Holomorphic Removability of Julia Sets

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Abstract

Let $f(z) = z^2 + c$ be a quadratic polynomial, with c in the Mandelbrot set M. Assume further that both fixed points of f are repelling, and that f is not renormalizable. Then we prove that the Julia set J_f of f is holomorphically removable in the sense that every homeomorphism of the complex plane to itself that is conformal off of J_f is in fact conformal on the entire complex plane. As a corollary, we deduce that M is locally connected at such c.

Contents

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Chapter 1

Statement of Main Theorem and Breakdown of Proof

1.1 Introduction

Let $f(z) = z^2 + c$ be a quadratic polynomial, with $c \in M$ (where M is the Mandelbrot set, defined in section 1.2). We consider two possible additional hypotheses on f:

1. Both of the fixed points of f are repelling, and f is not renormalizable;

2. All of the periodic cycles of f are repelling, and f is not infinitely renormalizable.

Under either of the two above hypotheses, there are the following theorems:

Theorem 1.1.1 (Yoccoz) *J_f is locally connected.*

Theorem 1.1.2 (Yoccoz) *M is locally connected at c.*

Theorem 1.1.3 (Lyubich; Shishikura) *J_f has measure 0.*

See [Yoc], [Lyu2]. See also Milnor [Mil2] and Hubbard [Hub] for expositions of Theorem 1.1.1 (and also Theorem 1.1.2 in the latter reference).

Definition 1.1.4 *We say that a compact subset J of $\mathbb{C}$ is* holomorphically removable (HR) *in an open neighborhood U of J if, for every topological embedding $h : U \to \mathbb{C}$, if $h|_{U-J}$ is conformal, then in fact $h|_U$ is conformal.*

Fact 1.1.5 *For each $K \geq 1$, $J \subset U$ is holomorphically removable if and only if J is removable for K-quasiconformal mappings, that is, for every topological embedding $h : U \to \mathbb{C}$, if $h|_{U-J}$ is K-quasiconformal, then in fact $h|_U$ is K-quasiconformal.*

For a proof, see section V.3 of [LV], where the conditions given here on h are just those to put it in Lehto and Virtanen's class $\mathcal{W}_2$ of functions.

Clearly, if $J \subset U \subset V$, and J is holomorphically removable in U, then it is holomorphically removable in V. Using Fact 1.1.5 above, it is easy to show that the converse is true, that J is HR in U if it is HR in V (assuming of course that J is compact). Thus we can suppress mention of the neighborhood and just assume $U = \mathbb{C}$.

The simplest example of a holomorphically removable set is a point. The next simplest is a piecewise smooth curve.

The purpose of this work is to prove the following theorem (with the same hypotheses):

Theorem 1.1.6 (Main Theorem) *J_f is holomorphically removable.*

In section 4.1 we give use Theorem 1.1.6 to give a quick proof of Theorem 1.1.2. **Throughout the first three chapters will we always assume the first hypothesis on f.** The proof with the weaker second hypothesis will be discussed in section 4.2. We mention that the critically non-recurrent cases (see chapter 3 for a definition) also follow from the work of Jones [Jon, CJY]. Speculations on further holomorphic removability results are discussed in section 4.3.

1.1.1 Acknowledgements

I would like to thank Curtis McMullen, Mikhail Lyubich, and Mitsuhira Shishikura for helpful conversations.

1.2 Yoccoz Partition

The main tool for proving all the above theorems is the Yoccoz partition, which we now describe [Mil2, Hub]. Let us first recall some basic theory and terminology for the dynamics of quadratic polynomials. Given a quadratic polynomial $f(z) = f_c(z) = z^2+c$, let $K(f) = \{z \mid f^n(z) \not\to \infty\}$. Then $J_c = J(f) = \partial K(f)$, and $K(f)$ is connected if and only if $0 \in K(f)$. Under hypothesis 1 (or 2) on f, $K(f) = J(f)$. Then the Mandelbrot set M is defined by

$$M = \{c \mid 0 \in K(f_c)\}.$$

If $K(f)$ is connected, then there exists a unique conformal isomorphism $\phi : \mathbb{C} - \overline{\Delta} \to \mathbb{C} - K(f)$ for which $\phi(z^2) = (\phi(z))^2 + c$. An *external ray* $R(\theta)$ is then defined by

$$R(\theta) = \phi(\{re^{2\pi i\theta} \mid 1 < r < \infty\}).$$

The map f acts as angle doubling modulo 1 on the external rays $R(\theta)$: $f(R(\theta)) = R(2\theta)$. We say that $R(\theta)$ *lands* at $z \in J_c$ if $\lim_{r\to 1} \phi(re^{2\pi i\theta}) = z$. We first recall [Mil1, Hub] two basic results about the landing of external rays:

Proposition 1.2.1 *If θ is periodic under doubling modulo 1, then $R(\theta)$ lands at a parabolic or repelling periodic cycle. Conversely, if $z \in J$ is a repelling (or parabolic) periodic point, then at least one periodic ray lands at it. In the case where z is a fixed point, then set of rays landing at z are cyclically permuted by the f.*

We are assuming both fixed points of f are repelling. The zero external ray lands at one of them, called the β fixed point, or just β. The other fixed point is called α. At least two rays land at α (because the only cycle of length 1 that is periodic under doubling is $\{0\}$, which lands at β). Form the connected 1-complex Γ_0 consisting of α, the portion of the rays landing at α with potential less than 1 (the *potential* of a point $z \in \mathbb{C} - K_f$ is defined as $\log|\phi^{-1}(z)|$), and the equipotential curve of potential 1. For $n \in \mathbb{Z}^+$, let $\Gamma_n = f^{-n}(\Gamma_0)$. A *piece* of level n is a bounded component of $\mathbb{C} - \Gamma_n$. Each piece is a Jordan domain. If $n < m$, then f^{m-n} maps every piece of level m to a piece of level n.

In Yoccoz's work [Mil2, Hub], Theorem 1.1.1 is proven by showing the following, which will be used in Chapter 3:

Theorem 1.2.2 *The diameter of all pieces of level n goes uniformly to zero as $n \to \infty$.*

Proof of 1.1.1: To show that J is locally connected at a given point $z \in J$ (with $f^n(z) \neq \alpha$ for all n), consider the pieces of all levels that contain that point. They are connected and open, and 1.2.2 above tells us that that they form a neighborhood base for z. (In the case where $f^n(z) = 0$ (so $z \in \Gamma_n$) for some n, the interior of the union of closures of pieces of level n that border on z is a connected open neighborhood of z, with diameter going to zero as $n \to \infty$). $\boxed{1.1.1}$

The theory used to prove 1.2.2 will be discussed in Chapter 3, where it will be used to show further results.

1.3 Quasiconformal Distortion Bounds

Let us now introduce the general concept of quasiconformal distortion bounds. Let $U \subset \mathbb{C}$ be a Jordan domain, and A a closed subset of U. Suppose there exists K such that for all embeddings $h : \overline{U} \to \mathbb{C}$ with $h|_{U-A}$ conformal, there exists an embedding $\tilde{h} : \overline{U} \to \mathbb{C}$ such that $\tilde{h}|_U$ is K-qc, and $h|_{\partial U} = \tilde{h}|_{\partial U}$. Then we let $\mathcal{QD}(A, U)$ be the least such K (and set $\mathcal{QD}(A, U) = \infty$ if there is no such K). In practice we will just be interested in establishing upper bounds for $\mathcal{QD}(A, U)$, or just showing that it is finite. We call such bounds *qc distortion bounds.*

For future reference, we include some basic facts about these distortion bounds:

Fact 1.3.1 *$\mathcal{QD}(A, U)$ is a conformal invariant: if $g : U \to V$ is a conformal isomorphism with $g(A) = B$, then $\mathcal{QD}(A, U) = \mathcal{QD}(B, V)$*

Proof: By Caratheodory's theorem[Mil1], g extends to homeomophism between $\overline{U}$ and $\overline{V}$. The result then follows immediately. $\boxed{1.3.1}$

Fact 1.3.2 *If $A \subset B \subset U$, then $\mathcal{QD}(A,U) \leq \mathcal{QD}(B,U)$.*

This is immediate.

The following fact shows that it can be sufficient to assume that h is only quasiconformal:

Fact 1.3.3 *Suppose $\mathcal{QD}(A,U) \leq K$, and $h : \overline{U} \to \mathbb{C}$ is an embedding with $h|_{U-A}$ L-qc. Then there exists an embedding $\tilde{h} : \overline{U} \to \mathbb{C}$ such that $\tilde{h}|_U$ is KL-qc, and $h|_{\partial P} = \tilde{h}|_{\partial P}$.*

Proof: Let the Beltrami coefficient μ on $h(U)$ be equal to the complex dilatation of h^{-1} on $h(U-A)$, and zero on A. Let $g : h(U) \to V$ be a quasiconformal map with complex dilatation μ. (The existence of g is guaranteed by the Measurable Riemann Mapping Theorem [ABer]. We can assume that V is a Jordan domain, and that g extends to a homeomorphism $g : h(\overline{U}) \to \overline{V}$). Then g is L-quasiconformal, and $g \circ h : \overline{U} \to \overline{V}$ is a homeomorphism that is conformal on $U - A$. Therefore there exists $\widetilde{(g \circ h)} : U \to V$ that is K-quasiconformal, and agrees with $g \circ h$ on ∂U. So let $\tilde{h} = g^{-1} \circ \widetilde{(g \circ h)}$: it has the required properties. $\boxed{1.3.3}$

Fact 1.3.4 *If there is a homeomorphism $g : \overline{U} \to \overline{V}$ such that $g|_U$ is L-qc, and $g(A) = B$, then $\mathcal{QD}(B,V) \leq L^2 \mathcal{QD}(A,U)$.*

Proof: Given $h : \overline{V} \to \mathbb{C}$ with $h|_{V-B}$ conformal, let $\tilde{h} = \widetilde{(h \circ g)} \circ g^{-1}$. Here $\widetilde{(h \circ g)}$ is as given from $h \circ g$ by Fact 1.3.3. $\boxed{1.3.4}$

We can also state and prove a more general fact:

Fact 1.3.5 *If there is a homeomorphism $g : \overline{U} \to \overline{V}$ with $g(A) = B$ and $g|_{U-A}$ L-quasiconformal, then $\mathcal{QD}(B,V) \leq L^2(\mathcal{QD}(A,U))^2$.*

Proof: Given an embedding $h : V \to \mathbb{C}$ with $h|_{V-B}$ conformal, we must find a $L^2(\mathcal{QD}(A,U))^2$-qc map $\tilde{h} : V \to \mathbb{C}$ with $h|_{\partial V} = \tilde{h}|_{\partial V}$. Now $h \circ g : U \to \mathbb{C}$ is an embedding that is L-qc on $U - A$, so by Fact 1.3.3 there exists a $L \cdot \mathcal{QD}(A,U)$-qc map $\widetilde{(h \circ g)} : U \to \mathbb{C}$ that agrees with $h \circ g$ on ∂U.

Now note that, by Fact 1.3.3, there exists a $L \cdot \mathcal{QD}(A,U)$-qc map $\tilde{g} : \overline{U} \to \overline{V}$ with $\tilde{g}|_{\partial U} = g|_{\partial U}$.

Then $\widetilde{(h \circ g)} \circ \tilde{g}^{-1} : \overline{U} \to \mathbb{C}$ is $L^2(\mathcal{QD}(A,U))^2$-qc (on U), and agrees with h on ∂U. It is the required map $\tilde{h}$. $\boxed{1.3.5}$

Fact 1.3.6 *If $A \subset U$ is compact, then $\mathcal{QD}(A,U) \leq \infty$.*

Proof: By the Riemann mapping theorem (and Caratheodory's theorem), we can assume U and $h(U)$ are both the unit disk, and $0 \in h(A)$. Then, using Schwartz reflection, we find that $h|_{S^1}$ is real-analytic, and, using Montel's theorem (or the Koebe distortion theorem), that $h'|_{S^1}$ is bounded. Likewise for $h^{-1}|_{S^1}$ (one checks that $h(A)$ always lies within some definite subdisk (depending only on A), because $h(U - A)$ has some fixed modulus). Therefore the map on the boundary is uniformly bi-Lipschitz, which is certainly enough to insure a uniformly quasiconformal extension (e.g. just cone it off, mapping (r, θ) to $(r, h(\theta))$). $\boxed{1.3.6}$

Fact 1.3.1 will be used in section 1.5; the others will be used in Chapter 2.

1.4 Uniform Distortion Bounds

The proof of the Main Theorem, 1.1.6, can be reduced to the following lemma:

Lemma 1.4.1 (Uniform Qc Bounds) *There exists a K, depending only on f, such that for all pieces P, $\mathcal{QD}(J \cap P, P) \leq K$.*

Assuming this Lemma, we can complete the proof of Theorem 1.1.6:

Proof: We will first show that there exists a K such that if $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, with $h|_{\mathbb{C}-J}$ conformal, then h is K-quasiconformal. This K will be independent of h.

Let K be as given in Lemma 1.4.1. We will show that h is K-quasiconformal by approximating it uniformly with K-qc maps. For each $n \in \mathbb{Z}^+$, we define $h_n : \mathbb{C} \to \mathbb{C}$ as follows: let $h_n = h$ on the unbounded component of $\mathbb{C} - \Gamma_n$, and for each piece P_i of level n, let $h_n = \tilde{h}_i$ on $\overline{P_i}$, where $\tilde{h}_i|_{\partial P_i} = h|_{\partial P_i}$, and $\tilde{h}_i|_{P_i}$ is K-qc. (The existence of the $\tilde{h}_i$'s are guaranteed by Lemma 1.4.1). Then h_n is K-qc. (Here we use the fact that Γ_n, a piecewise smooth 1-complex, is holomorphically removable). Now, because the diameters of the pieces goes to zero as $n \to \infty$, so do their images by h. But h_n maps every piece of level n to its image under h. Therefore $\|h - h_n\|_\infty \to 0$ as $n \to \infty$. So h is K-qc.

Now, the above fact (there exists K such that given $h : \mathbb{C} \to \mathbb{C}$ a homeomorphism, $h|_{\mathbb{C}-J}$ conformal, then h is K-quasiconformal) implies that J has zero area (thus we have also proven Theorem 1.1.3). For if not, one can take any Beltrami coefficient supported on J with dilatation (that is, essential supremum of pointwise dilatation) greater than K, and using the Measurable Riemann Mapping Theorem [ABer], integrate it to obtain a quasiconformal homeomorphism of $\mathbb{C}$ that is conformal off of J but has dilatation greater than K, a contradiction. We have thus shown so far that any homeomorphism $h : \mathbb{C} \to \mathbb{C}$ with $h|_{\mathbb{C}-J}$ conformal is (K-)quasiconformal, and conformal off of a set of measure 0. But then, we can conclude, as wanted, that it is conformal, by the following[Ah, LV]:

Theorem 1.4.2 *A quasiconformal mapping that is conformal off of a set of measure zero is conformal.*

$\boxed{1.1.6}$

1.5 Proof of the Uniform Distortion Bounds

There are two lemmas that form the basis of the proof of Lemma 1.4.1. (One will be proven in each of the following two chapters). The first is a non-uniform version, where we allow K to depend on the piece:

Lemma 1.5.1 (Piece-dependent Qc Bounds) *For all pieces P, there exists a $K(P)$ such that $\mathcal{QD}(J \cap P, P) \leq K(P)$.*

The second lemma breaks down each piece into copies of pieces at a fixed level:

Lemma 1.5.2 (Tiling Lemma) *There exists an $L \in \mathbb{Z}^+$ such that given any piece P of level greater than L, we can write*

$$P = T \cup R \cup \bigcup(\overline{Q_i} \cap P),$$

where T, R, and $\bigcup(\overline{Q_i} \cap P)$ are mutually disjoint; T is open, and $T \cap J = \emptyset$; R is compact and holomorphically removable; and each of the Q_i is a Yoccoz piece of level $q_i > L$, the Q_i are all mutually disjoint, and

$$f^{q_i - L}|_{Q_i}$$

is univalent.

Remark 1.5.3 *We allow either a finite or countable set of Q_i's, typically the latter.*

Remark 1.5.4 *Thus each Q_i is a univalent copy of a piece at level L, by a map (namely, an iterate of f) that maps Julia set to Julia set.*

Proof of Lemma 1.4.1, given Lemmas 1.5.1 and 1.5.2: Let L be as given by Lemma 1.5.2. Then let K in Lemma 1.4.1 to be the maximum of the $K(P)$'s of the (finitely many) pieces of level at most L (as given by Lemma 1.5.1). We will show that Lemma 1.4.1 holds for this choice of K.

This K works tautologically for all pieces of level at most L. Now let P be a piece of level greater than L. By Lemma 1.5.2, we may write

$$P = T \cup R \cup \bigcup(\overline{Q_i} \cap P).$$

Then, given $h : \overline{P} \to \mathbb{C}$, we define $\tilde{h}$ as follows:

For each Q_i,

$$\begin{aligned} \mathcal{QD}(J \cap Q_i, Q_i) &= \mathcal{QD}(f^{q_i-L}(J \cap Q_i), f^{q_i-L}(Q_i)) \\ &\qquad \text{(because } f^{q_i-L}|_{Q_i} \text{ is univalent)} \\ &= \mathcal{QD}(J \cap f^{q_i-L}(Q_i), f^{q_i-L}(Q_i)) \\ &\leq K(f^{q_i-L}(Q_i)) \leq K. \end{aligned}$$

So we can replace $h|_{\overline{Q_i}}$ by h_i, with $h_i|_{\partial Q_i} = h|_{\partial Q_i}$, and $h_i|_{Q_i}$ K-quasiconformal.

