julia set K. The Julia set ∂K is as usual denoted by J. Let $\psi : \overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \smallsetminus K$ be the unique conformal isomorphism, normalized as $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$, which conjugates the squaring map to f:

$$\psi(z^2) = f(\psi(z)). \tag{1}$$

(The inverse ψ^{-1} is often called the *Böttcher* coordinate.) By the *external ray* R_t we mean the image of the radial line $\{\psi(re^{2\pi it}) : r > 1\}$, where $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the *angle* of the ray. We say that R_t *lands* at $z \in J$ if $\lim_{r\to 1} \psi(re^{2\pi it}) = z$. A point $z \in J$ is called *accessible* if there exists an external ray which lands at z, and it is called *biaccessible* if z is the landing point of more than one ray. It can be shown that $z \in J$ is biaccessible if and only if $K \smallsetminus \{z\}$ is disconnected (see [**Mc**], p. 85).

Let us denote by $\gamma(t)$ the radial limit $\lim_{r\to 1} \psi(re^{2\pi it})$. According to a classical theorem of Fatou (see for example [**Ru**], p. 249), $\gamma(t)$ exists for almost every $t \in \mathbb{T}$ in the sense of the Lebesgue measure. For all such angles t, it follows from (1) that γ conjugates the doubling map to the action of f on the Julia set:

$$\gamma(2t) = f(\gamma(t)).$$

When K, or equivalently J, is locally-connected, it follows from the theorem of Carathéodory that γ is defined and continuous on the whole circle. In this case, the surjective map $\gamma : \mathbb{T} \to J$ is called the *Carathéodory loop*. Evidently the biaccessible points in J correspond to the points where γ fails to be one-to-one.

Whether or not J is locally-connected, the Lebesgue measure on the circle $\mathbb{T}$ pushes forward by γ to a probability measure μ on the Julia set. Complex analysts call μ the "harmonic measure" on J, but in the context of holomorphic dynamics, μ is called the *Brolin measure*. It has the following nice properties:

(i) The support of μ is the whole Julia set, with $\mu(J) = 1$.
(ii) μ is invariant under the 180° rotation $z \mapsto -z$, i.e., $\mu(-A) = \mu(A)$ for every measurable set $A \subset J$.
(iii) μ is f-invariant, i.e., $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set $A \subset J$. Moreover, μ is ergodic in the sense that for every measurable set $A \subset J$ with $f^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

All of these properties are immediate consequences of the corresponding properties of the Lebesgue measure and the angle-doubling map on the unit circle. Properties (ii) and (iii) are equivalent to the next property, which will be used repeatedly in this paper:

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(iv) $\mu(f(A)) = 2\mu(A)$ for every measurable set $A \subset J$ for which the restriction $f|_A$ is one-to-one.

Brolin proved that with respect to this measure μ the backward orbits of typical points have an asymptotically uniform distribution [**Br**]. Lyubich has proved that μ is the unique measure of maximal entropy $\log 2$. He has also constructed such invariant measures of maximal entropy for arbitrary rational maps of the Riemann sphere [**Ly**].

It follows from general plane topology that the set of points in J which are the landing points of more than two external rays is at most countable (see for example [**Po**], p. 36). On the other hand, the number of rays landing at a point is constant along an orbit, unless the orbit passes through the critical point. It follows from ergodicity that either μ-almost every point in the Julia set is the landing point of a unique ray, or else μ-almost every point is the landing point of exactly two rays.

As an example, for the *Chebyshev polynomial* $z \mapsto z^2 - 2$, the Julia set is the closed interval $[-2, 2]$ on the real line. Here every point is the landing point of exactly two rays except for the endpoints ± 2 where unique rays land. There are no other known examples of quadratic Julia sets with two rays landing at almost every point. In fact, as I heard from J. Hubbard and later M. Lyubich, it is conjectured that a polynomial Julia set has this property *only if* it is a straight line segment in which case the map is conjugate to a Chebyshev polynomial, up to sign. In this paper, we will confirm this conjecture for quadratic Julia sets which are locally-connected. The second part of this paper [**Za**], which is an expanded version of [**S-Z**], considers the Julia sets of quadratic polynomials with irrationally indifferent fixed points. By a completely different method we prove that every biaccessible point in J eventually maps to the critical point in the Siegel case and to the Cremer fixed point otherwise. As a byproduct, it follows that the set of biaccessible points in the Julia set has Brolin measure zero. This settles some cases that are not covered by Theorem 1 of this paper, since in the Cremer case J is certainly non locally-connected, and in the Siegel case J may or may not be locally-connected.

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§2. Basic Definitions. Let $f : z \mapsto z^2 + c$ be a quadratic polynomial whose filled Julia set K is locally-connected. As usual, the fixed points of f are denoted by α and β, where β is the more repelling fixed point. If α is attracting or $\alpha = \beta$ ($\Leftrightarrow c = 1/4$), the Julia set of f is a Jordan curve with a unique external ray landing at every point. Hence there are no biaccessible points at all and Theorem 1 below is trivially true. *So we may as well assume that $\alpha \neq \beta$ and α is not attracting.* It follows that either $\alpha \in J$, or else α is the center of a fixed Siegel disk for f.

By an *embedded arc* in K we mean any subset of K homeomorphic to the closed interval $[0, 1] \subset \mathbb{R}$. Since K is locally-connected, for any two points $x, y \in K$ there exists an embedded arc η in K which connects x to y. If K has no interior so that $J = K$ is full, then η is uniquely determined by the two endpoints x and y. If K does have interior, however, there is usually more than a choice for η. In what follows, we will show how to choose a canonical embedded arc between any two points in the filled Julia set.

Suppose that $int(K)$ is non-vacuous. Every component U of this interior is a bounded Fatou component whose closure $\overline{U}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ since K is locally-connected. According to Fatou and Sullivan (see for example [**Mi**]), every such component eventually maps to a periodic Fatou component which is either the immediate basin of attraction of an attracting periodic point, or an attracting petal for a parabolic periodic point, or a periodic Siegel disk. We refer to these cases simply as *hyperbolic*, *parabolic* and *Siegel* cases. Note that in the hyperbolic and parabolic cases the critical point 0 belongs to a central Fatou component which we denote by U_0. Also by our assumption on the α-fixed point, periodic Fatou components in the hyperbolic and parabolic cases form a cycle of period > 1.

