

On Fibers and Renormalization of Julia Sets and Multibrot Sets

Dierk Schleicher
Technische Universität München

Contents

1	Introduction	2
2	Tuning and Renormalization	3
3	Fibers of Multibrot Sets and Tuning	5
4	Fibers of Julia Sets and Renormalization	7
5	Quadratic Polynomials	16
6	References	22

Abstract

We continue the description of Mandelbrot and Multibrot sets and of Julia sets in terms of *fibers* which was begun in [S3] and [S4]. The question of local connectivity of these sets is discussed in terms of *fibers* and becomes the question of triviality of fibers. In this paper, the focus is on the behavior of fibers under renormalization and other surgery procedures. We show that triviality of fibers of Mandelbrot and Multibrot sets is preserved under tuning maps and other (partial) homeomorphisms. Similarly, we show for unicritical polynomials that triviality of fibers of Julia sets is preserved under renormalization and other surgery procedures, such as the Branner-Douady homeomorphisms. We conclude with various applications about quadratic polynomials and its parameter space: we identify embedded paths within the Mandelbrot set, and we show that Petersen's theorem about quadratic Julia sets with Siegel disks of bounded type generalizes from period one to arbitrary periods so that they all have trivial fibers and are thus locally connected.

1 Introduction

This is a continuation of the papers [S3] and [S4] in which we have introduced the concept fibers for Multibrot sets, for filled-in Julia sets and, more generally, for compact connected and full subsets of the complex plane. The idea of fibers is to use pairs of external rays landing at common points to cut such a set $K \subset \mathbb{C}$ as finely as possible into subsets (if the set K has interior points, then extra separations are needed to cut interior components apart). We call these subsets *fibers*; fibers will be defined precisely in Sections 3 respectively 4. A fiber is called *trivial* if it consists of a single point. In this case, the set K is locally connected at this point, but triviality of the fiber is a slightly stronger property. It is equivalent to “shrinking of puzzle pieces” (in the sense of Branner, Hubbard and Yoccoz) but independent of any particular choice of puzzles. It turns out that many proofs of local connectivity in holomorphic dynamics actually prove triviality of fibers. For Multibrot sets, there are new direct proofs of triviality of fibers and thus of local connectivity for all boundary points of hyperbolic components and for all Misiurewicz points [S4]. The investigation of fibers for Julia sets allows, among other things, to prove that the “impressions” of periodic rays are in many cases only points [S3]. We will build on these results and discuss how fibers behave under tuning and renormalization.

In Section 2, we review the definitions of Mandelbrot and Multibrot sets and discuss renormalization of Julia sets and tuning maps within Mandelbrot and Multibrot sets. In Section 3, we show that tuning maps preserve triviality of fibers: the fiber of any point in a Multibrot set is trivial if and only if the fiber of the corresponding point in a “little Multibrot set” is trivial. Any proof of local connectivity via shrinking of puzzle pieces for non-renormalizable parameters in Multibrot sets thus turns automatically into a proof for finitely renormalizable parameters. A similar result holds for Julia sets, but more is true: in Section 4, we show that whenever any renormalization of a polynomial with a single critical point has a locally connected Julia set, then the entire Julia set is locally connected (and conversely). For example, Petersen [Pt] has shown that Julia sets of quadratic polynomials with “bounded type Siegel disks” of period one are locally connected; it follows that this is true for arbitrary periods. This result is discussed in Section 5 together with further results which are known only for quadratic polynomials; among them is a discussion of how close the Mandelbrot set comes to being arcwise connected.

The definition of fibers depends of course on the collection of external rays used to cut the complex plane. For arbitrary compact connected and full set in \mathbb{C} , fibers may behave rather badly and there is no universal best choice of external rays. In the beginnings of Sections 3 and 4, we will give the exact definitions of fibers for Multibrot sets respectively for Julia sets together with a review of their most important properties. These two sections are independent of each other, while Section 5 builds on both of them.

ACKNOWLEDGEMENTS. As its continuation, this paper owes a lot to the same people which are already mentioned in the acknowledgements of [S3]. I would like to gratefully emphasize in particular the contribution and continuing support of Misha Lyubich, John Milnor and the Institute of Mathematical Sciences at Stony Brook.

2 Tuning and Renormalization

In this paper, we will be concerned with polynomials having only a single critical point of possibly higher multiplicity. Milnor has suggested to call these polynomials *unicritical*. They can have any degree $d \geq 2$, and up to normalization (by affine conjugation) they can always be written $z \mapsto z^d + c$. We will always suppose that our Julia sets are normalized in this way. The number c is a complex parameter which is uniquely determined up to multiplication by a $d - 1$ -st root of unity. The filled-in Julia set of such a polynomial always has exactly d -fold rotation symmetry (except if $c = 0$) and is connected if and only if it contains the only critical point.

The *Multibrot set* \mathcal{M}_d of degree $d \geq 2$ is the set of parameters c for which the Julia set of $z^d + c$ is connected. It is itself connected and has $d - 1$ -fold rotation symmetry (see [LS2] for pictures of various of these sets). The case $d = 2$ is the familiar Mandelbrot set. Similarly as [S4], the present paper can be read with the case $d = 2$ in mind throughout, but the higher degrees do not introduce any new difficulties. Certain results are known only in the quadratic case; they are collected in the final section of this paper.

Some of the results in this paper have been floating around the holomorphic dynamics community in some related form, sometimes without written proofs. We have included them here to provide references and proofs, as well as to generalize some of them from the quadratic case to the case of unicritical polynomials of arbitrary degrees.

We will make essential use of the theory of *polynomial-like maps* of Douady and Hubbard [DH2]. A polynomial-like map is a proper holomorphic map $f: U \rightarrow V$, where U and V are open and simply connected subsets of \mathbb{C} such that $\bar{U} \subset V$. Such a map always has a positive degree. Douady and Hubbard require their maps to have degrees at least two, and this is also the only case of interest to us. The *filled-in Julia set* of a polynomial-like map is the set of points $z \in U$ which remain in U forever under iteration of f ; it is always compact, and it is connected if and only if it contains all the critical points of f . The *Julia set* of f is the boundary of the filled-in Julia set.

The Straightening Theorem [DH2, Theorem 1] states that every polynomial-like map $f: U \rightarrow V$ of degree d is *hybrid equivalent* to a polynomial p of degree d restricted to an appropriate subset U' of \mathbb{C} such that $V' := p(U')$ contains \bar{U}' : that is, there exists a quasiconformal homeomorphism $\varphi: V \rightarrow V'$ conjugating f to p , i.e. $p \circ \varphi = \varphi \circ f$ on U ; this homeomorphism maps the filled-in Julia set of f onto the filled-in Julia set of p , and the complex dilatation of φ vanishes on this filled-in Julia set. When the filled-in Julia set of f is connected, then the polynomial p is unique up to affine conjugation.

Definition 2.1 (Renormalization)

A unicritical polynomial p of degree d is n -renormalizable if there exists a neighborhood U of the critical value such that the holomorphic map $p^{\circ n}: U \rightarrow p^{\circ n}(U)$ is polynomial-like of degree d and has connected Julia set. Any polynomial which is hybrid equivalent to this polynomial-like map is called a straightened polynomial of this renormalization. Such a polynomial is again unicritical of degree d , and in the normalization $z^d + c$ it is unique up to multiplication of the parameter c by a $d - 1$ -st root of unity.

The little Julia set of this renormalization is the filled-in Julia set of the polynomial-like map. The renormalization is simple if the little Julia set does not disconnect any of its images under $p, p^{\circ 2}, \dots, p^{\circ(n-1)}$.

REMARK. McMullen, in his general investigation of renormalization of quadratic polynomials [McM], distinguishes three types of renormalization: he says that a renormalization of period n with little Julia set K' is of

- **disjoint type** if K' is disjoint from $p(K'), p^{\circ 2}(K'), \dots, p^{\circ(n-1)}(K')$;
- **β -type** if K' intersects any of $p(K'), p^{\circ 2}(K'), \dots, p^{\circ(n-1)}(K')$ only at a point which corresponds to the landing point of a fixed ray of the straightened polynomial;
- **α -type** if K' intersects any of $p(K'), p^{\circ 2}(K'), \dots, p^{\circ(n-1)}(K')$ only at a point which corresponds to the unique fixed point of the straightened polynomial that is not the landing point of a fixed ray.

McMullen shows that any renormalization of a quadratic polynomial has one of these three types. A renormalization is simple if and only if it is of one of the first two types. The third type is also known as *crossed renormalization*, cf. [RS].

A quadratic polynomial with connected Julia set has two fixed points: one is the landing point of the unique fixed ray at angle 0; this fixed point is called the β -point. The other fixed point is called the α -point; thus the names of the renormalization types. These fixed points coincide exactly when the polynomial is conjugate to $z^2 + 1/4$.