Define $\tilde{h}$ by $\tilde{h}|_{\overline{Q_i}} = h_i|_{\overline{Q_i}}$, and $\tilde{h} = h$ off of $\bigcup \overline{Q_i}$. Then $\tilde{h}$ is well-defined and continuous on $\bigcup \overline{Q_i}$ (because $\tilde{h} = h$ on $\bigcup \partial Q_i$), and $\tilde{h}$ is continuous on $\overline{\bigcup Q_i}$, since the diameters of the Q_i (and their images under h, $\tilde{h}$) goes to zero as $i \to \infty$. So $\tilde{h}$ is continuous on $\overline{P}$. It is also injective, and hence is an embedding

We now just need to verify that $\tilde{h}$ is K-qc on P. First note that it is K-qc on $T \cup \bigcup Q_i$. Therefore it is K-qc on the open set $P - R = T \cup \bigcup(\overline{Q_i} \cap P)$, because $\bigcup(\partial Q_i \cap P)$ is a piecewise smooth locally finite 1-complex. Therefore it is K-qc on P, because the remaining set, R, is holomorphically removable. $\boxed{1.4.1}$

Chapters 2 and 3 give the proofs of Lemmas 1.5.1 and 1.5.2 repectively, thus completing the proof of Theorem 1.1.6.

Chapter 2

The Piece-dependent Bounds

In this chapter we prove the piece-dependent distortion bounds. We introduce a canonical model, and prove quasiconformal distortion bounds for it. Then, given an arbitrary Yoccoz puzzle piece P, we embed this canonical model into P in such a way as to imply qc distortion bounds for P.

In section 2.1 we define this canonical model and describe its role in the proof of piece-dependent distortion bounds. In section 2.2 we prove qc distortion bounds for it. In section 2.3 we describe how it is embedded into a given piece P. How all of this fits together to prove piece-dependent distortion bounds for P is also described in section 2.1.

2.1 The Role of the "Recursively Notched Square"

We first define the "recursively notched square" as the pair (S, N), which are defined as follows. Take the open square $S = (0,1)\times(-1/2,1/2)$, and divide it into nine equal-sized smaller squares in the obvious way. There are unique homotheties (i.e. direction-preserving similarities) from the large square to each of the smaller squares. Define N to be the smallest subset of the S such that N contains the central small square, and N contains its own image under the homotheties h_l to the middle-left square and h_r the middle-right one. (We have $h_l(z) = z/3$ and $h_r(z) = (z-1)/3+1$.) So if we let $(a_i)_{i=1}^n$ denote a sequence of l's and r's, then

$$N = \bigcup h_{a_1} \circ h_{a_2} \circ \ldots \circ h_{a_n}(\overline{S})$$

where the union ranges over all sequences of length $n \geq 0$. See figure 2.1. Note that $(\overline{S} - \mathrm{Int}\, N) \cap \mathbb{R} = \mathcal{C}$, where $\mathcal{C} \subset [0,1]$ denotes the middle-thirds Cantor set $\{\sum_{i=1}^{\infty} a_i 3^{-i} \mid a_i \in \{0,2\}\}$.

The key step toward showing Lemma 1.5.1 is the following:

Lemma 2.1.1 *The recursively notched square has quasiconformal distortion bounds:*

$$\mathcal{QD}(\overline{N}, S) = D_0 < \infty.$$

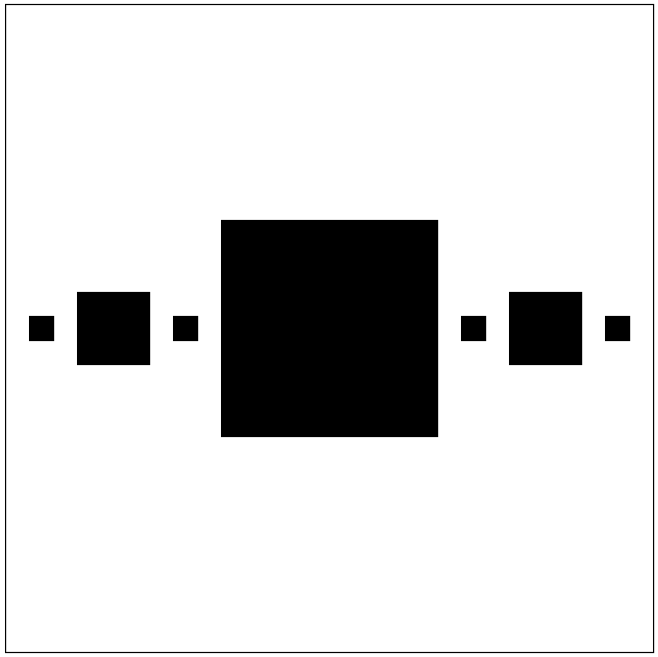

Figure 2.1: The recursively notched square

Now if P is any level n Yoccoz piece, $\partial P \cap J$ is a finite set because it is a subset of $f^{-n}(\alpha)$. We will show that we can cover a neighborhood of each point in this set by a copy of $(S, \overline{N})$. More precisely,

Lemma 2.1.2 *Given P, we can find $K, h_1, \ldots, h_m$ and open subsets $R_1, \ldots, R_m$ of P such that $h_i : \overline{S} \to \overline{R_i}$ is a homeomorphism with $h_i|_{S-\overline{N}}$ K-qc, $h_i(\overline{N}) \supset R_i \cap J$, the $\overline{R_i}$ are disjoint (and have holomorphically removable boundary), and $(P \setminus (\bigcup R_i)) \cap J$ is compactly contained in P.*

Given Lemmas 2.1.1 and 2.1.2 we can now prove Lemma 1.5.1, using the basic facts from section 1.3:

Proof of Lemma 1.5.1: We have that $\mathcal{QD}(J \cap R_i, R_i) < K^2 D_0^2$ by Facts 1.3.5 and 1.3.2. Therefore, given an embedding $h : \overline{P} \to \mathbb{C}$ with $h|_{P-J}$ conformal, we can replace h on each R_i with a $K^2 D_0^2$-quasiconformal map with the same boundary values, and thereby obtain an embedding that agrees with h on ∂P, and which is $K^2 D_0^2$ quasiconformal off of a compact subset of P, namely $E = (P - \cup R_i) \cap J$. By Fact 1.3.6, $\mathcal{QD}(E, P)$ is finite, say K_1. Then by Fact 1.3.3, there exists a $K_1 K^2 D_0^2$-quasiconformal map $\tilde{h} : \overline{P} \to \mathbb{C}$ with $\tilde{h}|_{\partial P} = h|_{\partial P}$. So

$$\mathcal{QD}(J \cap P, P) \leq K_1 K^2 D_0^2 < \infty.$$

$\boxed{1.5.1}$

2.2 Qc Distortion Bounds for the RNS

To prove quasiconformal distortion bounds for the recursively notched square, we will in fact show a stronger property, from which quasiconformal distortion bounds can be deduced. Let $U \subset \mathbb{C}$ be open. We denote by $W^{1,2}(U)$ the Sobolev space of functions $\xi : U \to \mathbb{R}$ (modulo constants) with one distributional derivative in L^2, with norm

$$\|\xi\|_U^2 = \|\xi\|_{U,1,2}^2 = \int\!\!\int \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 dx\,dy = 2i \int\!\!\int \left(\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \xi}{\partial \overline{z}}\right) dz\,d\overline{z}.$$

Remark 2.2.1 *The usual norm for $W^{1,2}(U)$ also includes the usual L^2 norm, obviating the need to mod out by constants. But what we need is the above.*

Remark 2.2.2 *Functions in this space are not necessarily continuous.*

The latter formula shows the norm is conformally invariant, in the sense that if $h : V \to U$ is conformal, then $\|\xi \circ h\|_V = \|\xi\|_U$. When there is no danger of confusion, we will omit the domain in the norm notation. Furthermore, the norm is quasiconformally quasi-invariant:

Fact 2.2.3 *If $h : V \to U$ is K-quasiconformal, then $\|\xi \circ h\|_V \leq K\|\xi\|_U$.*

Proof: Suppose ξ is C^1 with compact support; then $\xi \circ h \in W^{1,2}$ because h is in $W^{1,2}$ and is absolutely continuous for 2-dimensional Lebesgue measure (and thus the change-of-variable formula applies). An easy calculation then verifies the inequality in this case (see [Ah, Ch. 1, Sec. F]). But such ξ are dense in $W^{1,2}$, so the result follows. $\boxed{2.2.3}$

Now, suppose again we are given $A \subset U$ closed. We define $\mathcal{SD}(A, U)$ as the least K such that, for all $\xi : \overline{U} \to \mathbb{C}$ continuous, with $\|\xi|_{U-A}\|_{1,2} \leq 1$, there exists $\tilde{\xi} : \overline{U} \to \mathbb{C}$ continuous such that $\tilde{\xi}|_{\partial U} = \xi|_{\partial U}$, and $\|\tilde{\xi}|_U\| \leq K$.

Proposition 2.2.4 *For all K there exists K' such that for all $A \subset U \subset \mathbb{C}$, if $\mathcal{SD}(A, U) \leq K$, then $\mathcal{QD}(A, U) \leq K'$.*

Proof: We will use a result of Nag and Sullivan[NS], which states:

Theorem 2.2.5 *Suppose that X and Y are Jordan domains in $\mathbb{C}$, and $h : \partial X \to \partial Y$ is an orientation-preserving homeomorphism. Suppose there exists a C such that for all f continuous on $\overline{Y}$ with $\|f|_Y\|_{1,2} \leq 1$, there exists a continuous extension g (to $\overline{X}$) of $f|_{\partial Y} \circ h$ with $\|g|_X\| \leq C$. Then h has an extension $\tilde{h} : \overline{X} \to \overline{Y}$ such that $\tilde{h}|_X$ is C'-quasiconformal, with C' depending only on C.*

Now, given an embedding h of $\overline{U}$ that is conformal on $U - A$, let $V = h(U)$. For all continuous functions f on $\overline{V}$ with $\|f\|_{1,2} \leq 1$, we find that $\|f \circ h|_{U-A}\| \leq 1$, and therefore we can find g continuous on $\overline{U}$ with $g|_{\partial U} = f \circ h|_{\partial U}$, and with $\|g|_U\| \leq K$. Using the theorem above, we conclude that ∂h has a K'-quasiconformal extension, with K' depending only on K. $\boxed{2.2.4}$

To prove quasiconformal distortion bounds for the recursively notched square, we will show that $\mathcal{SD}(\overline{N}, S) \leq \infty$.

Now let $\mathcal{F} = \{z | 0 < \Im z < \pi\}$ be an infinite strip. Let us define $\hat{\mathcal{F}}$ as $\mathcal{F} \cup \partial\mathcal{F}$, where $\partial\mathcal{F}$ denotes the ideal boundary of S. Then $\hat{\mathcal{F}}$ may be identified with the closure of $\mathcal{F}$ in $\mathbb{C}$, plus two points, positive and negative (real) infinity, with the obvious neighborhood bases.

We will show:

Lemma 2.2.6 (Mapping lemma) *There is a quasiconformal homeomorphism $h : S - N \to \mathcal{F} - \overline{V}$, where V is a union of countably many vertical slits in $\mathcal{F}$ such that*

- *the imaginary part of each of the slits is bounded between $\pi/5$ and $4\pi/5$, and*
- *$\overline{V} \subset (V \cup M)$ (where $M = \{z | \Im z = \pi/2\}$ is the midline of $\mathcal{F}$).*

Moreover, h^{-1} extends continuously to a map $g : \hat{\mathcal{F}} - V \to \overline{S}$, and there exists $\tilde{g}$ such that $\tilde{g} : \hat{\mathcal{F}} \to \overline{S}$ is a homeomorphism with $\tilde{g}|_{\mathcal{F}}$ quasiconformal, and $\tilde{g}|_{\partial\mathcal{F}} = g|_{\partial\mathcal{F}}$

We will also show:

Lemma 2.2.7 *There exists a B such that for all continuous $f : \hat{\mathcal{F}} - V \to \mathbb{R}$ with $\|f|_{\mathcal{F}-\overline{V}}\|_{1,2} \leq 1$, there exists $\tilde{f}$ on $\hat{\mathcal{F}}$ with $\tilde{f} = f$ on $\partial\mathcal{F}$, $\tilde{f}$ harmonic on $\mathcal{F}$, and $\|\tilde{f}\| \leq B$.*

Remark 2.2.8 *The statement of this lemma is a little peculiar, because f is assumed continuous on a set that is neither open nor closed. It is certainly not enough to assume that f is continuous on $\hat{\mathcal{F}} \setminus \overline{V}$.*

Given these two lemmas, we can quickly prove:

Lemma 2.2.9

$$\mathcal{SD}(\overline{N}, S) < \infty.$$

Proof: If f is continuous on $\overline{S}$ with $\|f|_{S-N}\| \leq 1$, then $f \circ g$ (with g as in Lemma 2.2.6) satisfies the hypotheses of Lemma 2.2.7, so we can find $\widetilde{f \circ g}$ with $\|\widetilde{f \circ g}|_{\mathcal{F}}\| \leq B$, and then $\tilde{f} := \widetilde{f \circ g} \circ \tilde{g}^{-1}$ has universally bounded Sobolev norm on S (by Fact 2.2.3), and $\tilde{f}|_{\partial S} = f|_{\partial} S$. $\boxed{2.2.9}$

2.2.1 Proof of Sobolev bounds for the slitted strip $(\mathcal{F}, V)$.

Proof of Lemma 2.2.7: We first need to describe a formula for the $W^{1,2}$ norm of a harmonic function on $\mathcal{F}$. Let $\mathbf{H}$ denote the upper half plane. Suppose that $g : \mathbb{R} \cup \{\infty\} \to \mathbb{R}$ is continuous, and continuous at infinity, in the sense that $\lim_{t\to\infty} g(t)$ exists (and is independent of direction). Then there is a unique continuous harmonic extension $\tilde{g}$ of g to $\mathbf{H}$, and its Sobolev norm on $\mathbf{H}$ is given by

$$\|\tilde{g}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(g(s)-g(t))^2}{(s-t)^2} ds\, dt.$$

(In particular, $\tilde{g} \in W^{1,2}$ if and only if the double integral is finite.) This formula appears as equation (24) in [NS].

Now let f be a (real-valued) function on the ideal boundary $\partial\mathcal{F}$ of the infinite strip $\mathcal{F}$; for $t \in \mathbb{R}$ we let $f_0(t) = f(t)$, and $f_1(t) = f(t+i\pi$. We require that f is continuous; this is the same as saying that the f_i are continuous, and that $\lim_{t\to+\infty} f_i(t)$ and $\lim_{t\to-\infty} f_i(t)$ each exist and are independent of i. Using the conformal map $z \mapsto e^z$ from $\mathcal{F}$ to $\mathbf{H}$, we obtain the following formula:

Lemma 2.2.10 *Let f on $\partial\mathcal{F}$ be continuous; then f has a unique continuous harmonic extension $\tilde{f}$ to $\mathcal{F}$, whose Sobolev norm is given by*

$$\|\tilde{f}\|^2 = \sum_{i,j=0,1} \frac{I_{ij}(f)}{2\pi},$$

where

$$I_{ij}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(f_i(s)-f_j(t))^2}{(e^{\frac{s-t}{2}} - (-1)^{i+j}e^{-\frac{s-t}{2}})^2} ds\, dt$$

Note that each $I_{ij}(f)$ above is non-negative, so each I_{ij} must satisfy $I_{ij}(f) \le 2\pi\|\tilde{f}\|^2$

Suppose, as in Lemma 2.2.7, that f is defined and continuous on $\hat{\mathcal{F}} \setminus V$, and f has Sobolev norm at most 1 on $\mathcal{F} \setminus \overline{V}$. The conditions on V imply that

$$f(t+i\pi) - f(t-i\pi) = \int_{-\pi}^{\pi} \frac{\partial f(t+iv)}{\partial v} dv \qquad (*)$$

for almost every t. By the Lemma above, to find a bound for the Sobolev norm of the harmonic extension of $f|_{\partial\mathcal{F}}$, we just need to establish bounds for each $I_{ij}(f)$.

Let $v : \mathcal{F} \to \mathcal{F}$ be defined by $v(x+iy) = x + iy/5$. Then v is 5-qc, and $v(\mathcal{F})$ is a substrip $\mathcal{E}$ of $\mathcal{F}$ that lies below V ($\mathcal{E} = \{z | 0 < \Im z < \pi/5\}$). Then $f \circ v|_{\mathbb{R}} = f|_{\mathbb{R}}$, so $I_{00}(f \circ v) = I_{00}(f)$. By Fact 2.2.3, $\|f \circ v\|_{\mathcal{F}} \le 5\|f\|_{\mathcal{E}}$. So we obtain:

$$\frac{1}{2\pi} I_{00}(f) = \frac{1}{2\pi} I_{00}(f \circ v) \le \|f \circ v\|_{\mathcal{F}}^2 \le 25\|f\|_{\mathcal{E}} \le \|f\|^2,$$

and likewise for I_{11}.

So we just need to bound I_{01}. From the inequalities

$$(f_0(s) - f_1(t))^2 \le 2((f_0(s) - f_0(t))^2 + (f_0(t) - f_1(t))^2)$$

and

$$\frac{1}{(e^{\frac{s-t}{2}} - e^{-\frac{s-t}{2}})^2} \ge \frac{1}{(e^{\frac{s-t}{2}} + e^{-\frac{s-t}{2}})^2},$$

we obtain

$$I_{01} \le 2I_{00} + \int_{-\infty}^{\infty} \frac{1}{(e^{\frac{s}{2}} + e^{-\frac{s}{2}})^2} ds \int_{-\infty}^{\infty} (f_0(t) - f_1(t))^2 dt.$$

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} (f_0(t) - f_1(t))^2 dt &= \int_{-\infty}^{\infty} \left(\int_0^{\pi} \frac{\partial f(t+iv)}{\partial v} dv \right)^2 dt \\
&\quad \text{(by } (*)\text{)} \\
&\le \pi \int_{-\infty}^{\infty} \left(\int_0^{\pi} \left(\frac{\partial f(t+iv)}{\partial v} \right)^2 dv \right) dt \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
&\le \pi \|f\|^2.
\end{aligned}$$

Thus each I_{ij} is bounded in terms of $\|f\|^2$, so we have bounded $\|\tilde{f}\|$. $\boxed{2.2.7}$

2.2.2 Proof of mapping lemma

In order to prove Lemma 2.2.6, we first introduce another canonical object, the "recursively slitted square". We show that there is a map from the recursively notched square to the recursively slitted square with properties analogous to that described in Lemma 2.2.6. Then we describe a quasiconformal map from the recursively slitted square to the strip $\mathcal{F}$ that maps the slits of the recursively slitted square to a union $V \subset \mathcal{F}$ of slits with the properties described in Lemma 2.2.6.

