

On Fibers and Local Connectivity of Mandelbrot and Multibrot Sets

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Abstract

We give new proofs that the Mandelbrot set is locally connected at every Misiurewicz point and at every point on the boundary of a hyperbolic component. The idea is to show “shrinking of puzzle pieces” without using specific puzzles. Instead, we introduce *fibers* of the Mandelbrot set (see Definition 3.2) and show that fibers of certain points are “trivial”, i.e., they consist of single points. This implies local connectivity at these points.

Locally, triviality of fibers is strictly stronger than local connectivity. Local connectivity proofs in holomorphic dynamics often actually yield that fibers are trivial, and this extra knowledge is sometimes useful. We include the proof that local connectivity of the Mandelbrot set implies density of hyperbolicity in the space of quadratic polynomials (Corollary 3.6).

We write our proofs more generally for *Multibrot sets*, which are the loci of connected Julia sets for polynomials of the form $z \mapsto z^d + c$.

Although this paper is a continuation of [S3], it has been written so as to be independent of the discussion of fibers of general compact connected and full sets in \mathbb{C} given there.

1 Introduction

A great deal of work in holomorphic dynamics has been done in recent years trying to prove local connectivity of Julia sets and of many points of the Mandelbrot set, notably by Yoccoz, Lyubich, Levin, van Strien, Petersen and others. One reason for this work is that the topology of Julia sets and the Mandelbrot set is completely described once local connectivity is known. Another reason is that local connectivity of the Mandelbrot set implies that hyperbolicity is dense in the space of quadratic polynomials.

In [S3], we have introduced *fibers* for arbitrary compact connected and full subsets of the complex plane as a new point of view on questions of local connectivity, and we have applied fibers to filled-in Julia sets. In the present paper, we restrict to the case of the Mandelbrot set and its higher degree cousins. After a review of fundamental properties of the Multibrot sets, we define fibers of these sets and show that they are born with much nicer properties than fibers of general compact connected and full subsets of \mathbb{C} . Our goal here is to prove that Misiurewicz points and boundary points of hyperbolic components *have trivial fibers*, i.e. that their fibers consist of only one point. A consequence is local connectivity at these points, but triviality of fibers is a somewhat stronger property. Unlike in [S3], where the discussion was for more general subsets of \mathbb{C} , we try to be as specific as possible to Multibrot sets; we have repeated some key discussions about fibers in order to make the present paper as self-contained as possible.

We discuss polynomials of the form $z \mapsto z^d + c$, for arbitrary complex constants c and arbitrary degrees $d \geq 2$. These are, up to normalization, exactly those polynomials which have a single critical point. Following a suggestion of Milnor, we call these polynomials *unicritical* (or unisingular). We will always assume unicritical polynomials to be normalized as above, and the variable d will always denote the degree. We define the *Multibrot set* of degree d as the connectedness locus of these families, that is

$$\mathcal{M}_d := \{c \in \mathbb{C} : \text{the Julia set of } z \mapsto z^d + c \text{ is connected} \} .$$

In the special case $d = 2$, we obtain quadratic polynomials, and \mathcal{M}_2 is the familiar *Mandelbrot set*. All the Multibrot sets are connected, they are symmetric with respect to the real axis, and they also have $d - 1$ -fold rotation symmetries (see [LS2] with pictures of several of these sets). The present paper can be read with the quadratic case in mind throughout. However, we have chosen to do the discussion for all the Multibrot sets because this requires only occasional slight modifications – and because recently interest in the higher degree case has increased; see e.g. Levin and van Strien [LvS] or McMullen [McM].

One problem in holomorphic dynamics is that many results are folklore, with few accessible proofs published. In particular, many of the fundamental results about the Mandelbrot set, due to Douady and Hubbard, have been described in their famous “Orsay notes” [DH1] which have never been published. They are no longer available, and it is not always easy to pinpoint a precise reference even within these notes. This paper provides proofs of certain key results about the Mandelbrot sets, and more generally for Multibrot sets. Most of these results are known for $d = 2$, but several of our proofs are new and sometimes more direct than known proofs.

In a sequel [S4], we will give various applications of the concept of fibers which are related to the concepts of *renormalization* and *tuning*: triviality of fibers is preserved under tuning of Multibrot sets and under renormalization of Julia sets, and we will show that the Mandelbrot set comes quite close to being arcwise connected. These statements, in turn, have various further applications.

We begin in Section 2 by a review of certain important properties of Mandelbrot and Multibrot sets. We then define fibers for these sets in Section 3 using the same definition as in [S3], except that the definition simplifies because fibers of Multibrot sets generally behave quite nicely.

We also show that the fiber of an interior point is trivial if and only if it is in a hyperbolic component. We conclude the section by a proof that local connectivity is equivalent to triviality of all fibers, and both conditions imply density of hyperbolicity (using an argument of Douady and Hubbard; Corollary 3.6).

We then prove that every Multibrot set has trivial fibers and is thus locally connected at every boundary point of a hyperbolic component and at every Misiurewicz point. Boundaries of hyperbolic components are discussed in Section 4, except roots of primitive components: they require special treatment which can be found in Section 5. In Section 6 we prove that fibers of Misiurewicz points are trivial. This shows that fibers of Multibrot sets have particularly convenient properties. Finally, in Section 7, we compare fibers to combinatorial classes.

ACKNOWLEDGEMENTS. As a continuation of [S3], this paper inherits the acknowledgements mentioned there. I would like to repeat the contribution and support of Misha Lyubich in particular, as well as the continuing encouragement of John Milnor and the support by the Institute for Mathematical Sciences in Stony Brook.

2 Multibrot Sets

In this section, we review some necessary background about Multibrot sets, and we define fibers of these sets. We include a result from [S3]: whenever a fiber consists of a single point, then this implies local connectivity at this point.

All the Multibrot sets are known to be compact, connected and full. It is conjectured but not yet known that they are locally connected. However, it is known that many of its fibers are trivial. We will show that for certain particularly important fibers in Sections 4, 5 and 6.

We recall the definition of external rays: For any compact connected and full subset $K \subset \mathbb{C}$ consisting of more than a single point, there is a unique conformal isomorphism $\Phi: \overline{\mathbb{C}} - K \rightarrow \overline{\mathbb{C}} - \overline{\mathbb{D}}$ fixing ∞ , normalized so as to have positive real derivative at ∞ . Inverse images of radial lines in $\mathbb{C} - \overline{\mathbb{D}}$ are called external rays, and an external ray at some angle ϑ is said to *land* if the limit $\lim_{r \searrow 1} \Phi^{-1}(re^{i\vartheta})$ exists. The *impression* of this external ray is the set of all limit points of $\Phi^{-1}(r'e^{i\vartheta'})$ for $r' \searrow 1$ and $\vartheta' \rightarrow \vartheta$.

As in [S1] and [M3], we will denote external rays of the Multibrot sets by *parameter rays* in order to distinguish them from *dynamic rays* of Julia sets. All the parameter rays at rational angles are known to land (see Douady and Hubbard [DH1], Schleicher [S1] or, in the periodic case, Milnor [M3]).

A *ray pair* is a collection of two external rays (dynamic or parameter rays) which land at a common point. A dynamic ray pair is *characteristic* if it separates the critical value from the critical point and from all the other rays on the forward orbit of the ray pair. The landing point of a periodic or preperiodic dynamic ray pair is always on a repelling or parabolic orbit; if a preperiodic dynamic ray pair is characteristic, then its landing point is necessarily on a repelling orbit.

The following theorem will be used throughout this paper. The first half is due to Lavaurs [La]. Proofs of both parts can be found in [S2].

Theorem 2.1 (Correspondence of Ray Pairs)

For every degree $d \geq 2$ and every unicritical polynomial $z \mapsto z^d + c$, there are bijections

- *between the ray pairs in parameter space at periodic angles, separating 0 and c , and the characteristic periodic ray pairs in the dynamic plane of c landing at repelling orbits; and*
- *between the ray pairs in parameter space at preperiodic angles, separating 0 and c , and the characteristic preperiodic ray pairs in the dynamic plane of c .*

These bijections of ray pairs preserve external angles. □

We assume that the separating ray pairs do not go through the point c (the critical value or the parameter). For these boundary cases, we have the following. The critical value is never on a periodic ray pair because, depending on whether the corresponding Julia set is connected or not, it would either be periodic and thus superattracting and could not be the landing point of dynamic rays, or the periodic dynamic rays would bounce into a precritical point infinitely often and thus fail to land. Conversely, if the parameter c is on a parameter ray pair at periodic angles, the corresponding Julia set either has a dynamic ray pair which is the characteristic ray pair of a parabolic periodic orbit, or at least one of the rays on this ray pair bounces into a precritical point infinitely often. In the preperiodic case, if the parameter c is on a parameter ray pair at preperiodic angles, then in the corresponding dynamic plane, the dynamic rays at the corresponding angles form a dynamic ray pair landing at a preperiodic point in a repelling orbit. Moreover, this ray pair contains the critical value, and all its forward images are on the same side as the critical point. Conversely, if in some dynamic plane the critical value is on a characteristic preperiodic ray pair, then this ray pair lands on a repelling orbit and the parameter is on the parameter ray pair at corresponding angles. In the quadratic case, these results go back to Douady and Hubbard [DH1]; for proofs, see also [S1, Theorem 1.1] and, in the periodic case, [M3]. Proofs for higher degrees will be in [Eb].

