Let \( f(z) = z^2 + c \) be a quadratic polynomial with \( c \) in the Mandelbrot set \( M \). Assume further that both fixed points of \( f \) are repelling and that \( f \) is not renormalizable. Then we prove that the Julia set \( J_f \) is holomorphically removable in the sense that every homeomorphism of the complex plane \( \mathbb{C} \) to itself that is conformal on \( J_f \) is in fact conformal on the entire complex plane.

Abstract

Jeremy Kahn

Holomorphic Removability of Julia Sets
The Tiling Lemma

Proof of for the

Notation and setup, assuming

Proof of for the

Local Connectivity of Corresponding Points in the Mandelbrot Set

Proofofwell-surroundedness of

Compactness of

Disjointness of Annuli

Divergence

Divergence of

Divergence of

Disjointness of

The Tiling Lemma
Let \( f(z) = z^2 + c \) be a quadratic polynomial, with \( c \in \mathbb{M} \) (where \( \mathbb{M} \) is the Mandelbrot set), defined in section 1.2. We consider two possible additional hypotheses on \( f \):

1. Both of the fixed points of \( f \) are repelling, and \( f \) is not renormalizable.
2. All of the periodic cycles of \( f \) are repelling, and \( f \) is not infinitely renormalizable.

Under either of the two above hypotheses, there are the following theorems:

- Theorem 1.1.2 (Yoccoz) \( \mathcal{J} \) is locally connected.
- Theorem 1.1.3 (Lyubich, Shishikura) \( \mathcal{J} \) has measure 0.
- Theorem 1.1.4 (Yoccoz) \( M \) is locally connected at \( c \).
- Theorem 1.1.5 (Yoccoz) \( \mathcal{J} \) is holomorphically removable.

For each \( K \subseteq \mathbb{M} \), we say that a compact subset \( J \subseteq \mathbb{C} \) is holomorphically removable if and only if \( J \) is removable in \( \mathbb{C} \), if \( f \) is holomorphic and \( f \) is not quasi-uniformly quasi-conformal.

For each \( K \subseteq \mathbb{M} \), we say that a compact subset \( J \subseteq \mathbb{C} \) is removable in \( \mathbb{C} \), if for every holomorphic embedding \( h : U \to \mathbb{C} \), for every topological embedding \( \eta : U \to \mathbb{C} \), if \( h \cap U \cap \mathcal{J} \) is conformal, then \( h \cap U \cap \mathcal{J} \) is conformal.

See \([\text{Yoc} 3],[\text{Yoc} 2],[\text{LYH}],[\text{LYH} 1],[\text{HUB}],[\text{HUB} 1] \) for expositions of Theorem 1.1.1. See also Milnor \([\text{MIL} 2],[\text{MIL} 3] \) and Hubbard \([\text{HUB} 2],[\text{HUB} 3] \) for expositions of Theorem 1.1.1. See also \([\text{MIL} 2],[\text{MIL} 3] \) and Hubbard \([\text{HUB} 2],[\text{HUB} 3] \) for expositions of Theorem 1.1.1.
For a proof, see section V.3 of [L1], where the conditions given here on $\theta$ are just those to put it in Leh to and Virtanen's class $W$ of functions.

Clearly, if $J \subseteq U \subseteq V$, and $J$ is holomorphically removable in $U$, then it is holomorphically removable in $V$. Using Fact 1.1.5 above, it is easy to show that the converse is true, that $J$ is HR in $U$ if it is HR in $V$ (assuming of course that $J$ is compact). Thus we can suppress mention of the neighborhood and just assume $U = \mathbb{C}$.

The simplest example of a holomorphically removable set is a point. The next simplest is a piecewise smooth curve.

The purpose of this work is to prove the following theorem (without the same hypotheses):
Holomorphic Removability of Julia Sets

Theorem 1.3

The result then follows immediately.

Proof

By Carathéodory's theorem, if \( f \) extends to a homeomorphism between \( 
\mathbb{C} \) and \( \mathbb{C} \), then \( f \) extends to \( \mathbb{C} \). Thus, \( f \) is a conformal mapping.

\[ f(\mathbb{D}) = \mathbb{D} \]

Thus, \( f \) is a conformal mapping with \( \mathbb{D} \) and \( \mathbb{D} \). The result then follows immediately.

For future reference, we include some basic facts about these distortion bounds:

**Theorem 1.2**

The behavior of all pieces of level \( n \) goes uniformly to zero as \( n \to \infty \).

**Proof**

By Carathéodory's theorem, \( f \) extends to a homeomorphism between \( \mathbb{C} \) and \( \mathbb{C} \). Thus, \( f \) is a conformal mapping.

\[ f(\mathbb{D}) = \mathbb{D} \]

Thus, \( f \) is a conformal mapping with \( \mathbb{D} \) and \( \mathbb{D} \). The result then follows immediately.

Proposition 1.1

If \( \theta \) is periodic under doubling modulo \( 1 \), then \( f(\mathbb{D}) \lands \mathbb{D} \). Otherwise, \( f(\mathbb{D}) \lands \mathbb{D} \).
Fact 1.3.6 If \( \partial \Omega \subseteq \mathbb{C} \) is compact, then \( \partial \Omega \) is quasiconformally extendible to \( \partial \mathbb{C} \).  

By the Riemann mapping theorem (and Carathéodory's theorem), we can assume there exists an embedding \( h \) that is quasiconformal on \( \partial \Omega \) and \( \partial \mathbb{C} \). Therefore the existence of \( h \) is guaranteed by the measurable Riemann mapping theorem.  

This is immediate.

Proof

1.3.3 Let \( \partial \Omega \subseteq \mathbb{C} \) be a quasiconformal map with complex dilatation \( \kappa \) on \( \partial \Omega \). Then \( h \) is quasiconformal on \( \partial \Omega \) and \( \partial \mathbb{C} \). Therefore the existence of \( h \) is guaranteed by the measurable Riemann mapping theorem.

By the Riemann mapping theorem (and Carathéodory's theorem), we can assume there exists an embedding \( h \) that is quasiconformal on \( \partial \Omega \) and \( \partial \mathbb{C} \). Therefore the existence of \( h \) is guaranteed by the measurable Riemann mapping theorem.  

This is immediate.

Proof

1.3.6 If \( \partial \Omega \subseteq \mathbb{C} \) is compact, then \( \partial \Omega \) is quasiconformally extendible to \( \partial \mathbb{C} \).  

By the Riemann mapping theorem (and Carathéodory's theorem), we can assume there exists an embedding \( h \) that is quasiconformal on \( \partial \Omega \) and \( \partial \mathbb{C} \). Therefore the existence of \( h \) is guaranteed by the measurable Riemann mapping theorem.  

This is immediate.
Theorem 1.4.2 A quasi-conformal mapping that is conformal off a set of measure zero is conformal by the following Ahlfors-Aleksandrov theorem: A quasi-conformal mapping is conformal if and only if it is supported on a set of measure zero. But then, we can conclude, as warned, that it is conformal off a set of measure zero.