Any unicritical polynomial of degree $d \geq 2$ has d finite fixed points (counting multiplicities). If the Julia set is connected, then the $d - 1$ fixed rays land at $d - 1$ distinct fixed points, which are the analogues of the β -fixed points: if two rays were to land together, they would cut \mathbb{C} into two parts, one of which does not contain the critical point of the polynomial. This part would then have to map homeomorphically onto itself, which is a contradiction. The remaining fixed point is called α ; it may be attracting, indifferent, or repelling, and it may coalesce with one of the $d - 1$ other fixed points (this happens exactly at the $d - 1$ cusps of the unique hyperbolic component of period 1). There is thus at most one fixed point which is not the landing point of a fixed ray. Statement and proofs of McMullen's classification now generalize to unicritical polynomials of arbitrary degrees.

It turns out that the sets of simply n -renormalizable unicritical polynomials are organized in the form of little copies of Multibrot sets: disjoint types occur at *primitive copies* and β -types occur at *non-primitive* or *satellite* copies of Multibrot sets (see below). In order to describe this, we need to introduce tuning maps of Multibrot sets.

Definition 2.2 (Tuning Map)

A tuning map of period n of a Multibrot set \mathcal{M}_d is a homeomorphism $\Psi : \mathcal{M}_d \rightarrow \Psi(\mathcal{M}_d) \subset \mathcal{M}_d$ such that, for every $c \in \mathcal{M}_d$, the polynomial $z \mapsto z^d + \Psi(c)$ is n -renormalizable, and the corresponding polynomial-like map is hybrid equivalent to the polynomial $z \mapsto z^d + c$.

Douady and Hubbard have shown the following theorem in the case of the Mandelbrot set, but they have not published a complete proof. It can be found in Section 10 of the recent thesis [Ha] of Haïssinsky. Underlying is the theory of “Mandelbrot-like families” [DH2] which is in general known to work only for quadratic polynomials. However, the difficulties do not lie in the degree of the maps but in the number of independent

critical points. Therefore, their theory applies also to families of unicritical polynomials of arbitrary degrees.

Theorem 2.3 (Tuned Copies of Multibrot Sets)

For every hyperbolic component of period n of any Multibrot set \mathcal{M}_d , there exists a tuning map of period n sending the unique component of period 1 to the specified hyperbolic component. This tuning map is unique up to precomposition with a rotation by $k/(d-1)$ of a full turn, for some $k \in \{0, 1, 2, \dots, d-1\}$.

If the component is primitive, then the tuning map can be extended as a homeomorphism to a neighborhood of \mathcal{M}_d onto its image. If the component is non-primitive, then the tuning map can be extended as a homeomorphism to a neighborhood of \mathcal{M}_d minus one of the $d-1$ roots of the period one component. In both cases, the extension is no longer unique.

3 Fibers of Multibrot Sets and Tuning

The main result in this section is that triviality of fibers of Multibrot sets is preserved under tuning. Julia sets are discussed in the next section, and further similar statements for the Mandelbrot set will be given in Section 5.

We begin by defining fibers of Multibrot sets. What follows is a brief review of the definition in [S3], simplified using results from [S4, Section 7]. We will use parameter rays at periodic external angles; they are known to land at parabolic parameters. Two parameter rays at periodic angles are called a *ray pair* if they land at a common point. We say that two points in the Multibrot set are *separated* by this ray pair if they are different from the landing point and if they are in different connected components of \mathbb{C} minus ray pair and landing point. In order to be able to separate points within hyperbolic components, we also allow separations by two parameter rays at periodic angles which land at the boundary of a common hyperbolic component, together with a simple curve within this hyperbolic component connecting the two landing points. A *fiber* of a Multibrot set is an equivalence class of points which cannot be separated from each other. A fiber is called *trivial* if it consists of a single point.

This definition might seem special on two accounts: why are only hyperbolic components allowed for separation lines, excluding non-hyperbolic components, and why use only parameter rays at periodic angles, excluding preperiodic angles? In [S4], fibers have indeed been defined using parameter rays at periodic and preperiodic angles, and separation lines through arbitrary interior components were allowed. It turns out, however, that parameter rays at rational external angles never land at non-hyperbolic components by [S4, Corollary 7.2], so allowing them would change nothing except possibly confuse. As to parameter rays at preperiodic angles, they are simply not necessary: omitting them does not change fibers at all [S4, Proposition 7.6]. Since preperiodic parameter rays sometimes need special attention and separate arguments, it is simply a matter of convenience to exclude them.

Fibers of Multibrot sets are known to be compact, connected and full. If any fiber is trivial, then the Multibrot set is locally connected at this point; this is one of the fundamental properties which make fibers useful: see [S3, Proposition 3.5] or

[S4, Proposition 3.5]. All points on closures of hyperbolic components, as well as all Misiurewicz points, have trivial fibers. This includes all the landing points of parameter rays at rational angles. (For the Mandelbrot set, many more points are known to have trivial fibers; see Yoccoz' Theorem 5.4.)

For the purposes of this paper, there is another possible simplification: we can ignore separations through hyperbolic components and just restrict to periodic parameter ray pairs. Of course, this no longer allows to separate hyperbolic components, but all fibers on closures of hyperbolic components are trivial and we no longer have to discuss them. Any parameter which is not on the closure of a hyperbolic component can be separated from any hyperbolic component by a periodic ray pair, so its fiber can simply be constructed using periodic parameter ray pairs only. This will simplify some discussions below.

It is well known but not so well published that simply renormalizable parameters of any particular type in the Mandelbrot set are organized in the form of little Mandelbrot sets (compare Douady and Hubbard [DH2], Douady [Do1], McMullen [McM], Milnor [M2], Riedl [Ri] or Haïssinsky [Ha]). Each little Mandelbrot set comes with a parameter ray pair separating this little Mandelbrot set from the origin, and the angles of this ray pair are periodic with the same period n as the renormalization period. In fact, any little Mandelbrot set can be separated from the rest of \mathcal{M}_2 by this periodic parameter ray pair and a countable collection of parameter ray pairs. The same statement is true for all the Multibrot sets. Little Multibrot sets within \mathcal{M}_d have decorations attached only at tuned images of Misiurewicz points at external angles a/d^n for integers a and n . Here is a precise statement of the result we need. We will not include a proof here, although there is no precise classical reference; compare the references cited above.

Theorem 3.1 (Decorations at Little Multibrot Sets)

For any little Multibrot set \mathcal{M}' set within the entire Multibrot set \mathcal{M}_d , there is a countable set B of parameters with the following property: for any parameter $c \in \mathcal{M}_d - \mathcal{M}'$, there is a rational parameter ray pair landing at a point $c' \in B$ and separating c from $\mathcal{M}' - \{c'\}$. This set B consists of a single parabolic parameter and countably many Misiurewicz points.

Under the tuning map $\tau : \mathcal{M}_d \rightarrow \mathcal{M}'$, the parabolic point in B is the image of the landing point of the parameter ray at angle $k/(d - 1)$ for some integer k , and the Misiurewicz points in B are exactly the images of the landing points of the parameter rays at angles a/d^n , for any pair of positive integers a and n . No point in $\tau^{-1}(B)$ disconnects \mathcal{M}_d . □

Corollary 3.2 (Fiber Triviality of \mathcal{M}_d Preserved Under Tuning)

The fiber of any point in the Multibrot set is trivial if and only if the fiber of the image point under any and all of the tuning maps is trivial.

PROOF. With the right background about tuning as given in the previous theorem, this proof is quite easy. Let $\tau: \mathcal{M}_d \rightarrow \mathcal{M}'$ be a tuning homeomorphism and consider a point $c \in \mathcal{M}_d$ together with $c' := \tau(c) \in \mathcal{M}'$. By [S4, Sections 5 and 6], the landing points of all the rational parameter rays of \mathcal{M}_d have trivial fibers.

First observe that the fiber of any point in \mathcal{M}' must entirely be contained in \mathcal{M}' : by Theorem 3.1, any point in \mathcal{M}' is either the landing point of a rational parameter ray and its fiber is trivial, or it can be separated from any other point in $\mathcal{M}_d - \mathcal{M}'$ by a periodic parameter ray pair. Therefore, we have to show that c can be separated from any point in $\mathcal{M}_d - \{c\}$ if and only if c' can be separated from any point in $\mathcal{M}' - \{c'\}$. By the discussion at the beginning of this section, we only have to consider separations by periodic parameter ray pairs.