Next, we would like to choose a "center" $c(U)$ in every bounded Fatou component U subject only to the following conditions:

(C1) $c(-U) = -c(U)$,
(C2) If U contains the critical value $c = f(0)$, then $c(U) = c$,
(C3) If U contains the fixed point α, then $c(U) = \alpha$.

If follows from **(C1)** that whenever the critical point 0 belongs to the Fatou set, then it is the center of the corresponding Fatou component U_0: $c(U_0) = 0$. Also **(C3)** corresponds to the case where the α-fixed point is the center of a fixed Siegel disk U.

Given any bounded Fatou component U, there exists a homeomorphism $\phi : \overline{U} \xrightarrow{\simeq} \overline{\mathbb{D}}$ which is holomorphic in U with $\phi(c(U)) = 0$. An arc in $\overline{U}$ of the form $\phi^{-1}\{re^{i\theta} : 0 \leq a \leq r \leq b \leq 1\}$ is called a *radial arc*. Since ϕ is unique up to postcomposition with a rigid rotation of $\overline{\mathbb{D}}$, radial arcs in $\overline{U}$ are well-defined.

Following [**D-H**], we call an embedded arc I in K *regulated* if for every bounded Fatou component U, the intersection $I \cap \overline{U}$ is either empty or a point or consists of radial arcs in $\overline{U}$ (see also [**Do**], where he uses the word "legal" for regulated).

Lemma 1. *Given any two points $x, y \in K$, there exists a unique regulated arc I in K with endpoints x, y. Furthermore, if η is any embedded arc in K which connects x to y, then $I \cap J \subset \eta \cap J$.*

Proof. Take any embedded arc η in K with endpoints x, y. It is easy to see how one can deform η to a regulated arc I. Let U be a bounded Fatou component whose closure intersects η. Choose any parametrization $h : [0, 1] \to K$ with $\eta = h([0, 1])$, and define

$$t_0 = \inf\{t \in [0, 1] : h(t) \in \overline{U}\},$$

$$t_1 = \sup\{t \in [0, 1] : h(t) \in \overline{U}\}.$$

In other words, t_0 is the first moment η hits $\overline{U}$ and t_1 is the last moment η stays in $\overline{U}$. If $t_0 \neq t_1$, replace the subarc of η from $h(t_0)$ to $h(t_1)$ by the radial arc from $h(t_0)$ to $c(U)$ followed by the radial arc from $c(U)$ to $h(t_1)$ (see Fig. 1). If $h(t_0)$ and $h(t_1)$ happen to be on the same radial arc, simply connect the two by the radial arc between them.

Applying this construction to the intersection with every such Fatou component, we obtain a regulated arc I with endpoints x, y. Evidently we have the inclusion $I \cap J \subset \eta \cap J$.

To prove uniqueness, suppose that I and I' are both regulated, with the same endpoints x, y. If $I \neq I'$, then the complement $\mathbb{C} \smallsetminus (I \cup I')$ has a bounded connected component V. By the Maximum Principle, V is contained in some bounded Fatou component U. It follows that the boundary ∂V must be contained in a union of at most four radial arcs in $\overline{U}$. But a finite union of radial arcs cannot bound an open set in $\overline{U}$. Therefore, $I = I'$. $\square$

The regulated arc I given by the above lemma is denoted by $[x, y]$. The open arc (x, y) is defined by $[x, y] \smallsetminus \{x, y\}$, and similarly we can define the semi-open arc $[x, y)$.

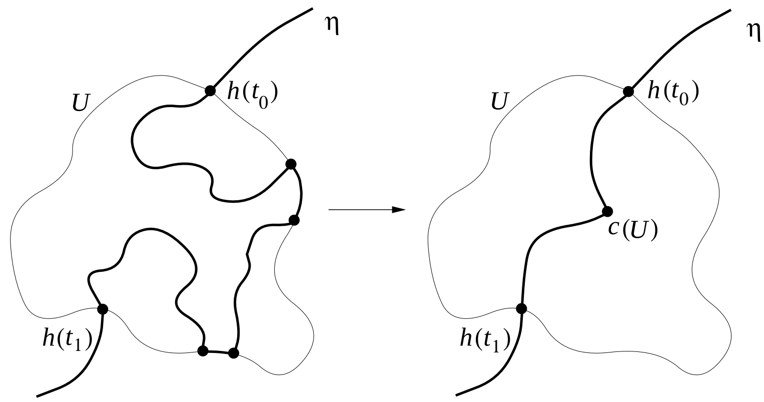

FIGURE 1. Deforming an embedded arc to a regulated arc.

More generally, given finitely many points $x, y, \dots, z$ in K, there is a unique smallest connected set $[x, y, \dots, z] \subset K$ made up of regulated arcs which contains all of these points. In fact this set is always a (finite) topological tree. We call $[x, y, \dots, z]$ the *regulated tree* generated by $\{x, y, \dots, z\}$. A vertex of this tree with exactly one edge attached to it is called an *end* of the tree. A point which is not an end is called an *interior point* of the tree. It follows easily from **(C1)** that

$$[-x, -y, \dots, -z] = -[x, y, \dots, z] \tag{2}$$

In the case of three distinct points, $[x, y, z]$ is either homeomorphic to a closed interval or to a letter Y. The first case occurs if and only if one of the points belongs to the regulated arc connecting the other two. In the second case, the three points x, y, z are ends of the tree $[x, y, z]$. In other words, there is a unique interior point $p \in [x, y, z]$ such that $[x, p] \cap [y, p] = [x, p] \cap [z, p] = [y, p] \cap [z, p] = \{p\}$ (see Fig. 2). In this case, we call $[x, y, z]$ a *tripod*. Point p is called the *joint* of this tripod.