Let us now define the recursively slitted square. Let S' denote the open square $(-1,1) \times (-1,1)$. We now define a set $V' \subset S'$, which is the union of a set of vertical slits. Let $\mathbf{Q}_2$ denote the set of dyadic rational points in the interval $(-1,1)$. For each $\alpha \in \mathbf{Q}_2$, let v_α be the minimal power of k such that $\alpha = p/2^k$. Let V_α be the vertical segment given by

$$x = \alpha; \quad |y| \le \frac{3}{5} 2^{-v_\alpha}.$$

Define

$$V' = \bigcup_{\alpha \in \mathbf{Q}_2} V_\alpha.$$

For future reference (in the proof of Lemma 2.2.14) , we note the following:

Fact 2.2.11 *If $x + iy \in V'$, then $|y/(1+x)| \leq \frac{3}{5}$ and $|y/(1-x)| \leq \frac{3}{5}$.*

Proof: We have $x = p2^{-k}$ with $-2^k < p < 2^k$, $p \in \mathbb{Z}$, and $|y| < \frac{3}{5}2^{-k}$. Therefore $x \geq (1-2^k)2^{-k}$, so $1 + x \geq 2^{-k}$, so $|y/(1+x)| \leq \frac{3}{5}$. Likewise $|y/(1-x)| \leq \frac{3}{5}$. $\boxed{2.2.11}$

Proposition 2.2.12 *There is a continuous map $\phi : \overline{S} - Int\, N \to \overline{S'}$ with the following properties:*

1. *$\phi(S - \overline{N}) = S' - \overline{V'}$, and $\phi : S - \overline{N} \to S' - \overline{V'}$ is a quasiconformal homeomorphism.*
2. *$\phi(S - N) = S' - V',$. and $\phi : S - N \to S' - V'$ is a homeomorphism. In particular, $(\phi|_{S-N})^{-1} : S' - V' \to S - N$ is continuous.*
3. *There is homeomorphism $\psi : \overline{S} \to \overline{S'}$ such that $\psi : S \to S'$ is quasiconformal, and $\psi|_{\partial S} = \phi|_{\partial S}$.*

From what one can tell from word of mouth and Yoccoz's lectures, a similar proposition is used by Yoccoz [Yoc] in his proof of Theorem 1.1.2 (local connectivity of M at c). Yoccoz uses it to prove a more limited version of the quasiconformal distortion bounds for the recursively notched rectangle. It seems to be a folk result: the author is unsure of its original discoverer. The idea of it was described to him by his advisor, Curtis McMullen. Since it does not appear in the present literature, we will give a complete proof of it.

Proof:

The idea of the proof is to divide $S - \overline{N}$ and $S' - \overline{V'}$ into a countable collection of similar (in the Euclidean geometry sense) regions organized in a tree-like fashion, and define a piecewise linear map from each region in $S - \overline{N}$ to the corresponding region in $S' - \overline{V'}$. We then check that these piecewise linear maps fit together to a quasiconformal map $\phi : S - \overline{N} \to S' - \overline{V'}$, and that ϕ extends continuously to $\overline{S} - \mathrm{Int}\, N$, and that the extension has the desired properties.

We say that a *marked rectangle* is a pair (A, B) where B is a rectangle and $A \subset \partial B$ is closed subinterval properly contained in a side of ∂B. We say that a pair (A', B') is a *slitted rectangle* if B' is a rectangle, and $A' \subset B'$ is a segment perpendicular to $\partial B'$ which intersects $\partial B'$ in a single point. (See figure 2.2).

The two combinatorially equivalent triangulations shown in figure 2.3 determine a piecewise affine (and hence quasiconformal) map $\alpha : B - A \to B' - A'$, defined by letting α on each triangle be the unique affine map mapping the triangle to the corresponding primed triangle. This PL map α will be the building block for the desired quasi-conformal map from $S - \overline{N}$ to $S' - \overline{V'}$.

Let X denote the union of horizontal lines in the plane of the form $y = \pm\frac{1}{2}3^{-n}, n > 0$. Figure 2.4 shows how the lines of X intersect $S - \overline{N}$ and partition it into connected components. Each component of the partition is a marked rectangle, and the components are in fact all similar to each other.

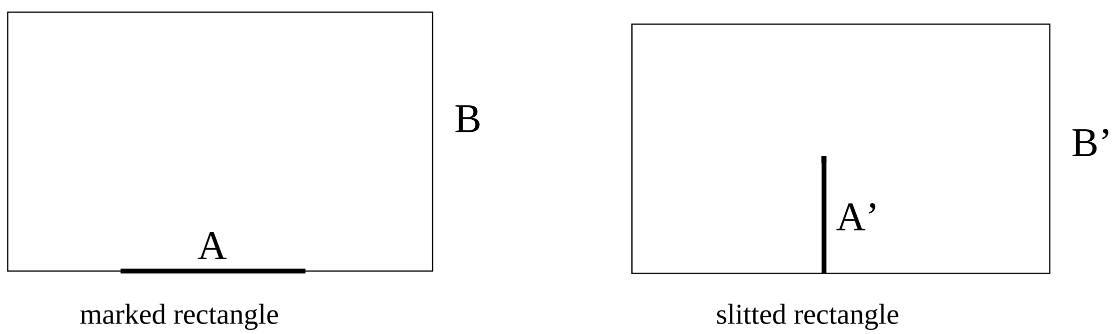

Figure 2.2: The marked rectangle and slitted rectangle.

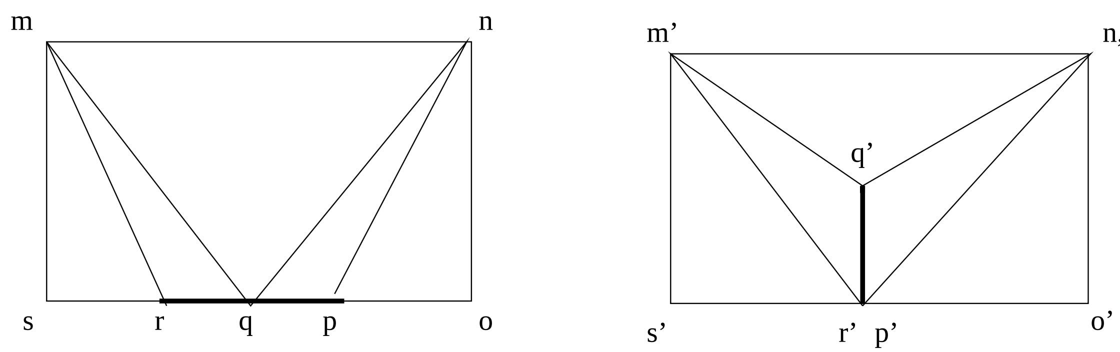

Figure 2.3: Combinatorially equivalent triangulation of the marked and slitted rectangles.

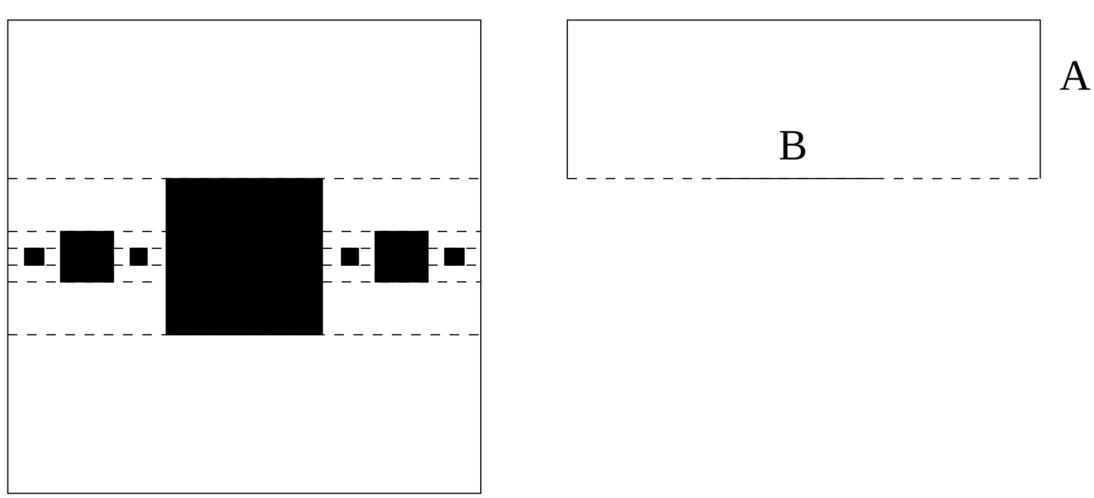

Figure 2.4: How the lines of X intersect $S - \overline{N}$.

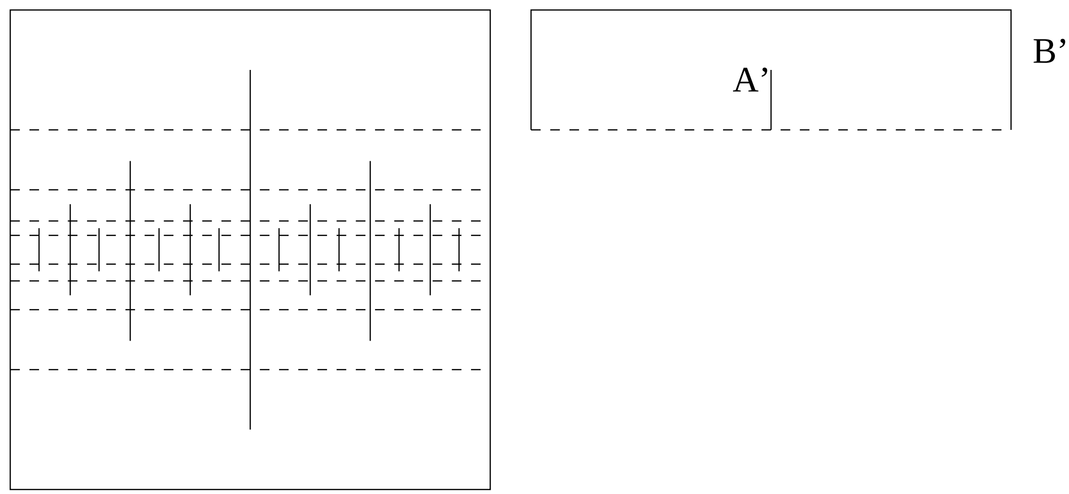

Figure 2.5: How the lines of X' intersect $S' - \overline{V'}$.

Let X' denote the union of horizontal lines of the form $y = \pm 2^{-n}$. Figure 2.5 shows how the lines of X' intersect $S - \overline{V'}$ and partition it into connected components. Each component is a slitted rectangle, and the components are all similar to each other.

The components of the partition of $S - (\overline{N} \cup X)$ correspond bijectively with the nodes of a pair of infinite binary trees—one for the top half of $S - \overline{N}$ and one for the bottom. Likewise for the components in the partition of $S - \overline{V'}$. In fact, the combinatorial structure of the two partitions is the same. The map α defined above extends, component by component, to give a piecewise linear, quasi-conformal, map $\phi : S - \overline{N} \to S' - \overline{V'}$. We will verify, in turn, that ϕ has the properties stated in Lemma 2.2.12.

Observe that ϕ extends continuously to the union of the closures of the marked rectangle components of $S - \overline{N} - X$. This union is equal to $\overline{S} - (\operatorname{Int} N \cup \mathbb{R})$. So we just need to check that it extends continuously to $\mathbb{R} - \operatorname{Int} N$, which is just the middle-thirds Cantor set $\mathcal{C}$.

The ends of the pair of binary trees for the partition $S - (\overline{N} \cap X)$ correspond to the middle thirds Cantor set in S, consisting of all points of the form $\sum_{i=1}^{\infty} a_i 3^{-i}$ with each $a_i \in \{0, 2\}$. Therefore ϕ, so far defined on $S - \operatorname{Int} N - \mathbb{R}$, extends continuously to this Cantor set subset of the reals as the Cantor function $\sum_{i=1}^{\infty} a_i 3^{-i} \mapsto \sum_{i=1}^{\infty} \frac{a_i}{2} 2^{-i}$.

Thus we have defined a continuous map $\phi : S - \operatorname{Int} N \to S'$, and we have already seen that property 1, that $\phi : S - \overline{N} \to S' - \overline{V}$ is a quasiconformal homeomorphism, is satisfied.

Property 2 then follows from the following simple lemma in point-set topology:

Lemma 2.2.13 *Suppose there exist X,Y,$f : X \to Y$,and $A \subset X$ such that*

1. *X, Y are compact metric spaces,*
2. *$f : X \to Y$ is continuous,*
3. *$f|_A$ is injective, and $f(A) \cap f(X - A) = \emptyset$.*

Then $f|_A : A \to f(A)$ is a homeomorphism.

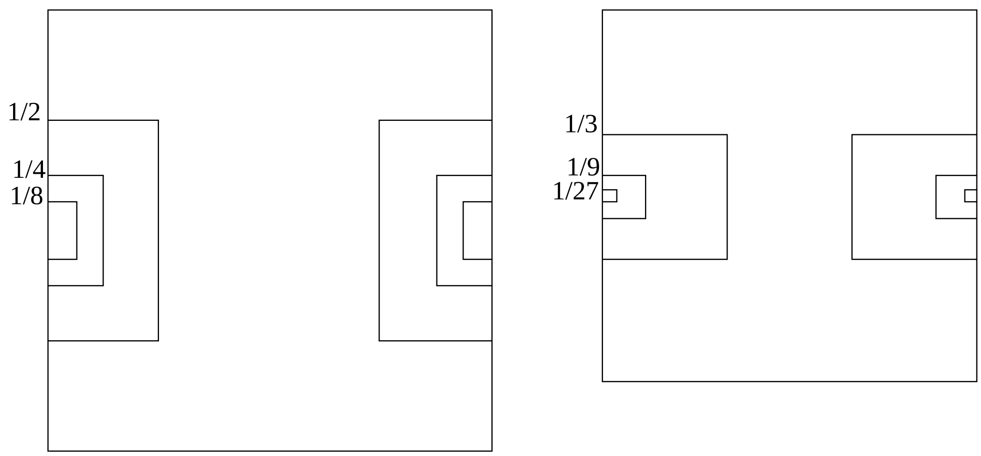

Figure 2.6: Combinatorially equivalent partitions of S and S'.

Note that we do not assume that A is a closed subset of X. We could drop the condition of metrizability, at the expense of using nets in the proof instead of sequences.

Proof: Note that f^{-1} is a well-defined function on $f(A)$. We just need to show that it is continuous, which is equivalent to showing that if $y_i, y \in f(A)$, with $\lim_{i\to\infty} y_i \to y$, then $f^{-1}(y_i) \to f^{-1}(y)$. It is enough to show that every subsequence of the y_i has a subsequence with the above property (that $f^{-1}(y_i) \to f^{-1}(y)$—here we follow the convention of not changing notation for passing to subsequences). So, given a subsequence of the y_i, pass to a further subsequence such that $f^{-1}(y_i) \to z$ for some $z \in X$ (possible by the compactness of X). But then $f(z) = y$ by the continuity of f, which implies that z is equal to $f^{-1}(y)$, the unique element of $f^{-1}(\{y\})$. $\boxed{2.2.13}$

So we just apply this Lemma to the case where $X = \overline{S} - \operatorname{Int} N$, $Y = \overline{S'}$, $f = \phi : \overline{S} - \operatorname{Int} N \to \overline{S'}$, and $A = \overline{S} - N$, and thereby conclude that $\phi : S - N \to S' - V'$ is a homeomorphism.

Finally, to show property 3, consider the combinatorially equivalent partitions of S and S' depicted in Figure 2.6. The crescent shaped sets of each partition (there are countably many—only finitely many are shown in the figure, of course) are equivalent to each other by Euclidean similarities. In S, their sizes decrease in powers of 3. In S', their sizes decrease in powers of 2. Each piece in the partition of S may be mapped to the piece in the partition of S', in the same combinatorial location, by a piecewise linear map, whose dilatation is independent of the choice of piece in the partition. The resulting map ψ is therefore quasi-conformal. It agrees with ϕ on ∂R, because the boundary values of both maps are piecewise linear maps (on $\partial S - \overline{N}$—note here that $\partial S \cap \overline{N}$ consists of just two points) that respect the same partitions of $\partial S - \overline{N}$.

This completes the proof of the proposition. $\boxed{2.2.12}$

We now describe a quasiconformal map from the recursively slitted square to a "ruler" (defined below), thus completing the proof of Lemma 2.2.6.

Let $\mathcal{F}$ be the infinite strip given by $|\Im z| \leq 1$. Let $V \subset \mathcal{F}$ be the union of a countable collection of vertical intervals, having the following properties:

1. For all $z \in V$, $|\Im z| < \frac{3}{5}$
2. The set V is symmetric with respect to reflection in the x-axis.
3. $\overline{V} \subset V \cup \mathbb{R}$

We say that the pair $(V, \mathcal{F})$ is a *ruler*. The value $\frac{3}{5}$ above is taken to correspond to the requirement that all the slits composing V in the statement of Lemma 2.2.6 have imaginary part between $\pi/5$ and $4\pi/5$ (in the strip defined by $0 < \Im z < \pi$.)