The main idea is to transfer separating ray pairs in \mathcal{M}_d to separating ray pairs in \mathcal{M}' and back. These ray pairs consist of parameter rays at periodic angles which land at parabolic parameters, and all their landing points have trivial fibers. A parabolic parameter remains parabolic after tuning, and the number of parameter rays landing there will always be one or two (one at co-roots, two at the root of any hyperbolic component of period at least two). Roots or co-roots map under tuning to roots respectively co-roots. It follows that periodic ray pairs preserve their separation properties: when a periodic ray pair separates c_1 and c_2 , then the ray pair landing at the tuned image of the landing point will separate $\tau(c_1)$ and $\tau(c_2)$ in \mathcal{M}' , and conversely.

We conclude the following: whatever separations are possible in \mathcal{M}_d , we will obtain the corresponding separations in \mathcal{M}' . The converse statement works in the same way. This is what we needed to prove. \square

REMARK. The nice thing about this proof is that, while proving that the fibers of arbitrary points in \mathcal{M}_d are trivial, we only have to worry about periodic parameter rays and their behavior under tuning, and these do not cause any difficulty. In a sense, we thus do not prove that new fibers are trivial, but we just show that triviality is “preserved” under certain maps. For example, it is sometimes possible to conclude certain results for real parameters, using purely real methods (compare the final section). We can then carry over these results to many non-real parameters.

The very same proof shows that the fiber of any point within some Julia set is trivial if and only if the fiber of the corresponding point within a renormalization of the Julia set is trivial. But for unicritical polynomials, much more is true: if the fibers of all the points within the “little Julia set” are trivial, then all the fibers within the entire Julia set are trivial. This will be the main conclusion in the next section.

4 Fibers of Julia Sets and Renormalization

In this section, we will discuss fibers of connected filled-in Julia sets of unicritical polynomials, always assuming them to be monic. To define fibers, it is easiest to first consider only a filled-in Julia set without interior. Let Q be the set of dynamic rays at rational angles. All these rays are known to land at repelling periodic respectively preperiodic points. We will only consider rays which land together with at least one more rational ray; a pair of rational rays landing at a common point is again called a *ray pair*. Any such ray pair cuts \mathbb{C} and the filled-in Julia set into two (relatively) open sets, and we say that points from different parts are *separated* by this ray pair. No ray pair separates its landing point from anything. The *fiber* of any point z is the collection

of all points in the filled-in Julia set which cannot be separated from z . The fiber is *trivial* if it consists of z alone.

If the filled-in Julia set has interior, then we have to extend the definitions. We will ignore the attracting or parabolic cases because such Julia sets are known to be locally connected anyway. The remaining case is that of a Siegel disk. Let Q be the set of dynamic rays at rational angles, plus the grand orbit (backward orbit of the entire forward orbit) of all the dynamic rays landing at the critical value, if any; their external angles are necessarily irrational. A *separation line* is either a pair of rays in Q landing at a common point, or a pair of rays in Q landing at the boundary of the same Fatou component, together with a simple curve within this Fatou component connecting the landing points of these two rays. (Even if two dynamic rays land on the boundary of the same Fatou component, it is not clear that it must be possible to connect their landing points by a curve within this Fatou component.) We now proceed as above: every separation line cuts the complex plane and the filled-in Julia set into two parts and separates the points on different sides. The fiber of any point z is then the collection of all points within the filled-in Julia set that cannot be separated from z , and it is trivial if it is the set $\{z\}$.

We have shown in [S3, Section 3] that the fiber of any point z in a filled-in Julia set is a compact, connected and full subset of \mathbb{C} and that the relation “is in the fiber of” is symmetric. The dynamics maps any fiber onto a unique image fiber, either homeomorphically or as a branched cover according to whether or not there is a critical point in the fiber. If the fiber of some point z is trivial, then the filled-in Julia set is locally connected at this point. In particular, triviality of all fibers implies local connectivity of the entire set. Conversely, if any unicritical filled-in Julia set is locally connected, then all its fibers as constructed above are trivial: the necessary separation lines need dynamic rays at rational angles and rays on the grand orbit of the unique ray landing at the critical value.

It is not always true that the relation “is in the fiber of” is an equivalence relation. This is true as soon as all the landing points of the rays used in the construction have trivial fibers, but this condition is not always satisfied. For example, it is not true when there is a Cremer point. The fiber of any repelling periodic or preperiodic point is trivial as soon as its entire forward orbit can be separated from the critical value and all periodic Siegel disks (if any) [S3, Theorem 3.5]. Important examples are infinitely renormalizable unicritical polynomials of any degree: fibers are constructed using rational rays, their landing points always have trivial fibers, and we have an equivalence relation.

The main goal in this section is to prove that triviality of fibers is preserved under renormalization: all fibers of any filled-in Julia set are trivial if and only if all the fibers of any of its renormalized Julia sets are trivial. The transfer of fiber triviality is done in two steps, corresponding to three sets: a “big” Julia set K , a “little” Julia set K' which is an invariant subset, and a “model” Julia set K_1 which is hybrid equivalent to the little Julia set; it is the filled-in Julia set of the renormalized polynomial. The first step shows that the big Julia set has trivial fibers whenever the little Julia set does, using separation lines from the big Julia set. The second step then relates triviality of fibers of the little Julia set to its model. These three sets are illustrated in Figure 1.

Proposition 4.1 (Trivial Fibers in Dynamic Subsets)

Let p be a unicritical polynomial with connected filled-in Julia set K . Consider a finite set of dynamic ray pairs at rational angles landing at repelling orbits and let U be one of the connected components of \mathbb{C} with these ray pairs removed. Fix any positive integer n . This defines a map $\tilde{p}: K \cap \overline{U} \rightarrow \mathbb{C}$ via $\tilde{p}(z) = p^{\circ n}(z)$. Let K' be the compact subset of points in $K \cap \overline{U}$ which never leave \overline{U} under iteration of the map \tilde{p} and suppose that K' contains the critical value. Let Q be a countable forward and backward invariant set of dynamic rays of p which contains the rays bounding U . Then, for this choice of dynamic rays, K has trivial fibers if and only if K' does.

REMARK. It may be helpful to rephrase the main conclusion of the proposition as follows: suppose that the dynamic rays (in Q) of K supply enough separation lines so as to disconnect the subset K' in such a way that all fibers are points. Then the rays in Q disconnect all of K into fibers consisting of single points.

The assumption that rays bounding U land on repelling orbits is largely for convenience. We could as well allow the rays to land on parabolic orbits. However, since all unicritical polynomials with parabolic orbits are known to be locally connected and thus to have trivial fibers anyway, we would not gain much.

PROOF. If K has trivial fibers, then any two of its points can be separated. In particular, any two points in K' can be separated, so all the fibers of K' are trivial.

The converse statement requires more work. We assume that K' has trivial fibers and contains the critical value. Let S be the finite set of landing points of the rational ray pairs from the statement and let S' be the union of the points in S with all their forward orbits, which is still a finite set.

First we assume that the polynomial p has no interior. Then fibers are constructed using pairs of dynamic rays at rational angles, and they land at repelling periodic or preperiodic points. Any repelling periodic point in K' can be separated from the critical value since the fiber of the critical value is trivial, and any repelling periodic point outside of K' can be separated from K' and thus from the critical value as well. Therefore, all the repelling periodic and preperiodic points have trivial fibers by the result mentioned above. This applies in particular to all the points in S .

Any point z within K' can be separated from any other point within K' because fibers of K' are trivial. The point z can also be separated from any point $z' \in K - K'$: if $z \in S$ or if z ever maps into S , then its fiber is trivial anyway; otherwise, there is a finite iterate of \tilde{p} which sends z' outside of \overline{U} , while it obviously leaves z within K' , away from the boundary of U . The boundary ray pairs of U thus separate the corresponding forward images of these two points, so z and z' are in different fibers. Therefore, all the points in K' have trivial fibers. Since the dynamics preserves triviality of fibers, all the points which eventually map into K' have trivial fibers.