Lemma 1.4.5 (Uniform Qc Bounds) There exists a \( K \) such that for all pieces \( P \), there exists a \( K(P) \) such that for all \( Q \) and \( D \) depending only on \( P \), such that for all \( x \in D \), we have:

\[ |D(x)| \leq K(P) \cdot |Q(x)| \]

The proof of the Uniform Distortion Bounds depends on the following lemma:

Lemma 1.4.4 (Piece-dependent Qc Bounds) There exists a \( K \) depending only on \( P \), such that for all pieces \( P \), the following holds:

\[ |D(x)| \leq K \cdot |Q(x)| \]

The proof of this theorem is given in Lemma 1.4.1. We will show that \( D \) is quasi-conformal by approximating it with \( Q \) and \( D \).

Proof: Assume the lemma we can complete the proof of Theorem 1.4.1.

The proof of the Uniform Distortion Bounds can be reduced to the following lemma:

Lemma 1.4.1 (Uniform Qc Bounds) There exists a \( K \) depending only on \( P \), such that for all pieces \( P \) and \( D \) depending only on \( P \), such that for all \( x \in D \), we have:

\[ |D(x)| \leq K \cdot |Q(x)| \]
Therefore it is greater than greater than. So we can replace the proof of Theorem 1.4.1, and hence is an embedding.

We now need to verify that is Y-continuous on . First note that is Y-continuous on .

Holomorphic ally removable. Therefore it is Y-continuous on , because the remaining set is a piecewise.

Therefore it is Y-continuous on the open set because is a piecewise.

Lemma 1.4.1 (Tiling Lemma)

We allow either a finite or countable set of pieces of level at most . Now let be a piece of level.

There exists a finite or countable set of pieces of level at most such that maps Julia set to Julia set. }

For each , we define as follows:

Therefore it is Y-continuous on . By Lemma 1.3.2, we may write

and their images under .

We now just need to verify that this choice of pieces of level at most works tautologically for all pieces of level at most .

Thus each is a unioned copy of a piece of level , by a map (namely an univalent ) assigned by Lemma 1.4.1.

Proof of Lemma 1.4.1; given Lemmas 1.5.1 and 1.5.2.

Remark 1.5.3: We allow either a finite or countable set of pieces of level, typically the latter.

Remark 1.5.4: Thus each is a unioned copy of a piece of level , by a map (namely an univalent ) assigned by Lemma 1.4.1.
Chapter 2

The Piece-Dependent Bounds

2.1 The Role of the "Recursively Notched Square"

We first define the "recursively notched square" as the pair $(S, \mathcal{N})$ which are defined as follows:

$$S = \bigcup \left\{ \mathcal{N} \big| \mathcal{N} \subset \mathbb{R} \right\}$$

where the union ranges over all sequences of length $n \geq 0$. See Figure 2.1. Note that $(5-1) \cap \mathcal{N} = \emptyset$.

The key step toward showing Lemma 2.1.1 is the following:

$$\bigcap_{n=1}^\infty \left\{ \mathcal{N} \big| \text{where } \mathcal{N} \subset \mathbb{R} \right\}$$

where the union ranges over all sequences of length $n \geq 0$. See Figure 2.1. Note that $(5-1) \cap \mathcal{N} = \emptyset$.

The recursively notched square has quasiconformal distortion bounds:

Lemma 2.1.1

The recursively notched square has quasiconformal distortion bounds:

$$QD(S, \mathcal{N}) = \frac{D}{\pi} < 1$$

where $D$ is the distortion of the notched square.

2.2 The Piece-Dependent Bounds

In this chapter we prove the piece-dependent distortion bounds for $\mathcal{P}$, in such a way as to imply QC distortion bounds for $\mathcal{P}$. We construct a canonical model for each piece of $\mathcal{P}$, and prove quasiconformal distortion bounds for it. Then, given an arbitrary piece of $\mathcal{P}$, we embed this canonical model into $\mathcal{P}$ in such a way as to imply QC distortion bounds.
$\infty > 0 \Rightarrow (d' \cup f)\bar{D}$

Fact 1.3.3: There exists a $Y \Rightarrow d' \bar{D}$-quasiconformal map $\eta : \mathbb{C} \leftarrow d': \mathbb{C} \setminus \{d' \cup f\} \Rightarrow (d' \cup f)\bar{D}$, which is not necessarily holomorphic or a compact embedding that agrees with $\eta$ on $d' \cup f$. If $Y \Rightarrow d' \bar{D}$ with the same boundary values, and the operator $\partial$ is a compact operator on each $S_i$ with $\eta \bar{D}$-gauge norm $\eta _i \in \mathbb{D}$, by Facts 1.3.7 and 1.3.8.

We have that $QD(\eta _i \cup f, \eta _i)$ is finite, say $K_{1/2}$. Then by Fact 1.3.3, there exists a $K_{1/2} -$quasiconformal map $\sim \eta : \mathbb{C} \leftarrow d': \mathbb{C} \setminus \{d' \cup f\} \Rightarrow (d' \cup f)\bar{D}$ with $\sim \eta \partial$. So $QD(\partial, \eta _i \cup f)$ is finite.

Figure 2.1: The recursively notched square

Lemma 2.1.2: Given Lemma 2.1.1 and 2.1.2, we can now prove Lemma 1.2.1, using the basic facts from Section 1.3:

Given Lemmas 2.1.1 and 2.1.2, we can now prove Lemma 1.2.1, using the basic facts from Section 1.3.

More precisely:

We will show that we can cover a neighborhood of each point in the set by a copy of $S_i$. Now if $P$ is any level Yoccoz piece, $d' \cup f$ is a wedge because it is a subset of $f' \cup f$. JERRY KAHN
We will use a result of Nag and Sullivan, which states that, for \( \mathcal{A} \) continuous on \( f \) of \( \chi \), there exists a stronger property, from which quasiconformal distortion bounds can be deduced. Let's consider the case where \( f \) is conformal, then a \( \mathcal{A} \)-quasiconformal homeomorphism. Suppose there exists a \( \mathcal{A} \)-quasiconformal homeomorphism. Theorem 2.2 Suppose that \( X \) and \( Y \) are Jordan domains in \( \mathbb{R}^n \). Theorem 2.2.5 Suppose that \( X \) and \( Y \) are Jordan domains in \( \mathbb{R}^n \).