Lemma 2.2.14 *There exists a quasi-conformal map $\phi : S' \to \mathcal{F}$ with the following properties:*

1. *ϕ is symmetric with respect to reflection in the coordinate axes.*
2. *ϕ takes (V', S') to a ruler $(V, \mathcal{F})$.*

Proof: Let Q be the "diamond" inscribed in S', whose vertices are at the midpoints of sides of S'. There is a simple quasi-conformal (in fact, piecewise-linear) map from S' to Q (See figure 2.7—we map $\triangle acd$ to $\triangle ac'd$ and $\triangle bcd$ to $\triangle bc'd$, and likewise in the other three corners. The map is in fact the identity on the convex hull of V'. Fact 2.2.11 implies that the convex hull of V' is indeed as shown in figure 2.7) Thus, we may work with the pair (V', Q) instead of with (V', S'). Let Q_- denote those points of Q with negative x coordinate. Likewise define Q_+.

Define $\rho_- : Q_- \to R'$ by the formula

$$\rho_-(x, y) = (\log(1 + x), y/(1 + x)).$$

Define $\rho_+ : Q_+ \to R'$ by the formula

$$\rho_+(x, y) = (-\log(1 - x), y/(1 - x)).$$

We compute the Jacobian:

$$D\rho_- = \begin{bmatrix} 1/(1+x) & -y/(1+x)^2 \\ 0 & 1/(1+x) \end{bmatrix}.$$

Multiplying by $1 + x$, we have:

$$(1 + x)D\rho_- = \begin{bmatrix} 1 & -y/(1+x) \\ 0 & 1 \end{bmatrix}.$$

Finally, by Fact 2.2.11, $|y/(1+x)| \leq 1$ in Q_-. Therefore (1+x)$D\rho_-$, stays within a compact subset of $\mathbf{GL_2}(\mathbf{R})$, so the dilatation of $D\rho_-$, which is equal to the dilatation of $(1 + x)D\rho_-$, is uniformly bounded. (In fact, $\|(1 + x)D\rho_-\| \leq \sqrt{3}$ (because the square of the norm of a matrix is less than the sum of the squares of its entries), and $(1 + x)D\rho_-$ is area-preserving, so the

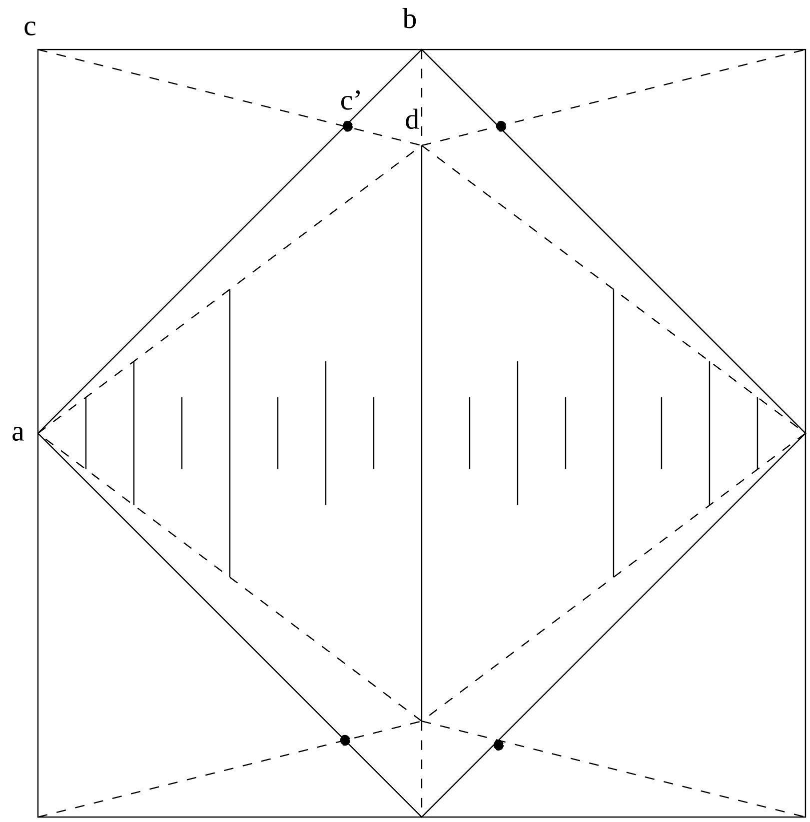

Figure 2.7: Piecewise linear map from S' to Q fixing V' pointwise

dilatation of $(1+x)D\rho_-$ is at most 3). A similar computation proves the same for ρ_+. Note that both these maps take vertical line segments to vertical line segments.

The union $\rho = \rho_- \cup \rho_+$ defined on $Q_- \cup Q_+$ is symmetric with respect to reflection in the y-axis, and hence extends to a quasiconformal map on all of Q.

Finally, we must check that all points in $\rho(V')$ have absolute value of imaginary part less than $\frac{3}{5}$. It is enough to check this for $\rho_-(V' \cap Q_-)$. Suppose $x + iy \in V' \cap Q_-$. Then $|\Im\rho_-(x+iy)| = |y/(1+x)| \leq \frac{3}{5}$ by Fact 2.2.11. $\boxed{2.2.14}$

So, to prove Lemma 2.2.6, simply follow the map given in Proposition 2.2.12 with the map given by Lemma 2.2.14, and then map the resulting strip by a Euclidean similarity to the one described for Lemma 2.2.6.

2.3 Covering J with the image of the recursively notched square.

The purpose of this section is to prove Lemma 2.1.2, which says roughly that we can embed copies of the recursively notched square into each piece P so as to cover the Julia set near the boundary of the piece with copies of $\overline{N}$. The embeddings are to be quasiconformal on $S - \overline{N}$. We first do so in the case where P is the top level piece containing the critical value, in which case we need only embed one copy of the RNR. We then use that embedding and the dynamics to get embeddings for all other pieces.

To get the embedding for the top level piece containing the critical value, we proceed as follows. Denote the two arguments of the external rays bounding that piece by A and D, with $A < D$. Then we find intervals $[A, B]$ and $[C, D]$ and a pair of monotonic maps $q_1 : \mathcal{C} \to [A, B]$ and $q_2 : \mathcal{C} \to [C, D]$ such that, for each $x \in \mathcal{C}$, the rays with arguments $q_1(x)$ and $q_2(x)$ land at the same point in J. Thus $\phi \circ e^{2\pi i q_i(\cdot)} : \mathcal{C} \to J$ is independent of i. The maps q_1, q_2 will also be such that $\phi \circ e^{2\pi i q_i}$ will extend to a quasiconformal map of $\overline{S} - \operatorname{Int} N$ into the dynamical plane. That map can then be easily extended to a map of all of S into the dynamical plane, such that N covers a neighborhood of ∂P in J.

2.3.1 Definitions and observations for external rays

We first require some basic definitions.

Recall that there exists a unique conformal isomorphism $\phi : \mathbb{C} - \overline{\Delta} \to \mathbb{C} - J$ such that, for all $z \in \mathbb{C} - \Delta$, $\phi(z^2) = (\phi(z))^2 + c$. Because J is locally connected (Theorem 1.1.1), Carathéodory's Theorem implies that ϕ extends continuously to a map

$$\phi : \mathbb{C} - \Delta \to \mathbb{C}$$

so that $\phi(\partial\Delta) = J$.

Recall also that a *external ray* (or just *ray*) $R(\theta)$ is defined by

$$R(\theta) := \{\phi(re^{2\pi i\theta}) \mid 1 < r < \infty\}.$$

Here we think of θ as an element of $\mathbb{R}/\mathbb{Z}$. Each such element has a unique representative in $[0, 1)$; we may sometimes denote the element by such a representative. The conjugacy properties of ϕ imply that $f(R(\theta)) = R(2\theta)$.

We say that a ray *lands* at $z \in J$ if

$$\lim_{r\to 1} \phi(re^{2\pi i\theta}) = z,$$

which is equivalent to saying that $\overline{R(\theta)} = R(\theta) \cup \{z\}$. Because J is locally connected, Carathéodory's theorem implies that every ray $R(\theta)$ lands. We denote the landing point of $R(\theta)$ by $l(R(\theta))$. Carathéodory's theorem also implies that $l(R(\theta))$ is a continuous function of θ. Note that $l(R(\theta)) = \phi(e^{2\pi i\theta})$. If $l(R(\theta_1)) = l(R(\theta_2))$, then we write $\theta_1 \simeq \theta_2$.

The term *combinatorial ray-pair* will denote a pair (θ_1, θ_2) such that $\theta_1 \neq \theta_2$ but $\theta_1 \simeq \theta_2$. The term *geometric ray-pair* will denote the union $R(\theta_1) \cup R(\theta_2) \cup \{z\}$, where (θ_1, θ_2) is a combinatorial ray-pair, and $z = l(R(\theta_1)) = l(R(\theta_2))$. We will denote the geometric ray-pair corresponding to the combinatorial ray-pair (θ_1, θ_2) by $\overline{R}(\theta_1, \theta_2)$. We will use the term *ray-pair* to refer to either a combinatorial or geometric ray-pair when the context makes it clear which one is being referred to.

Note that the continuity of f and conjugacy properties of ϕ imply that if $l(R(\theta)) = z$, then $l(R(2\theta)) = f(z)$. Therefore, if $\theta_1 \simeq \theta_2$ then $2\theta_1 \simeq 2\theta_2$. The converse, however, is not true: $2\theta_1 \simeq 2\theta_2$ does not necessarily imply $\theta_1 \simeq \theta_2$. We need to describe circumstances in which some sort of partial converse can be obtained.

Definition 2.3.1 *A* slice *is an open subset S of $\mathbb{C}$ such that the boundary of S has two components, each of which is a geometric ray-pair.*

Any two distinct geometric ray-pairs bound a unique slice. The slices S we will be interested in have the property that $l(R(0)) \notin S$. Such slices are called *vertical slices.* We can write the boundary of a vertical slice S as $\overline{R}(a, d) \cup \overline{R}(b, c)$, where $0 < a < b < c < d < 1$. In this case we write $S = S(a, b, c, d)$.

Lemma 2.3.2 *If $f^n : S(a, b, c, d) \to S(a', b', c', d')$ is univalent, and $2^n a \equiv a', 2^n b \equiv b', 2^n c \equiv c', 2^n d \equiv d'$ (all modulo 1), then $b' - a' = 2^n(b - a), d' - c' = 2^n(d - c)$, and, if x, y are such that $a < x < b$ and $c < x < d$, and $2^n x \simeq 2^n y$, then $x \simeq y$.*

Proof: We know that f^n is injective on the rays in $\overline{S(a, b, c, d)}$, so $z \mapsto 2^n z$ is injective on $[a, b] \cup [c, d]$, so we have $b' - a' = 2^n(b - a), d' - c' = 2^n(d - c)$.

We have $a' < 2^n x < b'$ and $c' < 2^n y < d'$. Therefore, $\overline{R}(2^n x, 2^n y) \subset S(a', b', c', d')$. Denoting the inverse of $f^n : S(a, b, c, d) \to S(a', b', c', d')$ by g, we find that $g(\overline{R}(2^n x, 2^n y))$ must be a geometric ray-pair $\overline{R}(u, v)$ with $2^n u \equiv 2^n x, 2^n v \equiv 2^n y$, and $a < u < b, c < v < d$. Since $b - a < 2^{-n}$ and $d - c < 2^{-n}$, we must have $u = x$ and $v = y$. Therefore $x \simeq y$. $\boxed{2.3.2}$

We can also state the analogous result when the slice is "flipped over" by f^q. (that is, when the "vertical orientation" is reversed).

Lemma 2.3.3 *If $f^n : S(a, b, c, d) \to S(a', b', c', d')$ is univalent, and $2^n a = c', 2^n b = d', 2^n c = a', 2^n d = b'$ (all modulo 1), then $b' - a' = 2^n(d - c), d' - c' = 2^n(b - a)$, and, if x, y are such that $a < x < b$ and $c < x < d$, and $2^n x \simeq 2^n y$, then $x \simeq y$.*

The proof is the same.

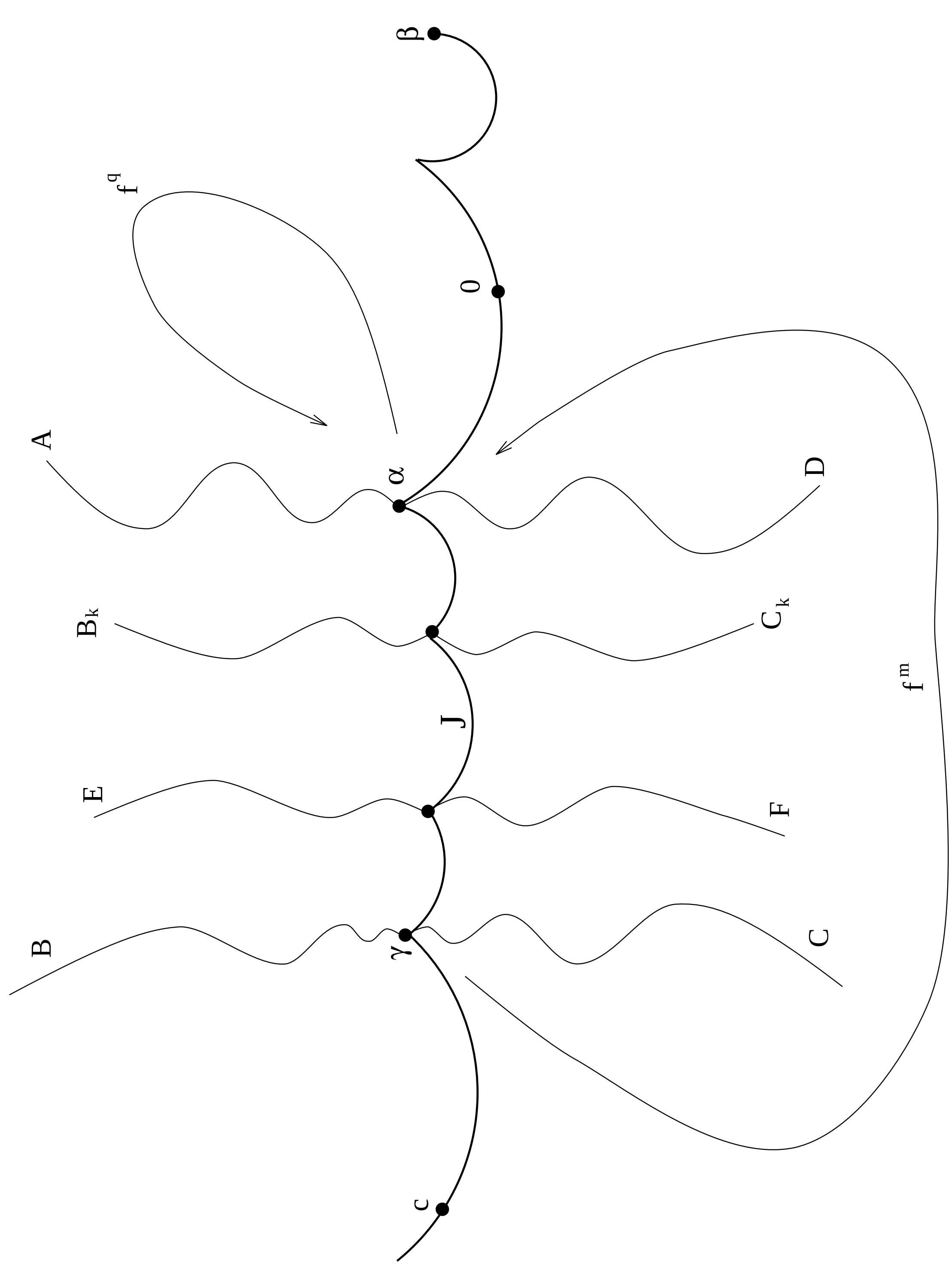

Figure 2.8: Slices in the dynamical plane

2.3.2 Getting univalent slice dynamics

In this subsection, the information shown in figure 2.8 is built up. The reader may wish to check with that figure while reading what follows.

Now let α denote the α fixed point, at which $q > 1$ rays land. Each ray is mapped to itself by f^q. These q rays union $\{\alpha\}$ divide $\mathbb{C}$ into q components; the boundary of each component is a single ray-pair containing α. One of these components (call it $\mathcal{C}$) contains the critical value $c = f(0)$. The β fixed point $\beta = l(R(0))$ will not be in this component, because β is in the component containing the critical point. Let the ray-pair bounding $\mathcal{C}$ be $\overline{R}(A, D)$, where $A < D$. (So $l(R(A)) = l(R(D)) = \alpha$.) Because $l(R(0)) \notin \mathcal{C}$, every ray $R(\theta)$ landing at a point in $\mathcal{C}$ will satisfy $A < \theta < D$.

For each $n \geq 0$, there is a level n Yoccoz puzzle piece P_n touching α (i.e. $\alpha \in \partial P$) and contained in this component $\mathcal{C}$. (So part of the boundary of P is $\overline{R}(A, D)$, cut off by an equipotential). If n is sufficiently large, then $c \notin \overline{P_n}$ (because the diameter of P_n goes to 0 as $n \to \infty$ (Theorem 1.2.2)). Choose the least such n. . The boundary of P_n consists of portions of an equipotential and a finite set of ray-pairs, cut off at that potential. One such ray-pair must separate α from c; we can denote it by $\overline{R}(B, C)$, with $A < B < C < D$. So then we have a vertical slice $S(A, B, C, D)$. Let $\gamma = l(R(B))(= l(R(C)))$, so $\partial(S(A, B, C, D) \cap J) = \{\alpha, \gamma\}$.