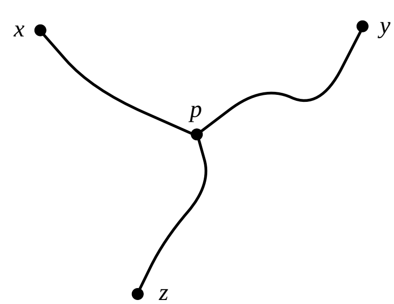

FIGURE 2. A tripod $[x, y, z]$ with joint p.

The regulated trees as defined above are not preserved by the dynamics of f. In fact, when K has interior, the center of a bounded Fatou component U is not necessarily mapped by f to that of $f(U)$. Hence regulated arcs in U do not map to regulated arcs in $f(U)$. This difficulty can be most conveniently overcome by deforming the polynomial f rel the Julia set into a new map F which respects the centers. To this end, it suffices to note that for every bounded Fatou component U, there is a homeomorphism between $\overline{U}$ and the cone over ∂U which sends $c(U)$ to the cone point and restricts to the identity map on ∂U. We can define F so as to preserve this cone structure on various bounded Fatou components. For example, for any component U and any $p \in \partial U$ take the Poincaré geodesic in U between $c(U)$ and p and define $F : \overline{U} \to f(\overline{U})$ so as to map this geodesic isometrically to the unique Poincaré geodesic between $c(f(U))$ and $f(p) \in \partial f(U)$. (Note that by our assumption $f(U) \neq U$ unless U is a fixed Siegel disk for which the α fixed point is the center. So in any case α is still a fixed point of F.) Apply this construction to every bounded Fatou component and let $F = f$ anywhere else. The map F will be the required modification of f which satisfies the following properties:

- **(F1)** $F(c(U)) = c(F(U))$ for every bounded Fatou component U. In particular, by **(C2)**, whether or not the critical point 0 belongs to the Fatou set, $F(0) = f(0) = c$ is always the critical value of f.
- **(F2)** $F = f$ on the closure of the basin of attraction of infinity.
- **(F3)** $F(z) = F(z') \Leftrightarrow z = \pm z'$.
- **(F4)** α and β are the only fixed points of F.

Also, since the support of the Brolin measure is the Julia set where f and F agree, it follows that properties (iii)-(iv) in section **§1** also hold for F. In other words,

- **(F5)** $\mu(F^{-1}(A)) = \mu(A)$ for any measurable set $A \subset \mathbb{C}$, and
- **(F6)** $\mu(F(A)) = 2\mu(A)$ for any measurable set $A \subset \mathbb{C}$ for which $F|_A$ is one-to-one.

Lemma 2. *Let $x, y, \ldots, z \in K$. Suppose that the critical point 0 is not an interior point of the tree $[x, y, \ldots, z]$. Then F maps $[x, y, \ldots, z]$ homeomorphically to $[F(x), F(y), \ldots, F(z)]$.*

In this case, we simply write

$$F : [x, y, \ldots, z] \xrightarrow{\simeq} [F(x), F(y), \ldots, F(z)].$$

Proof. First let us show that F restricted to $[x, y, \ldots, z]$ is injective. If not, it follows from **(F3)** that $[x, y, \ldots, z]$ contains a pair $\pm a$ of symmetric points. By (2), we see that $[a, -a] = -[a, -a]$. Hence the $180°$ rotation from the arc $[a, -a]$ to itself must have a fixed point, namely the critcal point 0. But this implies that 0 is an interior point of $[x, y, \ldots, z]$, contrary to our assumption.

Therefore, F restricted to $[x, y, \ldots, z]$ is injective. The image tree $F([x, y, \ldots, z])$ is evidently connected and contains all of the image points $F(x), F(y), \ldots, F(z)$. Since all the ends of $F([x, y, \ldots, z])$ are among $F(x), F(y), \ldots, F(z)$, we conclude that it is also minimal. To finish the proof, it is enough to show that the image of every regulated arc in $[x, y, \ldots, z]$ is a regulated arc. But this follows from **(F1)** since F preserves the centers hence the radial arcs in bounded Fatou components of f. $\square$

Definition. By the *spine* of the filled Julia set K we mean the unique regulated arc $[-\beta, \beta]$ between the β-fixed point and its preimage $-\beta$, which are the landing points of the unique external rays R_0 and $R_{1/2}$ respectively. By (2), the spine is invariant under the $180°$ rotation $z \mapsto -z$. In particular, the critical point 0 always belongs to the spine.

Let $z \in J$ be a biaccessible point, with a ray pair (R_t, R_s) landing at z and $0 < t < s < 1$. If $z \notin [-\beta, \beta]$, it follows that both t and s satisfy $0 < t < s < 1/2$ or $1/2 < t < s < 1$. Consider the orbit of the ray pair (R_t, R_s) under f. Since there exists an integer $n > 0$ such that $1/2 \leq 2^n s - 2^n t < 1$, the corresponding rays $f^{\circ n}(R_t)$ and $f^{\circ n}(R_s)$ must belong to different sides of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$ (see Fig. 3). Therefore $f^{\circ n}(z) \in [-\beta, \beta]$. This means that the set B of all biaccessible points in the Julia set is contained in the union of preimages of the spine:

$$B \subset \bigcup_{n \geq 0} f^{-n}[-\beta, \beta]. \tag{3}$$

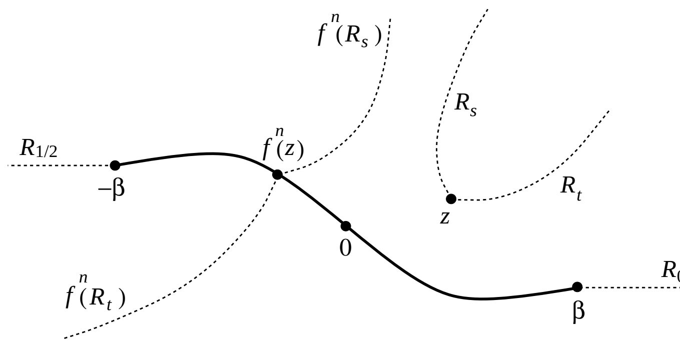

FIGURE 3

§3. Main Theorem and Supporting Lemmas. In this paper we will prove the following theorem:

Theorem 1. *If the Julia set J of the quadratic polynomial $f : z \mapsto z^2 + c$ is locally-connected, then the set of all biaccessible points in J has Brolin measure zero unless f is the Chebyshev polynomial $z \mapsto z^2 - 2$ for which the corresponding measure is one.*

By (3), it suffices to show that for every non-Chebyshev quadratic, the Brolin measure $\mu[-\beta, \beta]$ of the spine is zero.