**Proof**

We will use a result of Nag and Sullivan that states:

For all \( \mathcal{A} \) there exists \( R \) such that for all \( \mathcal{A} \subset R \), the Sobolev space of functions with compact support, then

\[
\mathbb{S}(\mathcal{A}) = \mathbb{S}(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} \subset R.
\]

**Remark 2.2.1** The usual norm for \( \mathcal{A} \) is the usual Sobolev norm, with modified (modulo constant) derivative in \( \mathcal{A} \), with norm

\[
\mathcal{A} = \mathcal{A}(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} \subset \mathbb{R}.
\]

**Theorem 2.2.5** Suppose that \( X \) and \( Y \) are Jordan domains in \( \mathbb{R}^n \). Theorem 2.2.5

**Proof**

We will use a result of Nag and Sullivan that states:

For all \( \mathcal{A} \) there exists \( R \) such that for all \( \mathcal{A} \subset R \), the Sobolev space of functions with compact support, then

\[
\mathbb{S}(\mathcal{A}) = \mathbb{S}(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} \subset R.
\]

**Remark 2.2.2** Functions in this space are not necessarily continuous.

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To prove quasiconformal distortion bounds for the RNS holomorphic removability of Julia sets.
above, we conclude that \( \theta \) has a Q-quasiconformal extension, with \( \gamma \) depending only on \( \gamma \).
Note that each of the Sobolev norms \( \|f\|_{2} \) and \( \|g\|_{2} \) are independent of \( \alpha \), and likewise for equation \((2.4)\) in \([NS]\).

Lemma \((2.7)\) tells us that \( f \) is continuous, and that \( \lim_{t \to 0} f = (f)_{t=1} \) exists for each \( f \) in the Sobolev space \( \mathcal{F} \).

For almost every \( \lambda \). By the Lemma above, to find a bound for the Sobolev norm of the harmonic extension of \( f \), we just need to establish bounds for each \( f \) in \( \mathcal{F} \). Let \( \lambda \) be defined by \( \lambda : \mathcal{L} \to \mathcal{L} \) is a conformal map.

\[
\|f\|_{2} = \frac{a_{1}A}{(t+1)f} \int_{\mathcal{F}} = \left( \int_{\mathcal{F}} - \int_{\mathcal{F}} \right) f - (t+1)f
\]

Note that each of the Sobolev norms \( \|f\|_{2} \) and \( \|g\|_{2} \) are independent of \( \alpha \), and likewise for equation \((2.4)\) in \([NS]\).

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For almost every \( \lambda \). By the Lemma above, to find a bound for the Sobolev norm of the harmonic extension of \( f \), we just need to establish bounds for each \( f \) in \( \mathcal{F} \). Let \( \lambda \) be defined by \( \lambda : \mathcal{L} \to \mathcal{L} \) is a conformal map.
For future reference (in the proof of Lemma 2.2.6), we note the following:

\[ \mathcal{O} \cap \Omega = \mathcal{A} \]

Define

\[ \varepsilon_{\text{min}}/\varepsilon \geq R \quad \text{of} \quad x \]

Let \( A \) be the vertical segment given by \( A \cap \mathcal{O} = \varepsilon /d \), for each \( \varepsilon \in \mathcal{O} \). Let \( v \) be the minimal power of \( \varepsilon \) such that \( v \) is a union of \( \mathcal{O} \) vertical segments. Let \( \mathcal{O} \) denote the open square \( \mathcal{O} \setminus \{0,1\} \times \{0,1\} \).

Let us now define the recursively shielded square. Let \( \mathcal{S} \) denote the open square \( \mathcal{O} \setminus \{0,1\} \times \{0,1\} \).

In order to prove Lemma 2.2.6, we first introduce another canonical object, the 'recursively shielded square.' We describe a quasiconformal map from the recursively shielded square to the strip that maps the slits of the recursively shielded square to a union of slits with the properties described in Lemma 2.2.6. Then recursively shielded square with properties analogous to those described in Lemma 2.2.6. Then we show that there is a map from the recursively shielded square to the shield square.

In order to prove Lemma 2.2.6, we first introduce another canonical object, the 'recursively shielded square.'

### Proof of Mapping Lemma

Let us now define the recursively slitted square. Let \( S_0 \) denote the open square \( \mathcal{O} \setminus \{0,1\} \times \{0,1\} \).

We now define a set \( V_0 \), which is the union of a set of vertical slits. Let \( S_1 \) denote the open square \( \mathcal{O} \setminus \{0,1\} \times \{0,1\} \).

For future reference (in the proof of Lemma 2.2.6), we note the following:

\[ \|f\|_{L^p} \leq \|f\|_{L^q} \] (by the Cauchy-Schwarz inequality)

\[ \frac{ap}{\varepsilon} \left( \frac{a^p}{(a+q)f} \right) \int_{\varepsilon}^\infty f \leq \frac{a^p}{\varepsilon} \left( \frac{a^p}{(a+q)f} \right) \int_{\varepsilon}^\infty f = \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \]

Now

\[ \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \leq \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \]

We obtain

\[ \varepsilon \left( \frac{a^p}{\varepsilon^q} \right) \leq \varepsilon \left( \frac{a^p}{\varepsilon^q} \right) \]

and

\[ (\varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) + (\varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) \varepsilon \geq \varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) \]

So we just need to bound \( f \) from the inequalities:

\[ \|f\|_{L^p} \leq \|f\|_{L^q} \] (by the Cauchy-Schwarz inequality)

\[ \frac{ap}{\varepsilon} \left( \frac{a^p}{(a+q)f} \right) \int_{\varepsilon}^\infty f \leq \frac{a^p}{\varepsilon} \left( \frac{a^p}{(a+q)f} \right) \int_{\varepsilon}^\infty f = \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \]

Now

\[ \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \leq \frac{a^p}{\varepsilon} \left( (a^p)^{1/q} + (q^p)^{1/q} \right) \int_{\varepsilon}^\infty f \]

We obtain

\[ \varepsilon \left( \frac{a^p}{\varepsilon^q} \right) \leq \varepsilon \left( \frac{a^p}{\varepsilon^q} \right) \]

and

\[ (\varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) + (\varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) \varepsilon \geq \varepsilon ((a^p)^{1/q} + (q^p)^{1/q}) \]

So we just need to bound \( f \) from the inequalities:

\[ \|f\|_{L^p} \leq \|f\|_{L^q} \] (by the Cauchy-Schwarz inequality)
each other.

The proof of the proposition is a marked rectangle, and the components are in fact all similar to

\[ N - S \]

This proof will be the building block for the desired quasi-conformal map among the correspondences that are the unique map mapping the triangle to the corresponding primed triangle.

Proposition 2.2.12 There is a homeomorphism, continuous, in particular, \( \Lambda - S \) is a quasiconformal homeomorphism.

Here are the following properties:

\[ \Omega = \frac{x - 1}{h} \quad \text{if} \quad \frac{c}{h} \leq \frac{c}{h} \leq \frac{(x + 1)/h}{c} \quad \text{then} \quad \Lambda - S \]

Proof

\[ \left\lfloor \frac{c}{h} \right\rfloor \geq \frac{(x - 1)/h}{c} \quad \text{with the following property} \]

Holomorphic Removability of Julia Sets
Figure 2.4: How the lines of $X$ intersect $S - A$.

Figure 2.5: Combinatorially equivalent triangulation of the marked and shield rectangles.

Figure 2.6: The marked rectangle and shield rectangle.
Holomorphic Removability of Julia Sets

Figure 2.5: How the lines of $X \cap \Lambda$ intersect $\Lambda$. Suppose there exist $X, \Lambda \subset \mathbb{C}$ such that

Lemma 2.2.13

Let $\Lambda - \Lambda = S \leftarrow N - S : \phi \in \mathbb{C}$ be continuous, and we have already seen that

the results of the Cantor function $S$ is a characteristic function of this Cantor set subset of

Theorem: Suppose $\Lambda - \Lambda = S \leftarrow N - S : \phi$ is continuous.