Not all rational parameter rays are organized in pairs. The number of parameter rays at preperiodic angles landing at a common point can be any positive integer. For parameter rays at periodic angles, this number is either 1 or 2; in the quadratic case, this number is always 2 (we count the parameter rays at angles 0 and 1 separately).

All parameter rays at periodic angles are known to land at parabolic parameters (where the critical orbit converges to a unique parabolic orbit). All parameter rays at preperiodic angles land at parameters where the critical value is strictly preperiodic:

such parameters are (somewhat unfortunately) known as “Misiurewicz points”. Another name for such points would be “critically preperiodic parameters”.

Any hyperbolic component has one *root* and $d - 2$ *co-roots* on its boundary: these are parameters with parabolic orbits whose periods are at most that of the attracting orbits within the component. The root is the landing point of two periodic parameter rays, while every co-root is the landing point of a single parameter ray at a periodic angle. The period of a hyperbolic component is also the period of the d parameter rays landing at root and co-roots of the component.

The *wake* of a hyperbolic component will be the region in the complex plane separated from the origin by the two rational parameter rays landing at the root of the component. *Subwakes* of a component will be wakes of components bifurcating directly from the component. The intersection of any wake or subwake with the Multibrot set will be called a *limb* or *sublimb*. Wakes and subwakes are open and connected, while limbs and sublimbs are connected and closed, except that one boundary point is missing (the “root” of the (sub-)limb, where it is attached to the rest of the Multibrot set). Every hyperbolic component has a unique *center*: a parameter where the critical orbit is periodic. These centers and Misiurewicz points together form the countable set of *postcritically finite* parameters. When treating them simultaneously, it will sometimes simplify language to identify the component with its center and speak, e.g. of the “wake of the center”, meaning of course the wake of the component containing this center.

All the rational rays landing at any Misiurewicz point cut the complex plane into as many open parts as there are rays. We will call these parts the *subwakes* of the Misiurewicz point. The subwake containing the origin will be called the *zero wake* (which is the entire complex plane minus the ray if there is only one ray), and the union of all the other wakes (together with any parameter rays between them) will be called the *wake* of the Misiurewicz point.

Of fundamental importance for the investigation of the Multibrot sets is the following Branch Theorem, which was first proved by Douady and Hubbard for the Mandelbrot set: this theorem is one of the principal results of their theory of “nervures” (Exposés XX–XXII in [DH1]; the Branch theorem is their Proposition XXII.3). Another proof can be found in [LS1, Theorem 9.1].

Theorem 2.2 (Branch Theorem)

For every two postcritically finite parameters c_1 and c_2 , either one of them is in the wake of the other, or there is a Misiurewicz point such that c_1 and c_2 are in two different of its subwakes, or there is a hyperbolic component such that c_1 and c_2 are in two different of its subwakes.

PROOF. We will work in the dynamical plane of c_1 , which we tried to sketch in Figure 2. Let α be one of the external angles of c_2 and let a_2 be the (pre-)periodic point at which the dynamic ray at angle α lands.

If the point c_1 is on the arc $[0, a_2]$, then two dynamic rays land at c_1 (if it is preperiodic) or at the dynamic root of the Fatou component containing c_1 (if it is periodic) and, by Theorem 2.1, the parameter rays at the same angles land at c_1 (resp. the root of the hyperbolic component containing c_1) and separate the parameter ray

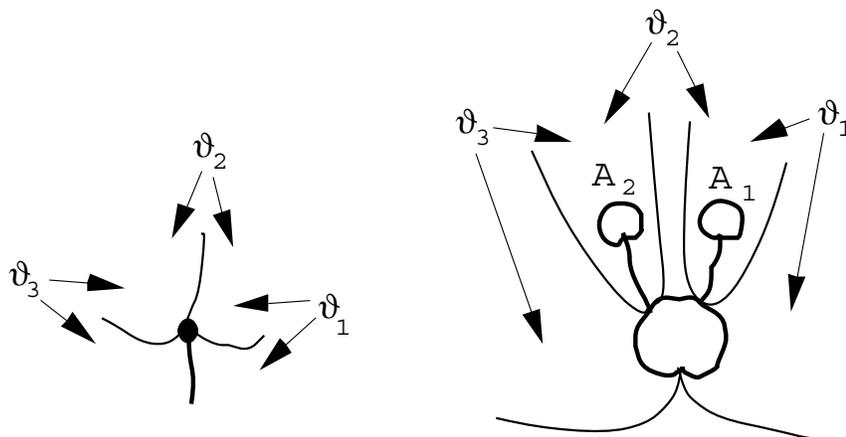


Figure 1: Illustration of the Branch Theorem. Left: a Misiurewicz point with three parameter rays landing, and the possible positions for the ϑ_i ; right: separation at a hyperbolic component.

at angle α and the point c_2 from the origin, so c_2 is in the wake of c_1 . Similarly, if a_2 is on the arc $[0, c_1]$, then c_1 is in the wake of c_2 . If none of these cases occurs, then the intersection of the arcs $[0, a_2]$ and $[0, c_1]$ is the arc $[0, b]$ for some point b which is a branch point in the Julia set. The union of forward images of the regular arcs $[0, c_1]$ and $[0, a_2]$ has the topological type of a finite tree; in particular, it has finitely many branch points. Since this tree is forward invariant, all these branch points are periodic or preperiodic (or critical, but then they have finite forward orbits by assumption); in particular, the forward orbit of b is finite.

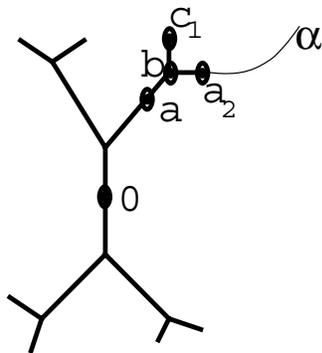


Figure 2: A sketch of the dynamic plane of the parameter c_1 as used in the proof of the Branch Theorem 2.2.

Now consider the set Z of characteristic (pre-)periodic points on $[0, b]$ and let a be the supremum of $Z - \{b\}$ on the arc $[0, b]$ with respect to the induced order on this arc. It is slightly easier to consider the case that c_1 is a Misiurewicz point, so every periodic and preperiodic point is on a repelling orbit. If c_1 is a center of a hyperbolic component, the point a cannot be on the superattracting periodic orbit because the Fatou component it is in separates the critical value from the origin, so a cannot be a characteristic point. Therefore, a is on a repelling orbit and thus the landing point

of at least two dynamic rays. For a similar reason, if the point b is characteristic, it cannot be on the superattracting orbit.

The case that a and b are two different points: if the arc $[a, b]$ contains periodic points on its interior, let z be one of lowest period; if there is no periodic point on the interior of this arc, let $z := b$. Then the point z is periodic or preperiodic; assuming for the moment that it not characteristic, then there is a first number s of iterations after which the point z maps behind itself. If z is periodic, then s must be smaller than the period. Under s iterations, the image of the arc $[a, z]$ contains the arc itself. By the intermediate value theorem, there is a point on $[a, z]$ which maps to itself after s iterations. This point cannot be z by construction, and neither can it be an interior point of the arc. It therefore follows that the point a is periodic, and its period is s or divides s . It is the landing point of at least two dynamic rays, and the two characteristic rays reappear as parameter rays landing at the root of a hyperbolic component by Theorem 2.1. Looking at external angles, it follows that the points c_1 and c_2 are contained in the wake of this component. If they are contained within the same subwake, the two parameter rays bounding this subwake would correspond to a periodic point in Z behind a , contradicting maximality of a .

If, however, z is characteristic, then $z = b$ and there is no periodic point on the arc $[a, b]$. The construction above goes through except if a and b are periodic and, after a finite number s of iterations, the arc $[a, b]$ covers itself homeomorphically. But then a and b cannot be both repelling as they must be.

The case that a and b are equal: in this case, at least three dynamic rays land at $a = b$, and these rays separate the points c_1 and a_2 from each other and from the origin. The point b is characteristic: otherwise, it would map behind itself after finitely many steps; by continuity, it could not be the limit of characteristic points in Z .

The point b is a limit of points in Z . If b is periodic, the dynamic rays landing at b would be permuted transitively by the first return map of b , and points in Z sufficiently close to b would be mapped behind b and thus behind themselves. This is a contradiction. Therefore, b is preperiodic, and Theorem 2.1 turns the three dynamic rays landing at b into three parameter rays landing at a common Misiurewicz point such that c_1 and c_2 are in two different of its subwakes because these subwakes contain the parameter rays associated to c_1 and c_2 .

This finishes the proof of the theorem. □

REMARK. In [S4], we show that the Mandelbrot set is “almost” arcwise connected. The Branch Theorem implies then that arbitrary “regular” arcs in the Mandelbrot set connecting postcritically finite parameters can branch off from each other only at Misiurewicz points or within hyperbolic components.

As a first corollary, we can describe how many rational parameter rays may land at the boundary of any interior component of a Multibrot set. All the known interior components of Multibrot sets are “hyperbolic” components in which the dynamics has an attracting periodic orbit. It is conjectured that non-hyperbolic (“queer”) components do not exist.