The proof depends on several lemmas which will be given in this section and section §4.

Lemma 3.

(a) *Any point in the Julia set J which belongs to the boundary of two Fatou components is necessarily biaccessible.*

(b) *Let η be any embedded arc in the filled Julia set K and z be a point in $\eta \cap J$ which is not an endpoint of η. Then either z is biaccessible or it belongs to the boundary of a unique bounded Fatou component.*

Proof. (a) Let U and U' be two such Fatou components, with $z \in \partial U \cap \partial U'$. Assume that z is not biaccessible. Then $K \smallsetminus \{z\}$ is connected, so there exists an embedded arc η in K between $c(U)$ and $c(U')$ which avoids z. By Lemma 1, $I \cap J \subset \eta \cap J$, where $I = [c(U), c(U')] = [c(U), z] \cup [z, c(U')]$ is the unique regulated arc between $c(U)$ and $c(U')$. It follows that η must contain z, which is a contradiction.

(b) If z is not biaccessible, then $K \smallsetminus \{z\}$ is connected. Hence there exists an embedded arc η' in K between the two endpoints of η which avoids z. Take a bounded connected component V of the complement $\mathbb{C} \smallsetminus (\eta \cup \eta')$ which contains z in its closure. By the Maximum Principle, V must be contained in a bounded Fatou component. Hence z belongs to the boundary of this bounded Fatou component. Uniqueness follows from part (a). $\square$

Corollary 1. *Let $f : z \mapsto z^2 + c$ have locally-connected Julia set. If the α-fixed point is not attracting and $\alpha \neq \beta$, then neither the β-fixed point nor any of its preimages can belong to the boundary of a bounded Fatou component of f.*

Proof. Otherwise there exists a bounded Fatou component U with $\beta \in \partial U$. Hence $\beta \in \partial U \cap \partial f(U)$. If $U = f(U)$, it must be a fixed Siegel disk by the assumption. But in this case $f|_{\partial U}$ is conjugate to an irrational rotation so it cannot have a fixed point. Therefore $U \neq f(U)$. By Lemma 3(a), β will be biaccessible. But this is impossible since the β-fixed point is always the landing point of the unique ray R_0. $\square$

Remark. In the non locally-connected case, it is *not* known if the β-fixed point can be on the boundary of any bounded Fatou component. In fact, it is not known if there are examples of quadratic polynomials with a fixed Siegel disk whose boundary is the whole Julia set. Any such quadratic would provide a counterexample to the above corollary in the non locally-connected case.

Lemma 4. *If $x \notin [-\beta, \beta]$, then $[-\beta, x, \beta]$ is a tripod.*

Proof. Otherwise, we must have $-\beta \in (x, \beta)$ or $\beta \in (x, -\beta)$. In either case, it follows that $-\beta$ or β belongs to the interior of an embedded arc in the filled Julia set. But β is the landing point of the unique ray R_0. Since the orbit $-\beta \mapsto \beta$ does not pass through the critical point, it follows that $-\beta$ is also the landing point of the unique ray $R_{1/2}$. By Lemma 3(b), either β or $-\beta$ must be on the boundary of a bounded Fatou component, which contradicts Corollary 1. $\square$

Here is a definition which will be used repeatedly in all subsequent arguments:

Definition. We define a projection $\pi : K \to [-\beta, \beta]$ as follows: For $x \in [-\beta, \beta]$, let $\pi(x) = x$. If $x \notin [-\beta, \beta]$, then $[-\beta, x, \beta]$ is a tripod by Lemma 4, and we define $\pi(x) \in (-\beta, \beta)$ to be the joint of this tripod.

Note that $\pi(x)$ can be described as the unique point in $[-\beta, \beta]$ such that for any y on the spine, $[x, \pi(x)] \subset [x, y]$. Set theoretically π a retraction from K onto its spine. However, when K has interior, π is not continuous.

For simplicity, we denote the regulated arc $[x, \pi(x)]$ by I_x. Since $\pi(-x) = -\pi(x)$, we have $I_{-x} = -I_x$.

Lemma 5. *The α-fixed point belongs to $(-\beta, 0)$.*

Proof. First we prove that $\alpha \in (-\beta, \beta)$. In fact, if α belonged to J and were off the spine, then the external rays which land at α would all belong to one side of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$. This would contradict the fact that the angle-doubling map on the circle has no forward orbit which is entirely contained in the interval $(0, 1/2)$ or $(1/2, 1)$. On the other hand, if α belonged to the Fatou set and were off the spine, then it would have to be the center of a fixed Siegel disk whose closure by **(C3)** touches $[-\beta, \beta]$ at the unique point 0. Take the external ray R_t which lands at the critical value c. Since the entire orbit of c is on one side of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$, the forward orbit of t under the doubling map must be entirely contained in one of the intervals $(0, 1/2)$ or $(1/2, 1)$, which is again a contradiction. Therefore, $\alpha \in (-\beta, \beta)$.