Finally, we have to consider the case that z is not in K' and never maps into K' . Let

$$K'' := \bigcup_{k=0}^{\infty} p^{\circ k}(K').$$

This set is compact and forward invariant under the dynamics of p : it is the union of all the finitely many forward images of K' . The set K'' contains only points which

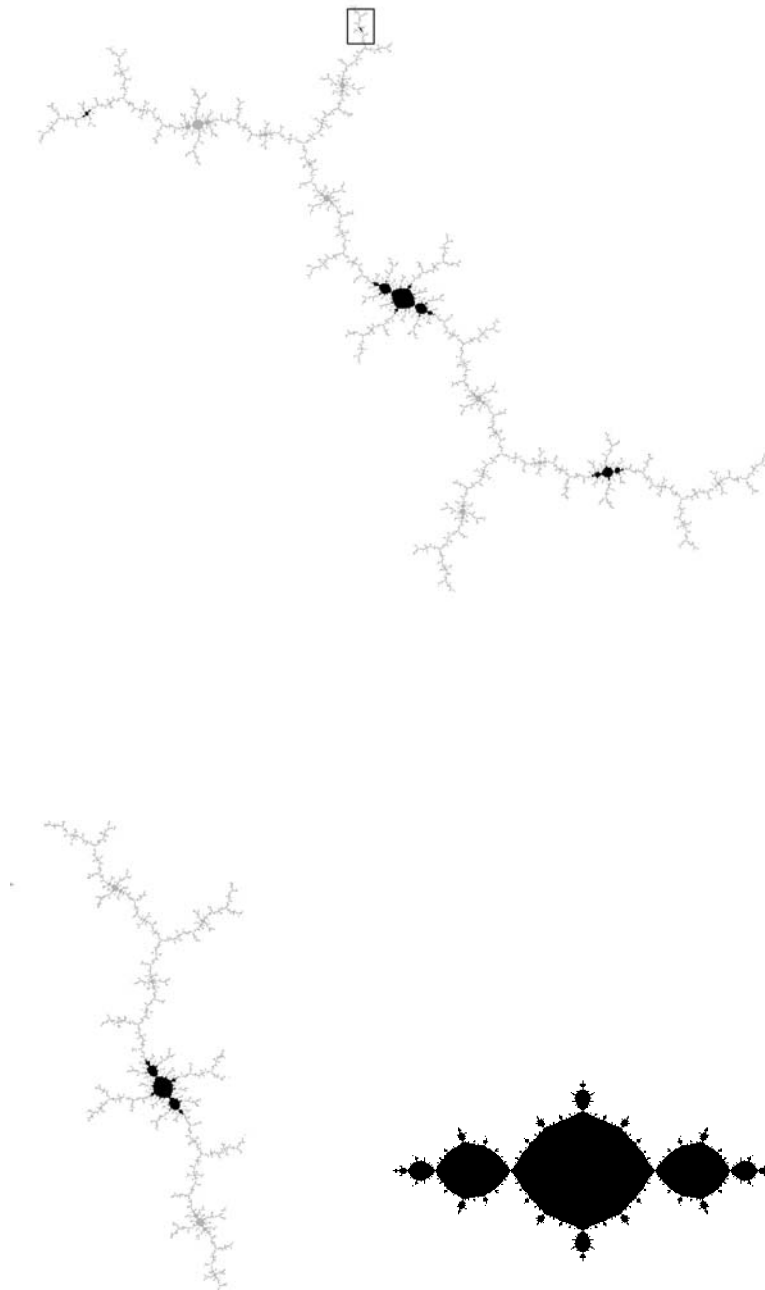


Figure 1: Illustration of Proposition 4.1. Top: a “big Julia set” K (grey) containing a “little Julia set” K' which is periodic of period 4. The little Julia set and its three periodic forward images are indicated in black; together, they form the set K'' . Bottom left: a blowup near the critical value, containing the little Julia set K' (black). Lower right: a model Julia set K_1 which is hybrid equivalent to K' .

eventually visit K' under iteration of p , so all fibers in K'' are trivial. The set $X := \mathbb{C} - K''$ is open and carries a unique normalized hyperbolic metric. Moreover, $X' := p^{-1}(X)$ is a proper subset of X , so at every point in X' the hyperbolic metric with respect to X' strictly exceeds the metric with respect to X . The map $p: X' \rightarrow X$ is an unbranched covering, hence a local hyperbolic isometry, so any branch of the inverse is again a local hyperbolic isometry. With respect to the hyperbolic metric of X in domain and range, any local branch of p^{-1} contracts the hyperbolic metric uniformly on compact sets.

Next we claim that the orbit of z has an accumulation point in X . Let

$$A := \bigcap_{N>0} \overline{\bigcup_{k \geq N} p^{\circ k}(z)}$$

be the accumulation set of z (i.e. its ω -limit set). Since z never maps into K' by assumption, the set A is obviously contained in $\overline{K - K''}$. If A does not contain a point in X , then the fact $\partial X \cap K \subset \cup_k p^{\circ k}(S)$ implies that A must consist entirely of points in S' , which is a finite set. But since the orbits of these points are repelling, this is possible only if the orbit of z falls exactly onto a point in S' , so it lands in K'' , contrary to our assumption.

Let then $a \in X$ be an accumulation point of the orbit of z . Then for every connected component of K'' , there is a ray pair in Q separating a from this connected component and avoiding the finitely many points in S' . These finitely many ray pairs will bound an open neighborhood of a , which we will denote by V . The closure of this neighborhood will not meet K'' , so the hyperbolic diameter of V in X is finite, hence less than some number $C < \infty$. For every integer $m > 0$, there is an integer $M > m$ such that the M -th iterate of z is in V , having mapped at least m times through V . Denoting the fiber of z by Y , then $f^{\circ k}(z) \in V$ implies $f^{\circ k}(Y) \subset V$. Therefore, the hyperbolic diameter of the corresponding forward image of Y is at most C . All the time, the fiber Y is mapped forward homeomorphically, and the pull-back from forward images of Y back to Y can only decrease hyperbolic distances. Therefore, the hyperbolic diameter of Y is at most C . Even better: every time Y maps through V , the hyperbolic metrics with respect to X' and X differ by a definite factor $\alpha < 1$, so the hyperbolic diameter of Y must in fact be less than $\alpha^m C$. Since this is true for any $m > 0$, the fiber Y must be a point, so the fiber of z is trivial.

It remains to consider the case that the filled-in Julia set has interior. We only have to consider the case that the polynomial p has a Siegel disk. Since the critical orbit remains in K'' and must accumulate at the boundary of the periodic Siegel disks, all the periodic Siegel disks are contained in K'' . The only difference to the first case lies in the fact that in order to prove that repelling periodic points have trivial fibers, it must be possible to separate them from the critical value as well as from all periodic Siegel disks.

First we show that no boundary of a Siegel disk can contain a periodic point. Indeed, if there is a repelling point on this boundary, then it is contained in K'' and its fiber is trivial by assumption. Then an internal ray of the Siegel disk must land at this periodic point, but this is impossible: if the ray is periodic, then the rotation number of the Siegel disk will be rational; otherwise, more than one internal ray has to land at the same periodic point of the Siegel disk, and both is a contradiction.

Since fibers in K'' are trivial, every repelling periodic point in K'' can be separated from all the boundaries of the periodic Siegel disks and from the critical value. This is obviously also true for repelling periodic points outside of K'' . Again, all the repelling periodic and preperiodic points have trivial fibers. The proof now proceeds exactly as above. \square

The previous proposition was written so that it captures the construction of renormalization of Julia sets. For other applications we have in mind, such as the Branner-Douady homeomorphisms, it will be necessary to phrase the statement somewhat more generally so that the set U consists of several pieces on which the map \tilde{p} is defined as a different iterate of p . This requires a compatibility condition for the dynamics, but the arguments are exactly the same. In fact, the proof above has been written so that it applies literally to the following variant of the proposition.

Proposition 4.2 (Trivial Fibers in Dynamic Subsets II)

Let p be a unicritical polynomial with connected filled-in Julia set K . Consider a finite set of dynamic ray pairs at rational angles landing at repelling orbits and let U_1, \dots, U_s be some of the connected components of \mathbb{C} with these ray pairs removed. Let U be the interior of $\cup \overline{U}_j$; it need not be connected. Further, let n_1, \dots, n_s be positive integers such that, whenever $z \in K \cap \partial U_j \cap \partial U_{j'}$, then $p^{n_j}(z) = p^{n_{j'}}(z)$. This defines a new map $\tilde{p}: K \cap \overline{U} \rightarrow \mathbb{C}$ by setting $\tilde{p}(z) := p^{n_j}(z)$ whenever $z \in K \cap \overline{U}_j$. Let K' be the compact subset of points in $K \cap \overline{U}$ which never leave \overline{U} under iteration of the map \tilde{p} . Suppose that K' contains the critical value. Let Q be a countable forward and backward invariant set of dynamic rays of p which contains the rays bounding the U_j . Then, for this choice of dynamic rays, K has trivial fibers if and only if K' does. \square

In both variants of the previous proposition, we have compared triviality of fibers of Julia sets to triviality of fibers of “little Julia sets” within these Julia sets. We will now relate this to triviality of fibers of renormalized Julia sets.

Theorem 4.3 (Trivial Fibers For Little Julia Sets)

Suppose, under the assumptions of Propositions 4.1 or 4.2, that \tilde{p} on K' is topologically conjugate to a unisingular polynomial p_1 on its filled-in Julia set K_1 . Then K has all its fibers trivial if and only if K_1 has the same property, for appropriate choices of the sets of dynamic rays of the polynomials p and p_1 used to define separation lines for K and K_1 .

REMARK. The conjugation assures that the unique critical orbit of p_1 (other than $\{\infty\}$) has bounded orbit, so K_1 and thus K' are connected. We do not require the conjugation to exist in neighborhoods of K' and K_1 , and we do not even require it to preserve the cyclic order of branch points.

