

ALMOST EVERY REAL QUADRATIC MAP IS EITHER REGULAR OR STOCHASTIC

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ABSTRACT. We prove uniform hyperbolicity of the renormalization operator for all possible real combinatorial types. We derive from it that the set of infinitely renormalizable parameter values in the real quadratic family $P_c : x \mapsto x^2 + c$ has zero measure. This yields the statement in the title (where “regular” means to have an attracting cycle and “stochastic” means to have an absolutely continuous invariant measure). An application to the MLC problem is given.

1. INTRODUCTION

The main goal of this paper is to prove hyperbolicity of the renormalization operator for all possible real combinatorial types, and to derive from it the well-known Regular or Stochastic Conjecture.

The Renormalization Conjecture was stated by Feigenbaum [F1, F2] and independently by Coulet & Tresser [CT, TC] in 1978, for the particular case of doubling combinatorics. In the works of Lanford, Epstein, Eckmann, Sinai, Sullivan, McMullen [La1, E, EE, VSK, S1, S2, McM2], among many others, a spectacular progress in this problem has been achieved (see [L4] for more historical details and references). However, until recently only the doubling case had been essentially resolved. In [L4] the Conjecture was proven for bounded combinatorics. In this work we will extend the conjecture (compare with Lanford’s Conjecture [La2] for circle maps) and prove it for all possible combinatorial types.

Let us introduce a few notations and state the results. Let us consider the real quadratic family $P_c : x \mapsto x^2 + c$, $c \in [-2, 1/4]$ (in this parameter range the maps P_c have an invariant interval). A map $f = P_c$ is called renormalizable if it has a periodic interval $L \ni 0$ of some “renormalization period” $p > 1$, i.e., $f^p L \subset L$. The set of renormalizable quadratics is the union of disjoint closed intervals $J_k \subset (-2, 1/4)$ called “renormalization windows” (see e.g., [MvS]). The renormalization period $p_k = p(J_k)$ is constant through the window.

Let $\mathcal{QL}_{\mathbb{R}}$ stand for the space of real quadratic-like maps, and let $\mathcal{H}_{\mathbb{R}}(f) \subset \mathcal{QL}_{\mathbb{R}}$ stand for the real hybrid class via $f \in \mathcal{QL}$ (see §2.1 for the definitions). By a renormalization strip $\mathcal{T}_{J_k} \subset \mathcal{QL}_{\mathbb{R}}$ we mean the union of hybrid classes $\mathcal{H}_{\mathbb{R}}(P_c)$, $c \in J_k$. On each renormalization strip one can define a real analytic renormalization operator $R_{J_k} : \mathcal{T}_{J_k} \rightarrow \mathcal{QL}_{\mathbb{R}}$, so that Rf is an appropriately restricted p_k -fold iterate of f . These maps can be organized in a single piecewise analytic map $R : \cup \mathcal{T}_{J_k} \rightarrow \mathcal{QL}_{\mathbb{R}}$.

Let Σ stand for the space of two-sided sequences of natural numbers, and ω stand the shift on this symbolic space. We are now ready to state the Renormalization Theorem for all real combinatorial types:

Theorem 1.1 (Full renormalization horseshoe). *There is a set $\mathcal{A} \subset \cup \mathcal{T}_{J_k}$ so that:*

- \mathcal{A} is R -invariant and $R|_{\mathcal{A}}$ is topologically conjugate to the two-sided shift ω ;
- The restriction $R|_{\mathcal{A}}$ is uniformly hyperbolic;
- Any stable leaf $W^s(f)$, $f \in \mathcal{A}$, coincides with the hybrid class $\mathcal{H}_{\mathbb{R}}(f)$ and has codimension 1;
- Any unstable leaf $W^u(f)$ is an analytic curve which transversally passes through all real hybrid classes except the cusp one (corresponding to $c = 1/4$);
- The renormalization operator has uniformly bounded non-linearity on the unstable leaves outside a neighborhood of the cusp class;
- The expansion factor of the branches R_{J_k} goes to ∞ as $k \rightarrow \infty$.

Remark. The above contraction and expansion properties hold with respect to a “Montel metric” induced by an appropriate Banach norm (see §2.1). The hyperbolicity is uniform in the following sense. The rate of contraction on the stable foliation is uniform on a subset of quadratic-like maps with a definite modulus ($\text{mod}(f) \geq \mu > 0$); the rate of expansion is uniform provided Rf stays outside a neighborhood of the cusp class (see Theorems 3.5 and 3.11).

A quadratic map $P_c : x \mapsto x^2 + c$ is called *regular* if it has an attracting cycle (i.e., a cycle whose multiplier has an absolute value less than 1). In this case, the attracting cycle is unique and attracts almost all orbits ([Si, G1]). It is called *stochastic* if it has an absolutely continuous invariant measure. In this case the measure is unique, weakly Bernoulli, and almost all orbits are asymptotically equidistributed with respect to it ([Le, BL2]).

Theorem 1.2 (Regular or stochastic). *Almost every real quadratic polynomial $P_c(z) = z^2 + c$, $c \in [-2, 1/4]$, is either regular or stochastic.*

Regular quadratic maps are also called (*uniformly*) *hyperbolic*, as they are uniformly expanding outside the basin of the attracting cycle [Fa, G1]. On the other hand, stochastic maps can also be called (*non-uniformly*) *hyperbolic* in the sense of the Pesin theory (as the invariant measure automatically has a positive characteristic exponent [BL2]). Thus one can say that *almost any real quadratic map is hyperbolic*.

Previously it was known that stochastic maps are observable with positive probability ([J, BC1]) but nowhere dense (as follows from the Yoccoz theorem, see [H]). On the other hand, the set of regular maps is open (obviously) and dense (see [L2] for the proof of this result and further reference comments). Our Regular or Stochastic Theorem completes the measure-theoretical picture of dynamics in the real quadratic family.

Let us remind the following topological decomposition of the parameter interval (see [MvS]): $[-2, 1/4] = \mathcal{R} \cup \mathcal{N} \cup \mathcal{I}$, where \mathcal{R} stands for the regular parameter values, \mathcal{N} stands for non-regular at most finitely renormalizable parameter values, and \mathcal{I} stands for infinitely renormalizable parameter values. The set \mathcal{S} of stochastic parameter values is contained in \mathcal{N} (this follows from a theorem that for $f \in \mathcal{I}$, almost all orbits converge to an attractor of measure 0 (see [G2, BL1, S2])). Thus Theorem 1.2 will follow from the following two results:

Theorem 1.3 (joint with Martens & Nowicki [L3, MN]). *Almost every non-regular real quadratic polynomial which is at most finitely renormalizable is stochastic: $\text{meas}(\mathcal{N} \setminus \mathcal{S}) = 0$.*

Namely, in our joint project, Martens and Nowicki gave a geometric condition for existence of an absolutely continuous invariant measure [MN], and the author has shown that this condition is satisfied almost everywhere [L3]. Note that it is known that the difference $\mathcal{N} \setminus \mathcal{S}$ is non-empty [Jo, HK, Bru].

Theorem 1.4. *The set of infinitely renormalizable real quadratics has zero Lebesgue measure: $\text{meas}(\mathcal{I}) = 0$.*

This result will be derived from Theorem 1.1. Let us give a few more applications of that Theorem. For any renormalization window J_k , there is a canonical map $\sigma : J_k \rightarrow [-2, 1/4)$ defined as the renormalization postcomposed with the straightening. Let $\{J_i^n\}$ stand for the collection of domains of definition of σ^n , that is the windows for the n -fold renormalization, and let $J_i^n(\epsilon) = \sigma^{-n}[-2, 1/4 - \epsilon] \cap J_i^n$.

Theorem 1.5. *The maps $\sigma^n : J_i^n(\epsilon) \rightarrow [-2, 1/4 - \epsilon]$ are uniformly quasi-symmetric (with the dilatation independent of n and i).*

Given a renormalizable map f , let $p(f)$ stand for the period of the first renormalization. The following result improves Theorem VIII of [L2]:

Theorem 1.6. *There is a number \bar{p} with the following property. If $f = P_c$ is a real quadratic map with $p(R^{n_k} f) \geq \bar{p}$ for a subsequence $n_k \rightarrow \infty$, then the Mandelbrot set is locally connected at c . Moreover, the corresponding little Mandelbrot sets M_{n_k} shrinking to c have a bounded shape.*

The last statement means that the canonical homeomorphisms of the sets M_{n_k} onto the whole Mandelbrot set M_* admit uniformly K -quasiconformal extensions to $(\epsilon \text{ diam } M_{n_k})$ -neighborhoods of the M_{n_k} (with absolute K and ϵ).

Let us now dwell on the main ingredients of the proof of Theorem 1.1. There are three types of combinatorics to take care of: bounded, essentially bounded and high. For bounded combinatorics, Sullivan [S2] and McMullen [McM2] constructed the renormalization horseshoe \mathcal{A} and its strong stable foliation. It was proven in [L4] that the renormalization horseshoe is hyperbolic. The idea of the proof is that in the complex analytic set up lack of hyperbolicity yields existence of “slowly shadowing orbits”. On the other hand, such orbits are ruled out by the Rigidity Theorem [L2]. Note that an important part of [L4] is supplying the space of quadratic-like germs (modulo affine conjugacy) with the complex analytic structure and demonstrating that the Douady & Hubbard hybrid classes [DH2] form a foliation of the connectedness locus with complex codimension 1 analytic leaves.

The unbounded combinatorics can be split into two types: “essentially bounded” and “high”. In the former case, the unboundedness is produced by the saddle-node behavior of the critical point (see [L2, LY]). This phenomenon can be analyzed by means of the parabolic bifurcation theory (see [D3]). Motivated by works of A. Epstein [Ep] and McMullen [McM2], Ben Hinkle has

proven a rigidity theorem for “parabolic towers” [Hi], geometric limits of dynamical systems generated by infinitely renormalizable maps with essentially bounded combinatorics. Using this result, we prove hyperbolicity of the renormalization operator with essentially bounded combinatorics. Note that McMullen’s argument for exponential contraction does not seem to work in this case, and instead we make use of the Schwarz Lemma in Banach spaces.

To treat high combinatorics we need an extensive analytic preparation on the geometry of the puzzle and parapuzzle which was done in [L2, L3]. The main geometric results of these works is linear growth of the conformal moduli of the “principal nest” of dynamical and parameter annuli. These imply that the image of a renormalization horizontal strip of high type is a narrow “vertical” strip close to the quadratic family. This yields strong hyperbolicity of the high type renormalization, with big contraction and expansion factors. Note that it is crucial for our argument that the results of [L2, L3] are proven for complex parameter values (even though in this work we are ultimately interested in the real quadratics).

Finally, the argument of [L4] (slowly shadowing orbit versus rigidity) glues the above ingredients together and yields Theorem 1.1.

This work completes a program of study the real quadratic family by complex methods carried in the series of papers [LM, L1, L2, LY, L3, MN, Hi, L4].

Let us finish with a couple remarks concerning more general settings. In the one-dimensional theory there are two natural ways to proceed: to higher degree polynomials and to C^2 -smooth maps. We expect the analogous “regular or stochastic” statement to be valid in generic one parameter families. There is still a lot of interesting work to be done in this direction.

One can also formulate an analogous conjecture for the complex quadratic family $z \mapsto z^2 + c$. Here absolute continuity of an invariant measure can be understood with respect to Sullivan’s conformal measure on the Julia set. “Almost all” in the parameter plane can be understood in the sense of Hausdorff dimension as “outside a set of strictly smaller dimension”. Of course, such a conjecture cannot be proven prior to the MLC conjecture (though can be disproven).

A general program in real higher dimensional situation was formulated by Palis (e.g., at the Paris/Orsay Symposium (1995)). Roughly speaking, it asserts that in a generic one parameter family, there is typically only finitely many attractors each of which carries an SBR measure and such that almost any orbit is equidistributed with respect to one of them. This program initiated by the work of Benedicks & Carleson [BC2] is now being intensively carried on (see Viana [V], Young [Y] and further references therein).

Notations. \mathbb{C} , \mathbb{R} , \mathbb{Z} and \mathbb{N} denote as usual the complex plane, the real line, and the sets of integer and natural numbers respectively;

$\mathbb{D}(a, r) = \{z : |z - a| < r\}$ is the open disk of radius r ,

$\mathbb{D}_r \equiv \mathbb{D}(0, r)$, $\mathbb{D} \equiv D_1$;

$\mathbb{T}_r = \partial\mathbb{D}_r$ is the circle of radius r , $\mathbb{T} \equiv \mathbb{T}_1$;

$U \Subset V$ means that U is *compactly contained* in V , that is, the closure \bar{U} is compact and is contained in V .

The closure of a set X will be denoted by \bar{X} .

When considering the space \mathbb{C}^2 , π_1 and π_2 will stand for the coordinate projections. Notation $\alpha \asymp \beta$ means as usual that the ratio α/β is bounded away from 0 and ∞ .

The quasi-conformality property will be often abbreviated as “qc”. Similarly, “qs” will stand for “quasi-symmetric”.

Let

$$\text{Dil}(h) = \text{ess-sup} \frac{\partial h + \bar{\partial} h}{\partial h - \bar{\partial} h}$$

stand for the dilatation of a qc map h .

Let $P_c(z) = z^2 + c$.

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2. QUADRATIC-LIKE GERMS, PUZZLE AND TOWERS

2.1. Space of quadratic-like germs. This section summarizes [L4, §§3,4]. Recall that a map $f : U' \rightarrow U$ is called *quadratic-like* if it is a double branched covering between topological disks U, U' such that either $U = U' = \mathbb{C}$ (and then f is a quadratic polynomial), or $U' \Subset U$. It has a single critical point which is assumed to be located at the origin 0, unless otherwise is stated.

The filled Julia set is defined as the set of non-escaping point: $K(f) = \{z : f^n z \in U, n = 0, 1, \dots\}$. Its boundary is called the *Julia set*, $J(f) = \partial K(f)$. The sets $K(f)$ and $J(f)$ are connected if and only if the critical point itself is non-escaping: $0 \in K(f)$. Otherwise these sets are Cantor.

The *fundamental annulus* of a quadratic-like map $f : U' \rightarrow U$ is the annulus between the domain and the range of f , $A = U \setminus U'$.

Any quadratic-like map has two fixed points counted with multiplicity. In the case of connected Julia set these two points can be dynamically distinguished. One of them, usually denoted by α , is either non-repelling or *dividing*, i.e., removing of it makes the Julia set disconnected. Another one, denoted by β , is always *non-dividing*.

For the details of the further discussion we refer to [L4]. First of all, we allow to change the domains (U, U') of a quadratic-like map without changing “its germ” near the Julia set. More precisely, let us say that two quadratic-like maps $f : U' \rightarrow U$ and $\tilde{f} : \tilde{U}' \rightarrow \tilde{U}$ represent the same *marked germ* if there is a string of quadratic-like maps $f_k : U'_k \rightarrow U_k$, representing the same germ at 0, with both points 0 and $f(0)$ contained in the same connected component W_k of $U_k \cap U_{k+1}$, $k = 0, 1, \dots, N - 1$, and such that $f_0 = f$, $f_N = \tilde{f}$. By [McM1, §5.4], a marked quadratic-like germ has a well-defined Julia set.

We will consider quadratic-like germs up to *affine* conjugacy (rescaling), so that near the origin they can be normalized as $f(z) = c + z^2 + \dots$. Marked quadratic-like germs modulo affine conjugacy will still be called briefly “quadratic-like germs”. We will not make notational difference between quadratic-like germs and quadratic-like maps representing them. Note also that any quadratic polynomial $P_c : z \mapsto z^2 + c$ determines a quadratic-like germ by restricting it to a sufficiently big round disk \mathbb{D}_r . These germs will still be called quadratic polynomials.

Let \mathcal{QL} stand for the space of quadratic-like germs, and \mathcal{C} be its *connectedness locus*, that is, the subset of germs with connected Julia set. We supply \mathcal{QL} with topology and complex analytic structure in the following fashion. Let \mathbb{V} be the ordered set of topological discs $V \ni 0$ with piecewise smooth boundary, with $U \succ V$ if $U \Subset V$. Let \mathcal{B}_V denote the space of normalized analytic functions $f(z) = c + z^2 + \dots$ on $V \in \mathbb{V}$ continuous up to the boundary supplied with sup-norm $\|\cdot\|_V$, and let $\mathcal{B}_V(g, \epsilon)$ stand for the ϵ -ball in this space centered at g .