We now find smaller slices and univalent maps with which to apply Lemmas 2.3.2 and 2.3.3.

First note that $\{f^i(0) \mid 0 < i \leq q\} \cap S(A, B, C, D) = \emptyset$, because $\{f^i(0) \mid 0 < i \leq q\} \cap \mathcal{C} = c$, and $c \notin S(A, B, C, D)$. Therefore, we can define a single-valued univalent branch g of f^{-q} on a neighborhood of $\overline{S(A, B, C, D)}$, with $g(\alpha) = \alpha$, $g(R(A)) = R(A)$, and $g(R(D)) = R(D)$. Then define $B', C' \in [0, 1)$ such that $g(R(B)) = R(B')$ and $g(R(C)) = R(C')$. Then $f^q : S(A, B', C', D) \to S(A, B, C, D)$ is a univalent map of vertical slices, mapping boundary rays to the corresponding boundary rays, and thus satisfies the hypothesis of Lemma 2.3.2

In fact, if we let B_k, C_k be such that $g^k(R(B)) = R(B_k)$, and $g^k(R(C)) = R(C_k)$ (so $B_1 = B'$, $C_1 = C'$), then we have a series of vertical slices $S(A, B_k, C_k, D)$, and, for each $k \geq 1$, $f^{kq} : S(A, B_k, C_k, D) \to S(A, B, C, D)$ is a univalent map satisfying the hypothesis of Lemma 2.3.2. Moreover, $B_k \to A$ and $C_k \to D$ as $k \to \infty$, so the diameters of $S(A, B_k, C_k, D) \cap J$ go to zero as $k \to \infty$.

Now $f^n(\gamma) = \alpha$, since γ belongs to the boundary of a level n puzzle piece. Furthermore, for some $m \in [n, n + q)$, $f^m(R(B)) = R(D)$, and $f^m(R(C)) = R(A)$. If k is large enough, then $\{f^i(0) \mid 0 < i \leq m\} \cap S(A, B_k, C_k, D) = \emptyset$. Then we can let h be the branch of f^{-m} defined on $S(A, B_k, C_k, D)$, such that h's extension (also called h) to $\overline{S(A, B_k, C_k, D)}$ satisfies $h(\alpha) = \gamma, h(R(A)) = R(C), h(R(D)) = R(B)$. Let E, F, $0 < E < B < C < F < 1$ be such that $h(R(B_k)) = R(F)$, and $h(R(C_k)) = R(E)$. Then $f^m : S(E, B, C, F) = S(A, B_k, C_k, D)$ satisfies the hypotheses of Lemma 2.3.3, and so does $f^{m+kq} : S(E, B, C, F) \to S(A, B, C, D)$.

As $k \to \infty$, $B_k - A$ and $D - C_k$ tend to 0, and therefore $B - E < D - C_k$ and $F - C < B_k - A$ also tend to 0. So we can choose k such that $B - E + B_k - A < B - A$ and $F - C + D - C_k < D - C$, and thereby obtain that $A < B_k < E < B < C < F < C_k < D$. We have now determined all of what is shown in figure 2.8.

2.3.3 Mapping the RNS into the slice

Let $\psi : \{z \mid \Im z \geq 0\} \to \mathbb{C}$ be defined by $\psi(z) = \phi(e^{2\pi i z})$. Given $y \in \mathbb{R}^+$, and a vertical slice S(a,b,c,d) we define the "cut-off slice" $CS(a,b,c,d,y)$ by $CS(a,b,c,d,y) := S(a,b,c,d) \cap \psi\{z \mid y > \Im z \geq 0\}$. So $CS(a,b,c,d,y)$ will be a bounded domain, and $J \cap CS(a,b,c,d,y) = J \cap S(a,b,c,d)$. Also, $\partial(CS(a,b,c,d,y))$ is piecewise smooth curve, and is thus holomorphically removable.

Our eventual goal is get a homeomorphism $\xi : \overline{S} \to \overline{CS(A,B,C,D,1/2)}$ such that $\xi|_{S-\overline{N}}$ is quasiconformal, and $\xi(\overline{S} - \overline{N}) \cap J = \emptyset$. This will be the embedding required by Lemma 2.1.2, in the case of the top-level piece containing the critical value.

Our plan now is to define embeddings

$$q_1 : (\overline{S} - \operatorname{Int} N) \cap \{z \mid \Im z \geq 0\} \to [A,B] \times [0,1/2],$$

and

$$q_2 : (\overline{S} - \operatorname{Int} N) \cap \{z \mid \Im z \leq 0\} \to [C,D] \times [0,1/2]$$

that are quasiconformal on the interior of their domains, that satisfy $\psi \circ q_1|_{\mathcal{C}} = \psi \circ q_2|_{\mathcal{C}}$ (recall $\mathcal{C} = (\overline{S} - \operatorname{Int} N) \cap \mathbb{R}$), and that have boundary values as shown in figure 2.9. Given such maps, we can then define

$$\psi \circ (q_1 \cup q_2) : \overline{S} - \operatorname{Int} N \to \overline{CS(A,B,C,D,1/2)} \subset \mathbb{C},$$

which we can then extend to $\overline{S}$ to get the desired embedding ξ.

Such maps q_1 and q_2 must have the property that for each $x \in \mathcal{C}$, $q_1(x) = (a,0)$ and $q_2(x) = (b,0)$, with $a \simeq b$. So our goal now is to identify a Cantor set of pairs (a,b). This is done via the dynamics of slices obtained in the previous section, which we now abstract as follows:

On the product of intervals $[A,B] \times [C,D]$, we then have the following linear dynamics. Define a linear isomorphism

$$l_1 : [A,B] \times [C,D] \to [A,B_k] \times [C_k,D] \subset [A,B] \times [C,D]$$

by the formula

$$l_1(A+x, D-y) = (A + 2^{-kq}x, D - 2^{-kq}y).$$

Then, by Lemma 2.3.2, if (a,b) is a combinatorial ray-pair, and $l_1(a,b) = (a',b')$, then (a',b') is a combinatorial ray-pair, and $g(\overline{R}(a,b)) = \overline{R}(a',b')$. Define another linear isomorphism

$$l_2 : [A,B] \times [C,D] \to [E,B] \times [C,F] \subset [A,B] \times [C,D]$$

by

$$l_2(x+A, D-y) = (B - 2^{-m-kq}y, C + 2^{-m-kq}x)$$

Then, by Lemma 2.3.3, if (a,b) is a combinatorial ray-pair, and $l_2(a,b) = (a',b')$, then (a',b') is a combinatorial ray-pair, and $h(\overline{R}(a,b)) = \overline{R}(a',b')$. Note that for l_2, a' depends linearly on

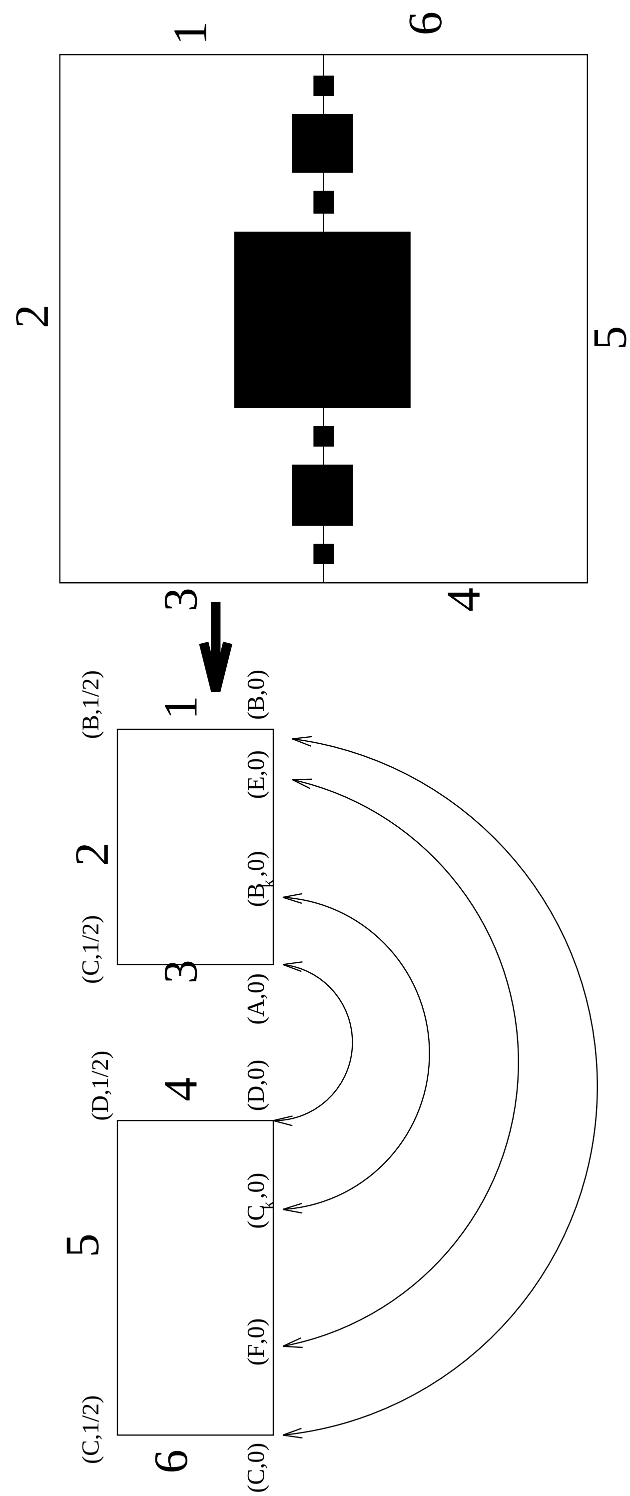

Figure 2.9: Boundary values for q_1 and q_2 (numbers 1-6 indicate corresponding sides)

b, and b' depends linearly on a. So we have two functions from $[A, B] \times [C, D]$ to itself, with disjoint images. (In fact, a point in the image of one function cannot share either coordinate with a point in the image of the other).

Now, note that for any finite sequence $i_1, i_2, \dots i_k$, with $i_j \in \{1, 2\}$, we have, by Lemmas 2.3.2 and 2.3.3 (as observed above), that $l_{i_1} \circ \dots \circ l_{i_k}(A, D)$ is a combinatorial ray-pair. Moreover, if $i_1, i_2, \dots$ is an infinite sequence with $i_j \in \{1, 2\}$, then $\lim_{k\to\infty} l_{i_1} \circ \dots \circ l_{i_k}(A, D)$ exists (because the l_i are contracting linear maps) and is also a combinatorial ray-pair. Let $T \subset [A, B] \times [C, D]$ denote the set of such pairs.

We now just need to define monotonic embeddings $q_1 : \mathcal{C} \to [A, B]$ and $q_2 : \mathcal{C} \to [C, D]$ such that for all $x \in \mathcal{C}$, $(q_1(x), q_2(x)) \in T$. (Then we will extend q_1 to a quasi-symmetric map $q_1 : [0, 1] \to [A, B]$, and then to the desired quasiconformal map $q_1 : [0, 1] \times [0, 1/2] \to [A, B] \times [0, 1/2]$ (which is then restricted to $(\overline{S} - \mathrm{Int}\, N) \cap \{z \mid \Im z \geq 0\}$). Likewise for q_2).

Now consider the following artificially constructed pair of linear isomorphisms, from $[0, 1]$ to subsets of itself. Firstly:

$$e_1 : [0, 1] \to [0, 1/3]$$

defined by

$$e_1(x) = (\frac{1}{3}x),$$

and secondly:

$$e_2 : [0, 1] > [2/3, 1]$$

defined by

$$e_2(x) = (1 - \frac{1}{3}x).$$

Then we can define q_1 and q_2 by

$$(q_1(x), q_2(x)) = \lim_{k\to\infty} l_{i_1} \circ \dots \circ l_{i_k}(A, D)$$

when

$$x = \lim_{k\to\infty} e_{i_1} \circ \dots \circ e_{i_k}(0)$$

where $(i_j)_{j=1}^{\infty}$ ranges over all possible sequences with $i_j \in \{1, 2\}$.

Then $q_1 : \mathcal{C} \to [A, B]$ is a monotonic embedding, and furthermore $q_1(\mathcal{C}) \subset [A, B]$ is a bounded geometry Cantor set, in the sense that for any sequence $(a_j)_{j=1}^{k}$, $a_j \in \{0, 2\}$, the ratio

$$q_1(s) - q_1(s + t) : q_1(s + t) - q_1(s + 2t) : q_1(s + 2t) - q_1(s + 3t)$$

is bounded, where $s = \sum_{j=1}^{k} a_j 3^{-j}$, and $t = 3^{-(k+1)}$. This is because the ratio will always be either $A - B_k : B_k - E : E - B$ or $C - E : E - C_k : C_k - D$. It follows that q_1 has a quasi-symmetric extension $q_1 : [0, 1] \to [A, B]$ (see the end of [Sul] for a discussion). We can likewise get a quasi-symmetric extension $q_2 : [0, 1] \to [C, D]$.

Now we need the following lemma:

Lemma 2.3.4 *If $q : [0,1] \to [0,1]$ is a quasisymmetric map, it has a continuous extension $Q : [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is quasiconformal on $(0,1) \times (0,1)$. (Here we identify $[0,1]$ with $[0,1] \times \{0\}$.) We can require Q to fix each side of the square $[0,1] \times [0,1]$ setwise (i.e. map each side of the square to itself).*

Proof: We start with $Q(x,0) = (q(x),0)$; we then want to define $Q : \partial([0,1] \times [0,1]) :\to \partial([0,1] \times [0,1])$ so that it is quasi-symmetric (with respect to arc-length); we can then extend Q to be quasiconformal on $(0,1) \times (0,1)$ (see [LV], section II.6). The right way to define Q on $\partial([0,1] \times [0,1])$ is by multiple reflection (cf. [LV], section II.7.2) That is, we set $Q(0,x) = (0,q(x))$, $Q(x,1) = (1-q(1-x),1)$, and $Q(1,x) = (1, 1-q(1-x))$. It is then easily checked that $Q : \partial([0,1] \times [0,1]) \to \partial([0,1] \times [0,1])$ is quasi-symmetric. $\boxed{2.3.4}$

Linearly rescaling the domain and range of 2.3.4 and applying it to $q_1 : [0,1] \to [A,B]$, we obtain the desired quasi-conformal extension $q_1 : [0,1] \times [0,1/2] \to [A,B] \times [0,1/2]$, and likewise $q_2 : [0,1] \times [-1/2,0] \to [C,D] \times [0,1/2]$.

So then we have

$$q_1 \cup q_2 : ([0,1] \times [0,1/2]) \cup ([0,1] \times [-1/2,0]) \to ([A,B] \times [0,1/2]) \cup ([C,D] \times [0,1/2]).$$

From this we can obtain, using the relation

$$\forall x \in \mathcal{C} : (\psi \circ q_1)(x,0) = (\psi \circ q_2)(x,0),$$

an embedding

$$\xi := \psi \circ (q_1 \cup q_2) : \overline{S} - \operatorname{Int} N \to \overline{CS(A,B,C,D,1/2)}$$

that sends ∂S to $\partial\overline{CS(A,B,C,D,1/2)}$, and that is quasiconformal on $S - \overline{N}$. (Note that $\xi|_{(\overline{S} - \operatorname{Int} N) \cap \{z | \Im z \geq 0\}} = q_1$, and $\xi|_{(\overline{S} - \operatorname{Int} N) \cap \{z | \Im z \leq 0\}} = q_2$, and $q_1|_{\mathcal{C}} = q_2|_{\mathcal{C}}$, so ξ is well-defined.) Then ξ applied to the boundary of any component of N is a Jordan curve in $\mathbb{C}$. We can then extend ξ continuously to each component of $\operatorname{Int} N$ via the Schoenflies theorem, to obtain the desired homeomorphism $\xi : \overline{S} \to \overline{CS(A,B,C,D,1/2)}$, with $\xi|_{S-\overline{N}}$ quasiconformal, and $\xi(\overline{S} - \overline{N}) \cap J = \emptyset$.

2.3.4 Embedding RNS's in a arbitrary piece P to cover ends of $J \cap P$.

For economy of space in what follows, let us denote $CS(A,B,C,D,1/2) = \xi(S)$, by $CS_{1/2}$. So now we have an embedding $\xi : \overline{S} \to \mathbb{C}$ such that $\xi(\overline{S}) = \overline{CS_{1/2}}$, and $\xi|_{S-\overline{N}}$ is quasiconformal, and $\xi(\overline{S} - \overline{N}) \cap J = \emptyset$. We wish to show Lemma 2.1.2 for all pieces P.

First note that $g^k : CS_{1/2} \to CS(A, B_k, C_k, D, 2^{-(kq+1)})$ is a homeomorphism (review subsection 2.3.2 for definitions of g^k, B_k, etc.). If we denote the top-level piece containing c by $P_c(0)$ (see section 3.3 for a general discussion of notation), then we observe that $J \cap (P_c(0) - CS(A, B_k, C_k, D, 2^{-(kq+1)}))$ is compactly contained in P. Moreover, for $k \geq 1$, and $0 \leq t < q$, we have $(f^t \circ g^k)(CS_{1/2}) \subset P_{f^t(c)}(0)$, and $J \cap (P_{f^t(c)}(0) - (f^t \circ g^k)(CS_{1/2}))$ is compactly contained in $P_{f^t(c)}(0)$.