Of course, when K' and K_1 are homeomorphic, then they are simultaneously locally connected. Triviality of fibers is somewhat stronger; the transfer from K or K' to K_1 is still easy. The reverse direction, showing that trivial fibers of K_1 imply trivial fibers of K' , is not immediate: the additional decorations of K' within K might make it difficult for dynamic rays to land in such a way as to obtain enough separation lines. Even if

we knew that the topological conjugacy and thus the homeomorphism extended to a neighborhood, the transfer would not be obvious: dynamic rays of K_1 would then turn into invariant curves of K avoiding K' , but they might run through $K - K'$. (Even hybrid equivalences between polynomial-like maps can have this problem. An insignificant problem, easily overcome by Lindelöf's Theorem (compare [Ah, Theorem 3.5] or [S3, Theorem A.5]), is that hybrid equivalences do not in general map dynamic rays to dynamic rays.) What we need is, for every dynamic ray of K_1 used in a separation line, a dynamic ray of K which lands at the correct point; if there are several rays land the same point in K_1 , then we need equally many rays of K (recall that we did not require our homeomorphism to preserve the order of branches).

As an example, consider the real quadratic polynomial with a superattracting cycle of period 3: it is $p(z) = z^2 + 1.75488\dots$. The third iterate is of course hybrid equivalent to the map $z \mapsto z^2$; the fixed ray at angle 0 of the latter map turns into a subset of the real line for p (under a symmetric hybrid equivalence), so part of the image of the ray is in the filled-in Julia set of p . There are two complex conjugate dynamic rays of p landing at the same point, which are both homotopic to the image of the 0-ray with respect to the "little Julia set".

PROOF. Suppose that all the fibers of K are trivial for some choice of dynamic rays. Then in particular all fibers in K' are trivial, and we want to show that all fibers in K_1 are trivial. If the conjugation between K' and K_1 extends to neighborhoods of these sets, then all the dynamic rays used in separation lines in K' transfer to invariant curves outside of K_1 landing at boundary points of K_1 . By Lindelöf's Theorem, there are dynamic rays landing at the same points through the same accesses. Curves within bounded Fatou components are of course preserved by the homeomorphism between K' and K_1 , so every separation line in K' has a counterpart in K_1 . If all fibers of K' are trivial, then all fibers of K_1 are trivial as well.

However, if the conjugation does not extend to neighborhoods of the filled-in Julia sets, for example if it does not respect the cyclic order of branch points, then we can still argue as follows: triviality of all fibers in K implies triviality of all fibers in K' and thus local connectivity of K' ; this turns into local connectivity of K_1 . But any locally connected unicritical filled-in Julia set has only trivial fibers, provided fibers are constructed using all dynamic rays at rational angles plus the grand orbit of the unique ray landing at the critical value in case there is a Siegel disk; compare [S3, Proposition 3.6].

The other direction of the theorem is less obvious: we assume that the fibers of K_1 are trivial for some collection Q_1 of rays and show that then an appropriate collection of dynamic rays of p makes all the fibers of K' trivial. We need to show that, for every dynamic ray in Q_1 landing at a point $z_1 \in K_1$, there are dynamic rays of p landing at the corresponding point z in K' and separating all the connected components of $K' - \{z\}$, which are equally many in number as the connected components of $K_1 - \{z_1\}$ (these numbers are known to be finite, and all the rays landing at a common point have the same period). Recall that if there is no Siegel disk, then we construct fibers using all dynamic rays at rational external angles; if there is a Siegel disk, then we have to add the grand orbit of the unique ray landing at the critical value.

As in the proof of Proposition 4.1, let K'' be the union of the forward images of K' . This is a compact and forward invariant subset of K , and any point in K'' will visit K'

after finitely many iterations.

We claim that, for an arbitrary point $z \in K'$, any two connected components K_1, K_2 of $K' - \{z\}$ must be in different connected components of $K - \{z\}$. Suppose the opposite were true. Since the filled-in Julia set of p is full, the connected components K_1 and K_2 must then be connected by a bounded Fatou component of p with z on its boundary. But such a Fatou component must be contained in K'' and even in K' , which is a contradiction.

The landing point of every periodic or preperiodic ray is on a repelling or parabolic orbit. Every periodic Fatou component is attracting, parabolic or a periodic Siegel disk. In the attracting or parabolic cases, one of the periodic components contains the critical value, so that all the periodic Fatou components are contained in K'' . Periodic Siegel disks are also contained in K'' : since K'' is forward invariant, it contains either the entire cycle of periodic Siegel disks or none at all, and in the latter case the boundary of U would separate the orbit of Siegel disks from the critical orbit, so the critical orbit could not accumulate on the boundary of the Siegel disks.

First consider a periodic angle $\vartheta_1 \in Q_1$. Denote its period by m and its landing point by z_1 . Then there is a corresponding periodic point z in K' which is repelling or parabolic. Any connected component of $K_1 - \{z_1\}$, of which there are finitely many, corresponds to a unique connected component of $K' - \{z\}$. All connected components of $K' - \{z\}$ are contained in different connected components of $K - \{z\}$. The finitely many dynamic rays of p landing at z separate any two of these connected components [S3, Lemma A.8]. Therefore, if any two points in K_1 can be separated by a ray pair landing at z_1 , then the corresponding points in K' can be separated by a ray pair landing at z . When there are separation lines for K_1 consisting of (pre-)periodic dynamic rays and running through bounded Fatou components, we get corresponding lines for K' .

If there is no Siegel disk, then triviality of fibers of K_1 implies triviality of fibers of K' and thus, by Proposition 4.1, triviality of all fibers of K . This finishes the proof of the theorem, except if there is a Siegel disk. In that case, we also have to consider the dynamic ray landing at the critical value of p_1 . Let ϑ_1 be its external angle. The goal is the same as above: we want to construct a dynamic ray of p which lands at the critical value of p and which has the same separation properties for K' as it does for K_1 . The hard part this time is to show the existence of an access outside K to the critical value. Lindelöf's theorem then supplies a ray landing at v , and it will automatically have the right separation properties because only one ray can land at the critical value.

Denote the critical values of p and p_1 by v and v_1 , respectively. We first show that the fiber of v in K is trivial, albeit using temporarily a different collection of separation lines. Since all the fibers of K_1 are trivial, for every point $z_1 \in K_1$ different from v_1 there is a curve in K_1 connecting two repelling periodic points such that this curve, together with two dynamic rays landing at these points, separates v_1 from z_1 . This separation line differs from usual separation lines in that its dynamic rays must land at repelling periodic points, but it is not required to traverse only a single Fatou component. For every point $z \in K'$ different from v , we now obtain a similar separation line separating z from v . Any point in $K - K'$ can easily be separated from v . Any point in $K - \{v\}$ can thus be separated from v by such a modified separation line. Since v is on the boundary of K , it is in the impression of the dynamic ray at some angle ϑ . This

impression cannot cross the modified separation lines just defined (similarly as in the proof of [S3, Lemma 2.5]), so the impression of the dynamic ϑ -ray is $\{v\}$ and the ray lands at v . At this point, there might be various rays landing at v , which will then be homotopic in the complement of K' . We can transport all these rays forward and backwards, so that in particular we obtain d dynamic rays landing at the critical point in a symmetric way (or even a multiple of d rays).

The set K_1 has trivial fibers for $Q_1 = \mathbb{Q}/\mathbb{Z}$, extended by the grand orbit of the ray landing at the critical value in case there is a Siegel disk (using separation lines in the usual sense). All these separation lines of K_1 have counterparts in K with the corresponding separation properties for K' . Therefore, all the separation lines made with dynamic rays landing at repelling and parabolic (pre-)periodic points of K' and at the grand orbit of the critical value (in case of a Siegel disk) suffice to make all the fibers of K' trivial. By the previous proposition, all the fibers of K are trivial. This finishes the proof of the theorem. (If there is a Siegel disk, we can now conclude that the critical value is the landing point of exactly one dynamic ray; see [S3, Corollary 3.7].)

□

We can now conclude that triviality of fibers of Julia sets is preserved under renormalization.

Corollary 4.4 (Triviality of Fibers Preserved Under Renormalization)

The Julia set for any polynomial in \mathcal{M}_d has trivial fibers if and only if any or all of its (simple) renormalizations have the same property.

PROOF. This conclusion is immediate once the description of simple renormalization is right. The usual definition for a polynomial $p \in \mathcal{M}_d$ to be n -renormalizable is that there exist two open subsets $U \subset V \subset \mathbb{C}$ such that $p^{on}: U \rightarrow V$ is polynomial-like of degree $d \geq 2$ with connected Julia set (Definition 2.1).