Now suppose that $\alpha \in (0, \beta)$. Then $[\alpha, \beta] \subset (0, \beta]$. Hence $F : [\alpha, \beta] \xrightarrow{\simeq} [\alpha, \beta]$ by Lemma 2. By **(F4)**, there is no fixed point of F in (α, β). Suppose that $[\alpha, \beta] \subset J$. Then f repels all points in $[\alpha, \beta]$ close to α and β. Since $f = F$ on the Julia set, the same must be true for F. Hence there has to be an attracting fixed point for F somewhere in

(α, β), which is a contradiction. Therefore $[\alpha, \beta]$ intersects a bounded Fatou component U. Passing to some iterate $f^{\circ n}(U) = F^{\circ n}(U)$, we may as well assume that U is periodic. Since F acts monotonically on $[\alpha, \beta]$, U must be fixed. Hence U is a Siegel disk with $c(U) = \alpha$. Now ∂U intersects $[\alpha, \beta]$ at a unique point p which is not the β-fixed point by Corollary 1. Clearly $F(p) = p$, which is a contradiction. This shows that $\alpha \in (-\beta, 0)$, and completes the proof. $\square$

Lemma 6. *There exists an F-preimage ω of 0 in $(-\beta, \alpha)$. The other preimage $-\omega$ is then in $(-\alpha, \beta)$.*

Proof. $F : [-\beta, \alpha] \xrightarrow{\simeq} [\beta, \alpha]$ by Lemma 2 since $0 \notin (-\beta, \alpha)$ by Lemma 5. Again by Lemma 5 we have $0 \in (\beta, \alpha)$, which shows there exists a unique $\omega \in (-\beta, \alpha)$ with $F(\omega) = 0$. $\square$

Let $<$ be the natural order between the points of the spine induced by any homeomorphism $h : [-\beta, \beta] \xrightarrow{\simeq} [-1, 1] \subset \mathbb{R}$, with $h(\pm\beta) = \pm 1$. In other words, for $x, y \in [-\beta, \beta]$ we have $x < y$ if and only if $h(x) < h(y)$.

Corollary 2. *Let $\pm\omega$ be the two F-preimages of 0 as in Lemma 6. Then we have the following order between the points on the spine (see Fig. 4):*

$$-\beta < \omega < \alpha < 0 < -\alpha < -\omega < \beta.$$

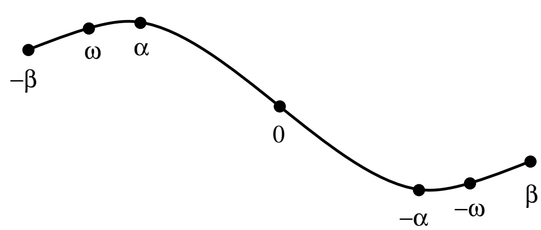

FIGURE 4

Lemma 7. *Let $c = f(0) = F(0)$ be the critical value. Then $\pi(c) \in [-\beta, \alpha]$. If $\pi(c) = -\beta$, then $c = -\beta$ in which case $f(z) = z^2 - 2$.*

Proof. By Lemma 2 we have $F : [0, \beta] \xrightarrow{\simeq} [c, \beta] = I_c \cup [\pi(c), \beta]$. Since $-\alpha \in [0, \beta]$, by **(F3)** and **(F4)** we must have $F(-\alpha) = \alpha \in [c, \beta]$. This is possible only if $\alpha \in [\pi(c), \beta]$, which is equivalent to $\pi(c) \in [-\beta, \alpha]$ (see Fig. 5).

If $\pi(c) = -\beta$, then $c = -\beta$ by Lemma 4. It is easy to see that $z \mapsto z^2 - 2$ is the only quadratic polynomial with the critical orbit $0 \mapsto c \mapsto \beta$. $\square$

Lemma 8. *Suppose that f is not the Chebyshev polynomial. Let $f(\xi) = F(\xi) = -\beta$. Then ξ does not belong to the spine $[-\beta, \beta]$. Furthermore, $\pi(\xi) \in [-\alpha, \alpha]$ and $F(\pi(\xi)) = \pi(c)$, with*

$$c \in [-\beta, \beta] \Leftrightarrow \pi(\xi) = 0.$$

Proof. First suppose that $\pi(\xi) \neq 0$. Replacing ξ by $-\xi$ if necessary, we may assume that $\pi(\xi) \in (0, \beta)$. Then $F : [\xi, \beta] \xrightarrow{\simeq} [-\beta, \beta]$, hence $-\alpha \in [\xi, \beta]$ which implies that $-\alpha \in [\pi(\xi), \beta]$, or equivalently, $\pi(\xi) \in (0, -\alpha]$. Also, since $0 \notin [\xi, \beta]$, c cannot belong to the spine $[-\beta, \beta]$. By Lemma 7, $\pi(c) \in (-\beta, \alpha]$. By Lemma 2 the set $[\xi, 0, \beta]$ maps homeomorphically

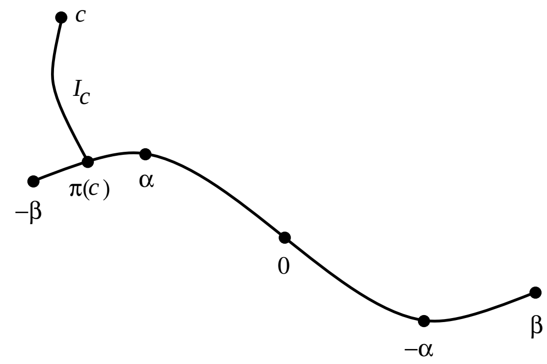

FIGURE 5

to the tripod $[-\beta, c, \beta]$, hence it must also be a tripod, with $\xi \notin [-\beta, \beta]$, and with the joint $\pi(\xi)$ mapped to $\pi(c)$ by F (see Fig. 6).