If $g \in \mathcal{B}_V$ is quadratic-like on V then all nearby maps $f \in \mathcal{B}_V$ are quadratic-like on a slightly smaller domain. Thus we have an embedding $\mathcal{B}_V(g, \epsilon) \rightarrow \mathcal{QL}$. This family of embeddings induces a topology and complex structure on \mathcal{QL} (see the Appendix).

Given a set $\mathcal{X} \subset \mathcal{QL}$, the intersections $\mathcal{X}_V = \mathcal{X} \cap \mathcal{QL}_V$ are called the *Banach slices* of \mathcal{X} .

By Lemma 5.4, compactness in \mathcal{QL} is equivalent to sequential compactness. Moreover, any compact set $\mathcal{K} \subset \mathcal{QL}$ locally sits in finitely many Banach slices \mathcal{QL}_V and possesses a *Montel metric* dist_M well-defined up to quasi-isometry.

Let $\mathcal{Q} = \{P_c : z \mapsto z^2 + c\} \approx \mathbb{C}$ stand for the quadratic family. It is a complex one-dimensional submanifold of \mathcal{QL} . By definition, the Mandelbrot set $M_* \subset \mathcal{Q}$ is equal to $\mathcal{Q} \cap \mathcal{C}$.

Given a marked germ f , let $\text{mod}(f) = \sup \text{mod}(A)$ where A runs over the fundamental annuli of quadratic-like representatives of f . For $\mu > 0$, let $\mathcal{QL}(\mu, \rho)$ stand for the set of normalized quadratic-like germs with $\text{mod}(f) \geq \mu$ and $|f(0)| \leq \rho$. Furthermore, let

$$\mathcal{QL}(\mu) = \{f \in \mathcal{QL} : \text{mod}(f) \geq \mu\}.$$

Given a set $\mathcal{X} \subset \mathcal{QL}$, let $\mathcal{X}(\mu) = \mathcal{X} \cap \mathcal{QL}(\mu)$.

Lemma 2.1 (Compactness). *For any $\mu > 0$ and $\rho > 0$, the sets $\mathcal{QL}(\mu, \rho)$ and $\mathcal{C}(\mu)$ are compact. Moreover, if $f_n \in \mathcal{QL}(\mu_n, \rho)$ with $\mu_n \rightarrow \infty$ then the limit points of the f_n are quadratic polynomials.*

Proof. See [McM1, Theorem 5.6] and [L4, Lemma 4.1]. □

Two quadratic-like germs f and g are called *hybrid equivalent* if they are quasi-conformally conjugate by a map h with $\bar{\partial}h = 0$ a.e. on $K(f)$. By the Douady-Hubbard Straightening Theorem [DH2], every hybrid class $\mathcal{H}(f)$ with connected Julia set intersects the quadratic family \mathcal{Q} at a single point $c = \chi(f)$ of the Mandelbrot set M_* . Thus the hybrid classes can be also labeled as \mathcal{H}_c , $c \in M_*$.

The hybrid classes can be supplied with the Teichmüller-Sullivan metric (see [S1]):

$$\text{dist}_T(f, h) = \inf_h \log \text{Dil}(h),$$

where h runs over all hybrid conjugacies between f and g . Let us also define $\text{dist}_{T,V}$ as a similar infimum as h runs over hybrid equivalences defined in V (warning: unlike dist_T , $\text{dist}_{T,V}$ is not a metric).

Lemma 2.2. *Let $f \in \mathcal{QL}_V$, $g \in \mathcal{H}(f)$, and $W \Subset V$. There exists an $\epsilon > 0$ such that if $\text{dist}_{T,V}(f, g) < \epsilon$ then g belongs to \mathcal{QL}_W and $\|f - g\|_W < \epsilon$.*

Vice versa, for any $\epsilon > 0$ there is a $\delta > 0$ such that if $g \in \mathcal{B}_V(f, \delta)$ then $\text{dist}_{T,W}(f, g) < \epsilon$.

Proof. The first statement follows from the fact that any normalized qc map h with $\text{Dil}(h) < \epsilon$ is uniformly close to id.

To prove the second one, observe that if $g \in \mathcal{B}_V(f, \delta)$, then f and g have $(1 + \epsilon)$ -qc equivalent fundamental annuli A_f and A_g in a slightly smaller domain, such that the corresponding qc map $h : A_f \rightarrow A_g$ respects dynamics on the inner boundaries of the annuli. Such an h extends to a hybrid equivalence between f and g in W with dilatation $(1 + \epsilon)$. \square

It is proven in [L4] that the hybrid classes \mathcal{H}_c , $c \in M_*$, are connected complex codimension one submanifolds of \mathcal{QL} . They form a foliation \mathcal{F} (or rather a “lamination”) called *horizontal*. This foliation is transversally quasi-conformal everywhere, and complex analytic on $\text{int } \mathcal{C}$.

Let us state the former result more precisely. Take two hybrid equivalent maps f_i , and two complex analytic transversals \mathcal{S}_i to the leaf $\mathcal{H} \equiv \mathcal{H}(f_i)$ via f_i . The holonomy γ from \mathcal{S}_1 to \mathcal{S}_2 along \mathcal{F} is called quasi-conformal if it admits a qc extension to a neighborhood of the f_i in the transversals (beyond the connectedness locus).

The local dilatation of γ at f_1 is

$$\inf_h \lim_{\epsilon \rightarrow 0} \text{Dil}(h | \mathbb{D}(f_1, \epsilon)),$$

where the infimum is taken over all local qc extensions h of γ .

Theorem 2.3 ([L4], Theorem 4.14). *Given two quadratic-like maps as above, the holonomy γ is quasi-conformal. If $\text{mod}(f_i) \geq \mu$ then the local dilatation of γ at f_1 is bounded by $K(\mu)$. Moreover, if additionally $\text{dist}_T(f_1, f_2) \leq \rho < 1$ then the above dilatation is $1 + O(\rho)$ with the constant depending on μ only.*

Let \mathcal{E} denote the space of real analytic expanding circle endomorphism of degree two considered up to rotation. By definition, any $g \in \mathcal{E}$ admits a double covering complex analytic extension $g : V \rightarrow V'$ to symmetric annuli neighborhoods of the circle \mathbb{T} (with piecewise smooth boundary) such that $V \Subset V'$ (where “symmetric” is understood in the sense of the involution $\tau : \bar{\mathbb{C}} \rightarrow \mathbb{C}$ about the circle \mathbb{T}).

There is a projection

$$\pi : \mathcal{QL} \rightarrow \mathcal{E}, \tag{2.1}$$

which associates to an $f \in \mathcal{QL}$ its external map $g = \pi(f) \in \mathcal{E}$ (see [DH2], [L4, §3.2]). The construction goes as follows. Take a quadratic-like representative $f : V' \rightarrow V$ and consider the fundamental annulus $A = V \setminus V'$, $\text{mod}(A) = \mu$. Using the map $f : I \rightarrow O$ from the inner to outer boundary of A , attach to the inner boundary of A an abstract annulus A_1 of modulus $\mu/2$. It comes together with a double covering $A_1 \rightarrow A$ which extends $f : I \rightarrow O$. Using this covering, attach in a similar way an annulus A_2 of modulus $\mu/4$ to the inner boundary of A_1 , etc. Taking the infinite union of these annuli together with $\mathbb{C} \setminus V$, we obtain a conformal punctured disk S and a double covering F between annuli neighborhoods of its ideal boundary. Let us uniformize it, $\phi = \phi_f : S \rightarrow \mathbb{C} \setminus \mathbb{D}$, and conjugate F by ϕ . This provides us with a double covering between outer annuli neighborhoods of \mathbb{T} . Reflecting it about the circle, we obtain the desired external map $g : V' \rightarrow V$, where V and V' are symmetric annuli neighborhoods of \mathbb{T} and $V' \Subset V$.

As the external map g is defined up to rotation so that it can be normalized by putting its fixed point at 1: $g(1) = 1$.

Note further that the uniformization ϕ_f provides a conformal isomorphism between the fundamental annuli $U \setminus U'$ and $V \setminus (V' \cup \mathbb{D})$, and conjugates f and g on their inner boundaries. Moreover, by means of dynamics it can be analytically extended to a domain containing the critical value $f(0)$ (and containing the critical point 0 on its boundary). Thus we can consider the image of the critical point under this map:

$$\xi(f) = \phi_f(f(0)) \quad (2.2)$$

(it is well-defined once g is normalized). The inverse map $\psi_f = \phi_f^{-1}$ will be called the *uniformization of f at ∞* .

Let us consider a real Banach space \mathcal{B}_V^s of \mathbb{T} -symmetric (i.e., commuting with τ) complex analytic maps $V \rightarrow \mathbb{C}$ continuous in \bar{V} . A sufficiently small Banach neighborhood $\mathcal{B}^s(f, \epsilon)$ consists of expanding circle endomorphisms, so that it is embedded into \mathcal{E} . This endows \mathcal{E} with the inductive limit topology and real analytic structure.

Restricted to any hybrid class \mathcal{H}_c , $c \in M_*$, the projection π becomes a homeomorphism. The inverse map $i_c : \mathcal{E} \rightarrow \mathcal{H}_c$ is provided by the “mating” of a circle map $g \in \mathcal{E}$ with the quadratic polynomial P_c (see [DH2, L4]). This allows us to transfer the complex analytic structure from the hybrid class \mathcal{H}_0 of $z \mapsto z^2$ to the space \mathcal{E} .

The Bers-Sullivan complex structure makes the projection $\pi : \mathcal{QL} \rightarrow \mathcal{E}$ and all the parametrizations $i_c : \mathcal{E} \rightarrow \mathcal{H}_c$, $c \in M_*$, complex analytic (see [L4, §4.3]). The fibers \mathcal{Z}_g , $g \in \mathcal{E}$, of π turn out to be complex analytic curves in \mathcal{QL} [L4, Theorem 4.18]. They are called *vertical fibers*.

The map ξ (2.2) provides a smooth extension (actually, real analytic) of the Riemann mapping $\mathbb{C} \setminus M_* \rightarrow \mathbb{C} \setminus \mathbb{D}$ (see [DH1]) to the complement of the connectedness locus. Moreover, this map is *vertically holomorphic*, i.e., it is holomorphic on the vertical fibers \mathcal{Z}_g [L4, Lemma 4.9],

Note finally that the Green function $G = \log |\xi| : \mathcal{QL} \setminus \mathcal{C} \rightarrow \mathbb{R}_+$ provides us with a dynamically natural way to measure the “distance” from an $f \in \mathcal{QL} \setminus \mathcal{C}$ to the connectedness locus. The level sets of the Green function are called *equipotentials* \mathcal{O}_r (of radius $r > 1$) in \mathcal{QL} . One can show that they are real codimension one smooth submanifolds in $\mathcal{QL} \setminus \mathcal{C}$ (since ξ is a smooth submersion).

2.2. Quadratic-like families. The reader is referred to [DH2, L3] for a discussion of quadratic-like families. Let us have a domain $\Lambda \Subset \mathbb{C}$. A domain $\mathbb{V} \subset \Lambda \times \mathbb{C}$ is called a *tube* over Λ if it is homeomorphic over Λ to a straight tube $\Lambda \times \mathbb{D}$. Let $V_\lambda = \pi_1^{-1}\{\lambda\}$ stand for the vertical fibers of a tube \mathbb{V} . We will assume that they are bounded by piecewise smooth curves and contain 0. Let $\mathbf{0} = \Lambda \times \{0\}$ stand for the zero section of \mathbb{V} . If $\Lambda' \subset \Lambda$ then let $\mathbb{V}|\Lambda' = \mathbb{V} \cap (\Lambda' \times \mathbb{C})$. Let $\partial^h \mathbb{V} = \cup_{\lambda \in \Lambda} \partial V_\lambda$ stand for the *horizontal boundary* of \mathbb{V} .

Given two tubes $\mathbb{V}' \subset \mathbb{V}$ as above, let us say that \mathbb{V}' is compactly contained in \mathbb{V} over Λ , $\mathbb{V} \Subset_{\Lambda} \mathbb{V}'$, if the relative closure of \mathbb{V}' in $\Lambda \times \mathbb{C}$ is bounded, contained in \mathbb{V} , and $\text{dist}(\partial^h \mathbb{V}', \partial^h \mathbb{V}) > 0$.

By definition, a map $\mathbf{f} : \mathbb{V}' \rightarrow \mathbb{V}$ between two tubes $\mathbb{V}' \subset \mathbb{V}$ over Λ is called a *quadratic-like family* over $\Lambda = \Lambda_{\mathbf{f}}$ if \mathbf{f} is a holomorphic endomorphism preserving the fibers, and such that every fiber restriction $f_\lambda : V'_\lambda \rightarrow V_\lambda$, $z \mapsto z^2 + c(\lambda) + \dots$, is a normalized quadratic-like map

with a critical point at 0. Clearly any quadratic-like family represents a holomorphic curve in \mathcal{QL} .

Let

$$M_{\mathbf{f}} = \{\lambda : 0 \in K(f_\lambda)\} = \mathbf{f} \cap \mathcal{C}$$

stand for the Mandelbrot set in \mathbf{f} . The straightening provides a canonical continuous map

$$\chi : M_{\mathbf{f}} \rightarrow M_* \quad (2.3)$$

(see Douady & Hubbard [DH2]). The family \mathbf{f} over Λ is called *full* if $M_{\mathbf{f}} \in \Lambda$. In this case the straightening properly maps $M_{\mathbf{f}}$ onto the whole Mandelbrot set M_* . A full family is called *unfolded* if the straightening (2.3) is injective (and hence is a homeomorphism).

Let $\phi(\lambda) = f_\lambda(0)$ stand for the critical value of f_λ , and let $\Phi(\lambda) = (\lambda, \phi(\lambda))$ stand for the “critical value section” $\Lambda \rightarrow \mathbb{V}$. The family \mathbf{f} is called *proper* if $\mathbb{V}' \underset{\Lambda}{\subseteq} \mathbb{V}$ and the map Φ is proper over Λ , i.e, for any $\mathbb{W} \underset{\Lambda}{\subseteq} \mathbb{V}$, we have $\Phi^{-1}\mathbb{W} \subseteq \Lambda$. In particular, for a proper family, the domain

$$\Lambda' \equiv \Lambda'_{\mathbf{f}} = \Phi^{-1}\mathbb{V}' = \{\lambda : f_\lambda(0) \in V'_\lambda\}$$

is compactly contained in Λ , so that this family is full.

For a full family, one defines the *winding number* $w(\mathbf{f})$ as the winding number of the curve $\lambda \mapsto \phi(\lambda)$ about the origin, as λ goes once anti-clockwise around a Jordan curve Γ enclosing $M_{\mathbf{f}}$. The family is unfolded if and only if it has winding number 1. In this case there is a single superstable parameter value $*$ (the root of ϕ) called the *center* of Λ .

Given a proper unfolded quadratic-like family $\mathbf{f} = \{f_\lambda\}$, let

$$\text{mod}(\mathbf{f}) = \min[\text{mod}(\Lambda \setminus \Lambda'), \text{mod}(\Lambda' \setminus M_{\mathbf{f}}), \inf_{\lambda \in \Lambda} \text{mod}(V_\lambda \setminus V'_\lambda)],$$

where $\text{mod}(\Lambda \setminus \Lambda')$ is understood as the extremal length of the family of curves in $\Lambda \setminus \Lambda'$ joining $\partial\Lambda$ with $\partial\Lambda'$, and similarly for $\text{mod}(\Lambda' \setminus M_{\mathbf{f}})$.

Let \mathcal{G} stand for the collection of proper unfolded quadratic-like families up to affine change of variable in λ . Such a family can be normalized so that superattracting parameter value $*$ sits at the origin and $\text{diam } M_{\mathbf{f}} = 1$. We will impose the *Carathéodory* topology on \mathcal{G} (compare [McM1]). In this topology a sequence of families \mathbf{f}_n over Λ_n converges to a family \mathbf{f} over Λ if $(\Lambda_n, *)$ Carathéodory converges to $(\Lambda, *)$ the domains $(\mathbb{V}_n, \mathbb{V}'_n, \mathbf{0})$ (with the preferred zero sections) Carathéodory converge to $(\mathbb{V}, \mathbb{V}', \mathbf{0})$, so that the convergence respects the fiberwise structure, and finally the \mathbf{f}_n converge to \mathbf{f} uniformly on compact sets.