For quadratic polynomials, it is well known (but admittedly I know of no precise reference; compare Milnor [M2], Lyubich [L, Section 2] or Haïssinsky [Ha, Section 10]) that then there are two rational ray pairs, one periodic and one preperiodic from the same orbit, which bound a subset $U' \subset U$ with the property that the little Julia set of the renormalization is exactly the set of points in \mathbb{C} with bounded orbits which never escape from \bar{U}' under iteration of p^{on} . (The set V for the polynomial-like map in the definition of renormalization above can be obtained from U' by restricting it to any positive equipotential and “thickening” it slightly near the parts of its boundary which are formed by dynamic rays and their landing points; the set U is then obtained from V by pulling it back n times under the dynamics of p .)

With this description of renormalization, Theorem 4.3 applies and shows that, whenever the filled-in Julia set of p has trivial fibers, then the fibers of any of its renormalizations are also trivial, and conversely.

For unicritical polynomials of higher degrees, the same reasoning works, but one needs $d - 1$ preperiodic ray pairs. □

REMARK. The same statement holds also for crossed renormalization, which has so far been described only in the quadratic case; see below. Quite generally, the same ideas

might give a statement like this: if an arbitrary polynomial is renormalizable (possibly using several regions U and V corresponding to different renormalization periods) so that these renormalizations absorb all the critical orbits, then all the fibers of the entire Julia set are trivial if and only if all the fibers of all the renormalized Julia sets are trivial.

5 Quadratic Polynomials

This section will contain further applications of the concepts introduced above which are in some sense specific to the case of quadratic polynomials and to the Mandelbrot set $\mathcal{M} = \mathcal{M}_2$. It seems quite possible that these statements have analogues for Multibrot sets of higher degrees, but the underlying theorems are so far known only for degree two. We will prove that triviality of fibers of both Julia sets and the Mandelbrot set are preserved under homeomorphisms like those arising in the context of crossed renormalizations or those introduced by Branner and Douady, and we will also conclude that quadratic Julia sets with Siegel disks of bounded type have trivial fibers.

Corollary 5.1 (Trivial Fibers Preserved under Crossed Renormalization)

The Julia set of a crossed renormalizable polynomial in \mathcal{M} has trivial fibers if and only if any of its crossed renormalizations has the same property.

PROOF. By [RS], any crossed renormalization which is not of immediate type (so that the little Julia sets intersect each other at a fixed point) is itself simply renormalizable in such a way that the renormalized Julia set is crossed renormalizable of immediate type. We can therefore restrict attention to the immediate case. For this, the argument is literally the same as for Corollary 4.4: the ray pairs bounding the set U' are constructed explicitly in [RS, Section 3.1]. \square

We will now discuss a homeomorphism which has been introduced by Branner and Douady [BD]: it maps the 1/2-limb of the Mandelbrot set homeomorphically onto a subset of the 1/3-limb. It was with this homeomorphism in mind that we added the extra notations in Proposition 4.2 and Theorem 4.3.

Corollary 5.2 (Trivial Fibers Preserved Under Branner-Douady Maps)

The Branner-Douady homeomorphism from the 1/2-limb into the 1/3-limb of the Mandelbrot set preserves the property of Julia sets that all fibers are trivial.

PROOF. As for tuning, this is an immediate conclusion with the right setup. It is more convenient to look at the inverse map from a subset of the 1/3-limb onto the 1/2-limb.

For any Julia set from the 1/3-limb, it is well known that the dynamic rays at angles $1/7$, $2/7$ and $4/7$ land at a single fixed point, which is called the α -fixed point. Similarly, the dynamic rays at angles $9/14$, $11/14$ and $1/14$ land at $-\alpha$. Denote the closures of the regions between the $1/7$ -ray and the $2/7$ -ray by Y_1 , between the $2/7$ -ray and the $4/7$ -ray by Y_2 and let $Z_1 := -Y_1$, $Z_2 := -Y_2$. Finally, let Y_0 be the “central

puzzle piece” in the complement of all these rays: the region containing the critical point. Then the critical value is always in Y_1 .

Now the set U consists of the interior of the pieces $Y_0 \cup Y_1 \cup Z_1$. More precisely, using the notation from Theorem 4.3, let $U_1 := Y_1$ with $n_1 = 2$, let $U_2 := Y_0$ with $n_2 = 1$ and $U_3 := Z_2$ with $n_3 = 1$. The dynamics of \tilde{p} , restricted to the filled-in Julia set, is then as follows: it sends $U_1 \cap K = Y_1 \cap K$ homeomorphically onto $(Y_0 \cup Z_1 \cup Z_2) \cap K$; $U_3 \cap K = Z_2 \cap K$ lands homeomorphically on the same image; and $U_2 \cap K = Y_0 \cap K$ is mapped in a two-to-one fashion onto $Y_1 \cap K$. Therefore, any point within $U \cap K$ which eventually escapes from U does so through Z_1 . It follows that a point in the $1/3$ -limb of the Mandelbrot set is in the range of the homeomorphism from the $1/2$ -limb iff the critical orbit avoids the piece Z_1 .

Let c_1 be any parameter in the $1/2$ -limb and let c be its image in the $1/3$ -limb under the Branner-Douady homeomorphism. Branner and Douady then show that the filled-in Julia set of c_1 is topologically conjugate to the subset K' of the filled-in Julia set of K consisting of points which stay in \overline{U} forever. Therefore, Theorem 4.3 applies and shows that the filled-in Julia sets of c and c_1 both have trivial fibers whenever one of them does. □

REMARK. By work of Levin and van Strien [LvS], and of Lyubich and Yampolsky [LY], it is known that all the Julia sets on the real axis of the Mandelbrot set are locally connected and thus have trivial fibers.

The construction of Branner and Douady gives, with only notational changes, homeomorphisms from the $1/2$ -limb into any $1/q$ -limb of the main cardioid of the Mandelbrot set. The theorem applies to all of them. We obtain the following corollary:

Corollary 5.3 (Julia Sets in Some Spines of the Mandelbrot Set)

There are topological arcs in the Mandelbrot set connecting the origin to the landing point of any parameter ray at angle $1/2^n$ and running only along parameters for which their Julia sets have trivial fibers. □

Using further homeomorphisms, one can extend this result to many more topological arcs. Such homeomorphisms are still covered by our arguments. Using results of Yoccoz, as well as the fact that triviality of fibers is preserved under tuning, one can arrive at this result in a different way. First we recall (and restate) a result of Yoccoz.

Theorem 5.4 (Trivial Fibers of the Mandelbrot Set at Yoccoz Points)

If a parameter in the Mandelbrot set is not infinitely renormalizable, then its fiber is trivial. □

REMARK. This theorem is a union of two results by Yoccoz, plus the triviality that fibers within hyperbolic components are trivial. For the case that all periodic orbits are repelling, Yoccoz’ result is [H, Theorem III] (only the non-renormalizable case is treated in detail there; the transfer to the finitely renormalizable case also follows from our Corollary 3.2). For the case of indifferent orbits, Yoccoz’ theorem is [H, Theorem I.B]; for a different proof, see [S4, Theorem 5.2]. The usual statement of Yoccoz’ theorems is that the Mandelbrot set is locally connected at these points. However, he proves the

stronger result that the fibers are trivial by proving shrinking of puzzle pieces, and we will need triviality of fibers.

Here is another lemma about the Mandelbrot set which will be needed.

Lemma 5.5 (Branch Points on the Real Axis)

If a Misiurewicz point in \mathcal{M} is on the real axis, or if it is a tuned image of a point $c \neq -2$ on the real axis, then it is not a branch point.

PROOF. Suppose that there is a Misiurewicz point on the real axis of \mathcal{M} which is a branch point. It is then the landing point of as many preperiodic parameter rays as there are branches at this point, at least three (in fact, because of symmetry, this number must be even). In the dynamic plane of this Misiurewicz point, there must then be at the same number of preperiodic dynamic rays landing at the critical value. The critical value is real, and by invariance of the real axis, there is then a repelling periodic real orbit such that at least four dynamic rays land at each of its points. This is impossible for various reasons: the combinatorial rotation number of this orbit must, by symmetry, be equal to its inverse, so it must be equal to $1/2$ and the various branches could not be permuted transitively. But this is always true if there are at least three rays (compare e.g. [M2, Lemma 2.4] or [S1, Lemma 2.4]). Another reason is that the wake of this orbit must be bounded by two parameter rays landing at a parabolic parameter with the same combinatorial rotation number. Again, symmetry of the real line requires this combinatorial rotation number to be $1/2$, so the only critical orbit could not visit all the parabolic basins, which is a contradiction.