Now suppose that $\pi(\xi) = 0$. Then by a similar argument, the set $[\xi, 0, \beta] = [\xi, \beta]$ still maps homeomorphically to the spine $[-\beta, \beta]$ since it does not contain a pair of symmetric points about the origin. In particular, c must belong to the spine. By Lemma 7, $c = \pi(c) \in (-\beta, \alpha]$. $\square$

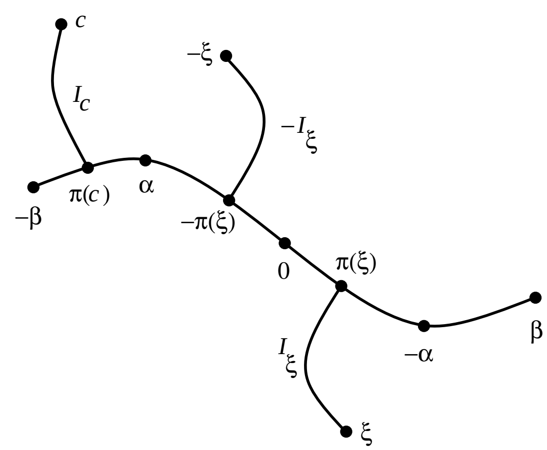

FIGURE 6

Corollary 3. *F maps $[0, \pm\pi(\xi)]$ to I_c and $\pm I_\xi$ to $[-\beta, \pi(c)]$ homeomorphically (see Fig. 6).*

Thus in all non-Chebyshev cases we have the situation illustrated in Fig. 6 (except that I_c may collapse to a point $\Leftrightarrow$ $[-\pi(\xi), \pi(\xi)]$ may collapse to a point, or alternatively $\pi(c)$ may coincide with α). Here

$$\pm\xi \overset{F}{\mapsto} -\beta \overset{F}{\mapsto} \beta,$$

$$\pm\pi(\xi) \overset{F}{\mapsto} \pi(c),$$

and

$$\pm\omega \overset{F}{\mapsto} 0 \overset{F}{\mapsto} c,$$

where ω lies somewhere between $-\beta$ and α.

Lemma 9. *Suppose that f is not the Chebyshev polynomial. Then the Brolin measure $\mu[-\beta,\beta]$ of the spine is zero if and only if $\mu(I_c)=0$.*

Note that the condition $\mu(I_c)=0$ is trivially satisfied if $c=\pi(c)$ belongs to the spine. The latter happens, for example, when the Julia set of $f(z)=z^2+c$ with $c\in\mathbb{R}$ is full. When the Julia set is full, it is conjectured that the critical value belongs to the spine if and only if c is real.

Proof. By Lemma 8, for one preimage ξ of $-\beta$, we have $\pi(\xi)\in[0,-\alpha]$, and then the other preimage $-\xi$ satisfies $\pi(-\xi)\in[\alpha,0]$. For simplicity, let $z_0=\pi(\xi)$ and $z_n=F^{\circ n}(z_0)$. It follows from Corollary 3 that

$$F^{-1}([-\beta,\beta]\cup I_c)=[-\beta,\beta]\cup I_\xi\cup -I_\xi. \tag{4}$$

By property (ii) in section **§1** and **(F5)**, we have

$$\mu(I_\xi)=\mu(-I_\xi)=\frac{1}{2}\mu(I_c). \tag{5}$$

Note that $z_1=\pi(c)\in(-\beta,\alpha]$ by Lemma 8 and Lemma 7. By Corollary 3, **(F6)** and (5),

$$\mu[-\beta,z_1]=\mu(F(I_\xi))=2\mu(I_\xi)=\mu(I_c). \tag{6}$$

If $\mu[-\beta,\beta]=0$, then $\mu[-\beta,z_1]=0$, hence $\mu(I_c)=0$ by (6). Conversely, if $\mu(I_c)=0$, then $\mu[-\beta,z_1]=0$.

To prove $\mu[-\beta,\beta]=0$, we distinguish two cases:

• *Case 1.* $z_1\in[\omega,\alpha]$. Then $\mu[-\beta,\omega]\le\mu[-\beta,z_1]=0$. Hence $\mu[0,\beta]=2\mu[-\beta,\omega]=0$, which by symmetry implies $\mu[-\beta,\beta]=0$.

• *Case 2.* $z_1\in(-\beta,\omega)$. Then $z_2=F(z_1)\in F(-\beta,\omega)=(0,\beta)$ and $\mu[z_2,\beta]=2\mu[-\beta,z_1]=0$. If $z_2\in[0,-\omega]$, then $\mu[-\omega,\beta]=0$ and it follows by an argument similar to *Case 1* that $\mu[-\beta,\beta]=0$. So let us assume that $z_2\in(-\omega,\beta)$. We can repeat the above argument by considering $z_3=F(z_2)\in(0,\beta)$. If $z_3\in[0,-\omega]$, we have $\mu[-\beta,\beta]=0$, otherwise $z_3\in(-\omega,\beta)$ and we continue. If this process never stops, it follows that $z_n\in(-\omega,\beta)$ and $(z_{n+1},\beta]\supset[z_n,\beta]$ for all n. The limit of the monotone sequence $\{z_n\}$ will then be a fixed point of F in $(-\omega,\beta)$, which contradicts **(F4)**. ☐

§4. The Proof. The idea of the proof of Theorem 1 is as follows: We consider the n-th iterate of $c=f(0)=F(0)$, $c_n=F^{\circ n}(c)$. Under the assumption $\mu(I_c)>0$, we show that c_n cannot belong to the spine and the Brolin measure of the arc I_{c_n} tends to infinity as $n\to\infty$, which is clearly impossible since $\mu(J)=1$. Hence we must have $\mu(I_c)=0$. By Lemma 9, this proves the theorem.

Definition. Let I_1 and I_2 be two regulated arcs in the filled Julia set K. We say that I_1 and I_2 *overlap* if the intersection $I_1\cap I_2$ contains more than one point. It follows that $I_1\cap I_2$ is a nondegenerate regulated arc I in K. We often say that I_1 and I_2 *overlap along* I.

It is not hard to check that for $x,y\in K\smallsetminus[-\beta,\beta]$, the arcs I_x and I_y overlap if and only if x and y belong to the same connected component of $K\smallsetminus[-\beta,\beta]$. In particular, we must have $\pi(x)=\pi(y)$.