Carathéodory convergence in the above definition means:

- For any tube $\mathbb{W} \ni \mathbf{0}$ compactly contained in \mathbb{V}_n , all the domains \mathbb{V}_n eventually contain $\bar{\mathbb{W}}$;
- Vice versa: Any domain \mathbb{W} as above compactly contained in infinitely many of the \mathbb{V}_n , is also contained in \mathbb{V} .

Note that the Carathéodory convergence of the families corresponds to the uniform on compact sets convergence of the corresponding holomorphic curves in \mathcal{QL} .

Let

$$\mathcal{G}_\mu = \{\mathbf{f} \in \mathcal{G} : \text{diam } \mathbb{V} \leq \mu^{-1}, \text{mod}(\mathbf{f}) \geq \mu\},$$

where the family \mathbf{f} above is meant to be normalized: $\text{diam } M_{\mathbf{f}} = 1$.

Lemma 2.4. *For any $\mu > 0$, the space \mathcal{G}_μ is compact.*

Proof. Let us have a sequence of normalized families \mathbf{f}_n over $(\Lambda_n, 0)$. Since $\text{mod}(\Lambda \setminus \Lambda') \geq \mu > 0$, $\text{mod}(\Lambda' \setminus M_{\mathbf{f}}) \geq \mu$, and $\text{diam } M_{\mathbf{f}} = 1$, the families of domains $\Lambda_n \equiv \Lambda_{\mathbf{f}_n}$ and $\Lambda'_n \equiv \Lambda'_{\mathbf{f}_n}$ are Carathéodory compact. Select a Carathéodory converging subsequence: $(\Lambda_n, 0) \rightarrow (\Lambda, 0)$ and $(\Lambda'_n, 0) \rightarrow (\Lambda', 0)$. Since the $\text{mod}(\Lambda_n \setminus \Lambda'_n)$ stay away from 0 and the $\text{diam } \Lambda_n$ are bounded, the limit domain Λ' is compactly contained in Λ .

Take a domain $\Omega \Subset \Lambda$. Let us now select a converging subsequence of the domains $(\mathbb{V}_n, \mathbf{0}) \rightarrow (\mathbb{V}, \mathbf{0})$ over Ω . To this end let us consider a family \mathcal{W} of domains $\mathbb{W} \ni \mathbf{0}$ such that some relative neighborhood of $\bar{\mathbb{W}}$ in $\Omega \times \mathbb{C}$ is compactly contained in infinitely many of the \mathbb{V}_n . This family is non-empty. Indeed, by normalization of the f_λ and the bound $\text{mod}(f_\lambda) \geq \mu > 0$, \mathcal{W} contains the round tubes $\Omega \times \mathbb{D}_\epsilon$ with sufficiently small $\epsilon > 0$.

The family \mathcal{W} has a countable basis \mathcal{W}^0 , i.e. a countable family of domains such that for any $\mathbb{W} \in \mathcal{W}$ there exists a $\mathbb{W}^0 \in \mathcal{W}^0$ such that $\mathbb{W}^0 \subset \mathbb{W}$ (for instance, take polygonal domains in \mathcal{W} with rational vertices). Select an exhausting sequence $\mathbb{W}_n \in \mathcal{W}^0$ so that $\mathbb{W}_1 \subset \mathbb{W}_2 \subset \dots$ and no $\mathbb{W} \in \mathcal{W}^0$ contains all the \mathbb{W}_n . Now select a sequence \mathbb{V}_n such that any \mathbb{W}_n is compactly contained in all the \mathbb{V}_k , $k \geq n$. This subsequence Carathéodory converges to $\mathbb{W} = \cup \mathbb{W}_n$ over Ω .

Now select an exhausting sequence of domains $\Omega \Subset \Lambda$. By means of the diagonal procedure we can find a subsequence of tubes \mathbb{V}_n converging over each Ω . Hence by definition it is Carathéodory converging over the whole Λ , $\mathbb{V}_n \rightarrow \mathbb{V}$.

Similarly select a further subsequence so that $\mathbb{V}'_n \rightarrow \mathbb{V}'$.

Furthermore, since the maps $\mathbf{f}_n = \{f_\lambda\}$ act fiberwise as branched double coverings and normalized as $z \mapsto z^2 + c(\lambda) + \dots$ at the origin, by the Koebe Theorem they form a pre-compact family on each compact subset of \mathbb{V}'_n . Hence we can select a converging subsequence $\mathbf{f}_n \rightarrow \mathbf{f}$. As the limit family is clearly proper and unfolded, we are done. \square

Let $f \in \mathcal{C}$ with $\chi(f) = c$, and let $P_f = D(i_c \circ \pi)_f$ be the projection of the tangent space $T_f \mathcal{QL}$ onto the tangent space $T_f \mathcal{H}_f$ to the leaf. The one-dimensional spaces

$$K_f = \text{Ker } D\pi_f = \text{Ker } DP_f \tag{2.4}$$

form a continuous subbundle $\mathcal{K} \subset T\mathcal{QL}$ complementary to the tangent subbundle $T\mathcal{F}$. For a vector $u \in T_f \mathcal{QL}$, let $u^h = P_f u$ and $u^v = (I - P_f)u$ stand for its ‘‘horizontal’’ and ‘‘vertical’’ projections.

If we take a Banach slice \mathcal{QL}_V whose tangent space contains K_f , then for $u \in T_f \mathcal{QL}_V$ we can define the angle between u and $T_f \mathcal{H}_f$ by letting

$$\text{tg}(\theta) = \|u^v\|/\|u^h\|. \tag{2.5}$$

Small angle means that v is almost tangent to the leaf.

If we have a family of tangent vectors belonging to finitely many Banach slices, we say that they are *uniformly transversal* to the foliation \mathcal{F} if their angles with the foliation stay away from 0.

Lemma 2.5. *Any family $\mathbf{f} \in \mathcal{G}_\mu$ is uniformly transversal to the foliation \mathcal{F} with the lower bound on the angle depending only on μ .*

Proof. Otherwise, by compactness (Lemma 2.4), there would be a family \mathcal{S} in \mathcal{G}_μ which were tangent to some leaf \mathcal{L} of \mathcal{F} . Let us take a Banach slice \mathcal{QL}_V locally containing \mathcal{S} and a curve $\gamma \subset \mathcal{L}$ tangent to \mathcal{S} . Then the slice \mathcal{L}_V is still tangent to \mathcal{S} in \mathcal{QL}_V .

By [L4, Lemma 4.12], the Banach slice \mathcal{F}_V of the foliation \mathcal{F} is a Banach foliation with codimension 1 complex analytic leaves. Moreover, it is transversally analytic over the $\text{int } \mathcal{C}_V$. Let us apply to it the results of the Appendix, §5.1.

By the Hurwitz Theorem, \mathcal{S} would have the same number of intersection points (counted with multiplicity) with all nearby leaves of \mathcal{F}_V . But by the Intersection Lemma, the intersection points with the nearby leaves of $\text{int } \mathcal{C}_V$ are simple, so that there would be more than one such a point. But unfolded families intersect every leaf of \mathcal{F} at a single point ([DH2]) - contradiction. \square

Let us now show that the quadratic-like families $\mathbf{f} \in \mathcal{G}_\mu$ “uniformly overflow” the connectedness locus. This can be measured in terms of the function $\xi : \mathcal{QL} \setminus \mathcal{C} \rightarrow \mathbb{C} \setminus \bar{\mathbb{D}}$ (2.2), and means that any family in question “goes beyond” an equipotential \mathcal{O}_r with $r = r(\mu)$.

Lemma 2.6. *There is an $r = r(\mu) > 1$ such that for any family \mathbf{f} over Λ , $|\xi(f_\lambda)| > r$ for $\lambda \in \partial\Lambda$.*

Proof. Let $f_\lambda : U'_\lambda \rightarrow U_\lambda$ and $g_\lambda : V'_\lambda \rightarrow V_\lambda$ be the corresponding external map. Since $f_\lambda 0 \in U_\lambda \setminus U'_\lambda$ for λ near $\partial\Lambda$ and $\xi(f_\lambda)$ corresponds to $f_\lambda(0)$ in the external model, we have: $\xi(f_\lambda) \in V_\lambda \setminus V'_\lambda$. Since

$$\text{mod}(V_\lambda \setminus V'_\lambda) = \text{mod}(U_\lambda \setminus U'_\lambda) \geq \mu,$$

there is an annulus of modulus at least $\mu/2$ which separates $\xi(f_\lambda)$ from the unit circle. Hence $|\xi(f_\lambda)| > r(\mu) > 1$ for λ near $\partial\Lambda$, and we are done. \square

If $\text{mod}(\mathbf{f})$ is big then the family \mathbf{f} is close to the quadratic family in the following sense:

Lemma 2.7. *For any $\epsilon > 0$ and r , there is a μ and a Banach space \mathcal{B}_V containing disk $\mathbb{D}_r = \{c : |c| < r\}$ in the quadratic family \mathcal{Q} with the following property. If \mathbf{f} is a full unfolded family with $\text{mod}(\mathbf{f}) > \mu$, then there is topological disk $\Delta \subset \mathbf{f}$ which belongs to \mathcal{B}_V and is represented in that space as a graph of an analytic function $\phi : \mathbb{D}_r \rightarrow E$ (where E is a complement of \mathcal{Q} in \mathcal{B}_V) with $\|\phi\| < \epsilon$.*

Proof. Take a domain V . If μ is big enough then all quadratic-like maps $f \in \mathcal{QL}(\mu)$ clearly belong to a Banach slice B_V . Select a complement E to \mathcal{Q} in this Banach space, and let $p : \mathcal{B}_V \rightarrow \mathcal{Q}$ be the projection of \mathcal{B}_V onto \mathcal{Q} parallel to E . By Lemma 2.1,

$$\|P_{c(\lambda)} - f_\lambda\|_V = \epsilon < \epsilon(\mu), \tag{2.6}$$

where $P_{c(\lambda)} = p(f_\lambda)$ and $\epsilon(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$.

Let us take a big ρ and consider the curve $\gamma = \{f_{\lambda(t)}\}$ in \mathbf{f} parametrized by a Jordan curve $\delta \subset \Lambda$ close to $\partial\Lambda$ (where \mathbf{f} is a family over Λ). Since \mathbf{f} is unfolded, the winding number of $\Gamma = p \circ \gamma$ around 0 is equal to 1.

Moreover, by (2.6), Γ encloses a disk $\mathbb{D}_r \subset \mathcal{Q}$ with a big $r = r(\mu)$. Indeed, if δ is sufficiently close to $\partial\Lambda$, then the critical value $f_\lambda(0)$ is arbitrary close to ∂U_λ for $\lambda \in \delta$. Hence $f_\lambda(0)$ can be separated from the Julia set $J(f_\lambda)$ by a fundamental annulus of modulus at least $\mu/2$. Hence the critical value $P_{c(\lambda)}0$ is separated from the Julia set $L(P_{c(\lambda)})$ by a fundamental annulus of modulus at least $\mu/2 - \delta(\epsilon)$ where $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. It follows that $|c(\lambda)| \geq r(\mu) \rightarrow \infty$ as $\mu \rightarrow \infty$.

Hence the winding number of Γ about the disk \mathbb{D}_r is equal to 1. It follows that the projection of \mathbf{f} onto \mathcal{Q} univalently covers this disk. Together with (2.6) this yields the desired statement. \square

Let us say that a Mandelbrot set $M_{\mathbf{f}}$ has a *K-bounded shape* if it is canonically homeomorphic to the standard set M_* by a map which admits a *K-qc* extension to a neighborhood D of M with $\text{mod}(D \setminus M) > 1/K$. Theorem 2.3 yields:

Lemma 2.8. *For any $\mathbf{f} \in \mathcal{G}_\mu$, the Mandelbrot set $M_{\mathbf{f}}$ has $K(\mu)$ -bounded shape.*

Let us say that a quadratic-like family $\mathbf{f} : \mathbb{V}' \rightarrow \mathbb{V}$ over $(\Lambda, *)$ is *equipped* if it is supplied with a holomorphic motion \mathbf{h} of the fundamental annulus over Λ , $h_\lambda : V_* \setminus V'_* \rightarrow V_\lambda \setminus V'_\lambda$, $\lambda \in \Lambda$. For instance, the quadratic family is equipped over a domain Λ bounded by any parameter equipotential. All primitive Mandelbrot copies in a full equipped family (\mathbf{f}, \mathbf{h}) are generated by equipped quadratic-like families (see [L3]).

Given a holomorphic motion \mathbf{h} over Λ , let

$$\text{Dil}(\mathbf{h}) = \sup_{\lambda \in \Lambda} \text{Dil}(h_\lambda).$$

For an equipped quadratic-like family (\mathbf{f}, \mathbf{h}) , let us call the pair of numbers $(\text{mod}(\mathbf{f}), \text{Dil}(\mathbf{h}))$ its *geometry*. We say that the geometry is bounded by over a collection of equipped families, if $\text{mod}(\mathbf{f}) \geq \mu > 0$ and $\text{Dil}(\mathbf{h}) \leq K$ for all families of the collection.

Let \mathcal{G}_μ^e denote the collection of equipped quadratic-like families (\mathbf{f}, \mathbf{h}) with $\text{mod}(\mathbf{f}) \geq \mu$, $\text{Dil}(\mathbf{h}) \leq \mu^{-1}$.

For $f \in \mathcal{QL}$, let us consider the following objects:

- the uniformization ψ_f defined after (2.2);
- the projection $\Pi : \mathcal{QL} \rightarrow \mathcal{H}_0$, $\Pi = (\pi|_{\mathcal{H}_0})^{-1} \circ \pi$, where π is the projection (2.1) and \mathcal{H}_0 is the hybrid class of $P_0 : z \mapsto z^2$;
- for $G \in \mathcal{H}_0$, we will use the notation $R_G \equiv \psi_G : \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus K(G)$ for the Riemann mapping with positive derivative at ∞ ;
- the map $\Psi_f = \psi_f \circ R_G^{-1}$, where $G = \Pi(f)$, which is another conformal representation of f “near ∞ ” .

Lemma 2.9. *The conformal representation $\Psi_f(z)$ analytically depends on f .*

Proof. By [L4, Lemma 4.10],

$$(f, z) \mapsto \Psi_f(z) \tag{2.7}$$

is continuous. In particular, this means that if Ψ_{f_0} is defined on a domain Ω_0 and $\Omega \Subset \Omega_0$, then all nearby maps $f \in \mathcal{U} \equiv \mathcal{B}_U(f_0, \epsilon)$ are defined on Ω . Let us show that (2.7) is analytic on $\mathcal{U} \times \Omega_0$.

Let us consider a holomorphic family $f_\lambda \in \mathcal{U}$ of quadratic-like maps over $(\Lambda, 0)$. For λ near the origin we can select a fundamental annulus A_λ holomorphically moving with λ in such a way that the corresponding holomorphic motion $h_\lambda : A_0 \rightarrow A_\lambda$ respects the boundary dynamics (see [DH2, Prop. 9] or [L4, Lemma 4.2]). Let us consider the corresponding holomorphic family of conformal structures $\mu_\lambda = h_\lambda^*(\sigma)$ on A_0 , where σ is the standard structure on A_λ . Pulling them back by the external map g_0 , we obtain a holomorphic family of conformal structures on the Riemann surface $S_0 \approx \mathbb{C} \setminus \bar{\mathbb{D}}$, (μ_λ is extended on $\mathbb{C} \setminus U_0$ as the standard structure). Let us extend these structures to $\bar{\mathbb{D}}$ as the standard ones as well. We obtain a holomorphic family of complex structures on \mathbb{C} which will be still denoted as μ_λ .