Corollary 2.3 (Interior Components of the Multibrot Sets)

Every hyperbolic component of a Mandelbrot or Multibrot set has infinitely many boundary points which are landing points of rational parameter rays, and all of them are accessible from inside the component. However, every non-hyperbolic component has at most one boundary point which is the landing point of a parameter ray at a rational angle

REMARK. In Section 7, we will strengthen the second statement by showing that no rational parameter ray can ever land on the boundary of a non-hyperbolic component (provided such a thing exists at all).

PROOF. The statement about hyperbolic components is well known at least in the quadratic case; see Douady and Hubbard [DH1], Milnor [M3, Section 7], or Schleicher [S1, Section 5]. For the general case, see [Eb].

Assume that two rational parameter rays land at different points on the boundary of a non-hyperbolic component. Denote their external angles by ϑ_1 and ϑ_2 and the landing points by c_1 and c_2 . For readers familiar with kneading sequences and their geometric interpretation as internal addresses [LS1], there is a brief argument: the external angles ϑ_i translate into kneading sequences and angled internal addresses. If the latter are different, then the landing points of the two parameter rays must be separated by a parameter ray pair at periodic angles (this argument, without using angles in the internal address, was used in Sections 3 and 4 of [S1] to establish the landing properties of parameter rays we are using here). However, if the two angled internal addresses coincide, then the two parameter rays land at a common point [LS1, Theorem 9.2].

Here we give a different proof, not involving internal addresses. If one of the ϑ_i is periodic, then replace the corresponding parabolic point c_i by the center of the component at the root or co-root of which the ray at angle ϑ_i lands. Since c_1 and c_2 are on the closure of the same non-hyperbolic component, there cannot be a Misiurewicz point or a hyperbolic component separating c_1 and c_2 from each other and from the origin. Therefore, by the Branch Theorem 2.2, one of these two points must be within the wake associated to the other point. Without loss of generality, assume that c_2 is within the wake of the Misiurewicz point c_1 or of the hyperbolic component with center c_1 . Take a third preperiodic angle ϑ_3 within the wake of c_1 but outside of the wake of c_2 , and so that its landing point c_3 is a Misiurewicz point different from c_1 and c_2 . If c_2 is within the wake of the Misiurewicz point c_3 , then two rays landing at this Misiurewicz point separate c_1 and c_2 , so these two points cannot be on the boundary of the same non-hyperbolic component. Otherwise the Branch Theorem supplies another Misiurewicz point or hyperbolic component separating c_2 and c_3 from each other and from the origin. If the angle ϑ_3 is chosen sufficiently closely to ϑ_2 , this separation point cannot be c_1 , so this new Misiurewicz point or hyperbolic component separates c_2 from c_1 , and again these two points cannot be on the boundary of a common non-hyperbolic component. \square

The most important application of the Branch Theorem is to relate local connectivity and density of hyperbolicity for Mandelbrot and Multibrot sets. We will do

that in Corollary 3.6, showing also that local connectivity and triviality of all fibers is equivalent.

3 Fibers of Multibrot Sets

We will now define fibers of Multibrot sets using parameter rays at rational angles; recall that all these rays are known to land. We first need to introduce separation lines.

Definition 3.1 (Separation Line)

A separation line is either a pair of parameter rays at rational angles landing at a common point (a ray pair), or a pair of parameter rays at rational angles landing at the boundary of the same interior component of the Multibrot set, together with a simple curve within this interior component which connects the landing points of the two rays. Two points c, c' in a Multibrot can be separated if there is a separation line γ avoiding c and c' such that c and c' are in different connected components of $\mathbb{C} - \gamma$.

Of course, the interior components used in this definition will always be hyperbolic components by Corollary 2.3.

It will turn out that points on the closure of a hyperbolic component of a Multibrot set have trivial fibers. For all other points, it will be good enough to construct fibers using ray pairs at rational external angles (in fact, periodic external angles suffice). We will justify this in Proposition 7.6.

Definition 3.2 (Fibers and Triviality)

The fiber of a point c in a Multibrot set \mathcal{M}_d is the set of all points $c' \in \mathcal{M}_d$ which cannot be separated from c . The fiber of c is trivial if it consists of the point c alone.

A fundamental construction in holomorphic dynamics is called the *puzzle*, introduced by Branner, Hubbard and Yoccoz. A typical proof of local connectivity consists in establishing *shrinking of puzzle pieces* around certain points. This is exactly the model for fibers: the fiber of a point is the collection of all points which will always be in the same puzzle piece, no matter how the puzzle was constructed. Our arguments will thus never use specific puzzles. The definition here is somewhat simpler than in [S3]. The reason is that the definition in [S3] is for arbitrary compact connected and full subsets of \mathbb{C} , while some conceivable difficulties cannot occur for Multibrot sets, due to the Branch Theorem 2.2 and its Corollary 2.3. (The equivalence of both definitions follows from this Corollary and [S3, Lemma 2.4].)

Proposition 3.3 (Fibers of Interior Components)

The fiber of a point within any hyperbolic component of a Multibrot set is always trivial. The fiber of a point within any non-hyperbolic component of a Multibrot set always contains the closure of its non-hyperbolic component.

PROOF. Let c be any interior point in a hyperbolic component. It can easily be separated from any other point within the same hyperbolic component by a separation

line consisting of two rational parameter rays landing on the boundary of this hyperbolic component, and a curve within this component; there are infinitely many such rays by Corollary 2.3.

Any boundary point of the hyperbolic component, and any point in \mathcal{M}_d outside of this component, can also be separated from c by such a separation line (here it is important to have more than two boundary points of the component which are landing points of rational parameter rays; if there were only two such boundary points, then these points could not be separated from any interior point).

Since any non-hyperbolic component has at most one boundary point which is the landing point of rational rays, no separation line can run through such a component, and the fibers of all such interior points contain at least the closure of this non-hyperbolic component. \square

Here are a few more useful properties of fibers. They are all taken from [S3, Lemma 2.4], using Corollary 2.3 to exclude certain bad possibilities. For convenience, we repeat most proofs in our context.

Lemma 3.4 (Properties of Fibers)

Fibers have the following properties:

1. *Fibers are always compact, connected and full.*
2. *The relation “is in the fiber of” is symmetric: for two points $c, c' \in \mathcal{M}_d$, either each of them is in the fiber of the other one, or both points can be separated.*
3. *The boundary of any fiber is contained in the boundary of \mathcal{M}_d , unless the fiber is trivial.*

PROOF. If c' is not in the fiber of c , then by definition these two points can be separated, and c is not in the fiber of c' . This proves the second claim.

No interior point of \mathcal{M}_d can be in the boundary of any non-trivial fiber: a hyperbolic interior point is a fiber by itself, and closures of non-hyperbolic components are completely contained in fibers. This settles the third claim.

Every fiber is compact because every separation line separates an open subset of \mathcal{M}_d . In the construction of the fiber of any point $c \in \mathcal{M}_d$, ray pairs always leave connected and full neighborhoods of c , and nested intersections of compact connected and full subsets of \mathbb{C} are compact connected and full. The only further possible separation lines run through hyperbolic components, but hyperbolic interior points have trivial fibers, and we only have to consider boundary points of hyperbolic components. These points can be treated easily; details can be found in [S3, Lemma 2.4]. (Equivalently, we will prove in Sections 4 and 5 that boundary points of hyperbolic components have trivial fibers.) \square

A key observation is that triviality of a fiber implies that \mathcal{M}_d is locally connected at this point. We repeat the argument from [S3, Proposition 2.9].

Proposition 3.5 (Trivial Fibers Yield Local Connectivity)

If the fiber of a point $c \in \mathcal{M}_d$ is trivial, then \mathcal{M}_d is openly locally connected at c . Moreover, if the parameter ray at some angle ϑ lands at a point c with trivial fiber, then for any sequence of external angles converging to ϑ , the corresponding impressions converge to $\{c\}$. In particular, if all the fibers of \mathcal{M}_d are trivial, then \mathcal{M}_d is locally connected, all external rays land, all impressions are points, and the landing points depend continuously on the angle.

PROOF. Consider a point $c \in \mathcal{M}_d$ with trivial fiber. If c is in the interior of \mathcal{M}_d , then K is trivially openly locally connected at c . Otherwise, let U be an open neighborhood of c . By assumption, any point c' in $\mathcal{M}_d - U$ can be separated from c such that the separation avoids c and c' . The region cut off from c is open; what is left is a neighborhood of c having connected intersection with \mathcal{M}_d . By compactness of $\mathcal{M}_d - U$, a finite number of such cuts suffices to remove every point outside U , leaving another neighborhood of c intersecting \mathcal{M}_d in a connected set. Removing the finitely many cut boundaries, an open neighborhood remains, and \mathcal{M}_d is openly locally connected at c . Similarly, if c is the landing point of the ϑ -ray, then external rays with angles sufficiently close to ϑ will have their entire impressions in \bar{U} (although the rays need not land). Finally, it is easy to see that the impression of any ray is contained in a single fiber (compare [S3, Lemma 2.5]). \square

Now, we show that the fiber of an interior point is trivial if and only if it is within a hyperbolic component, and that local connectivity of a Multibrot set is equivalent to all its fibers being trivial. Both properties will imply that every interior component is hyperbolic.