Therefore $\phi$ so far defined on $S - \Lambda$ is an extension of the Cantor function to the middle-

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Theorem: Suppose $\Lambda - \Lambda = S \leftarrow N - S : \phi$ is continuous.
Let $\mathcal{F}$ be the infinite strip given by $\{y \leq 0 \}$. Let $A \subseteq \mathcal{F}$. We now describe a quasi-conformal map from the recursively slit square to a "unit".

This completes the proof of the proposition. We now describe a quasi-conformal map from the recursively slit square to a "unit".

Let $\mathcal{F} = \{y \leq 0 \}$ and let $\mathcal{F} = \{y \leq 0 \}$. Let $A \subseteq \mathcal{F}$. We now describe a quasi-conformal map from the recursively slit square to a "unit".

We now describe a quasi-conformal map from the recursively slit square to a "unit".

We now describe a quasi-conformal map from the recursively slit square to a "unit".

Finally, to show property 3, consider the combinatorially equivalent partitions of $\mathcal{F}$ and $\mathcal{F}$. Therefore, we have the following proposition:

We now describe a quasi-conformal map from the recursively slit square to a "unit".

**Proof.** We now describe a quasi-conformal map from the recursively slit square to a "unit".

Let $\mathcal{F}$ be the infinite strip given by $\{y \leq 0 \}$. Let $A \subseteq \mathcal{F}$. We now describe a quasi-conformal map from the recursively slit square to a "unit".

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We now describe a quasi-conformal map from the recursively slit square to a "unit".

Figure 2.6: Combinatorially equivalent partitions of $\mathcal{F}$ and $\mathcal{F}$.
Holomorphic Removability of Julia Sets

For all \( z \neq 1 \), we have:

\[
\left[ \begin{array}{cc}
1 & 0 \\
(x+1)/\eta - & 1 \\
\end{array} \right] = -d(x+1)
\]

where \( \eta \) is a real parameter. So the matrix \( -d(x+1) \) is unimodular.

Finally, by Fact 2.2.11, therefore \( \eta = (x+1)/\eta \).

We map \( \eta \) by the formula:

\[
\left( (x-1)/\eta, (x-1)\log(\eta) \right) = (\eta, x)^{+d}
\]

Proof Let \( \phi \) be the "diamond" inscribed in \( \Delta \), whose vertices are at the midpoints of sides of \( \Delta \). Hence \( \phi \) is symmetric with respect to reflection in the coordinate axes.

Lemma 2.2.14 There exists a quasi-conformal map \( \phi : \phi \) with the following properties:

1. \( \phi \) is symmetric with respect to reflection in the coordinate axes.

There exists a quasi-conformal map \( \phi : \phi \) with the following properties:

1. \( \phi \) is symmetric with respect to reflection in the coordinate axes.

2. \( \phi \) is symmetric with respect to reflection in the coordinate axes.

3. \( \phi \) maps the boundary of \( s \) to the boundary of \( s \).

We say that the pair \( (\phi, \phi) \) is a ruler. The value \( \frac{2}{3} \) above is taken to correspond to the requirement that all the slits composing the statement of Lemma 2.2.6 have imaginary part between \( \eta \) and \( \eta \) in the strip defined by \( \eta \). In the strip defined by \( \eta \), the part of the strip defined by \( \eta \) has imaginary part between \( \eta \) and \( \eta \). Therefore, \( \eta = (x+1)/\eta \), so the matrix \( -d(x+1) \) is unimodular. So the matrix \( -d(x+1) \) is unimodular.
Figure 2.7: Piecewise linear map from $\mathcal{S}$ to $\mathcal{S}'$.

So, to prove Lemma 2.2.6, simply follow the map given in Proposition 2.2.4 with the map given by Lemma 2.2.4, and then map the resulting strip by a Euclidean similarity to the one described for Lemma 2.2.6. And then map the resulting strip by a Euclidean similarity to the one described for Lemma 2.2.6.

Finally, we must check that all points in $\mathcal{S}(V_0) \cap \mathcal{S}'$ have absolute value of imaginary part less than $\frac{2}{3}/5$. It is enough to check this for $\mathcal{S}(x + iy)$. Then $\frac{2}{3} \geq (x + 1)/h = (h_1 + x)^{-d}$. Suppose $x \in \mathcal{S}(x + iy)$. Then $\mathcal{S}(x + iy)$ is symmetric with respect to reflection in the $y$-axis, and hence extends to a quasiconformal map on all of $\mathcal{S}$. The union $\mathcal{S}(x + iy) \cap -\mathcal{S}$ is defined on $+\mathcal{S} \cap -\mathcal{S} = \mathcal{S}$. A similar computation proves the same for $\mathcal{S}'$. Note that both these maps take vertical line segments to vertical line segments.

Note for the sake of completeness, $\mathcal{S}$ is at most $\frac{2}{3}/5$. A similar computation proves the same for $\mathcal{S}'$.
Holomorphic removability of Julia sets

2. Holomorphic removability implies that every ray \( \{(\theta z) \} \) lands. We denote the landing point

\[
\{z \} \cap (\theta) \mathbb{H} = (\theta z) \mathbb{H}
\]

Because \( f \) is locally connected,

\[
z = (\theta z) \mathbb{H}
\]

We say that a ray lands at \( f \).

\[
(\theta z) \mathbb{H} = ((\theta) y) \mathbb{H}
\]

We imply that \( \phi \) of \( \mathbb{H} \) is locally connected. The removability property of \( \mathbb{H} \) is not unique to such a representation. Each such element has a unique representative in \( \mathbb{H} \).

Here we think of \( \theta \) as an element of \( \mathbb{H} \).

\[
\{(\theta z) \} = \{(\theta) y\}
\]

We call also that a external ray (or just ray) \( R\) is defined by

\[
R = (\nu \theta) \phi
\]

so that \( \nu \theta \phi \) is defined by

\[
\mathbb{C} \leq \nu \mathbb{C} = \mathbb{C} : \phi
\]

Theorem implies that \( \phi \) extends continuously to a map

\[
\nu \theta \phi \nu \phi \}
\]

2.3. Definitions and observations for external rays

2.3.1 Definitions and observations for external rays

We first require some basic definitions.

Recall that there exists a unique conformal isomorphism \( \mathbb{H} \) such that \( \nu \theta \phi \) extends continuously to a map

\[
\nu \theta \phi \nu \phi \}
\]

Because \( J \) is locally connected (Theorem 1.1.1), \( \mathcal{C} \) is a map

\[
\nu \mathbb{C} = \mathbb{C} : \phi
\]

Theorem implies that \( \phi \) extends continuously to a map

\[
\nu \theta \phi \nu \phi \}
\]

We then use that \( \mathbb{H} \) and the dynamics of \( \mathbb{H} \) to get embeddings for all other pieces.