By the Measurable Riemann Mapping Theorem, there is family of qc maps ω_λ holomorphically depending on λ which solves the Beltrami equations $(\omega_\lambda)_* \mu_\lambda = \sigma$. It maps $\mathbb{C} \setminus \bar{\mathbb{D}}$ onto $\mathbb{C} \setminus K(G)$ where $G = \Pi(f) \in \mathcal{H}_0$. Then $\Psi_\lambda = h_\lambda \circ \omega_\lambda^{-1}$, and we conclude that it analytically depends on λ . \square

For $G \in \mathcal{H}_0$, let $\mathcal{Z}_G = \Pi^{-1}\{G\}$ stand for the corresponding vertical fiber. Given a Banach neighborhood \mathcal{V} in $\mathcal{H}_{0,V}$, the set

$$\mathcal{T}_\mathcal{V} = \cup_{G \in \mathcal{V}} \mathcal{Z}_G \subset \mathcal{QL}$$

will be called a *vertical tube over \mathcal{V}* . Let us say that a tube $\mathcal{T} = \mathcal{T}_\mathcal{V}$ is *equipped* if there is a neighborhood $\mathcal{U} \subset \mathcal{T}$ of $\mathcal{T} \cap \mathcal{C}$ with the following properties

- The vertical fibers $\mathcal{U}_G = \mathcal{U} \cap \Pi^{-1}G$ represent full unfolded quadratic-like families;
- There is a holomorphically moving fundamental annulus $A_{G,\lambda}$, $G \in \mathcal{V}$, $\lambda \in \mathcal{U}_G$ such that the motion respect the boundary dynamics.

In other words, we equip all the vertical fibers \mathcal{Z}_G over \mathcal{V} in the way analytically depending on $G \in \mathcal{V}$.

Lemma 2.10. *Any $G_0 \in \mathcal{E}$ belongs to an equipped vertical tube $\mathcal{T}_\mathcal{V} \ni G_0$.*

Proof. External fibers represent full unfolded quadratic-like families by [L4, Corollary 4.19]. In order to equip them, let us use the conformal representations Ψ_f . By Lemma 2.9, Ψ_f holomorphically depends on f .

By [L4, Lemma 4.2], there is a Banach neighborhood $\mathcal{V} = \mathcal{H}_{0,V}(G_0, \epsilon)$ which can be equipped with a holomorphically moving fundamental annulus $A_G = V_G \setminus V'_G \subset \mathbb{C} \setminus K(G)$. Moreover, the conformal representation Ψ_f is well defined on $\mathbb{C} \setminus V'_G$, $G = \Pi(f)$, provided f varies in a certain neighborhood $\mathcal{U} \subset \mathcal{T}_\mathcal{V}$ of the connectedness locus $\mathcal{C}_\mathcal{V} = \mathcal{T}_\mathcal{V} \cap \mathcal{C}$. It follows that $B_f \equiv U_f \setminus U'_f = \Psi_f(A_G)$, $G = \Pi(f)$, is a holomorphically moving fundamental annulus of $f \in \mathcal{U}$.

Furthermore, \mathcal{U} is the union of $\mathcal{C}_\mathcal{V}$ and a domain where $a(f) \equiv \Psi_f^{-1}(f(0)) \in V$. By [L4, Theorem 3.4], for any $a \in V \setminus K(G)$ and $G \in \mathcal{V}$, there exists a unique $f = \theta(G, a) \in \mathcal{Z}_G$ such that $\Pi(f) = G$ and $a(f) = a$ (“mating” of G and a).

It follows that $\mathcal{U}_G \setminus \mathcal{C}$ is a topological annulus, so that \mathcal{U}_G is a topological disk representing an unfolded quadratic-like family. If $f \in \partial\mathcal{U}_G$ then by definition $a(f) \in \partial V$ and hence $f(0) \in \partial U_f$. Thus this family is proper. \square

2.3. Puzzle, parapuzzle and renormalization. The notion of *complex renormalization* was introduced by Douady and Hubbard [DH2, D2] in order to explain computer observable little Mandelbrot copies inside the Mandelbrot set (see [M, McM1, L2] for further discussion).

Let f be a quadratic-like map. Assume that we can find topological disks $U' \Subset U$ around 0 and an integer p such that $g = f^p : U' \rightarrow U$ is a quadratic-like map with connected Julia set. Assume also that the “little Julia sets” $f^k J(g)$, $k = 0, \dots, p-1$, are pairwise disjoint except, perhaps, touching at their non-dividing β -fixed points. Then the map f is called renormalizable (with period p) and the quadratic-like germ g considered up to rescaling is called a “renormalization” of f . The map f can be renormalizable with different periods, finitely or infinitely many. Accordingly it is called *at most finitely* or *infinitely renormalizable*. There is a canonical way to produce the *first* renormalization Rf of f , with the smallest period. It is provided by the *Yoccoz puzzle*.

The reader can consult [L2], §3, for a detailed discussion of the combinatorics of the Yoccoz puzzle. The main combinatorial object considered in that work is the *principal nest* of puzzle pieces $V^0 \supset V^1 \supset \dots$. There is a flexibility of the choice of the first puzzle piece V^0 . For the sake of this work (focused on the real combinatorics) it can be selected as the Yoccoz puzzle piece $Y^{(1)}$ of the first level (bounded by the external rays landing at the dividing fixed point α , the symmetric point α' , and some equipotential).

Then V^{n+1} is inductively defined as the pull-back of V^n corresponding to the first return of the critical point back to V^n . The corresponding return map $g_n : V^n \rightarrow V^{n-1}$ is a branched double covering. The return to level $n-1$ is called *central* if $g_n(0) \in V^n$. Let n_k count the non-central levels. If this sequence is infinite then the map f is non-renormalizable. Otherwise the principal nest ends up with an infinite central cascade $V^{n-1} \supset V^n \supset \dots$, and the map $g_n : V^n \rightarrow V^{n-1}$ (after perhaps little thickening of the domain and the range) is a quadratic-like map with connected Julia set. The germ of this map (up to rescaling) is called the first renormalization Rf of f .

Let us now state a little lemma which will be useful in what follows:

Lemma 2.11. *No quadratic polynomial P_c can be realized as the renormalization Rf of a quadratic-like map.*

Proof. Indeed, the renormalization Rf admits the analytic continuation to the domain of f^p as a branched covering of degree $2^p > 2$. This is certainly not compatible with the quadratic extension to the whole complex plane. \square

The number of the non-central levels in the principal nest is called the *height* of f .

The map $g_n : V^n \rightarrow V^{n-1}$ is a restriction of the full first return map $g_n : \cup V_i^n \rightarrow V^{n-1}$ (denoted by the same letter). Here $V_i^n \subset V^{n-1}$ are puzzle pieces with disjoint interiors, $V_0^n \equiv V^n$, and the restrictions $g_n : V_i^n \rightarrow V^{n-1}$ are univalent for $i \neq 0$.

Let us now consider the quadratic family $P_c : z \mapsto z^2 + c$. For any parameter value $c_0 \in M_*$ outside the main cardioid, there is a nest of *parapuzzle pieces* $\Delta^1(c_0) \supset \Delta^2(c_0) \supset \dots \ni c_0$

corresponding to the dynamical principal nest. For parameter values $c \in \Delta^n(c_0)$, the “combinatorics” of the first return maps to the puzzle piece V^{n-1} stay the same (see [L3] for the precise definition which, however, does not matter for the following discussion).

If P_c is non-renormalizable then the parapuzzle pieces $\Delta^n(c)$ shrink to c (Yoccoz, see [H], and [L3]). Otherwise the return maps $g_{n,c} = P_c^p : V^n \rightarrow V^{n-1}$, $c \in \Delta^n$, on the renormalization level form a quadratic-like family \mathbf{g} . In the primitive case (when the little Mandelbrot set $M_{\mathbf{g}}$ is not attached to the main cardioid, which is equivalent to saying that $n > 2$), \mathbf{g} is a full unfolded family. In the satellite case, \mathbf{g} is *almost* full and unfolded which means that the straightening χ homeomorphically maps $M_{\mathbf{g}}$ onto “unrooted” Mandelbrot set $* \setminus \{1/4\}$ (see [D2]).

The Mandelbrot set $M = M_{\mathbf{g}}$ encodes the combinatorial data of the renormalization: all maps P_c with $c \in M$ are “renormalizable with the same combinatorics”. The period of this renormalization is certainly constant throughout the copy, $p = p(M)$. Moreover, the little copies produced in this way are *maximal* in the sense that they are not contained in any other copy except for the whole set M_* .

A Mandelbrot copy is called *real* if it is centered on the real line. Let \mathcal{M} stand for the family of maximal real Mandelbrot copies, We say that the maps $f \in \mathcal{T}_M$ are renormalizable with *real* combinatorics.

Let $\mathcal{T}_M \subset \mathcal{QL}$ stand for the set of quadratic-like germs which are hybrid equivalent to the quadratic maps P_c with $c \in M$ (that is, the union of the hybrid classes via M). We call it a *renormalization strip*. The maps in the strip are renormalizable with the same combinatorics encoded by the little Mandelbrot set M . Thus the renormalization operator R is canonically defined on the union of all the renormalization strips. The restriction $R|_{\mathcal{T}_M}$ will also be denoted as R_M .

Lemma 2.12 (de Melo - van Strien [MvS]). *The renormalization operator $R : \cup \mathcal{T}_M \rightarrow \mathcal{QL}$ is injective.*

Moreover, renormalization is non-expanding with respect to the Teichmüller-Sullivan metric on the hybrid classes:

$$\text{dist}_T(Rf, Rg) \leq \text{dist}_T(f, g). \quad (2.8)$$

This immediately follows from the fact that an appropriate restriction of a hybrid conjugacy h between f and g provides a hybrid conjugacy between the renormalizations Rf and Rg . This observation was a starting point for Sullivan’s renormalization theory [S1].

For any $M \in \mathcal{M}$, there is a canonical homeomorphism $\sigma : M \rightarrow M_*$ defined as the composition of the renormalization and the straightening, $\sigma = \chi \circ R$ ([DH2, M]).

If a map $f \in \mathcal{QL}$ is renormalizable a few times, then its combinatorics is encoded by a sequence (finite or infinite) $\tau(f) = \{M_0, M_1, \dots\}$ such that $R^n f \in \mathcal{T}_{M_n}$, $n = 0, 1, \dots$. One says that an infinitely renormalizable map f has a *bounded type* if the periods $p(M_n)$ are bounded.

Note that any R_M admits a complex analytic extension to Banach neighborhoods of points $f \in \mathcal{T}_M$. Namely, if $R_M f = f^p : U' \rightarrow U$ is a quadratic-like renormalization of $f \in \mathcal{C}_V$ then any nearby map $g \in \mathcal{QL}_V$ admits a quadratic-like return map $g^p : U'_g \rightarrow U$ with the same range. We can call this map the renormalization of f even when it has Cantor Julia set. Since g^p

analytically depends on g , this provides us with the desired extension (see [L4, §5.3] for a more detailed discussion).

Let us say that a map f is *non-escaping* under the renormalization of type $\tau = \{M_0, M_1, \dots\}$ if all the maps

$$f_n = R_{M_n} \circ \dots \circ R_{M_0} f \quad (2.9)$$

are well-defined (where the R_{M_k} stand for the analytic extensions of the renormalizations) and $\text{mod}(f_n) \geq \epsilon > 0$, $n = 0, 1, \dots$

Lemma 2.13 ([L4], Lemma 5.7). *If a quadratic-like map f is non-escaping under the renormalization of type $\tau = \{M_0, M_1, \dots\}$ then it is infinitely renormalizable with type τ .*

Any equipped proper unfolded quadratic-like family (\mathbf{f}, \mathbf{h}) over a topological disk Λ can be tiled into the parapuzzles similarly to the quadratic family, depending on the initial choice of external rays but canonical on the Mandelbrot set (see [L3]). Let $f_0 \in \mathbf{f}$ be renormalizable with type M . Then as in the quadratic case, we have a full or almost full unfolded quadratic-like family $\mathbf{g}_n = \{g_{n,\lambda} : V_\lambda^n \rightarrow V_\lambda^{n-1}\}$ on the corresponding parapuzzle $\Delta^n \ni f_0$. The renormalization $R_M f_\lambda = g_{n,\lambda}$ is well defined and analytic on this parapuzzle, so that this provides us with the analytic continuation of R_M onto the parapuzzle Δ^n . It will be naturally denoted as $R_M \mathbf{f}$. Moreover, this family is also equipped with a holomorphic motion \mathbf{j} , so that we can write under these circumstances that $(\mathbf{g}, \mathbf{j}) = R_M(\mathbf{f}, \mathbf{h})$.

Let us consider a conformal map $f : \mathcal{S} \rightarrow \mathcal{T}$ between two Riemann surfaces supplied with conformal metrics. The *distortion* (or *non-linearity*) of f is defined as follows:

$$n(f) = \sup_{z, \zeta \in \mathcal{S}} \log \frac{\|Df(z)\|}{\|Df(\zeta)\|}.$$

The following statement shows the renormalization has bounded non-linearity on unfolded quadratic-like families:

Lemma 2.14. *Let us consider a family $(\mathbf{f} : \mathbb{V}' \rightarrow \mathbb{V}) \in \mathcal{G}_\mu$ over Λ , and let $\Lambda' = \{\lambda : f_\lambda(0) \in V'_\lambda\}$. Then R has a $K(\mu)$ -bounded nonlinearity on \mathbf{f} over Λ' with respect to the hyperbolic metric on Λ . Moreover, the non-linearity in the hyperbolic scale ϵ (i.e., within any hyperbolic disk in Λ of radius ϵ) is $O(\epsilon)$.*

Proof. This follows from the Koebe Distortion Theorem. □

2.4. Essentially bounded combinatorics. There is a special type of renormalization combinatorics related to the parabolic bifurcation which usually requires a special treatment. In this section we will describe this phenomenon.

Let f be a renormalizable map of period p with real combinatorics $M \in \mathcal{M}$. Let us consider a *central cascade*

$$V^m \supset V^{m+1} \supset \dots \supset V^{m+N} \quad (2.10)$$

meaning that all the levels $m, m+1, \dots, m+N-2$, are central: $g_{m+1}0 \in V^{m+N-1} \setminus V^{m+N}$. Then the quadratic-like map g_{m+1} is combinatorially close to either the Ulam-Neumann map

$z \mapsto z^2 - 2$, or to the parabolic map $z \mapsto z^2 - 1/4$ (see [L2], §8). In the former case the cascade is called *Ulam-Neumann*, while in the latter it is called *saddle-node*.

For $z \in \omega(0) \cap (V^m \setminus V^{m+1})$, let us consider the level k such that either $g_{m+1}z \in V^k \setminus V^{k+1}$ with $k < m + N$, or $g_{m+1}z \in V^{m+N}$ and then $k = m + N$. Let $d(z) = \max(k - m, m + N - k)$ stand for the “depth” of landing of z in the cascade. Finally, let $d_m = \sup d(z)$ as z runs through the set of points as above.

If (2.10) is a saddle-node cascade, then all the levels $m + i$ with $m + d_m < i < m + N - d_m$ will be called *neglectable*.

Let us remove from the orbit $\{f^n 0\}_{n=0}^{p-1}$ all the points whose first landing at some saddle-node cascade occurs on a neglectable level (we will refer to this procedure as “eliminating the neglectable part of the cascade”). The number of points which are left is called the *essential period* $p_e = p_e(f) = p_e(M)$ of the renormalization (see [L2, LY]).

If the period p is much bigger than the essential period p_e then the orbit of the critical point spends a lot of time near a “ghost” parabolic point. Since this saddle-node behaviour (also called *intermittency*) is well-understood (see [D3, DD]), such a combinatorial situation admits a thorough analysis.

We say that an infinitely renormalizable map has *essentially bounded combinatorics* if $p_e(R^n f) \leq \bar{p}_e$, $n = 0, 1, \dots$

Let us describe this phenomenon via the parameter plane. Let f be a quadratic polynomial with $p_e(f) \leq \bar{p}_e$. Let us consider the first cascade (2.10) with $m = 0$, and the first return $f^{l_0} 0 \in V^{N-1} \setminus V^N$ of the critical point to this cascade. By definition of the essential period, the return time l_0 is bounded in terms of p_e .

Take the parapuzzle piece Δ^m corresponding to the return map $g_1 = f^{l_0}$. This parapuzzle contains a little Mandelbrot copy M centered at the parameter value for which $g_{1,c} 0 = 0$. Let b be the cusp of this Mandelbrot set. If the above cascade is long enough then f is a small perturbation of f_b (moving out of the little copy M).

As the return time l_0 is bounded, the number of little copies M specified in this way is bounded (in terms of p_e). Let \mathcal{M}^0 stand for this family of little copies.