We use the ideas of the original proof of Douady and Hubbard [DH1, Exposé XXII.4].

Corollary 3.6 (Local Connectivity, Trivial Fibers and Hyperbolicity)

For the Mandelbrot and Multibrot sets, local connectivity is equivalent to triviality of all fibers, and both imply density of hyperbolicity.

REMARK. Local connectivity is definitely stronger than density of hyperbolicity: the former amounts to fibers being points, while the latter means only that fibers have no interior. This argument is taken from Douady [Do], where it is part of a sketch of an independent proof. In our context, it is just a restatement of the original proof in [DH1].

PROOF. We know from Proposition 3.5 that triviality of all fibers implies local connectivity. Moreover, any non-hyperbolic component would be contained in a single fiber, so triviality of all fibers implies that all interior components are hyperbolic. Since the exterior of \mathcal{M}_d is hyperbolic as well, triviality of all fibers of \mathcal{M}_d implies that hyperbolic dynamics is dense in the space of all unicritical polynomials of degree d .

It remains to show that local connectivity of a Multibrot set implies that all fibers are trivial. For this, we assume that the Multibrot set \mathcal{M}_d is locally connected. If there is a fiber which is not a singleton, or which even contains a non-hyperbolic component, denote it Y . The set Y is then uncountable and its boundary is contained in the boundary of \mathcal{M}_d by Lemma 3.4. Let c_1, c_2, c_3 be three boundary points which are

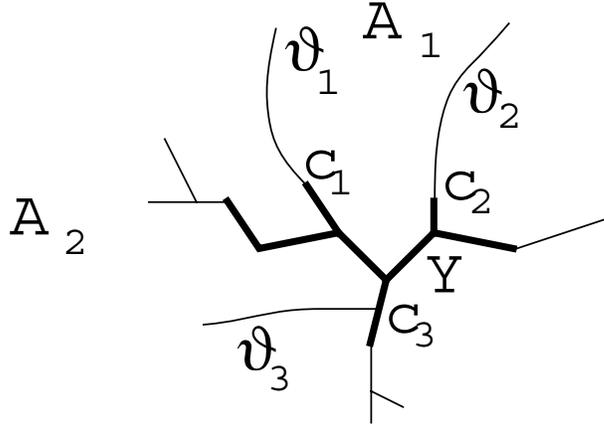


Figure 3: Illustration of the proof that local connectivity implies that all fibers are trivial. Highlighted is a non-trivial fiber Y , and three of its boundary points c_1, c_2, c_3 together with parameter rays landing there. The Branch Theorem, applied to the hyperbolic components \mathcal{A}_1 and \mathcal{A}_2 , provides then a contradiction.

not landing points of any of the countably many parameter rays at rational angles. By local connectivity and Carathéodory's Theorem, there are three parameter rays at angles $\vartheta_1, \vartheta_2, \vartheta_3$ landing at these points, and these three rays separate $\mathbb{C} - Y$ into three regions. Similarly, the three angles cut \mathbb{S}^1 into three open intervals. Pick one periodic angle from each of the two intervals which do not contain 0; the corresponding parameter rays land at roots or co-roots of two hyperbolic components \mathcal{A}_1 and \mathcal{A}_2 . The three parameter rays at angles ϑ_i , together with Y , separate these two components from each other and from the parameter ray at angle 0 and thus from the origin. Applying the Branch Theorem 2.2 to these two components, there must either be a Misiurewicz point or a hyperbolic component separating these two components from each other or from the origin. If it is a Misiurewicz point, then three rational rays landing at it must separate the three points c_1, c_2, c_3 from each other, which is incompatible with Y being a single fiber or a non-hyperbolic component. Similarly, if the separation is given by a hyperbolic component, then this component, together with the parameter rays forming its wake and the subwakes containing \mathcal{A}_1 and \mathcal{A}_2 , again separate the three c_i , yielding the same contradiction.

We conclude that, if a Multibrot set is locally connected, then its fibers are trivial, and every connected component of its interior is hyperbolic. \square

REMARK. In [S3, Proposition 2.10], we have shown that local connectivity of any compact connected and full subset of \mathbb{C} is equivalent to triviality of all fibers, provided that fibers are constructed using an appropriate collection of external rays. We have given an extra proof here in order to make this paper self-contained, and in order to show that the external rays at rational angles suffice. We will show in Section 7 that even parameter rays at periodic external angles are sufficient.

4 Boundaries of Hyperbolic Components

In this section, we will study boundary points of hyperbolic components, except roots of primitive components which will require special treatment (see Section 5). The following result was originally proved by Yoccoz using a bound on sizes of sublimbs (the “Yoccoz inequality”); see Hubbard [H, Section I.4]. That proof, as well as ours, uses in an essential way the fact that repelling periodic points in connected Julia sets are landing points of periodic dynamic rays. This fact is due to Douady or Yoccoz [H, Theorem I.A].

Theorem 4.1 (No Irrational Decorations)

Any point in \mathcal{M}_d within the wake of a hyperbolic component is either in the closure of the component or within one of its sublimbs at rational internal angles with denominator at least two.

REMARK. Sometimes, this theorem is phrased as saying that hyperbolic components have “no ghost limbs”: decorations are attached to hyperbolic components only at parabolic boundary points, but not at root, co-roots, or irrational boundary points.

PROOF. Let W be a hyperbolic component, let n be its period and let \tilde{c} be a point within the limb of the component but not on the closure of W . There are finitely many hyperbolic components of periods up to n , some of which might possibly be within the wake of W . We lose nothing if we assume \tilde{c} to be outside the closures of the wakes of such hyperbolic components, possibly after replacing \tilde{c} with a different parameter: if there was a hyperbolic component of period up to n in an “irrational sublimb” of W , then this same “irrational sublimb” is connected and thus contains points arbitrarily close to W . More precisely, we can argue as follows: the rational parameter rays landing on the boundary of W split into two groups according to whether they pass \tilde{c} to the “left” or “right”, and the region in the wake of W sandwiched between these rays is connected. (Moreover, it is quite easy to show that every hyperbolic component in the wake of W is contained in a subwake at rational internal angle; see [S2]).

Denote the center of W by c_0 . There is a path γ connecting c_0 to \tilde{c} within the wake of W and avoiding the closures of the wakes of all hyperbolic components of periods up to n within the wake of W , except W itself. This path need not be contained within the Multibrot set. For the parameter c_0 , the critical value is a periodic point. We want to continue this periodic point analytically along γ , obtaining an analytic function $z(c)$ such that the point $z(c)$ is periodic for the parameter c on γ . This analytic continuation is uniquely possible because we never encounter multipliers $+1$ for a period- n -orbit along γ . Therefore, we obtain a unique periodic point $z(\tilde{c})$ which is repelling. Since \tilde{c} is within the connectedness locus, the point $z(\tilde{c})$ is the landing point of finitely many dynamic rays at periodic angles.

At the parameter c_0 , all the periodic dynamic rays of periods up to n land at repelling periodic points of periods up to n . These periodic points can all be continued analytically along γ , they remain repelling and keep their dynamic rays because the curve avoided parameter rays of periods up to n which make up wake boundaries. Therefore, at the parameter \tilde{c} , there is no dynamic ray of period n available to land

at $z(\tilde{c})$, so that the rays landing at this point must have periods sn , for some $s \geq 2$. Therefore, at least s rays must land at $z(\tilde{c})$ (in fact, the number of rays must be exactly s). Now let \tilde{U} be the largest open neighborhood of \tilde{c} in which this periodic orbit can be continued analytically as a repelling periodic point, retaining all its s dynamic rays. This neighborhood is the *wake of the orbit* and is at the heart of Milnor's discussion in [M3]. This wake is bounded by two parameter rays at periodic angles, landing together at a parabolic parameter of ray period sn and orbit period at most n . Denote this landing point by \tilde{c}_0 . Obviously, \tilde{U} cannot contain the hyperbolic component W , so \tilde{U} must be contained within the wake of W .

The point \tilde{c}_0 is on the boundary of a hyperbolic component of period at most n within the wake of W . The wake of this component contains \tilde{U} and thus \tilde{c} , so this component can only be W by the assumptions on \tilde{c} . It follows that \tilde{U} is a sublimb of W at a rational internal angle. \square

REMARK. This result goes a long way towards proving local connectivity of the Multibrot sets at boundary points of hyperbolic components, as was pointed out to us by G. Levin.

Corollary 4.2 (Trivial Fibers at Hyperbolic Component Boundaries)

In every Multibrot set, the fiber of every boundary point of any hyperbolic component is trivial, except possibly at the root of a primitive component. The fiber of the root contains no point within the limb of the component.

PROOF. Any boundary point c of a hyperbolic component can obviously be separated from any point within the closure of the component and from every sublimb which it is not a root of. This shows that fibers are trivial for boundary points at irrational internal angles and at co-roots (which exist only for $d \geq 3$), because neither have sublimbs attached. If c is the root of a hyperbolic component W , then it can obviously be separated from any other point within the closure of W or from any rational sublimb, so its fiber contains no point within the limb of the component. If c is the point where the component W bifurcates from W_0 , then both arguments combine to show that the fiber of c is trivial. \square

The following corollary has been suggested by John Milnor. It will be strengthened in Corollary 5.4.