If a Misiurewicz point on a tuned image of the real axis is a branch point, then it must acquire more branches in the process of tuning. If c is the un-tuned Misiurewicz point, then it cannot disconnect \mathcal{M} by Theorem 3.1. Therefore, if it is on the real line, then it is the main antenna tip $c = -2$. □

Theorem 5.6 (The Mandelbrot Set is Almost Path Connected)

For every parameter in the Mandelbrot set which has a trivial fiber, or which is on a tuned image of the real line, there is an arc within the Mandelbrot set connecting this parameter to the origin.

PROOF. If the Mandelbrot set was locally connected, the claim would follow simply by the general fact that compact connected locally connected metric spaces are pathwise connected [M1, Section 16]. There are models for the Mandelbrot set which are locally connected: one of them is Douady’s “pinched disk model” (compare [Do2]). Another one is the quotient of the Mandelbrot set in which all fibers are collapsed to points (compare [S3, Section 2] or [S4, Section 7]). All locally connected models of the Mandelbrot set are of course homeomorphic. Another related locally connected model space is Penrose’s “abstract abstract Mandelbrot set” [Pe1],[Pe2], which is a parameter space of kneading sequences (however, it is not homeomorphic to the previous spaces because only a subset corresponds to realized kneading sequences, and many of its points correspond to more than one point in models of the Mandelbrot set. The necessary modifications of the proof below are only minor).

All these locally connected model spaces come with natural continuous projections from the actual Mandelbrot set to these spaces. Let \mathcal{M}^* be a locally connected model of \mathcal{M} and let $\pi: \mathcal{M} \rightarrow \mathcal{M}^*$ be such a projection. The inverse image $\pi^{-1}(c^*)$ over any point $c^* \in \mathcal{M}^*$ corresponds exactly to a fiber of the Mandelbrot set. Injectivity at any given point is then equivalent to this fiber being trivial.

Let c_0 be any point in \mathcal{M} with trivial fiber, or on a tuned image of the real axis. If c_0 is on a tuned image of the real line, we may replace it by a hyperbolic parameter on the same tuned image of the real line, so that the fiber of the new c_0 is trivial. It is easily possible to connect the old and new c_0 along the tuned image of the real axis, so nothing is lost by the assumption that the fiber of c_0 is trivial.

Let $\varphi: [0, 1] \rightarrow \mathcal{M}^*$ be an injective continuous map connecting the origin to c_0 within the locally connected model space \mathcal{M}^* . We may suppose that φ has the following property: for any little model Mandelbrot set $\mathcal{M}' \subset \mathcal{M}^*$ and two points c, c' on the tuned image of the real axis of \mathcal{M}' which are on the image of $\varphi([0, 1])$, the path connects these two points entirely along the tuned image of the real axis. (For example, all regular arcs have this property; see below.) We will show that this map lifts to a continuous map $\psi: [0, 1] \rightarrow \mathcal{M}$ such that $\varphi = \pi \circ \psi$.

For any $t \in [0, 1]$ such that the fiber of $\varphi(t)$ is trivial, the definition of $\psi(t) = \pi^{-1}(\varphi(t))$ is clear. In view of Yoccoz' theorem above, we only have to consider the case that $c^* = \varphi(t^*)$ is infinitely renormalizable. In fact, it is then infinitely simply renormalizable [McM] so that there is an infinite sequence of positive integers $n_1 < n_2 < \dots$ so that c^* is simply n_k -renormalizable and every n_k strictly divides n_{k+1} . Let \mathcal{M}'_k be the nested collection of corresponding embedded Mandelbrot sets. All of these sets contain c_0 in their wakes, and there are three possibilities for each \mathcal{M}'_k :

1. the little Mandelbrot set \mathcal{M}'_k may contain c_0 ;
2. a little Mandelbrot set \mathcal{M}'_k may contain c_0 in its wake, so that the “main antenna tip” of \mathcal{M}'_k (the tuned image of the point -2) separates c_0 from the origin (i.e. two parameter rays landing at the antenna tip do the separation);
3. the little Mandelbrot set \mathcal{M}'_k may contain c_0 in its wake, but not within itself and not so that its main antenna tip separates c_0 from the origin.

Suppose that there is some index k_0 so that \mathcal{M}'_{k_0} does not contain c_0 . Then there is a Misiurewicz point $B \in \mathcal{M}'_{k_0}$ which is the landing point of at least two parameter rays at preperiodic angles which separate c_0 from the rest of \mathcal{M}'_{k_0} (Theorem 3.1). Let c'_0 be the center of any hyperbolic component which is not separated from c_0 by the parameter rays landing at B ; such a component exists because only finitely many rays land at B and any periodic parameter ray will find such a component. As far as the desired path within \mathcal{M}'_{k_0} is concerned, we may replace c_0 by c'_0 .

We will now use internal addresses [LS1]. The internal address of any hyperbolic component is finite by definition (it is a strictly increasing sequence of integers starting with 1 and ending with the period of the component). Any hyperbolic component containing c'_0 in its wake but not within its subwake at internal angle $1/2$ must appear in the internal address of c'_0 by [LS1, Lemma 6.4]. This can happen only finitely often. (The reason for this is that upon entering the p/q -subwake of any such hyperbolic

component, a periodic orbit is created, and every point of this orbit is the landing point of q dynamic rays. Every hyperbolic Julia set has only finitely many such orbits with $q > 2$). All but finitely many little Mandelbrot sets \mathcal{M}'_k will thus contain c'_0 and c_0 within their sublimbs at internal angles $1/2$.

Let c_{-2} be the main antenna tip of \mathcal{M}'_{k_0} . We apply the Branch Theorem [S4, Theorem 2.2] to c_{-2} and c'_0 . These two points are either separated by a hyperbolic component or by a Misiurewicz point which must be on the tuned image of the real axis within \mathcal{M}'_{k_0} . However, Misiurewicz points on the interior of the tuned image of the real axis are never branch points by Lemma 5.5, so the separation takes place at a hyperbolic component. This hyperbolic component must then show up in the internal address of c'_0 for the same reason as above. It follows that the third case above can happen only finitely many times. Ignoring finitely many \mathcal{M}'_k , we can completely ignore the third case.

If at least one of the \mathcal{M}'_k realizes the second possibility, we are done: the tuning map of \mathcal{M}'_k is a homeomorphism from \mathcal{M} onto \mathcal{M}'_k and we can use its restriction to the real axis of the Mandelbrot set. This connects the root to the main antenna tip of \mathcal{M}'_k , and composition with the projection will connect the root to the main antenna tip of the projection of \mathcal{M}'_k . Therefore, φ is continuous in a neighborhood of t^* . By assumption on the map φ , the composition $\pi \circ \psi$ coincides with φ (possibly up to reparametrization) within the entire neighborhood of t^* in the tuned image of the real axis.

Otherwise, the first case must happen infinitely often. It follows that all \mathcal{M}'_k must contain c_0 as well as c^* , so both points are in the same fiber. Since we assumed that the fiber of c_0 is trivial, we have $\{c_0\} = \pi^{-1}(c^*)$. But we can then simply define $\psi(t^*) := c_0$.

This way, we have defined $\psi: [0, 1] \rightarrow \mathcal{M}$, and this map is continuous by construction. This finishes the proof. \square

This result seems to have been observed first by Kahn, at least for the case of dyadic Misiurewicz points. A sketch of proof, from which the idea of this proof is taken, can be found in Douady [Do2].

For the real line of the Mandelbrot set, there are many results known, most of which are quite recent: hyperbolicity is dense (Lyubich [L] and Graczyk and Świątek [GS]), and all the Julia sets are locally connected (Levin and van Strien [LvS], with simplifications by Sands, and Lyubich and Yampolsky [LY]). Therefore, we can draw the following conclusions:

Corollary 5.7 (Properties of Paths in Mandelbrot Set)

The paths in the Mandelbrot set, as defined in the previous theorem, run only through parameters with locally connected Julia sets, and hyperbolicity is dense on these paths.

\square

The Branch Theorem ([LS1, Theorem 9.1] or [S4, Theorem 2.2]) shows that the various paths within the Mandelbrot set starting at the origin and leading to different points can split only at Misiurewicz points or within hyperbolic components.

There is a concept of “regular arcs” or “legal arcs”, due to Douady and Hubbard [DH1] (see also [Do2]): a regular arc is an arc within the Mandelbrot set subject to the condition that it traverses any hyperbolic component only along the union of two

internal rays: it may enter the component along an internal ray towards the center, and then leave the component along another internal ray. It is easy to see that a regular arc in the Mandelbrot set is uniquely specified by its endpoints (except possibly for some freedom within non-hyperbolic components — but that will never happen for the arcs constructed here because hyperbolicity is dense on them). Branch points of regular arcs are necessarily postcritically finite.

Here are two more corollaries about the Mandelbrot set.

Corollary 5.8 (Crossed Renormalization and Trivial Fibers)

The fiber of any crossed renormalizable parameter in the Mandelbrot set is trivial if and only if the fiber of the crossed renormalized parameter is trivial.