Let us now wait until the first return of the critical point to the next cascade (2.10) of the principal nest (with $m = n_1 + 1$). Let it happens at moment l_1 . The combinatorics of the critical orbit until moment l_1 specifies finitely many sequences \mathcal{M}_i^1 of little Mandelbrot copies going to the cusps of the sets from \mathcal{M}^0 . Namely, the length of the first cascade specifies the element of the sequence. The sequence itself is specified by the combinatorics of the orbit after eliminating the neglectable part of the first cascade.

Let us now consider the motion of the orbit through the second cascade until its landing at the third one. Then we obtain finitely many Mandelbrot sequences \mathcal{M}_{ij}^2 accumulating to the cusps of the previous Mandelbrot sets. The sequence is specified by the combinatorics of the critical orbit after eliminating the neglectable parts of the first two cascades. The element of the sequence is specified by the pair of lengths (N_1, N_2) of the two cascade (so that formally speaking, it is a double sequence). If $N_1 \rightarrow \infty$ then the corresponding subsequence of the Mandelbrot copies accumulates on the cusps of the first family \mathcal{M}^1 (independently of N_2). If N_1 stays bounded while $N_2 \rightarrow \infty$, then the copies accumulate on the cusps of \mathcal{M}^2 .

Similarly we can consider the fourth cascade and the corresponding copies accumulating on the cusps of the first three families, etc.

Since the height of the principal nest is bounded in terms of \bar{p}_e , this procedure will terminate in a bounded number of steps, which altogether specifies a family $\mathcal{M}(\bar{p}_e)$ of little Mandelbrot copies M which contains all the copies with $p_e(M) \leq \bar{p}_e$ (see [Hi] for a more detailed and formal discussion of this situation).

2.5. Geometric bounds. An infinitely renormalizable map is said to have a priori bounds if $\text{mod}(R^n f) \geq \epsilon > 0$, $n = 0, 1, \dots$. We say that a map f is *close to the cusp* if it has an attracting fixed point with multiplier greater than $1/2$. Note that renormalizable maps are not close to the cusp.

Theorem 2.15 (A priori bounds [LS, LY]). *Let f be n times renormalizable real quadratic-like map with $\text{mod}(f) \geq \mu > 0$. Then*

$$\text{mod}(R^n f) \geq \nu_n(\mu) \geq \nu(\mu) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. Moreover, $\limsup \nu_n(\mu) \geq \nu > 0$, where ν is an absolute constant. Thus all real infinitely renormalizable maps have a priori bounds.

The following two geometric results are crucial for our study.

Theorem 2.16 (Big dynamical moduli [L2]). *Let $\text{mod}(f) \geq \mu > 0$. Then for any $M \in \mathcal{M}$,*

$$\text{mod}(R_M f) \geq \nu(\mu, M) \geq \bar{\nu}(\mu) > 0, \quad f \in M,$$

unless $p(M) = 2$ and f is close to the cusp. Moreover, $\nu(\mu, M) \rightarrow \infty$ as $p_e(M) \rightarrow \infty$.

Remark. A related result on moduli growth for real quadratics was independently proven by Graczyk & Swiatek [GS]. Note in this respect that our proof needs in a crucial way the above Theorem 2.16 for complex parameter values (even though in this paper we are ultimately interested in the real case).

The corresponding parapuzzle result is:

Theorem 2.17 (Parameter moduli [L3]). *Let $(\mathbf{f}, \mathbf{h}) \in \mathcal{G}_\mu^e$ be an equipped quadratic-like family over Λ , and let $M \in \mathcal{M}$, $p(M) > 2$. Then $(\mathbf{g}, \mathbf{j}) = R_M(\mathbf{f}, \mathbf{h})$ is an equipped quadratic-like family with*

$$\text{mod}(\mathbf{g}) \geq \nu = \nu(M, \mu) \geq \nu(\mu) > 0, \quad \text{Dil}(\mathbf{j}) \leq K(\mu).$$

Moreover, for any $\mu > 0$, $\nu(M, \mu) \rightarrow \infty$ as $p_e(M) \rightarrow \infty$.

Remark. The domain of definition of the renormalized family (\mathbf{g}, \mathbf{j}) can be chosen as a parapuzzle piece $\Delta_M \subset \Lambda$ bounded by certain parameter rays and equipotentials [L3]. Moreover, if we equip a vertical tube $\mathcal{T}_\mathcal{V}$ (see Lemma 2.10), then the parapuzzle pieces Δ_G in the vertical fibers \mathcal{Z}_G will holomorphically move with $G \in \mathcal{V}$. This motion is obtained by the holonomy along the extended foliation \mathcal{F} . Indeed, the parapuzzle pieces are specified by the coordinate of the critical value in the chart obtained by the straightening of the fundamental annulus. On the other hand, by definition (see the proof of [L4, Theorem 4.13]), this coordinate determines

the leaf of the extended foliation. Thus we obtain an equipped tube $\cup_{G \in \mathcal{V}} \Delta_G$ of the parapuzzle pieces to which the renormalization R_M analytically extends along the vertical fibers.

Corollary 2.18. *Let us consider an equipped quadratic-like family $(\mathbf{f}, \mathbf{h}) \in \mathcal{G}_\mu^e$ over Λ , and let $M_i = M_i(\mathbf{f}) \subset \mathbb{D}$ be the corresponding family of maximal real Mandelbrot copies except the doubling copy. Then the sets M_i have $K(\mu)$ -bounded shape and $\text{diam}(M_i) \rightarrow 0$ as $p(M_i) \rightarrow 0$ at rate depending only on μ .*

Proof. By Lemma 2.8, the Mandelbrot set $M_{\mathbf{f}}$ has a $L(\mu)$ -bounded shape. Hence it is enough to check shrinking of the M_i in the case of the quadratic family \mathcal{Q} . Moreover by the same Lemma and Theorem 2.17, the shapes of the sets M_i are bounded. Hence it is enough to have shrinking of their real traces $M_i \cap \mathbb{R}$. But these traces are pairwise disjoint as the M_i are maximal. \square

Remark. The doubling renormalization ($p(M) = 2$) produces an almost full unfolded quadratic-like family whose Mandelbrot set misses a single point, its cusp [DH2]. The set left after removing a neighborhood of the cusp is qc equivalent to the corresponding piece of the Mandelbrot set (with dilatation depending only on the geometry of the family (\mathbf{f}, \mathbf{h}) and the size of the removed neighborhood [L4]).

2.6. Renormalization limits. The previous results allow us to address the problem of possible renormalization limits.

Theorem 2.19. *Let us have a sequence of real renormalizable maps $f_k \in \mathcal{T}_{M_k}$ with $\text{mod}(f_k) \geq \mu$ and $p(M_k) \rightarrow \infty$. Take a limit $f = \lim Rf_k$ of their renormalizations. Then f is either a parabolic quadratic-like map, or a quadratic polynomial.*

Proof. Note first that by Theorem 2.16, the sequence Rf_k is pre-compact, so that we can always extract limit points. Furthermore, if the essential periods $p_e(M_k)$ are uniformly bounded then the renormalizations Rf_k must converge to the cusp points of the corresponding family $\mathcal{M}(\bar{p}_e)$ of little Mandelbrot copies (as described in §2.4). On the other hand, if $p_e(M_k) \rightarrow \infty$ along some subsequence, then by Theorem 2.16 the corresponding subsequence of the renormalizations must converge to a quadratic polynomial. \square

Note that parabolic cusp points with arbitrary combinatorics can be realized as above limits. Just take any little Mandelbrot copy M and a sequence of copies M_k going to its cusp c produced by increasingly long parabolic cascades (keeping $p_e(M_k)$ bounded). Then the limit points Rf_k , $f_k \in M_k$, have the combinatorics of the cusp c .

Also, take any real quadratic polynomial P_c without attracting points such that c is not a parameter of doubling bifurcation. Then P_c can be realized as one of the above limit. Indeed, it is easy to construct a sequence M_k converging to such a c , with $p_e(M_k) \rightarrow \infty$ (e.g., approximate c by Misiurewicz points and take little Mandelbrot copies nearby). Then by Theorem 2.16, $Rf_k \rightarrow P_c$ for $f_k \in M_k$.

2.7. Combinatorial rigidity.

Theorem 2.20 ([L2]). *Let f and g be two infinitely renormalizable quadratic-like maps with the same real combinatorial type $\tau = \{M_0, M_1, \dots\}$ (but not necessarily real), and with a priori bounds. Then f and g are hybrid equivalent.*

Together with a priori bounds (Theorem 2.15) the Rigidity Theorem yields:

Corollary 2.21. *For any real combinatorial type $\tau = \{M_0, M_1, \dots\}$, there is a single real quadratic P_c with this combinatorics.*

2.8. McMullen's towers. *McMullen's tower \bar{f} is a sequence $\{f_k\}_{k=l}^n$ of quadratic-like maps with connected Julia sets such that $f_k = Rf_{k+1}$. Combinatorial type $\tau(\bar{f})$ of such a tower is a sequence of maximal Mandelbrot copies $M_k \in \mathcal{M}$ such that $f_k \in M_k$. Let $p(\bar{f}) = \sup p(f_k)$ and $p_e(\bar{f}) = \sup p_e(f_k)$ stand respectively for the ‘‘period’’ and the ‘‘essential period’’ of the tower. One says that the tower has a bounded (or essentially bounded) combinatorics if $p(\bar{f})$ (respectively $p_e(\bar{f})$) is finite.*

The modulus $\text{mod}(\bar{f})$ of the tower is defined as $\inf \text{mod}(f_k)$. One says that a tower has a priori bounds if $\text{mod}(\bar{f}) > 0$. The space of towers is supplied with the weak topology: $\bar{g}_m = \{g_{m,k}\}_k \rightarrow \bar{f}$ as $m \rightarrow \infty$ if for each index k , $g_{m,k} \rightarrow f_k$. Compactness of $\mathcal{C}(\mu)$ yields:

Lemma 2.22. *The space of towers with uniformly bounded combinatorics and common a priori bounds is compact.*

Theorem 2.23 (Towers rigidity). *Two bi-infinite towers with the same bounded combinatorics and a priori bounds are affinely equivalent.*

Proof. By the Rigidity Theorem 2.20 two bi-infinite towers with the same combinatorics are quasi-conformally equivalent. By McMullen's Rigidity Theorem [McM2] qc equivalent towers are affinely equivalent. \square

Later on we will prove a similar rigidity theorem for towers with arbitrary real combinatorics (see Theorem 3.3).

2.9. Parabolic towers. Motivated by the works of C. McMullen [McM2] and A. Epstein [Ep], in [Hi] parabolic towers are introduced as geometric limits of McMullen's towers with uniformly bounded essential period. Fix a \bar{p}_e , and consider the family $\mathcal{M}(\bar{p}_e)$ of little Mandelbrot copies associated with \bar{p}_e -essentially bounded combinatorics (see §2.4). A *parabolic tower* \bar{f} with \bar{p}_e -essentially bounded combinatorics is a sequence of semigroups $\{G_n\}_{n=l}^n$ with two generators $G_n = \{f_n, g_n\}$. The map f_n is either renormalizable or parabolic. In the former case $g_n = \text{id}$ and $R_M f_n = f_{n-1}$ where $p_e(M) \leq \bar{p}_e$. In the latter case f_n has combinatorics of the cusp point of some $M \in \mathcal{M}(p_e)$, and g_n is the transit map between the Ecale-Voronin cylinders of f_n (see [D3, Hi]).

Two towers as above are called combinatorially equivalent if the maps f_n are either renormalizable with the same combinatorics (i.e., labeled by the same little Mandelbrot copy M_n) or parabolic with the same combinatorics. Two towers have a priori bounds if $\text{mod}(f_n) \geq \epsilon > 0$, $n = 0, 1, \dots$

Theorem 2.24 ([Hi]). *If two parabolic towers \bar{f} and \bar{g} with \bar{p}_e -essentially bounded combinatorics and a priori bounds are combinatorially equivalent then they are affinely equivalent.*

3. HYPERBOLICITY OF THE RENORMALIZATION OPERATOR

3.1. Uniformly exponential contraction.

Theorem 3.1. *Let f and g be two hybrid equivalent quadratic-like maps with modulus at least μ . Assume that f and g are n times renormalizable. Then*

$$\text{dist}(R^n f, R^n g) \leq C \rho^n,$$

where $\rho \in (0, 1)$ is an absolute constant, and $C > 0$ depends only on μ .

Proof. Let us fix a big \bar{p}_e . By Theorem 2.16, if f is a renormalizable map with $p_e(f) \geq \bar{p}_e$, then the renormalization Rf is ϵ -close to a quadratic map, where $\epsilon = \epsilon(\mu, \bar{p}_e)$. It follows that there is an absolute $\delta > 0$ such that the Banach ball $\mathcal{B}_V(f, \delta)$ is mapped into the Banach ball $\mathcal{B}_U(Rf, \epsilon)$ (here V and U are the appropriately chosen domains of f and Rf respectively). By the Schwarz Lemma in Banach spaces (see the Appendix), this map is uniformly ρ -contracting once $\epsilon < \frac{1}{2}\rho\delta$ (which is the case for sufficiently big \bar{p}_e).

Let us now show that there is an $N = N(\bar{p}_e)$ with the following property: If for $n \geq 2N$ consecutive renormalization iterates, the essential period stays bounded by \bar{p}_e , then R^N is contracting by $1/2$. Indeed, otherwise we can find a sequence of hybrid equivalent finite towers $\mathbf{F}_n = \{\mathbf{F}_m\}_{m=-l(n)}^{l(n)}$ and $\mathbf{G}_n = \{\mathbf{G}_m\}_{m=-l(n)}^{l(n)}$ of growing height $l(n) \rightarrow \infty$ but such that the F_0 and G_0 stay a definite distance apart (where n runs over a certain subsequence of \mathbb{N}). Passing to a geometric limit, we come to a contradiction with the Rigidity Theorem for parabolic towers 2.24.

Let us now consider the mixed case. Let $\epsilon > 0$. By Teichmüller non-expansion (see §2.1) and the relation between the Teichmüller and Banach metrics (Lemma 2.2), a Banach ball in any hybrid class of a sufficiently small radius $\delta > 0$ is mapped by all iterates R^n into a Banach ϵ -ball. By the Schwarz lemma, all the iterates R^m have a uniformly bounded norm on the hybrid classes.

Now, assume that we have $n \leq 2N$ renormalization iterates of essentially bounded type (i.e., with essentially period bounded by \bar{p}_e) followed by the renormalization of high type (i.e., with essential period greater than \bar{p}_e). If the quantifier \bar{p}_e is selected to be sufficiently big, then the contraction factor ρ of the last iterate suppresses the bounded expansion factor of R^n . Thus the whole composition is uniformly contracting.

On the other hand, if $n \geq 2N$ then we have a uniformly exponential contraction of the first n iterates. Indeed, every cascade of N consecutive iterates of essentially bounded type (except perhaps the tail cascade) contracts by $1/2$, while the tail cascade has a priori bounded norm. Altogether this yields the desired. \square

3.2. Cylinders. Let us consider an orbit $\{R^m f\}_{m=-l}^n$. Its (l, n) -itinerary is the sequence of the Mandelbrot copies $\{M_m\}_{m=-l}^n$ such that $R^m f \in \mathcal{T}_{M_m}$.

Lemma 3.2. *Let us have two points f and g with the same (l, n) -itinerary and such that $\text{mod}(R^k f) \geq \mu > 0$ and $\text{mod}(R^k g) \geq \mu > 0$, $-l \leq k \leq n$. Then $\text{dist}(f, g) < \epsilon = \epsilon(\mu, l, n)$, where $\epsilon \rightarrow 0$ as $l, n \rightarrow \infty$ (μ being fixed).*

Proof. By Theorem 3.1, there exist $\rho \in (0, 1)$ and N such that R^N is ρ -contracting on the foliation $\mathcal{F} \cap \mathcal{QL}(\mu)$.

Let $\chi(f) = P_c$ and $\chi(g) = P_b$. By Corollary 2.21, $|b - c| < \delta(n) \rightarrow 0$ as $n \rightarrow \infty$, so that f and g lie on the nearby leaves of the foliation \mathcal{F} . The same is applicable to $f_k \equiv R^k f$ and $g_k \equiv R^k g$, $k = -l, \dots, N$.