Corollary 4.3 (Roots Do Not Disconnect Limbs)

The limb of any hyperbolic component is connected. \square

REMARK. Note that we define the wake of a hyperbolic component to be open, so that the limb of the component (the intersection of the wake with the Multibrot set) is a relatively open subset of \mathcal{M}_d . Sometimes, wake and limb are defined so as to also contain the root. With that definition, the corollary says that the root does not disconnect the limb.

PROOF. Let W be the hyperbolic component defining the wake. Assume that its limb consists of more than one connected component, and let K be a connected component

not containing W . Any point within K must then, by Theorem 4.1, be contained within some rational sublimb of W , and then all of K must be contained within the same sublimb. But since the entire set \mathcal{M}_d is connected and the limb is obtained from \mathcal{M}_d by cutting along a pair of parameter rays landing at the root of the limb, it follows that the limb is connected as well. \square

Here is another corollary which has been found independently by Lavaurs [La, Proposition 1], Hubbard (unpublished) and Levin [Le, Theorem 7.3]; another proof is in Lau and Schleicher [LS1, Lemma 3].

Corollary 4.4 (Analytic Continuation Over Entire Wake)

If $c \in \mathcal{M}_d$ is a parameter which has an attracting periodic point z , then this periodic point can be continued analytically as an analytic map $z(c)$ over the entire wake of the hyperbolic component containing c , and it is repelling away from the closure of the component. Within any subwake at internal angle p/q , the point $z(c)$ is the landing point of exactly q dynamic rays with combinatorial rotation number p/q .

REMARK. For parameters within a hyperbolic component of period n , there are d distinguished periodic points: the point on the attracting orbit in the Fatou component containing the critical value, and $d - 1$ further boundary points of the same Fatou component with the same ray period (and possibly smaller orbit period). If $n > 1$, then exactly one of these boundary points is the landing point of at least two dynamic rays (this point is the “dynamic root” of the Fatou component), it is repelling and can be continued analytically over the entire wake of the component, retaining its dynamic rays (this is almost built into the definition of the wake). The remaining $d - 2$ distinguished boundary points of the Fatou component are landing points of one dynamic ray each (they are “dynamic co-roots”), and they also remain repelling and keep their rays throughout the wake. For the attracting orbit, this is not true: the rays landing at the orbit change, but the orbit nonetheless remains repelling away from the closure of the hyperbolic component.

PROOF. Denote the hyperbolic component by W , let c_0 be its root and let n be the period. The orbit $z(c)$ can be continued analytically over a neighborhood of $W - \{c_0\}$ within the wake of W , and it will be repelling there ([M3, Section 7], [S1, Section 5]). It will therefore remain repelling within any subwake of W at rational internal angles because that subwake is the wake where our repelling orbit has certain dynamic rays at specific angles landing. Within the wake of W , away from the closure of the component and outside of its rational subwakes, every Julia set is disconnected by Theorem 4.1 and every orbit is repelling. Therefore, $z(c)$ is repelling within the entire wake of W , except on the closure of the component, and can be continued analytically throughout the entire wake. The statement about the rays follows; compare for example [M3]. \square

REMARK. Traversing the wake of W outside of \mathcal{M}_d , the combinatorial rotation number of the orbit $z(c)$ behaves (locally) monotonically with respect to external angles, rotating around \mathbb{S}^1 $d - 1$ times. See also the appendices in [M3] and [GM].

5 Roots of Hyperbolic Components

In this section, we will prove triviality of fibers also at roots of primitive components. These roots are not handled by the arguments from the previous section, or by the Yoccoz inequality. We start with a combinatorial preparation from [S2].

Lemma 5.1 (Approximation of Ray Pairs, Primitive Periodic Case)

Let $\vartheta < \vartheta'$ be the two periodic external angles of the root of a primitive hyperbolic component. Then there exists a sequence of rational parameter ray pairs $(\vartheta_n, \vartheta'_n)$ such that $\vartheta_n \nearrow \vartheta$ and $\vartheta'_n \searrow \vartheta'$.

SKETCH OF PROOF. There are various combinatorial or topological variants of this proof. For example, it can be derived easily from the Orbit Separation Lemma (see [S1, Lemma 3.7] in the quadratic case and [Eb] in the general case). We will use Hubbard trees. They have been defined by Douady and Hubbard [DH1] for postcritically finite polynomials as the unique minimal trees within the filled-in Julia set connecting the critical orbit and traversing any bounded Fatou components only along internal rays.

Consider the dynamics at the center of the primitive hyperbolic component. The critical value is an endpoint of the Hubbard tree. Since the component is primitive, there is a non-empty open subarc γ of the Hubbard tree avoiding periodic Fatou components which ends at the Fatou component containing the critical value. We can suppose this arc short enough so that it meets no branch point of the Hubbard tree. Since the dynamics is expanding, the arc γ contains a point z which maps after some finite number n of steps onto the critical value. Choose n minimal and restrict γ to the arc between n and the periodic Fatou component containing the critical value. Then the n -th iterate of the dynamics map γ homeomorphically so that one end of γ lands at the critical value and the other end is a point on the critical orbit other than the critical value. Therefore, γ maps over itself in an orientation reversing way and must contain some point p which is fixed under the n -th iterate. This point must be a repelling periodic point, and it must be the landing point of two periodic dynamic rays at external angles $\vartheta_n < \vartheta'_n$. This construction can be done with arbitrarily short initial arcs γ , producing periodic points arbitrarily close to the periodic Fatou component containing the critical value. Alternatively, one can use the fact that one endpoint of γ is periodic (it is the landing point of the dynamic rays at angles ϑ and ϑ'), so that it is possible to pull γ back onto a subset of itself, creating a preperiodic inverse image of p closer to the critical value Fatou component. Both ways, it is easy to see that the external angles of the created periodic or preperiodic points have external angles converging to ϑ and ϑ' from below respectively from above.

Finally, Theorem 2.1 transfers these dynamic ray pairs into parameter space. \square

Lemma 5.2 (Fiber of Primitive Root)

Let c_0 be the root of any primitive hyperbolic component and let $\tilde{c} \neq c_0$ be some point in \mathcal{M}_d which is not in the limb of c_0 . Then there is a parameter ray pair at rational angles separating c_0 from \tilde{c} and from the origin. In particular, the fiber of the root of a primitive hyperbolic component contains no point outside of the wake of the component.

PROOF. The strategy of the proof is simple: if we denote the periodic parameter ray pair landing at c_0 by (ϑ, ϑ') , then Lemma 5.1 supplies a sequence of parameter ray pairs $(\vartheta_n, \vartheta'_n)$ converging to (ϑ, ϑ') , and in the dynamics of c_0 there are characteristic dynamic ray pairs at the same angles. If not all of these characteristic dynamic ray pairs exist for \tilde{c} , then \tilde{c} is separated from c_0 by one of the parameter ray pairs $(\vartheta_n, \vartheta'_n)$ (because these ray pairs bound the regions in parameter space for which the corresponding dynamic ray pairs exist). In this case, the two points c_0 and \tilde{c} can be separated as claimed and are thus in different fibers.

We may hence assume that all these dynamic ray pairs exist for the parameter \tilde{c} . We will then show that the limiting dynamic rays at angles ϑ and ϑ' also form a ray pair: this forces \tilde{c} to be in the closure of the wake of c_0 , contradicting our assumption. This finishes the proof of the lemma.

The only thing we need to prove, then, is the following claim: *assume that for the parameter \tilde{c} , the dynamic rays at angles ϑ_n and ϑ'_n land together for every n . Then the dynamic rays at angles ϑ and ϑ' also land together.*

To prove this claim, denote the period of ϑ and ϑ' by k . Since hyperbolic components and Misiurewicz points outside of the wake of c_0 are landing points of rational parameter rays, they can easily be separated from c_0 by the approximating parameter ray pairs, and we may thus suppose that \tilde{c} is neither a Misiurewicz point nor on the closure of a hyperbolic component.

We modify the sequence $(\vartheta_n, \vartheta'_n)$ as follows: let $(\vartheta_0, \vartheta'_0)$ be close enough to (ϑ, ϑ') so that, in the dynamics of c_0 , these two ray pairs do not enclose a postcritical point. Further, let the ray pairs $(\vartheta_n, \vartheta'_n)$ be the unique ray pairs in between satisfying $\vartheta - \vartheta_n = d^{-nk}(\vartheta - \vartheta_0)$ and $\vartheta'_n - \vartheta' = d^{-nk}(\vartheta'_0 - \vartheta')$. Denoting the regions bounded by (ϑ, ϑ') and $(\vartheta_n, \vartheta'_n)$ by U_n , then the pull-back for k steps of any U_n along the periodic backwards orbit of (ϑ, ϑ') will map univalently onto U_{n+1} , so the same branch of the pull-back fixing (ϑ, ϑ') will map $(\vartheta_n, \vartheta'_n)$ onto $(\vartheta_{n+1}, \vartheta'_{n+1})$.