So, given any level s Yoccoz piece $P(s)$, we have the branched covering map $f^s : P(s) \to P_{f^t(c)}(0)$ for some $0 \le t < q$. If k is sufficiently large (given $P(s)$), then

$$\{f^i(0) \mid 0 < i \le s\} \cap \overline{(f^t \circ g^k)(CS_{1/2})} \cap P_{f^t(c)}(0) = \emptyset.$$

Then for each point $z \in \partial P(s) \cap J$, $f^s(z) = \alpha$, and we can define a single-valued branch of f^{-s} on $(f^t \circ g^k)(CS_{1/2}))$ such that $f^{-s}(\alpha) = z$, and $f^{-s}((f^t \circ g^k)(CS_{1/2})) \subset P(s)$. Then we define the embedding $f_z : \overline{S} \to \overline{P(s)}$ by $f_z = f^{-s} \circ f^t \circ g^k \circ \xi$. It is then readily seen that $f_z(\overline{S}) \cap f_{z'}(\overline{S}) = \emptyset$ for $z, z' \in \partial P(s)$, $z \neq z'$, and that

$$J \cap (P(s) - \bigcup_{z \in \partial P(s)} f_z(\overline{S}))$$

is compactly contained in $P(s)$. Of course, $f_z|_{S-\overline{N}}$ is quasiconformal, and $f_z(\overline{S} - \overline{N}) \subset \mathbb{C} - J$, because ξ has these properties, and f_z is just ξ followed by (positive and negative) powers of f. Thus we have verified Lemma 2.1.2 for an arbitrary level s piece $P(s)$.

Chapter 3

The Tiling Lemma

All the numerical variables in this chapter (as opposed to object variables, like puzzle pieces) will denote integers.

Recall from Chapter 1 the statement of the Tiling Lemma, 1.5.2:

There exists an $L \in \mathbb{Z}^+$ such that given any piece P of level greater than L, we can write

$$P = T \cup R \cup \bigcup (\overline{Q_i} \cap P),$$

where T, R, and $\bigcup(\overline{Q_i} \cap P)$ are mutually disjoint; T is open, and $T \cap J = \emptyset$; R is compact and holomorphically removable; and each of the Q_i is a Yoccoz piece of level $q_i > L$, the Q_i are all mutually disjoint, and

$$f^{q_i - L}|_{Q_i}$$

is univalent.

In this chapter we break down the proof of Lemma 1.5.2 into three mutually exclusive cases for f, and settle each one by choosing the Q_i by a "greedy algorithm", and setting R to the leftover portion of the Julia set. The first case, in which $\exists n : f^n(0) = \alpha$, is trivial: we let $\{Q_i\} = \{P\}$, and $R = \emptyset$. This case must be eliminated in order to properly discuss the other two cases. In the second case, the critically non-recurrent case, the leftover set R comprises at most one point. In the third case, the critically recurrent case, the leftover set R is a Cantor set.

3.1 List of cases

Here are the three cases:

1. Some iterate of the critical point lands on the internal fixed point: $\exists n : f^n(0) = \alpha$.

2. The critical point is non-recurrent ($0 \notin \overline{\{f^n(0) \mid n > 0\}}$), but case 1 does not hold ($\forall n, f^n(0) \neq \alpha$).

3. The critical point is recurrent: $0 \in \overline{\{f^n(0) \mid n > 0\}}$.

We quickly treat case 1 in section 3.2. After introducing some notation in section 3.3 for cases 2 and 3, and making some basic observations, we take care of case 2 in section 3.4. We set up the proof for case 3 in section 3.5 and finish it in section 3.6.

3.2 Proof for the $\exists n : f^n(0) = \alpha$ case

In this case, $0 \in \Gamma_m$ for all $m \geq n$, so we can let $L = n$, and given any piece P of level $m > n$, $f^{m-n}|_P$ is univalent, so we can form the trivial decomposition of P, namely $T = R = \emptyset$, and $Q_1 = P$.

3.3 Notation and setup, assuming $\forall n, f^n(0) \neq \alpha$

For $z \in J$, denote by $P_z(n)$ the level n puzzle piece that contains z. This is well-defined if $f^n(z) \neq \alpha$, but if $f^n(z) = \alpha$, there will be no level n piece containing z, and more than one level n piece containing z in its closure. So this notation can only be used when we know already that $f^n(z) \neq \alpha$. In particular, our assumption here that $\forall n, f^n(0) \neq \alpha$, ensures that $P_0(n)$ will be defined for all n. We will call $P_0(n)$ the *critical piece* of level n.

Given z, n such that $z \in J$ and $f^n(z) \neq \alpha$, let $A_z(n)$ denote $P_z(n) - \overline{P_z(n+1)}$. We call such a $A_z(n)$ a *combinatorial annulus* (even though it is not necessarily an annulus). We will call $A_0(n)$ the *critical combinatorial annulus* of level n.

If we can prove Lemma 1.5.2 for the *critical* pieces of level greater than L, then we can prove it for all pieces: given a piece P of level greater than L, it maps univalently by some iterate of f either to some piece of level L or to a critical piece of level greater than L. In the former case, we can decompose P trivially, while in the latter case, we can pull back the decomposition of the critical piece to P.

3.4 Proof for the critically non-recurrent case, with $\forall n$, $f^n(0) \neq \alpha$

In this case the critical point forward orbit, $\{f^n(0) | n \in \mathbb{Z}^+\}$, is disjoint from some critical piece $P_0(N)$, because the diameters of the $P_0(N)$ go to zero. We then set $L = N$. Given a critical piece $P = P_0(m)$ with $m > N$, we let the $\{Q_i\}$ be $\{P_z(k) \mid k > m \text{ and } z \in A_0(k-1)\}$. Thus the the Q_i are the level k pieces that are subsets of $A_0(k-1)$. Note that such a $Q_i = P_z(k)$ is a subset of $P_0(k-1)$, but is not $P_0(k)$. If $f^t(P_z(k))$ were critical, for $t < k - L$, then $f^t(P_0(k-1))$ would be critical, in which case $f^t(0) \in P_0(k-t-1) \subset P_0(L)$, a contradiction. We let $R = \{0\}$, which is the intersection of all the critical pieces (by Theorem 1.2.2). Finally, let $T = P_0(m) \setminus (R \cup \bigcup \overline{Q_i})$. The only property of the T, R, Q_i left to verify is that $T \cap J = \emptyset$, which is equivalent to $P \cap J \subset R \cup \bigcup \overline{Q_i}$.

So note that $J \cap \bigcup_{z \in A_0(k-1)} \overline{P_z(k)} = J \cap \overline{A_0(k-1)}$, so $J \cap \bigcup \overline{Q_i} = J \cap (\overline{P_0(m)} - \bigcap_{l>m} P_0(l))$. But by Theorem 1.2.2 (diameters of pieces go to zero), $\bigcap_{l>m} P_0(l) = \{0\}$. Therefore $J \cap \bigcup \overline{Q_i} = J \cap (\overline{P_0(m)} - R)$, which is equivalent to $P \cap J \subset R \cup \bigcup \overline{Q_i}$.

We have thus shown Lemma 1.5.2 in the case where f is critically non-recurrent.

3.5 Notation and setup for the critically recurrent case

For the critically recurrent case of Lemma 1.5.2, we will need to let R be substantially more complicated. We say that $E \subset \mathbb{C}$ is *well-surrounded* if E is compact and there exists a collection $\mathbf{A}$ of disjoint annuli in $\mathbb{C} - E$ such that, if $x \in E$, the sum of the moduli of the annuli in $\mathbf{A}$ that surround x diverges.

Proposition 3.5.1 *If $R \subset \mathbb{C}$ is well-surrounded, then R is holomorphically removable.*

Proof:

We say that compact subset S of $\mathbb{C}$ has *absolute area zero* if, whenever S' is a compact subset of $\mathbb{C}$, and $h : \mathbb{C} - S \to \mathbb{C} - S'$ is a conformal isomorphism (that maps ∞ to ∞), then S' has measure 0. Then the proposition follows immediately from the following two results:

1. **Theorem 3.5.2 (McMullen)** *A well-surrounded set has absolute area zero.*

   This appears as Theorem 2.16 in [Mc].

2. **Theorem 3.5.3** *A set that has absolute area zero is holomorphically removable.*

   See [ABeu].

$\boxed{3.5.1}$

We also need some more facts about the Yoccoz partition. Here, as always, we assume that f is not renormalizable.

If $\overline{P_z(n+1)} \subset P_z(n)$, then $A_z(n)$ is a (geometric) annulus.

The following two statements can be found in the expositions of Milnor [Mil2] and Hubbard [Hub]:

Lemma 3.5.4 *There exists an n such that $A_0(n)$ is an annulus.*

In this case we call $A_0(n)$ a *critical annulus*. We say that $A_0(m)$ is a *critical descendant* of $A_0(n)$ if f^{m-n} maps $A_0(m)$ onto $A_0(n)$ as an unramified cover. Note that in this case, if $A_0(n)$ is a geometric annulus, then so is $A_0(m)$. If f^{m-n} has degree 2, we say that $A_0(m)$ is a *child* of $A_0(n)$.

Proposition 3.5.5 *If f is critically recurrent, then the sum of the moduli of the critical descendants of any critical annulus A_n diverges.*

We will assume for the rest of this section that f is critically recurrent. Let us now prove Lemma 1.5.2 for this case.

We call two critical descendants of the same critical annulus *fraternal* if neither one is a descendant of the other. Note that in this case they have no descendants in common . We will need the following lemma:

Lemma 3.5.6 *Every critical annulus $A_0(n)$ has at least two fraternal descendants $A_0(n_1)$ and $A_0(n_2)$.*

Proof: First note that, if $A_0(m)$ is a descendant of $A_0(n)$, and $A_0(l)$ is a descendant of $A_0(m)$, then $A_0(l)$ is a descendant of $A_0(n)$. It follows that either the above lemma is true for a given critical annulus $A_0(n)$, or the descendants of $A_0(n)$ form a sequence $A_0(k_1), A_0(k_2), A_0(k_3), \ldots$ such that $A_0(k_j)$ is a descendant of $A_0(k_i)$ whenever $j > i$. (So, in particular, $A_0(k_{i+1})$ is a descendant of $A_0(k_i)$). But since the modulus of a descendant of $A_0(m)$ is at most half the modulus of $A_0(m)$, the sum of the moduli of the descendants of $A_0(n)$ would in this case converge, a contradiction of Lemma 3.5.5. $\boxed{3.5.6}$

So, let N be the level of the non-degenerate critical annulus given by Lemma 3.5.4, and let $A_0(N_1)$ and $A_0(N_2)$ be two fraternal descendants of $A_0(N)$; their existence is guaranteed by Lemma 3.5.6. Set $L = \max(N_1, N_2) + 3$.

Now let $P = P_0(p)$, with $p > L$. We choose the Q_i by a kind of "greedy algorithm". First consider the set of all pieces Q contained in P such that $f^{q-L}|_Q$ is univalent, where q is the level of Q. Then let the Q_i be those elements of this set that are not a sub-piece of any other element. Then, by the Markov property of the Yoccoz partition, the Q_i are automatically mutually disjoint, and between them they cover as much of P as we could hope to cover. We let $R = (P \setminus \bigcup \overline{Q_i}) \cap J$, and let $T = P \setminus (R \cup \bigcup \overline{Q_i})$. Now we just need to verify is that R is compact and holomorphically removable. We will do so by showing that R is well-surrounded. In order to do this, we must of course define a set $\mathbf{A}$ of annuli.

We let

$$\mathbf{A} = \{A_z(n) \mid z \in R, n \geq p, \text{ and } f^{n-N} : A_z(n) \to A_0(N) \text{ is a covering map.}\}$$

(We will verify that $z \in R$ implies that $\forall n, f^n(z) \neq \alpha$, so $A_z(n)$ is well-defined).

We now need only the following:

Lemma 3.5.7 *The set R defined above is well-surrounded by $\mathbf{A}$. In particular,*

1. *the set R is compact,*
2. *the annuli in $\mathbf{A}$ are mutually disjoint, and disjoint from R, and*
3. *the sum of the moduli of the annuli in $\mathbf{A}$ that surround any given point in R diverges.*

To verify 1 and 2 we need just $L > N$; it is for one case of the verification of 3 that we need $L > \max N_i + 3$.

3.6 Proof of well-surroundedness of R for the critically recurrent case

Here are the verifications of the above three statements.

3.6.1 Compactness of R

Lemma 3.6.1 *If P is a piece, and $\eta \in \partial P \cap J$, then P has a subpiece P' of level k with $\eta \in \partial P'$ and f^{k-L} univalent on P'.*

Proof: Note that $\alpha \notin \overline{P_0(N+1)}$, because $\overline{P_0(N+1)} \subset P_0(N)$, and $\alpha \notin P_0(N)$. Therefore, no piece of level greater than N with α on its boundary can be critical. Now, for any piece $P_z(n)$, if $\alpha \in \partial P_z(n)$, then $\alpha \in \partial(f^k(P_z(n)))$, for all $0 < k \leq n$. Therefore, if z, n are such that $n > N$ and $\alpha \in \partial P_z(n)$, then $f^{n-N} : P_z(n) \to P_{f^{n-N}(z)}(N)$ is univalent. In other words, every piece with α on its boundary maps univalently (by an iterate of f) to a level N piece (and, hence, to a level L piece, since $L > N$).

Now, given $\eta \in \partial P \cap J$, chose m such that $f^m(\eta) = \alpha$. Note that η is not a critical point of f^m, due to our standing assumption in this section that $\forall n > 0, f^n(0) \neq \alpha$. Therefore, there is some level $k_0 \geq m$ such that f^m is univalent on every piece $P_z(k)$ of level $k \geq k_0$ with $\eta \in \partial P_z(k)$. But then $\alpha \in f^m(P_z(k))$, so if $k - m > L$, then $f^{(k-m)-L}|_{f^m(P_z(k))}$ is univalent, so, in any case, $f^{k-L}|_{P_z(k)}$ is univalent. So the desired subpiece P' is the unique piece of level k (say with $k = k_0$) that is contained in P and has η on its boundary. $\boxed{3.6.1}$

Note that the P' described above will be contained in some Q_i, because it maps univalently up to the level L. Note also that P' contains the intersection of P with some neighborhood of η.

Corollary 3.6.2 *If $x_i \to x$, and $x_i \in R$, then x is not the boundary of any piece.*

For if x were on the boundary of some piece, then we could choose a subsequence of the x_i to lie in one of the finitely many pieces of a given level that have x as a boundary point. But then the preceeding lemma provides a contradiction, because no points in that piece that are sufficiently close to x can be in R.

Corollary 3.6.3 *If $\eta \in \partial P$ for some piece P, then $\eta \notin R$.*

This is because $\eta \in \overline{P'} \subset \overline{Q}_i$ for some Q_i.

Corollary 3.6.4 *If $z \in R$, then z is not on the boundary of any piece, so $P_n(z)$ (and $A_n(z)$) is well-defined.*

This is an immediate consequence of the previous corollary.

Lemma 3.6.5 *If $x_i \in R$, and $x_i \to x$, and x is not on the boundary of a piece, then $x \in R$.*

Proof: If x were not in R, then there would be a piece containing it that maps univalently up to level L. But then a whole neighborhood of x would not be in R. $\boxed{3.6.5}$

We conclude from the above that R, in any piece, is compact (and in particular stays away from the boundary of that piece). We also conclude that $P_z(n)$ (and $A_z(n)$) is well-defined for all $n \geq 0$ and $z \in R$.

3.6.2 Disjointness of Annuli

Lemma 3.6.6 *No annulus in* $\mathbf{A}$ *can contain a point of* R *in its closure.*

Proof: Every annulus $A_z(k)$ in $\mathbf{A}$ is composed of pieces of level greater than p that map univalently to level N (because $A_z(k)$ is an unramified cover of $A_0(N)$, and an iterate of f, restricted to any piece, either has a critical point or is univalent), and we assume that $L \geq N+1$. So every point in $\overline{A_z(k)}$ lies in some $\overline{Q_i}$, and hence cannot be in R. $\boxed{3.6.6}$

Lemma 3.6.7 *Suppose two combinatorial annuli,* $A_z(k), A_w(l)$*, intersect. Then*

1. $z \in \overline{A_w(l)}$*, or*
2. $w \in \overline{A_z(k)}$*, or*
3. $A_z(k) = A_w(l)$.

Proof: Recall that $A_z(k) = P_z(k) - \overline{P_z(k+1)}$, and $A_w(l) = P_w(l) - \overline{P_w(l+1)}$. If $k = l$, then $P_z(k) = P_w(l)$, and either $P_z(k+1) = P_w(l+1)$ (case 3), or $P_z(k+1) \cap \overline{P_z(l+1)} = \emptyset$. If the latter holds, then we have (since pieces are open) $P_w(l+1) \subset P_z(k) - \overline{P_z(k+1)}$, and then $w \in P_w(l+1) \subset A_z(k)$. If $k > l$, then $\overline{P_z(k)} \subset P_w(l)$ (because $P_z(k) \cap P_w(l) \neq \emptyset$), and $P_z(k) \not\subset P_w(l+1)$, so in fact $P_z(k) \cap \overline{P_w(l+1)} = \emptyset$ by the Markov property for pieces, so $z \in P_z(k) \subset P_w(l) - \overline{P_w(l+1)} = A_w(l)$. $\boxed{3.6.7}$

Corollary 3.6.8 *No two distinct annuli in* $\mathbf{A}$ *can intersect.*

Proof: Suppose there were two distinct annuli $A_z(k), A_w(l) \in \mathbf{A}$, with $z, w \in R$, and $A_z(k) \cap A_w(l) \neq \emptyset$. Then by 3.6.7, either $z \in \overline{A_w(l)}$, or $w \in \overline{A_z(k)}$. But this contradicts Lemma 3.6.6. $\boxed{3.6.8}$

3.6.3 Divergence

For $z \in J$, let $\mathbf{A}_z$ denote all elements of the form $A_z(n)$ of $\mathbf{A}$, that is, all elements of $\mathbf{A}$ that surround z. Then our goal is to show, for each $z \in R$, that the sum of the moduli of the elements of $\mathbf{A}_z$ diverges. The first step is to determine which n are such that $A_z(n) \in \mathbf{A}_z$. This is done with the aid of the function $\tau_z(n)$, first defined by Shishikura (following Yoccoz) in his proof of Theorem 1.1.3. Then one property of $\tau_z(n)$ is abstracted in rise-and-drop functions, defined below. We prove certain lemmas about rise-and-drop functions. One such lemma is enough to deduce divergence of the sum of the moduli of the elements of $\mathbf{A}_z$ in the case where $\sup \tau_z(n)$ is infinite. In this case divergence is deduced from the divergence of $\mathbf{A}_0$, quoted as Lemma 3.5.5 in the previous section. In the other case (when $\sup \tau_z(n)$ is finite), we show that $\mathbf{A}_z$ contains infinitely many copies of one of the two $A_0(N_i)$, and hence the the sum of the moduli of the annuli in $\mathbf{A}_z$ diverges.