For any integer $k \in [-l, 0]$, let us consider a map $h_k \in \mathcal{H}(f_k)$ belonging to the vertical fiber via g_k , i.e., $\pi(h_k) = \pi(g_k)$. Then R^N ρ -contracts the distance between f_k and h_k . On the other hand, by Theorem 2.17, $R^N h_k$ and $R^N g_k$ belong to the same quadratic-like family of class \mathcal{G}_ν with $\nu = \nu(\mu)$. Since they have the nearby straightenings, Theorem 2.3 implies that $\text{dist}(R^N h_k, R^N g_k) < \delta_1(n) \rightarrow 0$ as $n \rightarrow \infty$.

Take an $\epsilon > 0$ and a $\rho' \in (\rho, 1)$, and find an n such that $\delta_1 = \delta_1(n) < \frac{(\rho' - \rho)\epsilon}{\rho + 1}$. If $\text{dist}(f_k, g_k) \geq \epsilon > 0$ then

$$\begin{aligned} \text{dist}(R^N f_k, R^N g_k) &\leq \text{dist}(R^N f_k, R^N h_k) + \text{dist}(R^N h_k, R^N g_k) \leq \\ &\rho \text{dist}(f_k, h_k) + \delta_1 \leq \rho(\text{dist}(f_k, g_k) + \delta_1) + \delta_1 < \rho' \text{dist}(f_k, g_k). \end{aligned} \quad (3.1)$$

Thus R^N uniformly contracts the distance between the f_k and g_k , while it stays greater than ϵ . Hence in bounded number of steps (depending on ϵ) this distance must become less than ϵ . \square

3.3. Realization and rigidity of general towers. Let us now prove that any real combinatorics $\bar{\tau} = \{M_k\}_{k=-\infty}^{\infty}$, $M_k \in \mathcal{M}_{\mathbb{R}}$, can be realized by a unique tower with a priori bounds.

Theorem 3.3. *For any real combinatorics τ there is a unique tower \bar{f} with this combinatorics and a priori bounds. Moreover, this tower is real and $\text{mod}(\bar{f}) \geq \nu$ with an absolute $\nu > 0$.*

Proof. By Theorem 2.15, there is an absolute $\nu > 0$ such that for any infinitely renormalizable quadratic polynomial $f = P_c \in \mathcal{I}$, $R^n f \in \mathcal{QL}(\nu)$, $n = 0, 1, \dots$.

Let us take a combinatorial sequence $\bar{\tau} = \{M_k\}$. For any $l \geq 0$, there is a real infinitely renormalizable quadratic polynomial $P_l \equiv P_{c_l}$ with combinatorics $\tau(P_l) = \{M_{-l}, \dots, M_0, \dots\}$. Let $f_{0,l} = R^l P_l$. These are infinitely renormalizable real quadratic-like maps with common combinatorics $\tau_0 = \{M_0, M_1, \dots\}$ and $\text{mod}(f_{0,l}) \geq \nu$. Since the set of such maps is compact, we can pass to a quadratic-like limit $f_0 = \lim_{l \rightarrow \infty} f_{0,l}$ (along a subsequence) with the same properties.

Let us now do the same thing for every $i \leq 0$. Let $f_{i,l} = R^{l+i} P_l$, and let $f_i = \lim_{l \rightarrow \infty} f_{i,l}$ be a limit point. The map f_i is real and has combinatorics $\tau_i = \{M_i, M_{i+1}, \dots\}$.

Selecting the above converging subsequences by means of the diagonal process, we construct a sequence of real infinitely renormalizable quadratic-like maps $\{f_i\}_{i=-\infty}^{\infty}$ such that $Rf_i = f_{i+1}$, $\chi(f_i) \in M_i$ and $\text{mod}(f_i) \geq \nu$. This sequence represents a real tower \bar{f} with combinatorics $\bar{\tau}$ and a moduli bound ν .

Thus any real combinatorics $\bar{\tau}$ is represented by a tower with a priori bounds. Moreover, this tower is unique. Indeed, if \bar{f} and \bar{g} are two such towers then by Lemma 3.2 $\text{dist}(f_0, g_0)$ is arbitrary small, so that $f_0 = g_0$. For the same reason $f_i = g_i$ for any i . \square

Let us now state a more general realization and rigidity theorem for one-sided towers.

Theorem 3.4. *For any real combinatorial past $\tau = \{M_k\}_{k=-1}^{-\infty}$ and any $c \in [-2, 1/4)$, there is a unique tower $\bar{f} = \{f_k\}_{k=0}^{-\infty}$ with a priori bounds such that $\chi(f_0) = c$ and $f_k \in M_k$ for $k < 0$. Moreover, this tower is real and $\text{mod}(\bar{f}) \geq \nu(\epsilon) > 0$, provided $c < 1/4 - \epsilon$.*

Proof. Theorem 2.15 for real finitely renormalizable quadratic-like maps yields the desired statement in the same way as for two-sided towers. \square

3.4. Renormalization horseshoe. Let us now consider the space Σ of all possible real combinatorial types $\hat{\tau} = \{M_k\}_{k=-\infty}^{\infty}$, where the $M_k \in \mathcal{M}$ are selected arbitrarily from the family of real maximal Mandelbrot copies. Supply Σ with the weak topology. Let $\omega : \Sigma \rightarrow \Sigma$ stand for the left shift on this space.

Let us say that an infinitely renormalizable map $f \in \mathcal{QL}$ is *completely non-escaping* under the renormalization if the full renormalization orbit $\{R^n f\}_{n=-\infty}^{\infty}$ is well-defined on f and $\text{mod}(f_n) \geq \mu = \mu(f) > 0$, $n \in \mathbb{Z}$. Note that in this case by Lemma 2.12 the backward trajectory $\{R^{-n} f\}_{n=0}^{\infty}$ is uniquely determined by f .

Let $\mathcal{A} \subset \mathcal{QL}$ stand for the set of completely non-escaping orbits with real combinatorics. We call this set the *(full) renormalization horseshoe* for the following reason:

Theorem 3.5. *There exist absolute $\nu > 0$ and $\rho \in (0, 1)$ with the following properties. The set \mathcal{A} belongs to $\mathcal{QL}_{\mathbb{R}}(\nu)$ and $R : \mathcal{A} \rightarrow \mathcal{A}$ is a homeomorphism. There exists a homeomorphism $\eta : \Sigma \rightarrow \mathcal{A}$ conjugating ω and $R|_{\mathcal{A}}$. Moreover, for any real infinitely renormalizable map f there exists a $g \in \mathcal{A}$ such that*

$$\text{dist}(R^n f, R^n g) \leq C\rho^n, \quad (3.2)$$

where C depends only on $\text{mod } f$.

Proof. Any completely non-escaping point f generates a bi-infinite tower $\{R^n f\}_{n=-\infty}^{\infty}$ with a priori bounds, and vice versa: zero coordinate of such a tower is non-escaping. By Theorem 3.3, any combinatorics $\tau \in \Sigma$ can be realized by unique such a tower. Thus we can define a map $\eta : \Sigma \rightarrow \mathcal{C}(\nu)$ by associating to a combinatorics $\tau \in \Sigma$ the zero coordinate f_0 of the tower $\bar{f} = \{f_i\}$ representing τ . This map is continuous by Lemma 3.2. Let \mathcal{A} be its image. Clearly, \mathcal{A} is R -invariant and η conjugates the shift ω and $R|_{\mathcal{A}}$. Moreover, by Lemma 2.12 this map is injective.

Since ω is a homeomorphism, $R : \mathcal{A} \rightarrow \mathcal{A}$ is bijective. Let us show that it is a homeomorphism. Let $J_k = M_k \cap \mathbb{R}$ be the real traces of the little copies $M_k \in \mathcal{M}$. Let us denote by $\mathcal{J} = \{J_k\}$ the family of these intervals (formally the same as \mathcal{M}).

For $J \in \mathcal{J}$, let $\mathcal{T}_J = \mathcal{T}_M \cap \mathcal{QL}_{\mathbb{R}}$ be the corresponding strip of real quadratic-like maps, and $\mathcal{A}_J = \mathcal{A} \cap \mathcal{T}_J$. As the boundary points of J are exactly once renormalizable, $\mathcal{A} \cap \partial\mathcal{T}_J = \emptyset$ for any $J \in \mathcal{J}$. Hence any map $f \in \mathcal{A}_J$ belongs to \mathcal{T}_J together with some neighborhood \mathcal{U} . Since every branch R_M of the renormalization is continuous, R is continuous at f .

Let us show that $R^{-1}|_{\mathcal{A}}$ is also continuous. Let $f \in \mathcal{A}$ and $R^{-1}f \in \mathcal{A}_J$. Let I be any other interval of family \mathcal{J} . Then by Lemma 2.12 $R\mathcal{A}_I \not\ni f$. Since the strip \mathcal{A}_I is compact, its image $R\mathcal{A}_I$ misses some neighborhood of f .

Let us show that these images cannot accumulate on f . Indeed, otherwise by Theorem 2.19, f is either parabolic quadratic-like map or a quadratic polynomial. But in the former case

the map is at most finitely renormalizable, while in the latter it is not anti-renormalizable (see Lemma 2.11).

Thus there is a neighborhood \mathcal{U} of f which misses all the images $R\mathcal{A}_I$ with $I \neq J$. Hence on this neighborhood $(R|\mathcal{A})^{-1} = (R|\mathcal{A}_J)^{-1}$. But the latter map is continuous since \mathcal{A}_J is compact.

Let us now show that $\eta : \Sigma \rightarrow \mathcal{A}$ is also a homeomorphism. The only thing to check is that the inverse map is continuous. Let $f \in \mathcal{A}$ be a map with itinerary $\{J_k\}_{k=-\infty}^{\infty}$. Let $n \geq 0$. Since $R^k f \in \text{int } \mathcal{T}_{J_k}$ for all k and $R|\mathcal{A}$ is a homeomorphism, all the maps $g \in \mathcal{A}$ near f have the same itinerary (J_{-n}, \dots, J_n) . But this exactly translates into continuity of η^{-1} .

Finally, for any real infinitely renormalizable quadratic-like map f , there is a map $g \in \mathcal{A}$ with the same combinatorics (by the Realization Theorem 3.3). By the Rigidity Theorem 2.20, f and g are hybrid equivalent, and Theorem 3.1 yields (3.2). \square

3.5. Periodic points of R . It is proven in [L4] that any infinitely renormalizable map f with periodic combinatorics is a hyperbolic periodic point for R . For what follows we will need the following weaker statement:

Lemma 3.6. *Periodic points of R are not attracting.*

Proof. Let $R^p f_0 = f_0$, $P_{c_0} = \chi(f_0)$. If f_0 is attracting than there is a neighborhood of the hybrid class $\mathcal{H}(f_0)$ attracted to the cycle of f_0 . By Lemma 2.13, all the maps in this neighborhood are infinitely renormalizable with the same combinatorics as f_0 . In particular, if c is nearby to c_0 then P_c is an infinitely renormalizable map with the same combinatorics as P_{c_0} contradicting Corollary 2.21. \square

Let DR_{tr} stand for the tangent action of DR in the one-dimensional quotient bundle $T\mathcal{QL}/T\mathcal{F}$ over \mathcal{C} . If $R^p f = f$ then the value $\lambda(f) = \|DR_{\text{tr}}^p(f)\|^{1/p}$ will be called *mean transversal multiplier* of f . Let

$$\bar{\lambda} = \inf_p \inf_{f: R^p f = f} \|DR_{\text{tr}}^p(f)\|^{1/p} \quad (3.3)$$

stand for the “smallest” mean transversal multiplier of the periodic points of R . By Lemma 3.6, $\bar{\lambda} \geq 1$. If $\bar{\lambda} > 1$ then we say that *the periodic points of R are uniformly hyperbolic* (this term is justified as by Theorem 3.1 R is uniformly contracting on the foliation \mathcal{F}).

3.6. Invariant cone field and line bundle.

Lemma 3.7. *For any $\mu > 0$ and $q \in (0, 1)$, there exist a $\delta > 0$ and a $c > 0$ with the following property. If $f \in \mathcal{QL}_{\mathbb{R}}$ is p times renormalizable with $\text{mod}(f) \geq \mu$, then*

$$\|DR_{\text{tr}}^p(f)\| \geq c(q\bar{\lambda}^\delta)^p,$$

where $\bar{\lambda}$ is the smallest mean transversal multiplier (3.3).

Proof. Let us have two points $f, g \in \mathcal{C}$ with the modulus at least μ lying on the same leaf of the foliation \mathcal{F} . Assume that $\text{dist}(f, g) \leq \epsilon$ (which is not necessarily small). Then

$$|DR_{\text{tr}}(g)| \geq q \|DR_{\text{tr}}(f)\|^\delta, \quad (3.4)$$

where $\delta > 0$, $q \in (0, 1)$, and $q \rightarrow 1$ as $\epsilon \rightarrow 0$. Moreover, there is a \bar{p} such that if $p(M) \leq \bar{p}$ then we can let $\delta = 1$, and otherwise we can let $q = 1$.

This follows from the fact that the holonomy from f to g is transversally qc (Theorem 2.3). Indeed, (3.4) is obviously true for any particular renormalization R_M with $\delta = 1$ and $q = q(M)$, as it is analytic. So let us take a Mandelbrot copy M with a big period $p(M)$. Then by Corollary 2.18, $\text{diam } M$ is small. Hence by bounded transversal non-linearity (Lemma 2.14), $\|DR_{\text{tr}}(f)\|$ is big.

Let us now take some full unfolded quadratic-like families $\mathcal{S}, \mathcal{X} \in \mathcal{G}_\sigma$ via f and g respectively. Let us take a disk $D \subset \mathcal{S}$ around f of size $\xi > 0$ whose image under the renormalization has size of order 1. By transversally bounded non-linearity, $\|DR_{\text{tr}}(f)\| \asymp \xi^{-1}$.

Furthermore, since the holonomy $\gamma : \mathcal{S} \rightarrow \mathcal{X}$ is qc, it is Hölder continuous with some exponent $\delta = \delta(\mu) > 0$ and an absolute constant. Hence $\text{diam}(\gamma D) = O(\xi^\delta)$, so that

$$\|DR_{\text{tr}}(g)\| \asymp (\text{diam}(\gamma D))^{-1} \geq c \|DR_{\text{tr}}(f)\|^\delta.$$

Finally, since $\|DR_{\text{tr}}(f)\|$ is big, we can kill the constant c by a small decreasing of the exponent. This yields (3.4).

Since by Corollary 2.14 transversal non-linearity of R is also bounded, we conclude that the same estimate holds under the assumption that f and g belong to the same renormalization strip and $\text{dist}(\chi(Rf), \chi(Rg)) < \epsilon$ (with the constants independent of the strip).

Given an f , let us consider the periodic point g of period p which has the same itinerary (M_0, \dots, M_{p-1}) as f . Then by Lemma 3.2 the orbit of $\{R^k g\}_{k=N}^{p-N}$ ϵ -shadows the corresponding orbit of f , where $N = N(\epsilon)$, and the desired estimate follows from (3.4) by the chain rule. \square

For $f \in \mathcal{C}_V$, let us define the θ -cone as follows;

$$C_f^\theta = \{u \in T_f \mathcal{QL}_V : \theta(u) > \theta\},$$

where the angle θ is defined by (2.5) (for notational convenience we skip dependence of the cone on the Banach slice).

Lemma 3.8. *There exist $\theta > 0$, N , and a choice of finitely many Banach slices $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$, $f \in \mathcal{A}$, such that $R^N C_f^\theta \subset C_{R^N f}^{2\theta}$.*

Proof. Let $\nu > 0$ be the absolute modulus bound ν from Theorem 2.15. There is an $\epsilon > 0$ such that if $f \in \mathcal{A}$ and U is a neighborhood of $K(f)$ with $\text{mod}(U \setminus K(f)) \geq \nu$, then U compactly contains the $(\epsilon \text{diam } K(f))$ -neighborhood $\Omega_\epsilon(f)$ of $K(f)$. By compactness, we can find a finite family of fundamental domains $V(f) \subset \Omega_\epsilon(f)$, $f \in \mathcal{A}$. By Theorem 2.15, there exists an N such that $\text{mod}(R^N f) \geq \nu$. Hence for small enough $\delta > 0$, we have: $R^N \mathcal{B}_{V(f)}(f, \delta) \subset \mathcal{B}_{V(R^N f)}$.

Let us consider the family of tangent cones $C_f^\theta \subset \mathcal{B}_f$.