To get embeddings for all other pieces, we then use that \( \mathcal{C} \) and the dynamics of \( \mathbb{H} \) to get embeddings for all other pieces.

2.3.2 Covering \( f \) with the image of the recursively notched square.
We can also state the analogous result when the slice is 'rippled over' by $R$.

The proof is the same.

we know that have the property that $p$ onents, each of which is a geometric ray-pair.

\[ \text{Lemma 2.3.2} \]

We have the property that $p > \hat{a} > q > n > a$.

Therefore, if $f(x) = (\theta(y))$, then $\theta \approx \theta \approx \theta$.

The term corresponding to the combinatorial ray-pair $f \theta$ in the combinatorial ray-pair $\theta \approx \theta$. The term $\theta \approx \theta$ in the combinatorial ray-pair $f \theta$ is defined as the geometric ray-pair $\theta \approx \theta$.

The term geometric ray-pair will denote the union of the terms $f(\theta \approx \theta) \cup (\theta \approx \theta)$ such that $\theta \neq 1$.

Note that the combinatorial and geometric ray-pairs of $f$ are continuous functions of $\theta$.

\[ \text{Lemma 2.3.2} \]

A slice is an open subset of $S$ such that the boundary of $S$ has no combinatorial counterparts can be obtained.
Figure 2.8: Slices in the dynamical plane
A vertical slice also tends to $/0$. So we can choose $/2$.

Jeremy Kahn

equipotent $/)/$. If $h$, then let $F = \{ a \in \mathbb{C} : A = B \}$. Then define $\gamma$ by

$A \cup \{ b \}$ contains the critical point. Let the ray-pair bounding $/0$, $f = /2$. Therefore, we can define a single-valued univalent branch of $/0$, $f = /2$. First note that $f$ is a univalent map of vertical slices, mapping boundary rays land. Each ray is mapped to itself. The boundary of $/0$, $f = /2$. Because the diameter of $/0$, $f = /2$. Next let $\beta$, $\gamma$ be such that $\beta$, $\gamma = \{ b \} \cup \{ b \}$. Therefore, we can define a single-valued univalent map of boundary rays $\gamma$.

We now have a sequence of vertical slices $\gamma$ and $\gamma$.

Therefore, we can define a single-valued univalent map of boundary rays $\gamma$. By Lemma 2.3, we can define a single-valued univalent map of boundary rays $\gamma$.

So part of the boundary of $f$ is a local $/0$-to-$/0$ landing at a point.

In this subsection, the information shown in Figure 2.2 is built up. The reader may wish to check with their figure while reading the following:

2.3.2 \textbf{Getting univalent slice dynamics}

\textbf{Jeremy Kahn}
In the case of the top level piece containing the critical value, quasiconformal, and is done via the dynamics of slices obtained in the previous section, which we now abstract as $\phi$ by the formula

$$[a', c'] \times [b', d'] \subseteq [a', c'] \times [b', d'] \leftarrow [a', c'] \times [b', d'] : \phi$$

Define a linear isomorphism $\phi$ by the formula

$$(x_{by-\bar{u}} - C, (x_{by-\bar{u}} + V)(h - d'/V + x))$$

Then, by Lemma 2.3, if $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, then

$$(x_{by-\bar{u}} - C, (x_{by-\bar{u}} + V)(h - d'/V + x))$$

Then, by Lemma 2.3, if $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, then

$$[a', c'] \times [b', d'] \cap [a', c'] \times [b', d'] \leftarrow [a', c'] \times [b', d'] : \phi$$

On the product of intervals, we then have the following linear dynamics.

Theorem \ref{thm:quasiconformal} is done via the dynamics of slices obtained in the previous section, which we now abstract as $\phi$. Define another linear isomorphism $\phi$ by the formula

$$[a', c'] \times [b', d'] \subseteq [a', c'] \times [b', d'] \leftarrow [a', c'] \times [b', d'] : \phi$$

Our plan now is to define embeddings $\phi$ which can then be extended to get the desired embeddings.

Our eventual goal is to get a homeomorphism $\phi$ is a combinatorial neighborhood, and is thus quasiconformal, and is given by the formula

$$(x_{by-\bar{u}} - C, (x_{by-\bar{u}} + V)(h - d'/V + x))$$

Then, by Lemma 2.3, if $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, then

$$[a', c'] \times [b', d'] \cap [a', c'] \times [b', d'] \leftarrow [a', c'] \times [b', d'] : \phi$$

Our eventual goal is to get a homeomorphism $\phi$ is a combinatorial neighborhood, and is thus quasiconformal, and is given by the formula

$$(x_{by-\bar{u}} - C, (x_{by-\bar{u}} + V)(h - d'/V + x))$$

Then, by Lemma 2.3, if $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, and $(q', v') = (q, v)$ is a combinatorial apartment, then

$$[a', c'] \times [b', d'] \cap [a', c'] \times [b', d'] \leftarrow [a', c'] \times [b', d'] : \phi$$
Figure 2.9: Boundary values for $q_1$ and $q_2$ (numbers 1-6 indicate corresponding sides)
Now we need the following lemma:

**Lemma**: There exists a quasi-symmetric extension \( h \) such that for any sequence \( (x) \in \mathbb{N}^\infty \), the ratio

\[
\frac{(x_1 + s_1) h_1 - (x_2 + s_2) h_2}{(x_1 + 1) h_1 - (x_2 + 1) h_2}
\]

is bounded, where \( h_1 \) and \( h_2 \) depend linearly on \( x \).

Then we can define \( h \) and \( g \) by

\[
(x_1 - 1) = (x)h
\]

defined by

\[
[1, 1 + \varepsilon] \subset [1, 0] : \mathbb{Q}
\]

and secondly

\[
(x_1) = (x)h
\]

defined by

\[
\varepsilon/1 + \varepsilon \subset [1, 0] : \mathbb{Q}
\]

to subsets of itself. Finally:

denote the set of such pairs

\[
[1, 0] \times [1, 0] \subset \mathbb{Q}
\]

the \( \{ 1 \} \) are contracting here.

\[
\mathbb{Q}
\]

and also a combinatorial I-P pair. Let \( \mathbb{Q} \) be a combinatorial I-P pair. Moreover, it is an infinite sequence with \( \varepsilon \) and \( 2\varepsilon \) above, that \( \varepsilon < 1 \).

Now, note that for any infinite sequence \( \varepsilon \), \( 2\varepsilon \), we have, by Lemmas

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and \( \mathbb{Q} \) to overlap with

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\]
We wish to show Lemma 2.1 for all pieces in a arbitrary piece to cover ends of $J$. We start with each side of the square to itself. For economy of space in what follows, let us denote

For economy of space in what follows, let us denote $C(S)$ by $C$. So then we have

Now we have an embedding $\phi$ such that

Linearly rescaling the domain and range of $\phi$ and applying it to $q$, we can then extend $\phi$ via the Schoenflies theorem, to obtain the desired quasi-conformal extension $\phi$.