Given $n \in \mathbb{N}, z \in J$ such that $f^n(z) \neq 0$, there is at most one $m \in [0, n]$ such that $f^{n-m}(P_z(n)) = P_0(m)$, and $f^{n-m}|_{P_z(n)}$ is univalent (so then $f^{n-m} : P_z(n) \to P_0(m)$ is an isomorphism). If such an m exists, then we set $\tau_z(n) = m$. If no such m exists, then $f^n|_{P_z(n)}$ is univalent, and $f^n(P_z(n))$ is not a critical piece. In this case we set $\tau_z(n) = -1$.

So now we can write

$$R = \{z \in J \cap P_0(p) \mid \forall n \geq p, \quad f^n(z) \neq \alpha \text{ and } \tau_z(n) > L\}.$$

We will be interested in the values of $\tau_z(n)$ for $z \in R$ and $n \geq p$. In particular, by our definition of R, $\tau_z(n)$ will be non-negative. In the statements that follow, we will have the standing assumption that $\tau_z(n)$ is non-negative, whenever n, z are mentioned in the hypothesis.

Lemma 3.6.9 *If $\tau_z(n) = m \geq 0$, and $\tau_z(n+1) = m+1$, then $f^{n-m} : A_z(n) \to A_0(m)$ is an isomorphism.*

Proof: We have that $f^{m-n} : \underline{P_z(n)} \to P_0(m)$ is an isomorphism, and $f^{m-n}(P_z(n+1)) = P_0(m+1)$, so $f^{m-n}(P_z(n) - \overline{P_z(n+1)}) = P_0(m) - \overline{P_0(m+1)}$. $\boxed{3.6.9}$

Corollary 3.6.10 *If $A_0(m)$ is a descendant of $A_0(N)$, and $\tau_z(n) = m \geq 0$, and $\tau_z(n+1) = m+1$ (for $n \geq p$), then $A_z(n) \in \mathbf{A}_z$ (and the modulus of $A_z(n)$ is equal to the modulus of $A_0(m)$).*

Proof: By Lemma 3.6.9, $f^{n-m} : A_z(n) \to A_0(m)$ is an isomorphism. By assumption, $f^{m-N} : A_0(m) \to A_0(N)$ is a covering map. Therefore $f^{n-N} : A_z(n) \to A_0(N)$ is a covering map, so $A_z(n) \in \mathbf{A}_z$. $\boxed{3.6.10}$

We now make a simple observation about the function $\tau_z(n)$:

Lemma 3.6.11 *For all n, z, $\tau_z(n+1) \leq \tau_z(n) + 1$*

Proof: For all ν, ζ, we have that $f^{\nu - \tau_\zeta(\nu)}$ is the greatest iterate of f that is univalent on $P_\zeta(\nu)$. Therefore $f^{n-\tau_z(n)}|_{P_z(n)}$ is univalent, and $P_z(n+1) \subset P_z(n)$, so $f^{(n+1)-(\tau_z(n)+1)}|_{P_z(n+1)}$ is univalent, and therefore $n + 1 - \tau_z(n+1) \geq n - \tau_z(n)$, that is, $\tau_z(n+1) \leq \tau_z(n) + 1$. $\boxed{3.6.11}$

This then motivates the following definition:

Definition 3.6.12 *A sequence of non-negative integers (a_n) is* rise-and-drop *if it is bounded below, and $\forall n, a_{n+1} \leq a_n + 1$. The sequences we will consider will either be finite in length or forward-infinite.*

So, by Lemma 3.6.11, $\tau_z(n)$ is rise-and-drop for all $z \in J$.

Definition 3.6.13 *A* step *is a pair $(m, m+1)$ of consecutive non-negative integers.*

Definition 3.6.14 *We say that a rise-and-drop sequence (a_n)* rises past *a step $(m, m+1)$ at time $(n, n+1)$ if $a_n = m$ and $a_{n+1} = m+1$*

Note that if $A_0(m)$ is a descendant of $A_0(N)$, and $\tau_z(n)$ rises past $(m, m+1)$, at time $(n, n+1)$, then $A_z(n) \in \mathbf{A}_z$, and $\operatorname{mod} A_z(n) = \operatorname{mod} A_0(m)$.

Lemma 3.6.15 (Intermediate value theorem for rise-and-drop sequences) *Suppose that $(a_i)_{i=1}^l$ is rise-and-drop. Then if $k \le l$ and $a_k \le m < m+1 \le a_l$, then $(a_i)_{i=1}^l$ rises past $(m, m+1)$.*

Proof: Let $s = \sup\{i \mid a_i \le m\}$. Then $a_s \le m$, $a_{s+1} \ge m+1$, and $a_{s+1} \le a_s + 1$, so $(a_s, a_{s+1}) = (m, m+1)$. $\boxed{3.6.15}$

We now present two further lemmas on rise-and-drop sequences. The first is for the case where $\sup \tau_z(n) = \infty$, and the second is for the case where $\sup \tau_z(n)$ is finite.

Lemma 3.6.16 *If $(a_n)_{n=k}^\infty$ is rise-and-drop, and $\sup(a_n)_{n=k}^\infty$ is infinite, then (a_n) rises past all but finitely many steps.*

Proof: The given sequence a_n is bounded below, so let $b = \inf a_n$. We will show that a_n rises past $(m, m+1)$ for all $m \ge b$. Since a_n is discrete-valued, its infinum is realized, so let k be such that $a_k = b$. Now, suppose we are given such an m. Then, since the sequence (a_i) has a discrete domain, $\limsup a_i = \sup a_i$, so $\exists l > k$ such that $a_l \ge m+1$. Then, by our "intermediate value theorem"(3.6.15), $\exists s$, with $k \le s < s+1 \le l$, such that $(a_s, a_{s+1}) = (m, m+1)$. $\boxed{3.6.16}$

Lemma 3.6.17 *Suppose $(a_n)_{n=k}^\infty$ is rise-and-drop, and $\sup(a_n)_{n=k}^\infty$ is finite. Then*

1. *(a_n) makes the same drop infinitely often: $\exists r, s$ with $r \ge s$ such that $a_n = r$ and $a_{n+1} = s$ for infinitely many n.*

2. *If m is given such that $s \le m < m+1 \le r$, then (a_n) rises past $(m, m+1)$ infinitely many times.*

Proof: In this case, a_n realizes only finitely many values, so there are only finitely many possible pairs of values (a_n, a_{n+1}) with $a_{n+1} \le a_n$, so at least one such pair of values must be realized infinitely often. So there is a monotonically increasing sequence $(n_i)_{i=1}^\infty$, with $a_{n_i} = r$ and $a_{n_i+1} = s$ for some $r \le s$. Then, given m with $r \le m < m+1 \le s$, we note that, for each $i \in \mathbb{N}$, $a_{n_i+1} = s$ and $a_{n_{i+1}} = r$, so, by our "intermediate value theorem"(3.6.15), there exist s_i, with $n_i + 1 \le s_i < s_i + 1 \le n_{i+1}$, such that $(a_{s_i}, a_{s_i+1}) = (m, m+1)$. $\boxed{3.6.17}$

With the help of Lemma 3.6.16, we can now settle the case where $\sup \tau_z(n) = \infty$.

Lemma 3.6.18 *If* $\sup \tau_z(n) = \infty$*, and* $z \in R$*, then the sum of the moduli of the annuli in* $\mathbf{A}_z$ *diverges.*

Proof: By Lemma 3.6.11, $(\tau_z(n))_{n=p}^{\infty}$ is rise-and-drop, so by Lemma 3.6.16, it rises past all but finitely many steps. By Corollary 3.6.10, for each time it rises past a step $(m, m+1)$ with $A_0(m)$ a descendant of $A_0(N)$, we get an element of $\mathbf{A}_z$ with modulus equal to the modulus of $A_0(m)$. Therefore, by Lemma 3.5.5, the sum of the moduli of the annuli in $\mathbf{A}_z$ diverges. $\boxed{3.6.18}$

The rest of this subsection is devoted to showing that the sum of the moduli of the annuli in $\mathbf{A}_z$ for $z \in R$ diverges when $\sup \tau_z(n)$ is finite. In this case the forward orbit of z does not accumulate on the critical point, and we cannot pull back a copy of each of the annuli surrounding the critical point. Instead, we will show that every time $\tau_z(n)$ fails to increase by 1 (when n increases by 1), it in fact "drops" past a step corresponding to the level of a critical descendant of N. The argument here is a little technical: we in fact show that it drops past the level of one of the two fraternal descendants $A_0(N_i)$ of $A_0(N)$.

Then we can conclude, using Lemma 3.6.17 and Lemma 3.6.9 (or Corollary 3.6.10), that $\mathbf{A}_z$ contains infinitely many conformal copies of a single critical descendant of $A_0(N)$. So in this case the series of moduli of elements of $\mathbf{A}_z$ contains infinitely many copies of the same number, and therefore diverges.

Lemma 3.6.19 *Suppose* $A_0(n)$ *is critical annulus. Because the critical point* 0 *of* f *is recurrent, there is some* $m > 0$ *such that* $f^m(0) \in P_0(n+1)$*; chose the least such* m*. Then* $f^m : A_0(n+m) \to A_0(n)$ *is a double cover, so* $A_0(n+m)$ *is a child of* $A_0(n)$*.*

Proof: We have $f^m(P_0(m+n+1)) = P_0(n+1)$, so $f^m(P_0(m+n)) = P_0(n)$, and $f^m|_{P_0(m+n+1)}$ is a degree 2 branched cover, so all we need check is that $f^m|_{P_0(m+n)}$ is degree 2. If not, then $f^i(P_0(m+n)) = P_0(m+n-i)$ for some $0 < i < m$. But then $m+n-i \geq n+1$, so we get $f^i(0) \in P_0(n+1)$, a contradiction. $\boxed{3.6.19}$

Corollary 3.6.20 *If* $A_0(n)$ *is a critical annulus, and* $f^k(0) \in P_0(n+1)$*, then* $A_0(n)$ *has a child* $A_0(t)$*, with* $t \leq n+k$*.*

Proof: Use Lemma 3.6.19: the m in Lemma 3.6.19 satisfies $m \leq k$, so the child $A_0(m+n)$ satisfies $n+m \leq n+k$. $\boxed{3.6.20}$

Lemma 3.6.21 *Suppose* $f^k(P_0(n+k)) = P_0(n)$*. Then for all* $l < n$ *, there exists* t *such that* $n+k > t \geq n$ *and* $A_0(t)$ *is a descendant of* $A_0(l)$*.*

Proof: Let m be the greatest integer such that $m < n$, and $A_0(m)$ is a descendant of $A_0(l)$. (We allow the possibility that $m = l$.) Then $f^k(0) \in P_0(n) \subseteq P_0(m+1)$, so by Corollary 3.6.20, $A_0(m)$ has a child $A_0(t)$ with $t \leq m+k < n+k$, but $t \geq n$ by our choice of m. So, in summary, $A_0(t)$ is a descendant of $A_0(l)$ (via $A_0(m)$), and $n+k > t \geq n$. $\boxed{3.6.21}$

Lemma 3.6.22 *If* $\tau_z(n+1) \le \tau_z(n)$*, then*

$$f^{\tau_z(n)-(\tau_z(n+1)-1)}(P_0(\tau_z(n))) = P_0(\tau_z(n+1)-1).$$

Proof: Let $a := \tau_z(n+1)$ and $b := \tau_z(n)$. Then $f^{n+1-a}(P_z(n+1)) = P_0(a)$, so $f^{n+1-a}(P_z(n)) = P_0(a-1)$. Also $f^{n-b}(P_z(n)) = P_0(b)$. Therefore $f^{(n+1-a)-(n-b)}(P_0(b)) = P_0(a-1)$. $\boxed{3.6.22}$

Lemma 3.6.23 *Suppose* $z \in R$*,* $a := \tau_z(n+1) \le b := \tau_z(n)$*, and* $a > L$*. Then there exists* m *such that* $a \le m < m+1 \le b$*, and* $A_0(m)$ *is a critical descendant of one of the two* $A_0(N_i)$*.*

Proof: We have $f^{b-(a-1)}(P_0(b)) = P_0(a)-1$ by Lemma 3.6.22. Then by Lemma 3.6.21, since both of the levels N_i are less than $a-1$ (by the definitions of L and R), there is a descendant $A_0(t)$ of each $A_0(N_i)$ with $a-1 \le t < t+1 \le b$. The two descendants are distinct, so one of the t's must satisfy $a \le t < t+1 \le b$. This t is the required m. $\boxed{3.6.23}$

Lemma 3.6.24 *If* $\limsup \tau_z(n) < \infty$*, and* $z \in R$*, then the sum of the moduli of the annuli in* $\mathbf{A}$ *that surround* z *diverges.*

Proof: By the first part of Lemma 3.6.17, there exists $a \le b$ such that for infinitely many n, $\tau_z(n+1) = a$ and $\tau_z(n) = b$. Then by Lemma 3.6.23 we can find a descendant $A_0(m)$ with $a \le m < m+1 \le b$. So then by the second part of Lemma 3.6.17 there are infinitely many q with $\tau_z(q) = m$ and $\tau_z(q+1) = m+1$. Then by Corollary 3.6.10, $A_z(q) \in \mathbf{A}$, and $\operatorname{mod} A_z(q) = \operatorname{mod} A_0(m)$. Thus there are infinitely many annuli in $\mathbf{A}_z$ with modulus $\operatorname{mod} A_0(m)$. So the sum of the moduli of the annuli in $\mathbf{A}_z$ diverges. $\boxed{3.6.24}$

Lemma 3.6.25 *For all* $z \in R$*, the sum of the moduli of the annuli in* $\mathbf{A}_z$ *diverges.*

Proof: This is just the conjunction of Lemmas 3.6.18 and 3.6.24. $\boxed{3.6.25}$

Chapter 4

Further Results

4.1 Local Connectivity of Corresponding Points in the Mandelbrot Set

In this section we will prove:

Theorem 4.1.1 *If $c \in \partial M$, and $f_c(z) = z^2 + c$ is not renormalizable and has no indifferent fixed point, then M is locally connected at c.*

The proof is by analyzing the behaviour of the graphs $\Gamma_n(c)$ as c varies. (Recall the definition of Γ_n (here written as $\Gamma_n(c)$ to emphasize its dependence on f_c) from section 1.2.)
Proof: We have $f_c = z^2 + c$. For $c \in M$ let $\beta(c)$ denote the landing point of the zero ray, and for $c \neq 1/4$ let $\alpha(c)$ denote the other fixed point. Then if $\alpha(c)$ is repelling let $\Gamma_n(c)$ denote the level n Yoccoz graph for f_c. We wish to show that M is locally connected at c if c is non-renormalizable.

The set of all c for which $\alpha(c)$ is repelling and has rotation number p/q is called the p/q limb of the Mandelbrot set, denoted $M_{p/q}$. There is a unique $c_{p/q}$ for which f_c has a parabolic fixed point of multiplier $e^{2\pi i p/q}$, and $M_{p/q}$ is one of the two components of $M - \{c_{p/q}\}$.

Theorem 4.1.2 *The arguments of the rays landing at a repelling periodic point of a polynomial (with connected Julia set) are stable under perturbation. Likewise, the arguments of rays landing at a non-ramified preiterate of a repelling periodic point are stable under perturbation. A non-ramified preiterate of α is a point z such that $f^n(z) = \alpha$, and $(f^n)'(z) \neq 0$.*

Suppose $c \in M_{p/q}$. Then we can define $\Gamma_n(c)$ for all n. For a given $n > 0$, if $f_c^n(0) \neq \alpha$ (here 0 is, of course, the critical point of f_c), then $\Gamma_n(c)$ remains constant on some neighborhood of c in M, in the sense that the information of which rays land in groups of q at points in $f^{-n}(\alpha)$ remains constant in that neighborhood. In other words, for any given n, the combinatorial information in $\Gamma_n(c)$ is locally constant in $M_{p/q} - \{c \mid f_c^n(0) = \alpha(c)\}$. (Note that $\{c \mid f_c^n(0) =$

$\alpha(c)\}$ is finite, since it is a subset of $\{c \mid f_c^n(0) = f_c^{n+1}(0)\}$). Therefore the information is constant on the finitely many components of $M_{p/q} - \{c \mid f_c^n(0) = \alpha(c)\}$, which are each open in M.

Now, suppose $c \in M_{p/q}$, and $\forall n$, $f_c^n(0) \neq \alpha(c)$. Then, for each $n > 0$, consider $M_{p/q}^n(c)$, the component of $M_{p/q} - \{c \mid f_c^n(0) = \alpha(c)\}$ that contains c. We claim that, if f_c is non-renormalizable, then the sets $M_{p/q}^n(c)$ form a neighborhood base for c in M.