By Theorem 3.1, R is uniformly exponential contracting in the \mathcal{F} -direction. On the other hand, by Lemmas 3.6 and 3.7, R can only slowly contract at the transversal direction: $\|DR_{\text{tr}}^n(f)\| \geq Cq^n$, $f \in \mathcal{A}$, with q arbitrary close to 1 (and $C = C(q)$).

Let us consider the tangent line bundle $\mathcal{K} = \{K_f\}$ complementing $T\mathcal{F}$ (see (2.4)). Recall that u^h and u^v stand for the horizontal and vertical projections of $u \in \mathcal{B}_f$. Since both subbundles $T\mathcal{F}$ and \mathcal{K} are continuous, $\|u^v\| \asymp \|u\|_{\text{tr}}$.

It follows that for N big enough, there exist $\rho' > \rho > 0$ with arbitrary small ratio ρ/ρ' such that

$$\|(R^N u)^h\| \leq \rho \|u^h\|, \quad \|(R^N u)^v\| \geq \rho' \|u^v\|, \quad (3.5)$$

where the norms are taken in the corresponding Banach spaces \mathcal{B}_f .

Let now $u \in \partial C_f^\theta$ with a small θ , $r \in (\rho, \rho')$. Then

$$\|(R^N u)^h\| = \|R^N u^h + (R^N u^v)^h\| \leq \rho \|u^h\| + O(\theta \|u^h\|) \leq r \|u^h\|, \quad (3.6)$$

provided θ is sufficiently small. By the second inequality of (3.5) and (3.6), $R^N u \in C_{R^N f}^{2\theta}$. \square

We are ready to construct the unstable-to-be line bundle \mathcal{O}^u over \mathcal{A} .

Lemma 3.9. *The renormalization operator has a continuous invariant tangent line field $\mathcal{E}^u = \{E_f^u \subset T_f \mathcal{QL}\}$ over the horseshoe \mathcal{A} transversal to \mathcal{F} .*

Proof. This is a standard construction by going backwards and pushing forward the cones: Let $C_f^{\theta,n} = R^n C_{R^{-n}f}^\theta$ and $E_f = \bigcap_{n \geq 0} C_f^{\theta,n}$. Note that since $f \in \mathcal{A}$, $R^{-n}f \in \mathcal{QL}(\nu)$ for all $n \geq 0$, the cones $C_f^{\theta,n}$ are well-defined and nested by the previous lemma.

Let us consider the projective cone \hat{C}_f^θ , i.e., the space of lines in C_f . It can be realized as the cross-section of C_f by the hyperplane $\{u : u^v = \text{const}\}$.

Supply the projective cones with the projective distance as follows: For $\hat{u}, \hat{v} \in \hat{C}_f^\theta$, consider the line interval $I(\hat{u}, \hat{v}) = \{w = \hat{u} + t\hat{v} \in \hat{C}_f^\theta\}$, and view it as the one-dimensional hyperbolic line \mathbb{H}^1 . Then the projective distance between \hat{u} and \hat{v} is defined as the hyperbolic distance between \hat{u} and \hat{v} in $I(\hat{u}, \hat{v})$.

The embedding $C_f^{2\theta} \rightarrow C_f^\theta$ uniformly contracts the projective distance on these cones, while the differential $DR^n : C_{R^{-n}f}^\theta \rightarrow C_f^\theta$ is at least simply contracting. Thus $DR^n : C_{R^{-n}f}^\theta \rightarrow C_f^\theta$ is uniformly contracting.

It follows that the projective cones $C_f^{\theta,n}$ uniformly exponentially shrink to some projective points. These points represent the tangent lines E_f^s transversal to \mathcal{F} . This line bundle is clearly invariant. It is also continuous: indeed the cone field $\{C_f^{\theta,n}\}$ is continuous for any given n , and well localizes the line field for n big enough. \square

3.7. Slow shadowing and hyperbolicity. We will now prove in the similar way as [L4] that $R|\mathcal{A}$ is uniformly hyperbolic. The idea is to construct (assuming the contrary) an orb(g) which slowly shadows some orb(f) on \mathcal{A} , which contradicts the Rigidity Theorem 2.20.

Let \mathcal{E}^s stand for the tangent bundle to \mathcal{F} over \mathcal{A} (the horizontal subbundle) and \mathcal{E}^u denote as above the transversal line bundle given by Lemma 3.9.

Theorem 3.10. *The renormalization operator $R : \mathcal{A} \rightarrow \mathcal{A}$ is uniformly hyperbolic with \mathcal{E}^s and \mathcal{E}^u serving for the stable and unstable subbundles.*

Proof. Due to Lemma 3.7, it is enough to prove uniform hyperbolicity of the periodic points. Assume the contrary: $\bar{\lambda} = 1$.

Let us consider a family of Banach slices $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$, $f \in \mathcal{A}$, from Lemma 3.8. By [L4, Lemma 4.12], the slice $\mathcal{F}_f \equiv \mathcal{F}_{V(f)}$ of the foliation \mathcal{F} admits a local extension to a Banach

neighborhood $\mathcal{U}_f \subset \mathcal{B}_f$ of f , which will still be denoted as \mathcal{F}_f . Since $\bar{\mathcal{A}}$ is compact in \mathcal{QL} , the size of \mathcal{U}_f is uniform over $f \in \mathcal{A}$. The leaf of \mathcal{F}_f via g will be denoted as $\mathcal{L}_f(g)$, $\mathcal{L}(f) \equiv \mathcal{L}_f(f)$.

Let $E_f^h(g) \subset \mathcal{B}_f$ stand for the tangent plane to the leaf $\mathcal{L}_f(g)$ of \mathcal{F}_f via g , and $E_f^v(g) \subset \mathcal{B}_f$ stand for the complementary line via g parallel to $T_f \mathcal{Z}_f \equiv E_f^v(f) \equiv E_f^v$. Given a tangent vector $u \in T_g \mathcal{B}_f$, its projections to $E_f^h(g)$ and $E_f^v(g)$ will be respectively denoted as u^h and u^v . Let us define the angle $\gamma = \gamma(u)$ by $\text{tg}(\gamma) = \|u^v\|/\|u^h\|$. Let

$$\Lambda_f(g) = \{u \in T_g \mathcal{B}_f : \gamma(u) > \pi/4\}$$

stand for the $\pi/4$ -cone at g .

Let us consider a topological bidisk $Q_f \subset \mathcal{B}_f$ centered at f of such a kind. Take a vertical topological disk $\mathcal{S}_f \subset E_f^v$ containing f , and consider its motion $\mathcal{S}_f \rightarrow \mathcal{S}_f(g)$ under the holonomy along \mathcal{F}_f , where g runs over a neighborhood $\mathcal{V}_f \subset \mathcal{L}(f)$ of f . The disks $\mathcal{S}_f(g)$ will be called the *vertical cross-sections* of Q_f . Let

$$\partial^h Q_f = \bigcup_{g \in \mathcal{V}_f} \partial \mathcal{S}_f(g) \quad \text{and} \quad \partial^v Q_f = \bigcup_{g \in \partial \mathcal{V}_f} \mathcal{S}_f(g)$$

stand respectively for the horizontal and vertical boundaries of the bidisk Q_f .

Let Υ_f stand for the family of complex analytic curves Γ in Q_f with $\partial \Gamma \subset \partial^h Q_f$ and with the tangent lines $T_g \Gamma$ belonging to the cones $\Lambda_f(g)$.

We can select the family of bidisks Q_f in such a way that for some N it satisfies the following properties:

- *Horizontal contraction:* There is a $\rho < 1$ such that if $g \in Q_f$ and $R^N g \in Q_{R^N f}$ then for any $u \in T_g Q$, $\|(DR^N v)^h\| \leq \rho \|v^h\|$;
- *Invariance of the cone fields:* If $g \in Q_f$ and $R^N g \in Q_{R^N f}$ then $R^N \Lambda_g \subset \Lambda_{R^N g}$;
- *Overflowing property for high periods:* There exist $\mu > 0$ and \bar{p} such that if $p(f) \geq \bar{p}$ then for any $g \in Q_f^s$, $R^N \mathcal{S}_g$ is a full family of class \mathcal{G}_μ and $R^N \mathcal{S}_g \cap Q_{R^N f} \in \Upsilon_{R^N f}$;
- *Definite vertical size on every strip:* For any M , there is an $\epsilon = \epsilon(M) > 0$ such that any vertical cross-section \mathcal{S}_g of Q_f contains a round disk of radius ϵ centered at g .

To make such a selection, let us first take a $\delta > 0$ so small that $\Pi \mathcal{B}_f(f, \delta) \subset \mathcal{H}_{W(f)}(\Pi(f), \epsilon)$, $f \in \mathcal{A}$, where the vertical tube over $\mathcal{H}_{W(f)}(\Pi(f), \epsilon)$ can be equipped with a holomorphically moving fundamental annulus (see Lemma 2.10).

Now, all further selections will be made in such a way that $\text{diam} Q_f < \delta$, so that all the bidisks Q_f will belong to the corresponding vertical tubes. Let us select the horizontal section \mathcal{V}_f of a bidisk Q_f as a Banach ball in the leaf \mathcal{L}_f of radius $\delta/4$. The choice of the vertical cross-sections \mathcal{S}_f depends on the renormalization strip $\mathcal{T}_M \ni f$ in the following fashion.

Let $\mathbb{D}_f(g, \epsilon) \subset E_f^u(g)$ stand for the round disk of radius ϵ centered at g . Let $\mathcal{M}_f(g) = E_f^u(g) \cap \mathcal{C}$ stand for the Mandelbrot set in the complex line $E_f^u(g)$ considered as a quadratic-like family. By Corollary 2.18, the little Mandelbrot copies M_i in the quadratic family \mathcal{Q} shrink. As by Theorem 2.3 the foliation \mathcal{F} is transversally quasiconformal, $\mathcal{M}_f(g) \subset \mathbb{D}_g(\delta/4)$, provided $p(M) \geq \bar{p}$ with big enough \bar{p} . For $p(M) < \bar{p}$, we let $\mathcal{S}_f = \mathbb{D}_f(\delta^K)$, where $K > 1$ is a bound on the qc dilatation of the holonomy along \mathcal{F} . Then the sections $\mathcal{S}_f(g)$ obtained from \mathcal{S}_f by the holonomy have diameter at most δ (since K -qc maps are Hölder continuous with exponent $1/K$).

For $p(M) \geq \bar{p}$, let us consider the vertical fiber \mathcal{Z}_f via f . By the above choice, it is a full unfolded equipped quadratic-like family with a definite geometry. By Theorem 2.17, the renormalization R_M analytically extends to a parameter puzzle piece $\tilde{\Delta}_f(g) \subset \mathcal{Z}_f(g)$ bounded by a κ -quasicircle and such that $\text{mod}(\tilde{\Delta}_f(g)) \asymp \nu > 0$, with absolute $\kappa > 0$ and $\nu > 0$. Let us shrink these domains a bit, $\Delta_f(g) \Subset \tilde{\Delta}_f(g)$, so that both $\text{mod}(\tilde{\Delta} \setminus \Delta)$ and $\Delta_f \setminus M_f$ are still definite, where $M_f = \mathcal{Z}_f \cap \mathcal{C}$ (in order to provide us with a Koebe space for the renormalization).

Let $\mathcal{S}_f(g) \subset E_f(g)$ be the image of Δ_f by the holonomy $\mathcal{Z}_f \rightarrow E_f^u(g)$. It follows that these are ρ -quasidisks with a definite space around the corresponding Mandelbrot sets: $\text{mod}(\mathcal{S}_f(g) \setminus \mathcal{M}_f(g)) \asymp \mu > 0$, where ρ and μ are absolute. Moreover, the corresponding bidisks Q_f have diameter less than δ if \bar{p} is sufficiently big. Hence these bidisks belong to the vertical tubes of Lemma 2.10, so that by Theorem 2.17 (and the Remark afterwards) the renormalization admits analytic continuation to them.

Furthermore, if $\delta > 0$ and $\kappa > 0$ are sufficiently small then there exist $0 < \rho < \rho' < 1$ with arbitrary small ratio ρ/ρ' , and an N such that for any $f \in \mathcal{A}$ and $g \in Q_f$ with $\text{dist}(R^N f, R^N g) < \kappa$,

$$\|DR^N(v)\| \leq \rho\|v\|, \quad v \in E_g^s, \quad (3.7)$$

$$\|DR^N(v)\| \geq \rho'\|v\|, \quad v \in E_g^u. \quad (3.8)$$

These estimates follow from the contraction in the hybrid classes (Theorem 3.1), almost repelling in the transversal direction (Lemma 3.7), and uniformly bounded non-linearity on the vertical cross-sections (according to the choice of the boxes). Indeed, these immediately imply (3.8), and also imply (3.7) by means of a simple estimate similar to (3.1) and the Schwarz Lemma.

Estimates (3.7) and (3.8) yield the first two desired properties of the family of boxes: horizontal contraction and invariance of the cone field. The last property, definite vertical size on the renormalization strips, is obvious from the definition of the boxes.

Let us check the third property, overflowing. By (3.7) and (3.8), the images $R^N \mathcal{S}_g$ of the horizontal cross-sections have the vertical slope at most 1 in scale κ , and the point $R^N g$ stays distance at least $(1 - \rho)\delta$ from the vertical boundary $\partial^v Q_{R^N f}$. Hence the overflowing property is satisfied, once the vertical size of all boxes is selected to be smaller than some sufficiently small $\epsilon > 0$.

As in [L4], let us now consider the fiber action \bar{R} of the renormalization on the space $\mathcal{A} \times \mathcal{QL}$ fibered over \mathcal{A} . The above boxes Q_f are naturally embedded into the fibers of this space. Let \mathcal{Y} stand for the union of the embedded boxes.

For $\tau \in (0, 1)$ near 1, let us consider a fiberwise contraction $T_\tau : \mathcal{Y} \rightarrow \mathcal{Y}$ (linear on the fibers in the above local charts). Consider the perturbation L_τ of \bar{R}^N by postcomposing \bar{R}^N with this contraction: $L_\tau = T_\tau \circ \bar{R}^N$. Assume that a periodic point $f = f_\tau$ of period p becomes attracting under this perturbation. Then by [L4, Lemma 2.1], there is an l and a point

$$g = g_\tau \in \partial^u Q_{L_\tau^l f} \quad (3.9)$$

such that

$$g \in \Omega_0 = \{h : L_\tau^k h \in Q_{L^{(k+l)}f}, k = 0, 1, \dots, \text{ and } L_\tau^n h \rightarrow \text{orb}(f) \text{ as } n \rightarrow \infty\}. \quad (3.10)$$

Moreover, the overflowing property implies that $L_\tau \partial^h Q_f \cap Q_{R^N f} = \emptyset$ if $p(f) \geq \bar{p}$. Hence f_τ with the shadowing point g_τ satisfying (3.9) and (3.10) may belong only to finitely many renormalization strips \mathcal{T}_M with $p(M) \leq \bar{p}$, and $\text{dist}(f_\tau, g_\tau) \geq \delta_0 > 0$.

Since R^N is transversally non-singular, $\text{dist}(R^N f_\tau, L_\tau g_\tau) \geq \delta_1 > 0$. Since the vertical diameter of the Q_f goes to 0 as $p(f) \rightarrow \infty$, $R^N f_\tau$ can belong to only finitely many possible strips \mathcal{T}_M . Repeating this argument for all further iterates, we conclude that for any $k \geq 0$, $R^{Nk} f_\tau$ may belong to only finitely many strips \mathcal{T}_M (depending on k). It follows that any limit of the maps f_τ is infinitely renormalizable.

Since we assume that the periodic points are not uniformly hyperbolic ($\bar{\lambda} = 1$), for any $\tau \in (0, 1)$ there is an attracting periodic point f_τ and the corresponding shadowing point g_τ . As by Lemma 2.1 the space $\mathcal{QL}(\mu, \rho)$ is compact, we can pass to limits $f = \lim f_{\tau_k}$ and $g = \lim g_{\tau_k}$ as $\tau_k \rightarrow 1$. As we have just shown, the first function is infinitely renormalizable. The second one shadows the first: $\bar{R}^{Nk} g \in D_{R^{Nk}f}$, $k = 0, 1, \dots$, and hence is non-escaping. By Lemma 2.13, g is infinitely renormalizable as well.