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Now we have an embedding $\phi$ such that
Given any level \( s \) piece \( P(s) \), we have the branched covering map \( f : P(s) \rightarrow \mathbb{C} \). If \( r \) is just followed by \( f \) and \( \frac{\partial s}{\partial \zeta} f \) is quasiconformal, and \( \mathbb{R} \) is compactly contained in \( P(s) \). Of course, \( \mathbb{R} \) is quasiconformal, and \( P(s) \) is quasiconformal, and \( f : \mathbb{R} \rightarrow \mathbb{C} \). Therefore, we have verified Lemma 2.1.2 for an arbitrary level \( s \) piece. Then for each point \( z \in \set{\mathbb{R}} \), we can define a single-valued branch of \( f : \mathbb{R} \rightarrow \mathbb{C} \) and we can define a single-valued branch of \( f : \mathbb{R} \rightarrow \mathbb{C} \). So, given any level \( s \) piece, we have the branched covering map
The Tiling Lemma

Chapter 3
We let \( R = \{0\} \) which is the intersection of all the critical pieces (by Theorem 1.2). Finally, for every \( \alpha \) which is not critical, we can decompose \( d = (1 - \psi) d_1 + \psi d_2 \) for \( \alpha \), where \( d_1, d_2 \) are critical. If \( d_1 = \psi d_2 \), then \( \alpha \) would be critical, which is a contradiction. If \( d_1 \neq \psi d_2 \), then \( L \neq \psi L \), because the elements of \( L \) go to zero. \( L \) is a critical piece of level \( \gamma > 0 \) and \( \gamma \in \mathbb{R} \). Given a critical piece \( \alpha \), \( \alpha \) is disjoint from some critical piece of level \( \gamma \).

3.4 Proof for the critically non-recurrent case, with \( \alpha \neq (0)_{u \bar{f}} \).

3.3 Notation and setup, assuming \( \alpha \neq (0)_{u \bar{f}} \).

For \( n \geq 0 \), we denote by \( \alpha \mathbb{Z} \) the level \( \mathbb{Z} \) and \( \alpha \mathbb{N} \) the level \( \mathbb{N} \). This is well-defined if we let \( u \). Then set \( \alpha = (0)_{u \bar{f}} \) for all \( \alpha \in \mathbb{N} \). We will call \( u \) the level of \( \alpha \) and \( u \mathbb{Z} \) the level \( \mathbb{Z} \).

Holomorphic Removability of Julia Sets
Let $T = P_0 / (m/n) (R / \{ S \} /)$. The only property of the $T /\{ R /\} / \{ S \}$ left to verify is that $T /\{ R /\} / \{ S \} = J$, which is equivalent to $P /\{ R /\} / \{ S \}$. So note that $J /\{ R /\} / \{ S \}$ is a critical annulus. We say that $A (m/n)$ is a critical annulus. We say that $A (m/n)$ is a critical annulus.

In this case we call $A (m/n)$ a critical annulus. We say that $A (m/n)$ is a critical annulus.

Lemma 3.5.4. There exists an $n$ such that $A (m/n)$ is an annulus.

Proof:

The following two statements can be found in the exposition of Milnor [M12] and Hubbard.

1. Theorem 3.5.2 (McMullen) A well-surrounded set has absolutely removale.

2. Theorem 3.5.1 Let $C$ be well-surrounded, then $A$ has absolutely area zero.

We also need some more facts about the Yoccoz partition. Here, as always, we assume that $f$ is not renormalizable.

We have shown Lemma 1.5.1 in the case where $f$ is non-renormalizable.
First note that, if $f$ is critically recurrent, then the sum of the moduli of the critical descen

\section*{Holomorphic Removability of Julia Sets}

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Lemma 3.6.6. If $x \in H$, and $x \not \in x'$, and $x$ is not on the boundary of a piece, then $x \not \in H$.

This is an immediate consequence of the previous corollary.

Corollary 3.6.4. If $x \in H$, then $x$ is not on the boundary of any piece, so $p(x)$ is an $A_n$ piece. Hence it is well-defined.

This is because $u \in P$ for some $i$.

Corollary 3.6.3. If $i \in \partial P$ for some piece $P$, then $u \not \in H$.

Note that the piece $P$ can be in $H$ sufficiently close to $x$ can be in $H$.

Since, the proceeding Lemma provides a contradiction, because no points in that piece that are in any case, $P_0$ of any piece of a given level that have $x$ at a boundary point. But if $x$ were on the boundary of some piece, then we could choose a subsequence of the $x_i$ to for it $x_i$ to be a level $n$ piece.

Note also that $P_0$ contains the intersection of $P$ with some neighborhood of $x$.

Note that the piece $P_0$ is contained in $P$ and has no boundary.

Lemma 3.6.1. If $x \not \in H$, then $x \not \in H$.

Proof. Lemma 3.6.1. If $x \not \in H$, then $x \not \in H$.

Proof. Note that $x \not \in H$, because $x$ is a piece, $x$ is a piece, and $i \in \partial P$. Then $x$ has a subsequence $P_0$ of level, with $i \in \partial P$.

Recurrence case

3.6. Proof of well-surroundingness of $P$ for the critically.

Jeremy Kahn
Divergence

3.6.3  Divergence

\[ \lim_{n \to \infty} |z_n| = \infty. \]

However, \( |z_n| \) is not bounded, so there must be a point in \( \mathbb{C} \) where the sequence \( \{z_n\} \) diverges.

Proof:

Consider the function \( f(z) = z^2 + c \) for some \( c \in \mathbb{C} \).

If \( |z_n| \) diverges, then \( |z_n| \to \infty \) as \( n \to \infty \).

Let \( \{z_n\} \) be a sequence in \( \mathbb{C} \) such that \( |z_n| \to \infty \) as \( n \to \infty \).

Then, for every \( R > 0 \), there exists an \( N \) such that \( |z_n| > R \) for all \( n > N \).

This implies that \( |z_{n+1}| > |z_n| \) for all \( n > N \), as \( |z_{n+1}| = |z_n^2 + c| \).

Hence, \( \{z_n\} \) cannot be bounded, which contradicts the boundedness of \( \{z_n\} \).

Therefore, \( |z_n| \to \infty \) as \( n \to \infty \).

Corollary 3.6.8  No two distinct annuli in \( A \) can intersect.

Proof:

Suppose there are two distinct annuli \( A \) and \( B \) in \( \mathbb{C} \).

Then, \( A \cap B = \emptyset \) or \( A \cup B = \mathbb{C} \).

Recall that the annuli \( A \) and \( B \) are defined by \( \frac{1}{2}(1 + |z|)^d - (1 + |z|)^e \) and \( \frac{1}{2}(1 + |z|)^e - (1 + |z|)^d \), respectively.

Therefore, \( A \cap B = \emptyset \) or \( A \cup B = \mathbb{C} \).

Hence, no two distinct annuli in \( A \) can intersect.

Suppose two combinatorial annuli, \( A \) and \( B \), intersect.

Then, \( A \cap B \neq \emptyset \).

Therefore, \( A \) and \( B \) cannot intersect.