All this works in the dynamics of c_0 . For the parameter \tilde{c} , denote the landing points of the two dynamic rays at angles ϑ and ϑ' by w and w' , respectively. Then the critical value cannot be separated from w or w' by any rational ray pair (α, α') : there is no such ray pair for c_0 , and the largest neighborhood of \tilde{c} where such a ray pair exists will be bounded by parameter rays at rational angles. Such a neighborhood exists because \tilde{c} is neither a Misiurewicz point nor on the closure of a hyperbolic component.

Similarly, for any $m > 0$, the m -th forward image of the critical value cannot be separated from the landing points of the dynamic rays at angles $2^m\vartheta$ and $2^m\vartheta'$, or we could pull back such a separating ray pair, obtaining a separation between the critical value and the dynamic rays at angles ϑ and ϑ' . It follows that all the $(\vartheta_n, \vartheta'_n)$ (for $n \geq 1$) are separated from the critical orbit by other such ray pairs with smaller or larger n .

Now let X be the complex plane minus the postcritical set for the parameter \tilde{c} ; then X is the open set of all points in \mathbb{C} which have a neighborhood that is never visited by the critical orbit. Let $p(z) = z^d + \tilde{c}$ and set $X' := p^{-1}(X)$; since the postcritical set is forward invariant, we have $X' \subset X$, and $p: X' \rightarrow X$ is an unbranched covering, i.e., a local hyperbolic isometry.

Let u, u' be two points on the dynamic rays at angles ϑ and ϑ' , respectively, and construct a simple piecewise analytic curve γ_0 as follows: connect u along an equipotential to the dynamic ray at angle ϑ_0 (the “short way”, decreasing angles), then continue along this ray towards its landing point, then out along the dynamic ray at angle ϑ'_0 up to the potential of u' , and connect finally along this equipotential to u' (increasing angles). This curve runs entirely within X , so it has finite hyperbolic length ℓ_0 , say.

Now we pull this curve back along the periodic backwards orbit of ϑ and ϑ' , yielding a sequence of curves $\gamma_1, \gamma_2, \dots$ after $k, 2k, \dots$ steps. The branch of the pull-back fixing ϑ will also fix ϑ' : we had verified that in the dynamics of c_0 , and this is no different for \tilde{c} because the critical value is always on the same side of all the ray pairs.

Denote the hyperbolic lengths of γ_n in X by ℓ_n . Since $X' \subset X$, this sequence is strictly monotonically decreasing. The two endpoints of the curves in the sequence obviously converge to the two landing points w and w' of the dynamic rays at angles ϑ and ϑ' . Now there are two possibilities: either the sequence (γ_n) has a subsequence which stays entirely within a compact subset of X , or it does not. If it does, then the hyperbolic length shrinks by a definite factor each time the curve is within the compact subset of X , so that we can connect points (Euclideanly) arbitrarily closely to w and w' by (hyperbolically) arbitrarily short curves at bounded (Euclidean) distances from the postcritical set, and this implies $w = w'$ as required. On the other hand, if these curves converge to the boundary, then their Euclidean lengths must shrink to zero, and we obtain the same conclusion. The proof of the lemma is complete. \square

Corollary 5.3 (Triviality of Fibers at Hyperbolic Components)

The fiber of every point on the closure of any hyperbolic component of a Multibrot set \mathcal{M}_d is trivial.

PROOF. In Corollary 4.2, we have already done most of the work; the remaining statement is exactly the content of the previous lemma. \square

The next corollary strengthens Corollary 4.3 and has also been suggested by Milnor.

Corollary 5.4 (Roots Disconnect Multibrot Sets)

Every root of a hyperbolic component of period greater than 1 disconnects its Multibrot set into exactly two connected components.

PROOF. Let c_0 be the root of a hyperbolic component of period greater than 1. It is the landing point of exactly two parameter rays at periodic angles. This parameter ray pair disconnects \mathbb{C} into exactly two connected components. The component not containing the origin is the *wake* of the component; its intersection with \mathcal{M}_d is the limb. The limb is connected by Corollary 4.3. Now let $\tilde{c} \in \mathcal{M}_d$ with $\tilde{c} \neq c_0$ be any parameter not in the limb of c_0 .

We first discuss the case that the component is primitive. By Lemma 5.2 above, there is a rational parameter ray pair S separating c_0 from \tilde{c} and from the origin. This ray pair S disconnects \mathbb{C} into two connected components. Let \mathcal{M}' be the closure of the connected component of the origin intersected with \mathcal{M}_d . Then \mathcal{M}' itself is connected and

contains the origin and \tilde{c} . It follows that \tilde{c} and 0 are in the same connected component of $\mathcal{M}_d - \{c_0\}$, so c_0 cuts \mathcal{M}_d into exactly two connected components.

In the non-primitive case, the parameter c_0 is the root of one hyperbolic component and on the boundary of another one, say W_0 . Then by Theorem 4.1, there are three cases: the point c_0 may be outside of the limb of W_0 so that the two parameter rays landing at the root of W_0 separate c_0 and \tilde{c} ; the point c_0 may be on the closure of W_0 ; or it may be in a sublimb of W_0 at rational internal angles, but not in the sublimb with c_0 on its boundary (which is the wake of c_0). In all three cases, it is easy to see that \tilde{c} must be in the same connected component of $\mathcal{M}_d - \{c_0\}$ as the origin.

Therefore, c_0 disconnects \mathcal{M}_d into exactly two connected components in the primitive as well as in the non-primitive case. □

Corollary 5.5 (Rays at Boundary of Hyperbolic Components)

Every boundary point of a hyperbolic component at irrational internal angle is the landing point of exactly one parameter ray, and the external angle of this ray is irrational (and in fact transcendental).

Every boundary point at rational internal angle is the landing point of exactly two parameter rays, except co-roots: these are the landing points of exactly one parameter ray. The external angles of all these rays are periodic and in particular rational.

In no case is a boundary point of a hyperbolic component in the impression of any further parameter ray.

PROOF. Every boundary point of a hyperbolic component at irrational internal angle is in the boundary of \mathcal{M}_d and thus in the impression of some ray. Since the fiber of this boundary point is trivial, the ray must land there and its impression is a single point [S3, Lemma 2.5]. And since all the parameter rays at rational rays land elsewhere, the external angle of the ray must be irrational (and in fact transcendental; see [BS] for a proof in the quadratic case). If this point is in the impression of a further ray, this ray must land there, too, and these two rays separate some open subset of \mathbb{C} from the component. Since not all rays between the two landing rays can land at the same point (the rational rays land elsewhere), there must be some part of \mathcal{M}_d between these two rays, and this contradicts Theorem 4.1.

We know that boundary points at rational internal angles are parabolic points, and the statements about the landing properties of rational parameter rays are well known (compare Section 2). If such a point is in the impression of an irrational parameter ray and thus its landing point, we get a similar contradiction as above, except if the parabolic point is the root of a primitive component and the extra parameter ray is outside of the wake of the component. But that case is handled conveniently by Lemma 5.1, supplying lots of parameter ray pairs which separate any parameter ray outside of the wake of the component from the component, its root, and all of its co-roots. □

6 Misiurewicz Points

Now we turn to Misiurewicz points, proving triviality of fibers in a somewhat similar way. Again, we need some combinatorial preparations from [S2]. They are preperiodic analogs to Lemma 5.1.

Lemma 6.1 (Preperiodic Critical Orbits Have Trivial Fibers)

Let c_0 be a Misiurewicz point and let z be the any periodic point on the forward orbit of the critical value. Then there are finitely many rational ray pairs separating z from the entire critical orbit except $\{z\}$. The separating ray pairs can be chosen so that their landing point is not on the grand orbit of the critical point.

SKETCH OF PROOF. Like Lemma 5.1, this result has various proofs. For example, it is a consequence of [S3, Theorem 3.5]. Another variant uses the fact that the Julia set is locally connected, so its fibers are trivial when constructed using rational ray pairs [S3, Proposition 3.6].

Our proof uses similar ideas as in Lemma 5.1. Again, we consider the Hubbard tree of the Julia set (the unique minimal tree connecting the critical orbit within the Julia set). It suffices to prove the result for any point on the periodic orbit of z .

Let γ be a short arc in the Hubbard tree starting at z towards the critical value c . By expansion, there must be a point z_n on γ which maps to c after some finite number n of iterations, and we may suppose n minimal. Restrict γ to the arc between z and z_n . The n -th iterate will then map γ forward homeomorphically such that a subarc of γ maps over γ . If n is not a multiple of the period of z , then the n -th iterate does not fix any end of γ , and there must be an interior point p of γ which is fixed under this iterate. If n is a multiple of the period of z , the n -th iterate of γ connects z to c homeomorphically. We keep mapping this arc forward until it crosses the critical point for the first time. The next iterate will then provide a periodic point p unless the point z is back again. If this happens all the time, then the itineraries of z and c must coincide, but this is not the case: the first is periodic, the second preperiodic.