Consider the continuum $\bigcap_{n=1}^{\infty} \overline{M_{p/q}^n(c)}$. We just need to show that it is degenerate (i.e. a single point). If it is non-degenerate, then we can find $c' \neq c$ such that $\forall n$, $f_{c'}^n(0) \neq \alpha(c')$ (and such that $c' \neq c_{p/q}$). Then $c' \in M_{p/q}^n(c)$ for all n, so $\Gamma_n(c) = \Gamma_n(c')$ for all n. But then $c = c'$ by the following:

Theorem 4.1.3 *Suppose $c, c' \in M_{p/q}$ for some $0 < p/q < 1$ in lowest terms. If f_c and $f_{c'}$ are combinatorially equivalent (i.e. $\forall n$, $\Gamma_n(c)$ and $\Gamma_n(c')$ have the same rays landing in groups of q) and non-renormalizable, then $f_c = f_{c'}$.*

Proof:
Step 1. If c, c' are combinatorially equivalent and non-renormalizable, then there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h \circ f_c = f_{c'} \circ h$ on $\mathbb{C}$, $h(J_c) = J_{c'}$, and $h|_{\mathbb{C}-J_c}$ is conformal.
Step 2. From the above and the holomorphic removability of J_c we can immediately conclude that h is conformal, and hence f_c and $f_{c'}$ are conformally and thus affinely conjugate, so $c = c'$.

Proof of Step 1. For all n, since $\Gamma_n(c)$ and $\Gamma_n(c')$ have the same combinatorics, there is a canonical homeomorphism from the one to the other (off of $\alpha(c)$ and its preiterates, it factors through the two Riemann maps). That homeomorphism can be extended conformally outside of $\Gamma_n(c)$, and arbitrarily on the bounded components of the complement, to form a homeomorphism h_n from $\mathbb{C}$ to $\mathbb{C}$. The h_n are eventually constant on the complement of J_c (and converge uniformly on any compact subset of the complement), and are uniformly bounded on J_c. Any pointwise limit h_∞ of the h_n is conformal off of J_c, and, because the diameter of the pieces of $\Gamma_n(c)$ and $\Gamma_n(c')$ go to zero as $n \to \infty$, h_∞ is continuous (sufficiently nearby points in the domain lie in the same or adjacent small pieces, and therefore have nearby images), injective (distinct points eventually lie in distinct and non-adjacent pieces, and hence the lim inf for the distance between their images under the h_n is positive), and proper, and therefore it is a homeomorphism. It is a conjugacy off of J_c, and therefore on all of $\mathbb{C}$. $\boxed{4.1.3}$

$\boxed{4.1.1}$

4.2 Finitely Renormalizable Quadratic Polynomials

4.2.1 Definitions

Suppose $f_c(z) = z^2 + c$ has both fixed points repelling. Then we can form the Yoccoz graph Γ_n. for f. Suppose futher that $f^n(0) \neq \alpha$ for all $n > 0$. Then $P_0(n)$ is well defined.

We say that f is *combinatorially renormalizable* (with period $n > 1$) if $\exists k, n$ such that $f^n : P_0(k+n) \to P_0(k)$ is a degree two branched cover, and $f^{tn}(0) \in P_0(k+n)$ for all $t > 0$. We call the map $f^n : P_0(k+n) \to P_0(k)$ a *combinatorial renormalization* (with period n). If such an n exists for f, it will be unique, and in fact, for all $k' > k$, $f^n : P_0(k'+n) \to P_0(k)'$ will have the same properties (mentioned above) as $f^n : P_0(k+n) \to P_0(k)$. The set $K_{\mathcal{R}f} := \{z \in P_0(k+n) \mid \forall t > 0, f^{nt}(z) \in \overline{P_0(k+n)}\} = \bigcup_i \overline{P_0(i)}$ is called the filled-in Julia set of the combinatorial renormalization for f. $K_{\mathcal{R}f}$ does not depend on the choice of k, so it is a well-defined object (given a renormalizable map f).

Following Douady and Hubbard, we define a *quadratic-like map* as a holomorphic degree 2 branched cover $g : U' \to U$, where $U', U \subset \mathbb{C}$ are topological disks, and $\overline{U'} \subset U$. We define the filled-in Julia set K_g of g by $K_g = \{z \in U' \mid \forall t > 0, g^t(z) \in U'$. We say that g is *non-trivial* if the critical point of g lies in K_g. In this case, K_g is connected. Given $f_c(z) = z^2 + c$, we say that f is *geometrically renormalizable* (with period n) if there exist U', U and n such that $f^n : U' \to U$ is a non-trivial polynomial-like map, and $0 \in U'$. (Note then that 0 is the unique critical point of f^n in U'.)

We have the following theorem[Mil2, Hub], which is part of the Yoccoz theory:

Theorem 4.2.1 (Straightening Theorem) *If f is combinatorially renormalizable with period n, then f is geometrically renormalizable with period n, and the Julia set of the combinatorial renormalization is the same as that of the geometric renormalization.*

The converse is also true (but we will not need it), if we assume that f is *simply* (geometrically) renormalizable. For a definition of simple renormalization, and a discussion, see [Mc]. The geometric renormalizations that arise from the above theorem will always be simple renormalizations.

We will use the term *renormalizable* to mean combinatorially renormalizable, which, by the above, is equivalent to being (simply) geometrically renormalizable.

We will require the following theorem of Douady and Hubbard[DH]:

Theorem 4.2.2 *If $g : U' \to U$ is a quadratic-like map, then there exists a quasiconformal embedding $h : U \to \mathbb{C}$ and a map $f_c(z) = z^2 + c$ such that $h(g(w)) = f_c(h(w))$ for all $w \in U'$. It follows then that $h(K_g) = K_f$. Moreover, c is unique if we require that g is non-trivial, and that the dilatation of h be zero a.e. on K_g.*

(Note that the last condition is trivially satisfied if K_g has measure 0.) In the case where g is a (geometric) renormalization of f, we will call the f_c given above the *straigtened renormalization* of f.

Suppose all periodic cycles of $f = f_c$ are repelling. Then if $f^n : U' \to U$ is a (geometric) renormalization of f, then all of its periodic cycles are repelling. It follows that all the periodic cycles of the map the straightened renormalization $f_{c'}$ given by the preceeding theorem (so $h \circ f^n = f_{c'} \circ h$ on U') must also be repelling, because repelling periodic cycles are preserved under quasiconformal conjugacy. Now, with the same supposition on f, we will say that f is m

($m > 1$) times renormalizable if its straightened renormalization is $m-1$ times renormalizable. (We say that f is once renormalizable if f is renormalizable). We say that f is *infinitely renormalizable* if f is m times renormalizable for all m. If f is not infinitely renormalizable, then there exists a series of maps $f_0 = f, f_1, \dots f_m$ (all of the form $f(z) = z^2 + c$) such that f_{i+1} is the straightened renormalization of f_i, and f_m is not renormalizable.

4.2.2 Renormalization and Holomorpic Removability

Suppose that f is renormalizable, with period n. Let q be the number of external rays that land at α. We will consider two cases:

1. *Primitive* renormalization: $n > q$
2. *Satellite* renormalization: $n = q$

In Case 1, there exists a nondegenerate critical annulus $A_0(N)$, just as in the non-renormalizable case [Mil2], and furthermore we can find $M \geq N$ such that $f^n : P_0(M+n) \to P_0(M)$ is a non-trivial quadratic-like map [Mil2] (and therefore $J_{\mathcal{R}f} \subset P_0(k)$ for all k). In Case 2, $\alpha \in \partial P_0(k)$ for all k, and $\alpha \in J_{\mathcal{R}f}$. This will make Case 2 a little harder to handle in what follows.

We will first prove:

Proposition 4.2.3 *Suppose f is primitively renormalizable, and $J_{\mathcal{R}f}$ is holomorphically removable. Then J_f is holomorphically removable.*

Proof: In this case, $\alpha \notin \overline{P_0(N+1)}$ (where $A_0(N)$ is non-degenerate, as mentioned above), so if P is the piece of level N such that $\alpha \in \partial P$, and $P \subset P_c(0)$ (where $c = f(0)$ is the critical value), then $c \notin P$, and then we can proceed as in Subsection 2.3.2, and in fact the entire argument of Section 2.3 applies, so Lemma 2.1.2 applies, and therefore Lemma 1.5.1 applies, i.e, for all pieces P of the Yoccoz puzzle for f, $\mathcal{QD}(J \cap P, P) < \infty$.

We can also prove the Tiling Lemma, 1.5.2, for f, as follows. Let $L = M + n$ (where M is mentioned above). Then given $P_0(p)$, with $p > L$, we let $R = J_{\mathcal{R}f}$, and let Q_i be the pieces of level $q_i > p$ such that $Q_i \subset A_0(q_i - 1)$. Now since $f^n : P_0(M+n) \to P_0(M)$ is a degree two branched cover, $f^n : A_0(r+n) \to A_0(r)$ is a degree two (unbranched) cover for all $r \geq M$. Therefore, by induction, $f^{tn} : A_0(r+tn) \to A_0(r)$ is an unbranched cover (of degree 2^t) for all $r \geq M$. It follows that f^{L-q_i} is univalent on q_i, since we can find tn such that $M + n > q_i - tn \geq M$, and then $f^{tn} : A_0(q_i - 1) \to A_0(q_i - 1 - tn)$ is a covering, so f^{tn} is univalent on any pieces Q_i with $Q_i \subset A_0(q_i - 1)$. Letting $T = P_0(P) - R \cup \bigcup \overline{Q_i}$ we find that $T \cap J = \emptyset$ because $R = \bigcap_{q>p} P_0(q) = P_0(p) - \bigcup_{q>p} A_0(q-1)$, and $A_0(q-1) \cap J = (\bigcup_{q_i=q} Q_i) \cap J$. This completes the proof of the tiling lemma.

Now we can apply the proof of Lemma 1.4.1 verbatim, and conclude that there exists a K such that $\mathcal{QD}(J \cap P, P) \leq K$ for all pieces P. We can then apply the proof of Main Theorem,1.1.6, given in section 1.4, but there is one minor detail: the diameter of the pieces for

f do not go to zero. However, given a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h|_{\mathbb{C}-J_f}$ is conformal, we can still find a sequence of quasiconformal mappings $h_n : \mathbb{C} \to \mathbb{C}$ such that $h_n = h$ on Γ_n and on the unbouounded component of $\mathbb{C}-\Gamma_n$. Then, by the compactness of K-quasiconformal mappings[Ah, LV], we can find a uniform limit h_∞ of a subsequence of the h_n. Then $h_n = h$ on $\mathbb{C} - J$ and also on $\{z \mid \exists n : f^n(z) = \alpha\}$, which is dense in J. So $h_\infty = h$ and h_∞ is K-quasiconformal, so we conclude that h is always K-quasiconformal, where K depends only on f. Then we can conclude, as in section 1.4, that J_f is holomorphically removable. $\boxed{4.2.3}$

Corollary 4.2.4 *Suppose f is finitely renormalizable (with all periodic cycles repelling), so there exists a sequence $f_0 = f, f_1, \ldots, f_m$ where f_{i+1} is the straightened renormalization for f_i, and f_m is non-renormalizable. Suppose that each renormalization is primitive. Then J_f is holomorphically removable.*

Proof: We prove this by backwards induction on i. Certainly J_{f_i} is holomorphically removable if $i = m$. If $J_{f_{i+1}}$ is holomorphically removable, then $J_{\mathcal{R}f_i}$ is too, because there is a quasiconformal map from one to the other. Then by the proposition, J_{f_i} is holomorphically removable. $\boxed{4.2.4}$

4.2.3 Satellite Renormaliztion

We must now consider the case of Case 2, i.e. satellite renormalization. In this case our goal is still to first prove piece-dependent distortion bounds. We observe that a sufficient hypothesis for Lemma 2.1.2 (cf. Subsection 2.3.3) is the following:

Hypothesis 4.2.5 *There exists mappings $q_1, q_2 : \mathcal{C} \to S^1$ such that $\forall x \in \mathcal{C}, q_1(x) \simeq q_2(x)$, and q_1, q_2 extend to quasisymmetric mappings $q_1 : [0,1] \to [A,B], q_2 : [0,1] \to [C,D]$ (where q_2 is orientation-reversing), where $CS(A,B,C,D)$ is a vertical slice, as in section 2.3.3.*

We then prove the following lemma:

Lemma 4.2.6 *Suppose that f is finitely renormalizable (with all periodic cycles repelling). Then Hypothesis 4.2.5 is satisfied for J_f.*

Proof: If f is primitively renormalizable, then, as discussed previously (in the proof of 4.2.3), the hypothesis holds. If f is satellite renormalizable, we will need the following lemma:

Lemma 4.2.7 *Suppose f has a satellite renormalization, and let $\hat{f}$ be the straightened renormalization of f. Then J_f satisfies Hypothesis 4.2.5 if $J_{\hat{f}}$ does.*

Proof: There are two things to prove:

1. If J_g satifies Hypothesis 4.2.5, then it also does for a slice $CS(A',B',C',D')$ with $\beta \in \partial CS(A',B',C',D')$ (so $A' = D' = 0$).

2. If $J_{\hat{f}}$ satisfies the above conclusion, then J_f satisfies Hypothesis 4.2.5

To prove the first, we first apply g^{-1} to get the slice in $P_\beta(0)(= P_0(0))$. Then, we can keep applying the branch of g^{-1} that fixes β to get a series of slices limiting on β. Then we can map the first half of the Cantor set to the first slice, and the third quarter of $\mathcal{C}$ to the second slice, and the seventh eighth of $\mathcal{C}$ to the third slice, and so forth, and then map the last point of $\mathcal{C}$ to β.

To prove the second, we need a folk result, which relates the combinatorial ray-pairs for $J_{\hat{f}}$ to those of J_f[Mil3]. It states that there exists a pair (a_0, a_1) of binary strings, such that if $t_1 \equiv t_2$ on $J_{\hat{f}}$, then $E(t_1) \equiv E(t_2)$, where E is defined by $E(.d_1 d_2 d_3 \ldots) = .a_{d_1} a_{d_2} a_{d_3}$. From this, the second step easily follows, after we note that $E(.000\ldots)$ and $E(.111\ldots)$ are both rays landing at α, in the case where f is primitively renormalizable. $\boxed{4.2.7}$

The result then follows by induction on the number of times that f is renormalizable. $\boxed{4.2.6}$

Now given this lemma, we observe that if f is finitely renormalizable, then Lemma 2.1.2 holds for the pieces of the Yoccoz puzzle for f, and hence so does Lemma 1.5.1. Now, we wish to proceed as in the case of primitive renormalization, assuming that $J_{\hat{f}}$ is holomorphically removable, and proving that J_f is. As before $J_{\mathcal{R}f}$ is qc equivalent to $J_{\hat{f}}$ and is hence holomorphically removable, but we run into a minor glitch in proving the Tiling Lemma, because now $J_{\mathcal{R}f} \not\subset P_0(k)$ for any k, so $J_{\mathcal{R}f} \cap P_0(k)$ is not compact. We can get around this by proving the following lemma:

Lemma 4.2.8 *Suppose that $A \subset \mathbb{C}$ is compact and holomorphically removable, and $U \subset \mathbb{C}$ is open, and ∂U is locally holomorphically removable, and $\mathcal{QD}(A \cap U, U) < \infty$. Then if $h : \overline{U} \to \mathbb{C}$ is an embedding, and $h|_{U-A}$ is conformal, then $h|_U$ is conformal.*

(A closed subset $B \subset \mathbb{C}$ is *locally holomorphically removable* if, for all open sets $V \subset \mathbb{C}$, if $h : B \to \mathbb{C}$ is an embedding, and $h|_{V-B}$ is conformal, then $h|_B$ is conformal.)

Proof: Given h as above, we can find $\tilde{h} : \overline{U} \to \mathbb{C}$ such that $\tilde{h}|_U$ is quasiconformal, and $\tilde{h}|_U = h|_U$. Then $\tilde{h}^{-1} \circ h$ maps $\overline{U}$ homeomorphically to itself and is the identity on ∂U; extend it by the identity to a homeomorphism $g : \mathbb{C} \to \mathbb{C}$. Then g is quasiconformal on $\mathbb{C} - A \cup \partial U$, and therefore is quasiconformal on $\mathbb{C} - A$, because ∂U is locally holomorphically removable. Then g is qc on all of $\mathbb{C}$, so $h|_U = \tilde{h} \circ g$ is qc, and hence conformal (since A must have measure 0). $\boxed{4.2.8}$

Note that the boundary of any Yoccoz puzzle piece is locally holomorphically removable, since every point has a neighborhood that is a smooth curve, or the union of a point and something locally holomorphically removable. So then we can apply the above lemma with $A = J_{\mathcal{R}f}$ and $U = P_0(k)$, and then the proof of the tiling lemma goes through (with $R = J_{\mathcal{R}f} \cap P_0(k)$—it's okay that R is not compact, since it's still HR in $P_0(k)$, as in the above lemma), and then we can proceed just as in the primitive case, to get the analog of Lemma 4.2.3 (and hence Lemma 4.2.4). This completes the proof of holomorphic removability of Julia sets of finitely renormalizable quadratic polynomials.

4.3 Conjectures on Holomorphic Removability

The techniques used here to show holomorphic removability could conceivably have much wider application. Boundaries of John domains have been shown already to be holomorphically removable[Jon]; it seems that these techniques could provide a different, and in some ways more elementary, proof. A careful examination of how distortion bounds are obtained for the canonical model mentioned above suggest that such bounds could be shown for much more general models, and then some very general criterion for holomorphic removability could be described, perhaps in terms of the capacities of certain sets. There are also further possible dynamical applications. Certainly it should be possible to apply these techniques to obtain holomorphic removability for all higher degree polynomial Julia sets where the Yoccoz theory yields local connectivity. It seems likely that holomorphic removability can also be shown for Julia sets of certain infinitely renormalizable quadratic polynomials for which local connectivity and area zero are known [Lyu1, Yar]. Finally, I conjecture that the boundary of M is itself holomorphically removable, and that it can be proved to be so by this technique of cutting neighborhoods of the set into pieces and showing distortion bounds for those pieces [Kah].

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