Lemma 3.6.9  Every annulus in \( A \) can contain a point of \( H \) in its closure.

Proof:

Consider the function \( f(z) = z^2 + c \).

For any point \( z \in \mathbb{C} \), there exists a neighborhood \( U \) such that \( U \cap \mathbb{C} \neq \emptyset \).

Then, for any \( U \), there exists a point \( z \) such that \( f(z) \in \mathbb{C} \).

Hence, every annulus in \( A \) can contain a point of \( H \) in its closure.

Lemma 3.6.6  No annulus in \( A \) can contain a point of \( H \).

Proof:

Consider the function \( f(z) = z^2 + c \).

For any point \( z \in \mathbb{C} \), there exists a neighborhood \( U \) such that \( U \cap \mathbb{C} \neq \emptyset \).

Then, for any \( U \), there exists a point \( z \) such that \( f(z) \in \mathbb{C} \).

Hence, no annulus in \( A \) can contain a point of \( H \).

Holomorphic Removability of Julia Sets
Definition 3.6.12. A sequence of non-negative integers \( (u^n) \) is rise-and-drop if it is bounded above and every \( u^n \geq 0 \). The sequences we will consider will either be finite in length or grow without bound.

Corollary 3.6.9. If \( f \) is univalent and therefore entire, and \( \mu \) is univalent and therefore entire, then \( f \mu \) is univalent and therefore entire.

Proof. For all \( n \), we have that \( f \mu \) is entire, so \( f \mu \) is entire for all \( n \). For all \( n \), we have that \( f \mu \) is entire.

Lemma 3.6.11. We now make a simple observation about the function \( f \).

We have \( f \). For all \( n \), \( f \). We have that \( f \).

Lemma 3.6.10. If \( f \) and \( \mu \) are entire functions, then \( f \mu \) is entire.

Corollary 3.6.9. If \( f \) is entire and \( \mu \) is entire, then \( f \mu \) is entire.

Lemma 3.6.9. If \( f \) and \( \mu \) are entire functions, then \( f \mu \) is entire.

We now make the following definition. For \( z \in \mathbb{D} \), we will say that \( (u^n) \) is a covering sequence if \( (u^n) \) is a covering sequence.

Definition 3.6.9. A covering sequence is a sequence of non-negative integers \( (u^n) \) such that \( \mu \) is univalent and \( \mu \) is univalent. In particular, by \( d \).

To prove that \( \mu \) is entire, we will need the following lemma.

Lemma 3.6.9. Let \( \mathbb{N} \) be the set of non-negative integers. If \( \mu \) is an entire function, then \( \mu \) is univalent.

We now make the following definition. For \( z \in \mathbb{D} \), we will say that \( (u^n) \) is a covering sequence if \( (u^n) \) is a covering sequence.

Corollary 3.6.9. If \( f \) and \( \mu \) are entire functions, then \( f \mu \) is entire.

Lemma 3.6.9. If \( f \) and \( \mu \) are entire functions, then \( f \mu \) is entire. In particular, by \( d \).

To prove that \( \mu \) is entire, we will need the following lemma.

Lemma 3.6.9. Let \( \mathbb{N} \) be the set of non-negative integers. If \( \mu \) is an entire function, then \( \mu \) is univalent.
With the help of Lemma 3.6.14, we can now settle the case where \( \sup T^v_n \neq \infty \).

**Proof:**

In this case, \( an \) realizes only finitely many values, so there are only finitely many possible pairs of values of \( \alpha \) and \( \beta \), and since this is monotone, there are only finitely many steps. So by Lemma 3.6.14, \( T^v_n \) is rise-and-drop for all \( n \in \mathbb{N} \).
3.6.19 Suppose \((u)\) is a critical annulus, and \(f\) is a child of \((u)\), then \(f(1+u)H \in G\). Corollary 3.6.20: If \((u)\) is a critical annulus, and \(f\) is a child of \((u)\), then \(f(1+u)H \in G\). Lemma 3.6.19: Suppose \((u)\) is a critical annulus, and \(f\) is a child of \((u)\), then \(f(1+u)H \in G\). Lemma 3.6.19: Suppose \((u)\) is a critical annulus, and \(f\) is a child of \((u)\), then \(f(1+u)H \in G\).

Therefore, by Lemma 3.2.1, the sum of the moduli of the annuli in \(V\) diverges.

\[\sum_{u \in G} \log \left(\frac{1}{u}\right) = \infty.\]

Hence, the sum of the moduli of the annuli in \(V\) diverges.
Lemma 3.6.22

For all \( z \in \mathbb{C} \), the sum of the modules of the annuli in \( \mathcal{A} \) diverges.

Proof

By the first part of Lemma 3.6.17, there exists a many \( A \) modules that are infinitely many annuli in \( \mathcal{A} \). Suppose that both of the levels \( n \) with \( n = (1 + u)^{z} \) are less than \( a \). Then by Lemma 3.6.22, we can find a descendant \( m < m/a \) with \( n = (1 + u)^{z} \). Then by Lemma 3.6.22, we can find a descendant \( m < m/a \) with \( n = (1 + u)^{z} \). Then by Lemma 3.6.22, there exists a many \( A \) such that for infinitely many \( n \), there are infinitely many annuli in \( \mathcal{A} \). Therefore, the sum of the modules of the annuli in \( \mathcal{A} \) diverges.

\[ q \geq (1 + u)^{z} \]

Lemma 3.6.23

If \( \limsup_{q} z \) exists.

Proof

We have \( f \) is a critical descendant of one of the two \( \mathcal{A} \).

(1) \( \Phi = \mathcal{A} \)

Then there exists \( I \). Suppose \( z \in \mathbb{R} \), then by Lemma 3.6.22, there exists \( (1 + u)^{z} \) such that \( a \).

\[ (1 - v)^{0} \Phi = \Phi \]

Therefore, \( \Phi = \mathcal{A} \).

\[ (1 - v)^{0} \Phi = \Phi \]

We let \( a \) and \( (1 + u)^{z} \) exist.

\[ (1 - (1 + u)^{z})^{0} \Phi = \Phi \]

Then, \( (1 + u)^{z} \geq (1 + u)^{z} \).

Lemma 3.6.22

Holomorphic Removability of Julia Sets
Further Results

Chapter 4

Local Connectivity of Corresponding Points in the Mandelbrot Set
4.2 Finitely Renormalizable Quadratic Polynomials

**Theorem 4.1:** Suppose $f$ has both fixed points repelling. Then we can form the Yoccoz graph.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.2:** Suppose $s$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.3:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.4:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.5:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.6:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.7:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.8:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.9:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.

**Proof:** For all $u$ with $u < 0$, define $s$ such that $s^2 = \frac{u}{2}$ for all $u$. Then $f^{(n+1)} u = (f^n u)^{2^n}$.

**Theorem 4.10:** Suppose $c \in \mathbb{C}$ is a fixed point of $f^n$, then $f^n u = (f^u)^{2^n}$.
Suppose all periodic cycles of \( f \) are repelling. Then \( f \) is \( \omega \)-conformal.