PROOF. According to [RS, Section 3], the subset of the Mandelbrot set corresponding to any particular type of crossed renormalization is homeomorphic to a sublimb of the Mandelbrot set, and it is separated from the rest of the Mandelbrot set by a countable collection of rational parameter ray pairs, just like in Theorem 3.1. Therefore, the fiber of any crossed renormalizable parameter is contained within the crossed renormalization locus of the same type. Within this locus, the same argument as in Corollary 3.2 applies.

□

Corollary 5.9 (Branner-Douady Homeomorphisms and Trivial Fibers)

The Branner-Douady homeomorphism from the 1/2-limb into the 1/3-limb of the Mandelbrot set preserves triviality of fibers of \mathcal{M} .

SKETCH OF PROOF. The proof is essentially the same as above, so we just sketch the main steps. The first step is again to show that the image of the 1/2-limb within the 1/3-limb is bounded by rational parameter ray pairs, and this follows from the corresponding properties in the dynamic planes by the same transfer arguments as for simple and crossed renormalization.

We can then conclude that the fiber of any point in the image of the homeomorphism is entirely contained within the image. Finally, we show as before that any separation line in the 1/2-limb carries over to a separation line in the 1/3-limb with the corresponding separation properties in the image, and conversely. □

REMARK. Another way to say this is like this: the Branner-Douady homeomorphism is compatible with tuning, so that it maps little Mandelbrot sets to little Mandelbrot sets. Points which are not within those have trivial fibers by Yoccoz’ result, and for points within little Mandelbrot sets, the Branner-Douady image is the same as the image under tuning maps. Extra decorations are attached within the 1/3-limb to the image of the 1/2-limb. These are separated from the little Mandelbrot set by rational parameter ray pairs, so fibers cannot get larger under this homeomorphism. However, this argument requires the Yoccoz theorem and is therefore restricted to degree two only.

Finally, we give a result about Siegel disks. The usual choice of rational external angles in the definition of fibers ($Q = \mathbb{Q}/\mathbb{Z}$) had to be extended by the grand orbit of

the ray landing at the critical value precisely when there is a Siegel disk. We will now give a use of this extra work. First we recall a recent theorem of Petersen [Pt], reproved by Yampolsky. A Siegel disk is called of *bounded type* if its multiplier $\mu = e^{2\pi i\vartheta}$ is such that the continued fraction expansion of ϑ has bounded entries.

Theorem 5.10 (Local Connectivity of Period One Siegel Disks)

For every quadratic polynomial with a period one Siegel disk of bounded type, the Julia set is locally connected.

We can now remove the condition on the period.

Corollary 5.11 (Trivial Fibers and Bounded Type Siegel Disks)

For every quadratic polynomial with a Siegel disk of bounded type, the Julia set has trivial fibers and is thus locally connected.

PROOF. The Siegel disk of period one is locally connected and thus has trivial fibers for an appropriate choice of Q [S3, Proposition 3.6]. Any quadratic Siegel disk of any period $n \geq 1$ has its parameter on the boundary of a hyperbolic component of period n , so it is the image of a parameter on the boundary of the main cardioid of the Mandelbrot set under a tuning map of period n . The dynamics is thus n -renormalizable. The result now follows from Corollary 4.4. □

6 References

- [Ah] Lars Ahlfors: *Conformal invariants*. McGraw-Hill (1973).
- [BD] Bodil Branner, Adrien Douady: *Surgery on complex polynomials*. In: R. Gomez-Mont (ed.), *Holomorphic dynamics*, Lecture Notes in Mathematics 1345. Springer Verlag (1986), 11–72.
- [Do1] Adrien Douady: *Algorithms for computing angles in the Mandelbrot set*. In: M. Barnsley, S. Demko (eds.), *Chaotic dynamics and fractals, Atlanta 1984*. Academic Press (1986), 155–168.
- [Do2] Adrien Douady: *Compacts in \mathbb{C}* . In: *Topological methods in modern mathematics*. Publish or Perish, Houston (1992), 429–265.
- [DH1] Adrien Douady, John Hubbard: *Etude dynamique des polynômes complexes*. Publications mathématiques d’Orsay 84-02 (1984) (première partie) and 85-04 (1985) (deuxième partie).
- [DH2] Adrien Douady, John Hubbard: *On the dynamics of polynomial-like mappings*. Ann. Scient. Ec. Norm. Sup. 18 (1985), 287–343.
- [GS] Jacek Graczyk, Grzegorz Świątek: *Generic hyperbolicity in the logistic family*. Ann. of Math. **146** (1997), 1–56.
- [H] John Hubbard: *Three theorems of J.-C. Yoccoz*. In: *Topological methods in modern mathematics*. Publish or Perish, Houston (1992), 467–511.

- [Ha] Peter Haïssinsky: *Applications de la chirurgie holomorphe, notamment aux points paraboliques*. Thèse, Université de Paris-Sud (1998).
- [La] Pierre Lavaurs: *Une description combinatoire de l'involution définie par M sur les rationnels à dénominateur impair*. Comptes Rendus Aca. Sci. Paris (4) **303** (1986) 143–146.
- [LS1] Eike Lau, Dierk Schleicher: *Internal addresses in the Mandelbrot set and irreducibility of polynomials*. Preprint, Institute for Mathematical Sciences, Stony Brook **19** (1994).
- [LS2] Eike Lau, Dierk Schleicher: *Symmetries of fractals revisited*. The Mathematical Intelligencer, **18** (1996), 45–51.
- [LvS] Genadi Levin, Sebastian van Strien: *Local connectivity of the Julia set of real polynomials*. Annals of Mathematics, **147** (1998), 471–541.
- [L] Mikhail Lyubich: *Dynamics of quadratic polynomials I–II*. Acta Math. **127** (1997), 185–297.
- [LY] Mikhail Lyubich, Mikhail Yampolsky: *Dynamics of quadratic polynomials: complex bounds for real maps*. Preprint, Mathematical Sciences Research Institute, Berkeley **034** (1995). To appear in Ann. Inst. Fourier.
- [M1] John Milnor: *Dynamics in one complex variable: introductory lectures*. Preprint, Institute for Mathematical Sciences, Stony Brook **5** (1990).
- [M2] John Milnor: *Periodic orbits, external rays and the Mandelbrot set; an expository account*. Manuscript (1997), to appear in *Astérisque*.
- [McM] Curt McMullen: *Complex dynamics and renormalization*. Annals of Math. Studies 135, Princeton Univ. Press (1994).
- [Pe1] Chris Penrose: *On quotients of the shift associated with dendrite Julia sets of quadratic polynomials*. Thesis, University of Coventry (1990).
- [Pe2] Chris Penrose: *Quotients of the shift associated with dendrite Julia sets*. Manuscript, submitted for publication (1994).
- [Pt] Carsten Petersen: *Local connectivity of certain Julia sets with Siegel disks*. Acta Math. **177** (1997), 163–224.
- [Ri] Johannes Riedl: *Renormierung in der komplexen Dynamik, oder: Woher die kleinen Mandelbrotmengen in der großen kommen*. In: *Proceedings der 6. Jahrestagung über nichtlineare Dynamik München*. FakTUM **13** (1997).
- [RS] Johannes Riedl, Dierk Schleicher: *On the locus of crossed renormalization*. In: S. Morosawa (ed.), *Proceedings of the Kyoto workshop on complex dynamics*, Kyoto (1997).
- [S1] Dierk Schleicher: *Rational external rays of the Mandelbrot set*. Preprint, Institute for Mathematical Sciences, Stony Brook **13** (1997). To appear in *Astérisque*.
- [S2] Dierk Schleicher: *On the structure of the Mandelbrot set*. Manuscript, in revision.
- [S3] Dierk Schleicher: *On fibers and local connectivity of compact sets in \mathbb{C}* . Preprint, Institute for Mathematical Sciences, Stony Brook **12** (1998).

- [S4] Dierk Schleicher: *On fibers and local connectivity of Mandelbrot and Multibrot Sets*. Preprint, Institute for Mathematical Sciences, Stony Brook **13a** (1998).
- [TL] Tan Lei: *Voisinages connexes des points de Misiurewicz*. Ann. Inst. Fourier, Vol. 42, No. 4 (1992), 707–735.
- [T] William Thurston: *On the geometry and dynamics of iterated rational maps*. Preprint, Princeton University (1985).
- [Ya] Mikhail Yampolsky: *Complex bounds for renormalization of critical circle maps*. Manuscript (1997) (an earlier version appeared as preprint #1995/12 of the IMS Stony Brook).

Dierk Schleicher
Fakultät für Mathematik
Technische Universität
Barer Straße 23
D-80290 München, Germany
dierk@mathematik.tu-muenchen.de