Lemma 10. *Let $x \in K \setminus [-\beta, \beta]$. Then one and only one of the following cases occurs, as illustrated in Figures 7, 8, 9:*

(a) *I_x and I_ξ (or $-I_\xi$) overlap along an arc I_y. Then F maps $[x, y]$ homeomorphically to $I_{F(x)} = [F(x), F(y)]$.*

(b) *$\pi(x) \in (-\pi(\xi), \pi(\xi))$. Then F maps I_x homeomorphically to the arc $F(I_x) = [F(x), F(\pi(x))]$. In this case, $I_{F(x)}$ and I_c overlap along $I_{F(\pi(x))} = [F(\pi(x)), \pi(c)]$.*

(c) *$\pi(x) \notin (-\pi(\xi), \pi(\xi))$ and I_x and $\pm I_\xi$ do not overlap. Then F maps I_x homeomorphically to $I_{F(x)}$.*

Proof. (a) If $x \in I_\xi$ or $-I_\xi$, then $y = x$ and the result it trivial. Otherwise, $[\xi, x, \pi(x) = \pm\pi(\xi)]$ maps homeomorphically to $[-\beta, F(x), \pi(c)]$ (see Fig. 7). Hence $F(y) = \pi(F(x))$ and the result follows.

(b) If $\pi(x) \in (-\pi(\xi), \pi(\xi))$, then $F(\pi(x)) \in I_c \setminus \{\pi(c)\}$, hence $I_{F(x)}$ and I_c overlap along $I_{F(\pi(x))}$ (see Fig. 8).

(c) Since $\pi(x) \notin (-\pi(\xi), \pi(\xi))$, $F(\pi(x)) \in [-\beta, \beta]$. So the claim is proved once we show that $\pi(F(x)) = F(\pi(x))$. If these two points are distinct, then the nondegenerate arc $I = [\pi(F(x)), F(\pi(x))] \subset [-\beta, \beta]$ is contained in $[F(x), F(\pi(x))]$ (see Fig. 9). Hence $F^{-1}(I)$ will be a nondegenerate arc in $I_x \cap I_\xi$ or $I_x \cap -I_\xi$, which contradicts our assumption. $\square$

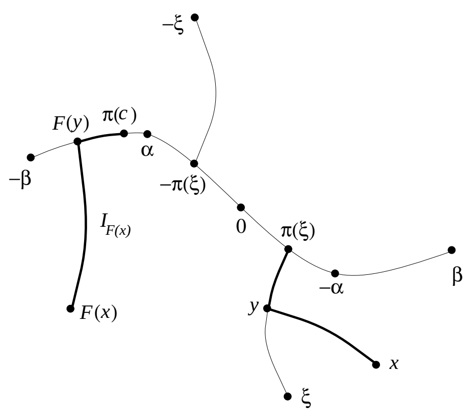

FIGURE 7

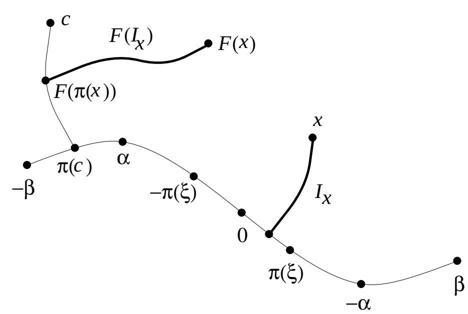

FIGURE 8

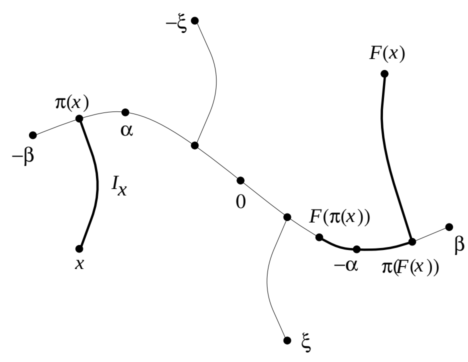

FIGURE 9

Let us put $m = \mu(I_c)$. By (5), we have $\mu(\pm I_\xi) = m/2$.

Corollary 4. *If $x \in K \setminus [-\beta, \beta]$ and $\mu(I_x) \geq 2m$, then $\mu(I_{F(x)}) \geq \dfrac{3}{2}\mu(I_x)$.*

Proof. By Lemma 10 one and only one of the cases (a)-(c) occurs. In case (b), we have $\mu(I_{F(x)}) = \mu(F(I_x)) + \mu(I_{F(\pi(x))}) \geq \mu(F(I_x)) = 2\mu(I_x)$ and in case (c), $\mu(I_{F(x)}) = 2\mu(I_x)$. In case (a),

$$\begin{aligned}\mu(I_{F(x)}) &= \mu[F(x), F(y)] = 2\mu[x, y] \\ &= 2(\mu(I_x) - \mu(I_y)) \\ &\geq 2(\mu(I_x) - \mu(\pm I_\xi)) \\ &= 2\mu(I_x) - m \\ &\geq (3/2)\mu(I_x),\end{aligned}$$

which proves the corollary. □

Proof of Theorem 1. Consider the orbit of the critical value $\{c = c_0, c_1, c_2, \dots\}$, where $c_n = F^{\circ n}(c)$. Let $m = \mu(I_c) > 0$, and apply Lemma 10 to the point $x = c$. Clearly the only possible cases are (a) and (c), since $\pi(c) \notin (-\pi(\xi), \pi(\xi))$.

In case (c) we obtain the estimate $\mu(I_{c_1}) \geq 2m$. This, by repeated application of Corollary 4, will lead to the estimate $\mu(I_{c_{n+1}}) \geq (3/2)^n \mu(I_{c_1})$ which tends to infinity as $n \to \infty$ and therefore is impossible.

In case (a), I_c and $-I_\xi$ overlap along some I_y with $F(y) \in (-\beta, \pi(c))$ and $\mu(I_{c_1}) = \mu[c_1, F(y)] = 2\mu[c, y] \geq m$. Apply Lemma 10 this time to $x = c_1$. Note that the only possible case is (c), since $\pi(c_1) \in [-\beta, \pi(c))$. This gives the estimate $\mu(I_{c_2}) = 2\mu(I_{c_1}) \geq 2m$. Hence successive applications of Corollary 4 will give the estimate $\mu(I_{c_{n+1}}) \geq (3/2)^n \mu(I_{c_2})$, which again contradicts the fact that the Brolin measure of the Julia set is finite.