Once we have periodic points close to z , we have them at all branches of z because local branches at periodic points are permuted transitively, except possibly if there are exactly two branches which are fixed by the first return map of z (compare [M3, Lemma 2.4] or [S1, Lemma 2.4]). In that case, we have periodic points close to any point on the periodic orbit of z at least from the side towards the critical point, and it is not hard to verify that the dynamics cannot always preserve these sides. Therefore, mapping forward along the periodic orbit, we get periodic points arbitrarily close to z from all sides. If the approximating periodic points are chosen close enough to z , they cannot be from the periodic orbit of z . \square

Lemma 6.2 (Approximation of Ray Pairs, Preperiodic Case)

Consider a Misiurewicz point and let $\{\vartheta_i\}$ be the finite set of its preperiodic external angles (i.e., the external angles of the critical value in its dynamic plane). Then, for every $\varepsilon > 0$, there are finitely many rational parameter ray pairs separating the Misiurewicz point from every parameter ray whose external angle differs at least by ε from

all of the ϑ_i : one can take one such rational ray pair for every connected component of $\mathbb{S}^1 - \{\vartheta_i\}$. All these ray pairs can be chosen to be periodic.

SKETCH OF PROOF. From the previous lemma, we can conclude that any periodic point z on the critical orbit can be separated from the remaining critical orbit by periodic dynamic ray pairs. These ray pairs can be pulled back along the repelling periodic orbit of z , at least if they were close enough to z , and we get dynamic ray pairs arbitrarily close to the rays landing at the critical value. We need to transfer them into parameter space. This is easily done using Theorem 2.1 for the rays separating the critical value from the critical point. For the others, it requires a more precise description of the combinatorics of Multibrot sets; see [S2]. The idea is to use Theorem 2.1 not in the dynamic plane of the Misiurewicz point, but of hyperbolic components in the various wakes of the Misiurewicz point. A sketch of the quadratic case is in Douady [Do]. \square

REMARK. The statement essentially says that every external ray which is not landing at a given Misiurewicz point can be separated from this point by a rational ray pair. The approximating ray pairs can be required at will to land at Misiurewicz points, or at roots of primitive hyperbolic components, or at roots of non-primitive hyperbolic components (in the latter case, such a non-primitive hyperbolic component will usually be a bifurcation by a factor two). See [S2].

The following obvious corollary is an analogue to Corollary 5.5.

Corollary 6.3 (No Irrational Rays at Misiurewicz Points)

No Misiurewicz point is in the impression of a parameter ray at an irrational external angle. \square

The following theorem is well known for the Mandelbrot set. As far as I know, it was first proved by Yoccoz using his puzzle techniques. A variant of this proof for most cases is indicated in Hubbard's paper [H, Theorem 14.2]. Tan Lei has published another proof in [TL].

Theorem 6.4 (Misiurewicz Points Have Trivial Fibers)

The fiber of any Misiurewicz point in any Multibrot set is trivial.

PROOF. Let c_0 be a Misiurewicz point and denote its various subwakes by U_0, U_1, \dots, U_s , for some integer $s \geq 0$, such that U_0 contains the origin. Let $\vartheta_0, \dots, \vartheta_s$ be the external angles of the Misiurewicz point. We have to show that, for every $\tilde{c} \in U_i \cap \mathcal{M}_d$, there is a rational ray pair separating \tilde{c} from c_0 .

If \tilde{c} is on the closure of a hyperbolic component, then one of the approximating ray pairs will separate the entire hyperbolic component from c_0 , and we are done. Similarly, we are done if \tilde{c} is a Misiurewicz point. If \tilde{c} is in some non-hyperbolic component (if that ever occurs), then we may replace \tilde{c} by some other point (different from c_0) on the boundary of the same non-hyperbolic component and thus on the boundary of the Multibrot set: a ray pair separating a boundary point of a non-hyperbolic component from c_0 will separate the entire component from c_0 . We may therefore suppose that \tilde{c}

is a boundary point of the Multibrot set, that all periodic orbits are repelling, and that the critical orbit is infinite.

In the dynamic plane of c_0 , let z be the first periodic point on the orbit of the critical value. In a parameter neighborhood of c_0 , this periodic point can be continued analytically as a periodic point $z(c)$. We know from Lemma 6.1 that there are finitely many rational ray pairs separating $z(c_0)$ from its entire forward orbit, such that the neighborhood of $z(c_0)$ cut out by these rays together with any equipotential, when pulled back along the periodic backwards orbit of $z(c_0)$, shrinks to the point $z(c_0)$ alone. The same will then be true for the point $z(c)$ for parameters c from a sufficiently small neighborhood V of c_0 because the finitely many bounding ray pairs will persist under perturbations, and the critical value will not be contained in the regions to be pulled back. The point $z(c)$ can thus be separated from any other point in the Julia set by a rational ray pair (in other words, the fiber of $z(c)$ is hence trivial when fibers of the Julia set are constructed using rational external rays. Another way to arrive at this same conclusion is to use [S3, Theorem 3.5].)

Let $z'(c)$ be the analytic continuation of the repelling preperiodic point which equals the critical value for the parameter c_0 . This analytic continuation is possible sufficiently closely to c_0 ; we may assume it to be possible throughout V , possibly by shrinking V . Since the fiber of c_0 is connected, we may suppose that $\tilde{c} \in V$, subject to the same restrictions as above, so that $z(\tilde{c})$ and thus $z'(\tilde{c})$ have trivial fibers.

Since $z'(\tilde{c})$ is different from the critical value \tilde{c} , there is a rational ray pair $(\tilde{\vartheta}, \tilde{\vartheta}')$ separating the critical value from $z'(\tilde{c})$ and from all the dynamic rays at angles ϑ_i , which will land at $z'(\tilde{c})$. The angles of this separating ray pair will differ by some $\varepsilon > 0$ from all the ϑ_i .

This separating dynamic ray pair will persist under sufficiently small perturbations of \tilde{c} . Since we had assumed $\tilde{c} \in \partial\mathcal{M}_d$, such a perturbation is possible into the exterior of \mathcal{M}_d . The external angle of the perturbed parameter must then differ by at least ε from all the ϑ_i , no matter how small the perturbation. Now we use Lemma 6.2 to get finitely many rational parameter ray pairs separating c_0 from all the parameter rays the angles of which differ from all the ϑ_i by at least $\varepsilon/2$. This separation must then also separate \tilde{c} from c_0 , so these two parameters are in different fibers. It follows that the fiber of c_0 is trivial. \square

Corollary 6.5 (Misiurewicz Points Disconnect)

Every Misiurewicz point disconnects its Multibrot set in exactly as many connected components as there are rational parameter rays landing at it.

PROOF. All the parameter rays landing at a Misiurewicz point have preperiodic external angles by Corollary 6.3, and they obviously disconnect the Multibrot set in at least as many connected components as there are such rays.

By Lemma 6.2, between any two adjacent external rays of a Misiurewicz point there is a collection of rational parameter ray pairs exhausting the interval of external angles in between, so any extra connected component at the Misiurewicz point cannot have any external angles. It must therefore be contained within the fiber of the Misiurewicz point, but this fiber is the Misiurewicz point alone. \square

7 Fibers and Combinatorics

Now that we have trivial fibers at all the landing points of rational rays, it follows that fibers of any two points of \mathcal{M}_d are either equal or disjoint. This is Lemma 2.7 in [S3]; we repeat the proof in order to make this paper self-contained.

Theorem 7.1 (Fibers of \mathcal{M}_d are Equivalence Classes)

The fibers of any two points in \mathcal{M}_d are either equal or disjoint.

PROOF. The relation “ c_1 is in the fiber of c_2 ” is always reflexive, and it is symmetric for \mathcal{M}_d by Lemma 3.4. In order to show transitivity, assume that two points c_1 and c_2 are both in the fiber of c_0 . If they are not in the fibers of each other, then the two points can be separated by a separation line avoiding c_1 and c_2 . If such a separation line can avoid c_0 , then these two points cannot both be in the fiber of c_0 . The only separation between c_1 and c_2 therefore runs through the point c_0 , so c_0 cannot be in the interior of \mathcal{M}_d and rational rays land at c_0 . Therefore, the fiber of c_0 consists of c_0 alone. Any two points with intersecting fibers thus have indeed equal fibers. \square

The theorem allows to simply speak of *fibers* of \mathcal{M}_d as equivalence classes of points with coinciding fibers, as opposed to “fibers of c ” for $c \in \mathcal{M}_d$. Another consequence shown in Lemma 2.7 in [S3] is that there is an obvious map from external angles to fibers of K via impressions of external rays. This map is surjective onto the set of fibers meeting ∂K .

The following corollary is obvious and just stated for easier reference. In Corollary 2.3, we had only been able to show that at most one parameter ray at a rational angle can land at a non-hyperbolic component.

Corollary 7.2 (Non-Hyperbolic Components Rationally Invisible)

No rational parameter ray lands on the boundary of a non-hyperbolic component. \square

Combinatorial building blocks of the Multibrot sets which are often discussed are combinatorial classes. We will show that they are closely related to fibers.

Definition 7.3 (Combinatorial Classes and Equivalence)

We say that two connected Julia sets are combinatorially equivalent if in both dynamic planes external rays at the same rational angles land at common points. Equivalence classes under this relation are called combinatorial classes.

REMARK. In the language of Thurston [T], combinatorially equivalent Julia sets are those having the same rational lamination. The definition is such that topologically conjugate (monic) Julia sets are also combinatorially equivalent (for an appropriate choice of one of the $d - 1$ fixed rays to have external angle 0). In particular, all the Julia sets within any hyperbolic component, at its root and at its co-roots are combinatorially equivalent.