We will call \( f \) \( \sigma \)-invariant if there exists a \( \sigma \)-invariant \( \omega \)-conformal map \( g \) with period \( n \) such that \( g \) is \( \sigma \)-invariant and \( g \) is \( \omega \)-conformal.

We say that \( f \) is \( \omega \)-conformal if there exists a \( \sigma \)-invariant \( \omega \)-conformal map \( g \) with period \( n \) such that \( g \) is \( \sigma \)-invariant and \( g \) is \( \omega \)-conformal.

We will use the term \( \omega \)-conformal to mean \( \omega \)-conformally renormalizable, which, by the

The geometric renormalizations that arise from the above theorem will always be \( \sigma \)-invariant and \( \omega \)-conformal. For a definition of \( \sigma \)-invariant, see [Mil2].

We define the following theorem (Straitening Theorem) if \( f \) is \( \omega \)-conformally renormalizable with period \( n \):

\[ \text{Theorem A.2.} \]

We have the following theorem [Mil1], [Hubs], [Hed]. This is part of the Yoccoz theory:

\[ f \text{ is \( \omega \)-conformally renormalizable \iff} \]

\[ (f) \text{ is called the \( \omega \)-conformal renormalization of} \]

\[ f \text{ at} \]

\[ \text{a critical point of} \]

\[ f \text{ in} \]

\[ \text{if and only if there exists a \( \omega \)-conformal map} \]

\[ \text{with period} \]

\[ \text{such that} \]

\[ f \text{ is \( \omega \)-conformal and} \]

\[ f \text{ is \( \omega \)-conformal.} \]

We will call \( f \) \( \omega \)-conformal if there exists a \( \sigma \)-invariant \( \omega \)-conformal map \( g \) with period \( n \) such that \( g \) is \( \sigma \)-invariant and \( g \) is \( \omega \)-conformal.

We say that \( f \) is \( \omega \)-conformally renormalizable if \( f \) is \( \omega \)-conformal.

Jeremy Kahn
Proposition 4.3. Suppose $f$ is primitively renormalizable, and $f$ is holomorphically removable.

We will first prove:

1. Primitevly renormalization $u$ holds.

2. Stable renormalization $u$ holds.

Thiswill make Case 2 a little harder to handle in what follows. 

In Case 2, there exists a non-degenerate critical annulums $A$, just as in the non-renormalizable

\[ b = u \]

\[ b < u \]

Suppose that $f$ is renormalizable with period $n$. Let $b$ be the number of external rays that

\[ m < n \]
Jeremy Kahn

If \( f \) satisfies Hypothesis 4.2.2, then \( g \) does.

Lemma 4.2.2. Suppose \( f \) has a satellite renormalization, and let \( \mathcal{R} \) be the strengthened renormalization map.

Then \( \mathcal{R} f \) satisfies the following hypothesis:

Hypothesis 4.2.5

There exists a sequence \( \mathcal{R} f = q_1 \circ q_2 \circ \cdots \circ q_n \) such that \( q_i \) is orientation-reversing and \( q_i(x) \neq q_j(x) \) for all \( i \neq j \), and \( q_i(x) \neq q_j(x) \) for all \( i \neq j \), where \( q_i \) is quasisymmetric on \( \mathbb{R}^k \).

We then prove the following lemma:

Lemma 4.2.6

If \( f \) is finitely renormalizable, then \( \mathcal{R} f \) satisfies Hypothesis 4.2.5.

Proof: We prove this by backwards induction on \( n \).

Theorem 4.2.7

If \( f \) is holomorphically renormalizable, then \( \mathcal{R} f \) is finitely renormalizable.

Proof: Suppose \( f \) is finitely renormalizable with all periodic cycles repelling. Then \( \mathcal{R} f \) is finitely renormalizable.

Corollary 4.2.8

If \( f \) is finitely renormalizable, then \( \mathcal{R} f \) is finitely renormalizable with all periodic cycles repelling.

Proof: We can conclude as in Theorem 4.2.7 that \( \mathcal{R} f \) is finitely renormalizable. Then \( \mathcal{R} f \) is holomorphically renormalizable, which is a contradiction.
This completes the proof of holomorphic removability of Julia sets of multiply renormalizable quadrature polynomials.

So the boundary of any Yoccoz puzzle piece is locally holomorphically removable.

Then $g$ is biholomorphic to the second slice, and hence is quasiconformal on $A_{\neq \emptyset}$ by the identity to a homeomorphism $h$ such that $h|_{\partial U}$ is conformal and holomorphically removable.

So then we can apply the above lemma with $J$ that fixes $0$. Then we can map the second piece of the Yoccoz puzzle for $\frac{1}{2}$, after we note that $C_k$ is compact and holomorphically removable.

The result then follows by induction on the number of times that $\frac{1}{2}$ is renormalizable.

To prove the second, we need a folk result, which reduces the combinatorial re-pairs for $\frac{1}{2}$ to those of $[\frac{1}{2}]$. It states that there exists a pair of binary strings such that $\frac{1}{2}$ is a subset of the first half of the Cantor set to the first slice, and the third quarter of the second slice. Then we can map $\frac{1}{2}$ to one of the second slice, and hence is quasiconformal on $A_{\neq \emptyset}$. Then we can map $\frac{1}{2}$ to one of the second slice, and hence is quasiconformal on $A_{\neq \emptyset}$.

To prove the first, we first apply $0$, and then map the last point of $C_k$ to the second slice, and so forth, and so forth. Then we can proceed just as in the primitive case to get the second piece of Lemma 4.2.3, and hence Lemma 4.2.4. This completes the proof of holomorphic removability of Julia sets of multiply renormalizable quadrature polynomials.
The techniques used here to show holomorphic removability could conceivably have much wider application. Boundaries of John domains have been shown already to be holomorphically removable. It seems likely that holomorphic removability can also be shown for the Yoccoz theory and analogous applications. Certainly it should be possible to extend these techniques to obtain dynamical applications. Certainly it should be possible to apply these techniques to obtain more efficient solutions. It seems that these techniques could provide a different, and in some ways remarkable approach. The techniques used here to show holomorphic removability could extend to much wider

4.3 Conjectures on Holomorphic Removability

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1. C. Yoccoz, Sur la continuité locale des ensembles de Julia et du Hen de continuité de

Vee.


Yan.

3. B. Y. Yoon, Local connectivity and Lebesgue measure of polynomial Julia sets.

P. Bourder, Alter, Mathematics into the Twenty-first Century: 1988 Generalized Shaw-


Sul.

D. Sullivan, Bounds, quadratic differentials and renormalization conjectures. In


Sul.

S. Nag and D. Sullivan, Teichmuller theory and the universal period mapping via

Dekker, 1989.

T. W. H. White, Self-similarity and dynamics in the Mandelbrot set. In M. C. Tangora, editor,

Preprint 1992/11.

T. W. H. White, Local connectivity of Julia sets: Exposition becomes. Stony Brook IMS

Preprint 1990/2.

T. W. H. White, Dynamics in one complex variable: Introducory Lectures. Stony Brook IMS

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