The contradiction shows that $m = \mu(I_c)$ must be zero, and this completes the proof of Theorem 1 by Lemma 9. □

It has been shown recently that the Julia set of a *real* quadratic polynomial in the Mandelbrot set is locally-connected [**L-S**]. These correspond to quadratics $f : z \mapsto z^2 + c$ with $-2 \leq c \leq 1/4$. Therefore, we have the following corollary of Theorem 1:

Corollary 5. *Let $f : z \mapsto z^2 + c$ be a real quadratic polynomial with $-2 < c \leq 1/4$. Then, with respect to the Brolin measure on the Julia set of f, almost every point is the landing point of a unique external ray.*

§5. Further Discussion. Finally, we consider the following result, which is a consequence of Theorem 1 as well as the fact that the Julia set has no compact forward-invariant proper subsets of positive Brolin measure.

Theorem 2. *Let $f : z \mapsto z^2 + c$ be a quadratic polynomial with locally-connected filled Julia set K. If we exclude the Chebyshev case and the cases where the α-fixed point of f is attracting or $\alpha = \beta$, then every embedded arc in K has Brolin measure zero.*

The exceptional cases correspond respectively to $c = -2$ where the Julia set is a straight line segment, c in the "main cardioid" of the Mandelbrot set where the Julia set is a quasicircle, and $c = 1/4$ where the Julia set is a Jordan curve but not a quasicircle. Roughly speaking, the theorem says that in any other case, embedded arcs are buried in the filled Julia set so that they are almost invisible from the basin of infinity.

We need the following elementary observation for the proof:

Lemma 11. *Let $A \subset J$ be forward-invariant under f, i.e., $f(A) \subset A$. Then either $\mu(A) = 0$ or $\mu(A) = 1$. In particular, if A is compact and $A \neq J$, then $\mu(A) = 0$.*

Proof. Let $\gamma : \mathbb{T} \to J$ be the Carathéodory loop and $E = \gamma^{-1}(A)$. Then E is forward-invariant under the doubling map $d : \mathbb{T} \to \mathbb{T}$ defined by $d(t) = 2t \pmod 1$. We prove that $\ell(E) = 0$ or $\ell(E) = 1$, where ℓ denotes the Lebesgue measure on $\mathbb{T}$. Let $\ell(E) > 0$ and let x be a point of density of E. Given an $\varepsilon > 0$, we can find an $n > 0$ and an interval $S \subset \mathbb{T}$ centered at x such that $\ell(S) = 2^{-n}$ and $\ell(S \cap E) \geq (1 - \varepsilon)\ell(S)$. Apply the n-th iterate $d^{\circ n}$ on S and use $d^{\circ n}(E) \subset E$ to estimate

$$1 - \varepsilon \leq 2^n \ell(S \cap E) = \ell(d^{\circ n}(S \cap E)) \leq \ell(\mathbb{T} \cap E) = \ell(E).$$

Since this is true for every $\varepsilon > 0$, we must have $\ell(E) = 1$. □

Corollary 6. *Still assuming that K is locally-connected, the Brolin measure of the union of the boundaries of bounded Fatou components of f is zero unless the α-fixed point is attracting or $\alpha = \beta$ in which case the corresponding measure is one.*

Proof. Since every bounded Fatou component eventually enters a cycle of Fatou components of the form $U_1 \mapsto U_2 \mapsto \ldots \mapsto U_p \mapsto U_1$, it suffices to prove that $\mu(A) = 0$, where $A = \bigcup_{j=1}^{p} \partial U_j$. This set is compact and forward-invariant under f, so by Lemma 11 if $\mu(A) > 0$, then $A = J$ must be the case. But this implies that f has only p bounded Fatou components. It is easy to see that this can happen only if $p = 1$, in which case the component is either the immediate basin of attraction for an attracting fixed point or the attracting petal for a parabolic fixed point. □

As an illustrative example, consider a quadratic polynomial f whose α-fixed point is the center of a Siegel disk U with rotation number θ of constant type (an example is provided by $f : z \mapsto z^2 - 0.3905408 - 0.5867879i$, where $\theta = (\sqrt{5} - 1)/2$ is the golden mean). By [**Pe**], the filled Julia set is locally-connected. The critical point $0 \in \partial U$ is the landing point of exactly two rays $(R_s, R_{s+1/2})$, where

$$s = \sum_{0<p/q<\theta} 2^{-(q+1)}.$$

Since the orbit of 0 is dense on ∂U, the set of angles t for which $\gamma(t) \in \partial U$ coincides with the closure of the orbit of s under the doubling map on the circle. This set is known to be an invariant Cantor set C of measure zero in the interval $[s, s + 1/2] \subset \mathbb{T}$ (see [**B-S**]). It follows that the set of all t for which $\gamma(t)$ belongs to the boundary of a bounded Fatou component is the countable union of Cantor sets consisting of C and all its preimages under the doubling map. This set has Lebesgue measure zero, hence the union of the boundaries of all bounded Fatou components will have Brolin measure zero.

Proof of Theorem 2. Let $\eta \subset K$ be any emdedded arc. Let B be the set of biaccessible points in J and B' be the set of all points in J which belong to the boundary of a bounded Fatou component. By Theorem 1 and Corollary 6, we have $\mu(B) = \mu(B') = 0$. On the other hand, by Lemma 3(b), every $z \in \eta \cap J$ is either an endpoint or it belongs to $B \cup B'$. Hence, $\mu(\eta) = \mu(\eta \cap J) \leq \mu(B \cup B') = 0$. □

Corollary 7. *A locally-connected quadratic Julia set is not a countable union of embedded arcs unless it is a straight line or a Jordan curve.*

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Department of Mathematics, SUNY at Stony Brook, NY 11794

E-mail address: zakeri@math.sunysb.edu