With the given definition, the combinatorial class of a hyperbolic component also includes its boundary points at irrational angles, although the dynamics will be drastically different there. Therefore, one might want to refine the definition of combinatorial equivalence accordingly.

REMARK. There are homeomorphic Julia sets which are not topologically conjugate and which are thus not combinatorially equivalent: as an example, it is not hard to see that the Julia set of $z^2 - 1$ (known as the “Basilica”) is homeomorphic to any locally connected quadratic Julia set with a Siegel disk of period one. However, this homeomorphism is obviously not compatible with the dynamics, and the Basilica is in a different combinatorial class than any Siegel disk Julia set.

Proposition 7.4 (Combinatorial Classes and Fibers)

Hyperbolic components together with their roots, co-roots and irrational boundary points form combinatorial classes. All other combinatorial classes are exactly fibers. In particular, if there is any non-hyperbolic component, then its closure is contained in a single combinatorial class and a single fiber.

PROOF. The landing pattern of dynamic rays changes upon entering the wake of a hyperbolic component, so any parameter ray pair at periodic angles separates combinatorial classes. Similarly, the landing pattern is different within all the subwakes of any Misiurewicz point: in a neighborhood of the Misiurewicz point, the rays landing at the critical value will have the same angles, but the d preperiodic inverse images will carry different angles according to which subwake of the Misiurewicz point the parameter is in. Hence parameter ray pairs at preperiodic angles also separate combinatorial classes, and combinatorial classes are just what fibers would be if separations were allowed only by rational ray pairs, excluding separation lines containing curves through interior components. Since fibers have more separation lines, they are contained in combinatorial classes. The closure of any non-hyperbolic component is contained in a single fiber (Corollary 2.3), so it is also contained within a single combinatorial class.

We know that the rational landing pattern is constant throughout hyperbolic components and at its root and co-roots, as well as at its irrational boundary points. Hyperbolic components with these specified boundary points are therefore in single combinatorial classes; since every further point in the Multibrot set is either in a rational subwake of the component or outside the wake of the component, the combinatorial class of any hyperbolic component is exactly as described.

Finally, we want to show that every non-hyperbolic combinatorial class is a single fiber. If this was not so, then two points within the combinatorial class could be separated by a separation line. Unless these points are both on the closure of the same hyperbolic component, such a separation line can always be chosen as a ray pair at rational angles, and we have seen above that such ray pairs bound combinatorial classes. □

The following corollary is closely related to the Branch Theorem 2.2.

Corollary 7.5 (Three Rays at One Fiber)

If three parameter rays of a Multibrot set accumulate at the same fiber, then the three rays are preperiodic and land at a common Misiurewicz point. If three parameter rays accumulate at a common combinatorial class, then this combinatorial class is either a Misiurewicz point or a hyperbolic component, and the three rays land.

PROOF. We will argue similarly as in Corollary 3.6. Denote the three external angles by $\vartheta_1, \vartheta_2, \vartheta_3$ in increasing order and let Y be their common fiber. These angles separate \mathbb{S}^1 into three open intervals. Two of these intervals do not contain the angle 0. Choose a preperiodic rational angle from both of them. The corresponding parameter rays land at two Misiurewicz points. Applying the Branch Theorem 2.2 to these two points, we find either that one of these two points separates the other from the origin, or there is a Misiurewicz point or hyperbolic component which separates both from each other and from the origin. In all cases, it is impossible to connect the three parameter rays at angles $\vartheta_{1,2,3}$ to a single fiber, unless these three rays land at the separating Misiurewicz point or hyperbolic component. In the Misiurewicz case, all three angles must be preperiodic by Corollary 6.3, and in the hyperbolic case the three parameter rays must still land at a common point because of fiber triviality on closures of hyperbolic components. This is impossible.

If, however, the three rays are only required to land at a common combinatorial class, then this combinatorial class may either be a Misiurewicz point or a hyperbolic component, and both cases obviously occur. \square

We have defined fibers of a Multibrot set \mathcal{M}_d using parameter rays at periodic and preperiodic angles (where the dynamics as usual multiplication by the degree d). However, it turns out that only periodic angles are necessary.

Proposition 7.6 (Fibers Using Periodic Parameter Rays)

Fibers for a Multibrot set \mathcal{M}_d remain unchanged when they are defined using only parameter rays at periodic angles, rather than at all rational angles.

PROOF. Preperiodic parameter rays never land on the boundary of an interior component of a Multibrot set, so separation lines using preperiodic rays are always ray pairs. We only have to show that if any two parameters can be separated by a preperiodic ray pair, then there is a periodic ray pair separating them as well.

First we show that the fiber of any Misiurewicz point is trivial even when only periodic parameter rays are used. Indeed, let c_0 be a Misiurewicz point and let $c \in \mathcal{M}_d - \{c_0\}$ be a different parameter. By triviality of the fiber of c_0 , there is a separation line between c_0 and c . If this separation line is a preperiodic ray pair, then there is a periodic ray pair separating c_0 from the preperiodic ray pair (see Lemma 6.2) and thus also from c . Therefore, c_0 can be separated from any parameter in $\mathcal{M}_d - \{c_0\}$ even if only parameter rays at periodic angles are allowed in the construction of fibers.

Finally, let c_1 and c_2 be two parameters of \mathcal{M}_d in different fibers. If they are separated by a parameter ray pair at preperiodic angles, let c_0 be the Misiurewicz point at which

this ray pair lands. Then there is a periodic ray pair separating c_0 and c_1 . Since the rays landing at c_0 separate c_1 and c_2 , this periodic ray pair must also separate c_1 and c_2 and we are done. \square

It follows that combinatorial classes are exactly the pieces that we can split Multibrot sets into when using only periodic ray pairs as separation lines (except that we have to declare what we want combinatorial classes to be on the boundary of hyperbolic components). This is exactly the partition used to define internal addresses, which was introduced in [LS1] mostly for postcritically finite parameters. Now we see that combinatorial classes are the objects where internal addresses live naturally. The following is a restatement of [LS1, Theorem 9.2] and the remark thereafter.

Corollary 7.7 (Internal Addresses Label Combinatorial Classes)

Angled internal addresses label combinatorial classes completely: points within different combinatorial classes have different angled internal addresses, and points within the same combinatorial class have the same angled internal address. \square

In order to label fibers, we only have to distinguish points within hyperbolic components. This is most naturally done by the multiplier of the unique non-repelling cycle. For degrees $d > 2$, every hyperbolic component contains $d - 1$ points with equal non-repelling cycles. They are distinguished by their “sectors” within the component, and these sectors are reflected in both the kneading sequences and the angled internal addresses with sectors [LS1, Section 10].

Theorem 7.8 (The Pinched Disk Model of Multibrot Sets)

The quotient of any Multibrot set in which every fiber is collapsed to a point is a compact connected locally connected metric Hausdorff space. In particular, the quotient is pathwise connected.

PROOF. Every fiber of \mathcal{M}_d is closed. In fact, the entire equivalence relation is closed: suppose that (z_n) and (z'_n) are two converging sequences in \mathcal{M}_d such that z_n and z'_n are in a common fiber for every n . The limit points must then also be in a common fiber: if they are not, then they can be separated by either a ray pair at periodic angles or by a separation line through a hyperbolic component, and the separation runs in both cases only through points with trivial fibers. In order to converge to limit points on different sides of this separation line, all but finitely many points of the two sequences must be on the respective sides of the separation line, and z_n and z'_n cannot be in the same fiber for large n .

It follows that the quotient space is a Hausdorff space with respect to the quotient topology. It is obviously compact and connected, and local connectivity is easy: the same proof applies which shows that trivial fibers imply local connectivity (see Proposition 3.5): it makes sense to speak of fibers of the quotient space, and they are all trivial.

The quotient inherits a natural metric from \mathbb{C} : the distance between any two points is the distance between the corresponding fibers. Since fibers are closed, different points always have positive distances. Symmetry and the triangle inequality are inherited

from \mathbb{C} . As a compact connected locally connected metric space, the quotient space is pathwise connected (see Milnor [M1, §16]). \square

This quotient space is called the “pinched disk model” of the Multibrot set; compare Douady [Do]. It comes with an obvious continuous projection map π from the actual Multibrot set, and inverse images of points under π are exactly fibers of \mathcal{M}_d . The name “pinched disk model” comes from Thurston’s lamination model [T]: start with the closed unit disk $\overline{\mathbb{D}}$; for any periodic ray pair at angles ϑ and ϑ' , connect the corresponding boundary points of $\overline{\mathbb{D}}$, for example by a geodesic with respect to the hyperbolic metric in \mathbb{D} (or simply by a Euclidean straight line). This connecting line is called a “leaf”. Whenever two boundary points ϑ, ϑ' are limits of two sequences $(\vartheta_n), (\vartheta'_n)$ such that $(\vartheta_n, \vartheta'_n)$ is a leaf for every n , then the boundary points ϑ and ϑ' in $\overline{\mathbb{D}}$ have to be connected as well to form a leaf. Now we take the quotient of $\overline{\mathbb{D}}$ in which each leaf is collapsed to a point: the disk $\overline{\mathbb{D}}$ is “pinched” along this leaf. The quotient space is homeomorphic to our space constructed above.

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