By the Rigidity Theorem 2.20, g must be hybrid equivalent to f . But on the other hand, (3.9) implies that g stays on positive distance from $\mathcal{H}(f)$ - contradiction. \square

3.8. Global unstable foliation. Let us now show that the unstable foliation of the horseshoe \mathcal{A} goes through all real combinatorial classes except the cusp.

Let us consider a family of disjoint complex analytic curves γ in \mathcal{QL}_V . Given a map $f \in \mathcal{QL}_V$, denote the curve passing through f by γ_f . Let us say that such a family of curves is *normal* if for any $f_0 \in \mathcal{QL}_V$ there is a ‘‘horizontal-vertical’’ local chart $\mathcal{U} = \mathcal{U}^h \times \mathcal{U}^v$ such that for any nearby f the curve $\gamma_f \cap \mathcal{U}$ is a graph of an analytic function $U^v \rightarrow U^h$.

Theorem 3.11. *Take any $\epsilon > 0$. Then there is a family $\mathcal{W}^u = \mathcal{W}_\epsilon^u$ of complex analytic leaves $W^u(f)$, $f \in \mathcal{A}$ satisfying the following properties.*

- $R^{-1}W^u(f)$ is well-defined and $R^{-1}W^u(f) \subset W^u(R^{-1}f)$;
- If $g \in W^u(f)$ then $\text{dist}(R^{-n}g, R^{-n}f) \leq C\rho^n$ with absolute $C > 0$ and $\rho \in (0, 1)$;
- Every unstable leaf $W^u(f)$ transversally intersects at a single point any hybrid class \mathcal{H}_c with $c \in [-2, 1/4 - \epsilon]$;
- The family of the unstable leaves $W^u(f)$ is normal;
- The renormalization R has uniformly bounded non-linearity on all the leaves;
- The straightening $\chi : W^u(f) \rightarrow \mathcal{Q}$ is uniformly quasi-conformal;
- The real traces $W^u(f) \cap \mathcal{QL}_\mathbb{R}$ of the leaves are pairwise disjoint.

Proof. Let us consider the family of boxes Q_f , $f \in \mathcal{A}$, constructed in the proof of Theorem 3.10. We can now add to the properties of this family listed therein (horizontal contraction, invariance of the cone field, etc.) the property of *uniform vertical expansion* (stated similarly to the horizontal contraction). It follows from the hyperbolicity of the horseshoe and uniformly bounded vertical distortion.

Let us now take a small number $q \in (0, 1)$ and scale all the boxes vertically by this factor. We obtain a family of boxes $\tilde{Q}_f \subset Q_f$. Let us consider a family \mathcal{Y}_f of complex analytic curves $\gamma \subset Q_f$ via f whose tangent lines stay within the corresponding family of cones. Consider also a similar family $\tilde{\mathcal{Y}}_f$ in \tilde{Q}_f but with additional assumption that these curves spread over the whole cross-section \tilde{Q}_f^u . If $\gamma \in \tilde{\mathcal{Y}}_f$ and $R^k \gamma \in \mathcal{Y}_{R^k f}$, $k = 0, \dots, l$, then $R^k \gamma$ is a curve of $\mathcal{Y}_{R^k f}$, over a vertical domain $D_k \subset Q_{R^k f}^u$. Moreover, by vertically bounded distortion, the domains D_k are quasi-disks with bounded shape. By the vertical expansion, the size of these quasi-disks exponentially grows with k .

It follows that there exists an l such that $R^k \gamma$ goes outside $Q_{R^k f}$ for some $k \leq l$. By bounded shape, at this moment $R^k \gamma$ overflows $\tilde{Q}_{R^k f}$, i.e., $R^k \gamma \cap \tilde{Q}_{R^k f} \in \tilde{Y}_{R^k \gamma}$.

Now we can construct the local unstable foliation \tilde{W}_{loc}^u in the usual way by letting

$$\tilde{W}_{\text{loc}}^u(f) = \lim_{k \rightarrow \infty} R^k \gamma_{-k},$$

where γ_{-k} is an arbitrary curve of $\mathcal{Y}_{R^{-k} f}$, and for $\gamma \in \mathcal{Y}_g$, $R\gamma$ is understood as the ‘‘cut-off’’ iterate $R\gamma \cap Q_{Rg}$. By the overflowing property just established, these manifolds are spread over the whole vertical cross-sections \tilde{Q}_f^u . Note that the horizontal contraction yields the usual property:

$$\tilde{W}^u(f) = \{g \in Q_f : R^{-n}g \in Q_{R^{-n}f}, n = 0, 1, \dots\}. \quad (3.11)$$

Let now $W_{\text{loc}}^u(f) = R\tilde{W}_{\text{loc}}^u(R^{-1}f)$. By the overflowing property for high periods stated in the proof of Theorem 3.10, the vertical sizes of these manifolds are bounded away from 0. It follows that this family is normal.

Globalize the $W_{\text{loc}}^u(f)$ by iterating them forward. Then for some n all the leaves of $R^n \mathcal{W}_{\text{loc}}^u$ will intersect all the hybrid classes \mathcal{H}_c with $c \in [-2, 1/4 - \epsilon]$. Indeed, by Theorem 3.4, any combinatorial past $\tau_- = \{\dots, J_{-2}, J_{-1}, c\}$ with $J_i \in \mathcal{J}$ and arbitrary $c \in [-2, 1/4 - \epsilon]$ can be realized by a one-sided tower $\{\dots, g_{-1}, g_0\}$. On the other hand, take a two-sided tower $\{f_k\}_{k=-\infty}^{\infty}$ with combinatorics $\{J_k\}_{k=-\infty}^{\infty}$ which has the same combinatorial past as τ_- . By Lemma 3.2, $\text{dist}(f_{-k}, g_{-k}) < \epsilon$ for all $k \geq n = n(\epsilon)$. Together with (3.11) and the overflowing property for high periods this implies that $g_{-n} \in W_{\text{loc}}^u(f_{-n})$, as was claimed.

Transversality between \mathcal{W}^u and \mathcal{F} follows from the corresponding property for the local unstable leaves and transversal non-singularity of R [L4, Lemma 5.3]. Uniqueness of the intersection point follows from the uniqueness of the one-sided tower with a given combinatorics (Theorem 3.4).

Since R is transversally non-singular and the local unstable foliation is normal, the family $\mathcal{W}^u = R^n \mathcal{W}_{\text{loc}}^u$ of global leaves will also be normal. Now bounded non-linearity follows from the Koebe Theorem (compare Lemma 2.14), while the bounded dilatation follows from Theorem 2.3. Disjointness of the real traces of the leaves (or better: disjointness of the intersections $W^u(f) \cap \mathcal{C}$) follows from Lemma 2.12. \square

4. Consequences

4.1. Proof of Theorem 1.4. Let us take any infinitely renormalizable parameter value $c \in \mathcal{I}$. By Lemma 3.5, there is a point $f \in \mathcal{A}$ with $\chi(f) = c$. Let $I_n(f) = R^{-n}W^u(R^n f) \subset W^u(f)$. By Theorem 3.11, $\text{diam } I_n(f) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, the same theorem implies by means of the standard hyperbolic estimate of the distortion that

$$R^n : I^n(f) \rightarrow W^u(R^n f) \quad (4.1)$$

has a uniformly bounded distortion.

For $g \in \mathcal{A}$, let us consider the interval $L(g) = (\chi|W^u(g))^{-1}(-3/4, 1/4 - \epsilon)$ on the unstable manifold $W^u(g)$ consisting of maps with attracting fixed point. Since the straightening $\chi : W^u(g) \rightarrow (-2, 1/4 - \epsilon)$ is uniformly quasi-symmetric (by Theorem 3.11), $\text{diam } L(g)/\text{diam } W^u(g) \geq \delta > 0$ for all g .

Let now $S_n(f) = R^{-n}L(R^n f) \subset I_n(f)$. Since the distortion of (4.1) is bounded, it follows that $\text{diam } S_n(f)/\text{diam } I_n(f) \geq \delta_1 > 0$ for all f and n . But the maps in $S_n(f)$ are only n times renormalizable. Hence the set of infinitely renormalizable maps has definite gaps in arbitrary small scales on $W^u(f)$ near f . Using once more that the straightening is uniformly quasi-symmetric we conclude that the same property holds in the real quadratic family $(-2, 1/4 - \epsilon)$ near c . Thus c is not a density point of \mathcal{I} , and the conclusion follows. \square

4.2. Proof of Theorem 1.5. Let $J = J_i^n(\epsilon)$. As in the proof of Theorem 1.4 we can find an interval $I = I^n(f) \subset W^u(f)$, $f \in \mathcal{A}$, such that $\chi(I) = J$ (see (4.1)). Then

$$\sigma^n|J = \chi \circ R^n \circ \chi^{-1}|I.$$

As $R^n|I$ has bounded distortion and χ is uniformly qs, the conclusion follows. \square

4.3. Proof of Theorem 1.6. Since by Theorem 3.11, the family \mathcal{W}^u of unstable leaves is normal, there is a neighborhood $\Omega \subset M_*$ of $[-2, 1/4 - \epsilon]$ in the Mandelbrot set covered by the straightenings $\chi(W^u(f))$ of all leaves. On the other hand, by Lemma 2.18, the maximal Mandelbrot copies $M \in \mathcal{M}$ shrink as $p(M) \rightarrow \infty$. Hence there is a \bar{p} such that $\chi(W^u(f)) \supset M$ for any $f \in \mathcal{A}$ and any $M \in \mathcal{M}$ with $p(M) \geq \bar{p}$.

Take a map $f \in \mathcal{A}$ with $\chi(f) = c$. Let

$$M(f) \equiv M^1(f) \supset M^2(f) \supset \cdots \ni f$$

stand for the nest of the Mandelbrot copies in the unstable leaf $W^u(f)$ containing f . We have shown that if $p(R^n f) \geq \bar{p}$ then $M(R^n f) \subset W^u(R^n f)$. But $M^n(f) = R^{-n}M(R^n f)$, and the map R^{-n} is contracting on the unstable foliation. It follows that $\text{diam } M^n(f) \rightarrow 0$, provided there is a subsequence $n_k \rightarrow \infty$ such that $p(R^{n_k} f) \geq \bar{p}$.

The Mandelbrot sets M^{n_k} have bounded shape because on the unstable foliation the renormalization iterates R^{-n} have bounded non-linearity and the straightening χ has bounded dilatation.

\square

5. APPENDIX: COMPLEX STRUCTURES MODELED ON FAMILIES OF BANACH SPACES

This Appendix is a brief version of [L4, Appendix 2] included for the reader's convenience. We refer to [L4] for the proofs and more details. We assume familiarity with the standard theory of manifolds modeled on Banach spaces (see e.g., [D1, Lang]).

5.1. Analytic functions theory in Banach spaces. Given a Banach space \mathcal{B} , let $\mathcal{B}_r(x)$ stand for the ball of radius r centered at x in \mathcal{B} , and $\mathcal{B}_r \equiv \mathcal{B}_r(x)$.

Cauchy Inequality. *Let $f : (\mathcal{B}_1, 0) \rightarrow (\mathcal{D}_1, 0)$ be a complex analytic map between two Banach balls. Then $\|Df(0)\| \leq 1$. Moreover, for $x \in \mathcal{B}_1$,*

$$\|Df(x)\| \leq \frac{1}{1 - \|x\|}.$$

The Cauchy Inequality yields:

Schwarz Lemma. *Let $r < 1/2$ and $f : (\mathcal{B}_1, 0) \rightarrow (\mathcal{D}_r, 0)$ be a complex analytic map between two Banach balls. Then the restriction of f onto the ball \mathcal{B}_r is contracting: $\|f(x) - f(y)\| \leq q\|x - y\|$, where $q = r/(1 - r) < 1$.*

Let us state a couple of facts on the intersection properties between analytic submanifolds which provide a tool to the transversality results.

Let \mathcal{X} and \mathcal{S} be two submanifolds in the Banach space \mathcal{B} intersecting at point x . Assume that $\text{codim } \mathcal{X} = \dim \mathcal{S} = 1$. Let us define the *intersection multiplicity* σ between \mathcal{X} and \mathcal{S} at x as follows. Select a local coordinate system (w, z) near x in such a way that $x = 0$ and $\mathcal{X} = \{z = 0\}$. Let us analytically parametrize \mathcal{S} near 0: $z = z(t), w = w(t), z(0) = 0, w(0) = 0$. Then by definition, σ is the multiplicity of the root of $z(t)$ at $t = 0$.

Hurwitz Theorem. *Under the above circumstances, let us consider a submanifold \mathcal{Y} of codimension 1 obtained by a small perturbation of \mathcal{X} . Then \mathcal{S} has σ intersection points with \mathcal{Y} near x counted with multiplicity.*

As usual, a foliation of some analytic Banach manifold is called analytic (smooth) if it can be locally straightened by an analytic (smooth) change of variable.

Intersection Lemma. *Let \mathcal{F} be a codimension one complex analytic foliation in a domain of a Banach space. Let \mathcal{S} be a one-dimensional complex analytic submanifold intersecting a leaf \mathcal{L}_0 of the foliation at a point x with multiplicity σ . Then \mathcal{S} has σ simple intersection points with any nearby leaf.*

Corollary 5.1. *Under the circumstances of the above lemma, \mathcal{S} is transversal to \mathcal{L}_0 at x if and only if it has a single intersection point near x with all nearby leaves.*

Let $X \subset \mathbb{C}$ be a subset of the complex plane. A *holomorphic motion* of X over a Banach ball $(\mathcal{B}_1, 0)$ is a family of injections $h_\lambda : X \rightarrow \mathbb{C}$, $\lambda \in \mathcal{B}_1$, with $h_0 = \text{id}$, holomorphically depending on $\lambda \in \mathcal{B}_1$ (for any given $z \in X$). The graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in X$, form a foliation \mathcal{F} (or rather a lamination as it is partially defined) in $\mathcal{B}_1 \times \mathbb{C}$ with complex codimension 1 analytic leaves. This is a “dual” viewpoint on holomorphic motions.

Given two complex one-dimensional transversals \mathcal{S} and \mathcal{T} to the lamination \mathcal{F} in $\mathcal{B}_1 \times \mathbb{C}$, we have a partially defined holonomy $\mathcal{S} \rightarrow \mathcal{T}$. We say that this map is locally quasi-conformal if it admits local quasi-conformal extensions near any $(\lambda, z) \in \mathcal{S}$.

Given two points $\lambda, \mu \in \mathcal{B}_1$, let us define the hyperbolic distance $\rho(\lambda, \mu)$ in \mathcal{B}_1 as the hyperbolic distance between λ and μ in the one-dimensional complex slice $\lambda + t(\mu - \lambda)$ passing through these points in \mathcal{B}_1 .

λ -Lemma. *Holomorphic motion h_λ of a set X over a Banach ball \mathcal{B}_1 is transversally locally quasi-conformal. The local dilatation K of the holonomy from $(\lambda, z) \in \mathcal{S}$ to $(\mu, \zeta) \in \mathcal{T}$ depends only on the hyperbolic distance ρ between the points λ and μ in \mathcal{B}_1 . Moreover, $K = 1 + O(\rho)$ as $\rho \rightarrow 0$.*

5.2. Inductive limits. Let (\mathbb{V}, \succ) be a partially ordered set. Recall that such a set is called *directed* if any two elements have a common majorant. We assume that \mathbb{V} has a countable base, i.e., there is a countable subset $\mathbb{W} \subset \mathbb{V}$ such that any $V \in \mathbb{V}$ has a majorant $W \in \mathbb{W}$.

Let us have a family of Banach spaces \mathcal{B}_V labeled by the elements of \mathbb{V} . An ϵ -balls in \mathcal{B}_V centered in an $f \in \mathcal{B}_V$ will be denoted $\mathcal{B}_V(f, \epsilon)$. Elements of the \mathcal{B}_V will be called “maps” (keep in mind further applications to quadratic-like maps). For every pair $U \succ V$, let us have a continuous linear *injection* $i_{U,V} : \mathcal{B}_V \rightarrow \mathcal{B}_U$. We assume the following properties:

- C1. *Density:* the image $i_{U,V}\mathcal{B}_V$ is dense in \mathcal{B}_U ;
- C2. *Compactness:* the $i_{U,V}$ is compact, i.e., the images $i_{U,V}\mathcal{B}_V(f, R)$ of balls are pre-compact in \mathcal{B}_U .

These properties yield:

- Lemma 5.2.**
- If $U, W \succ V$, $f \in \mathcal{B}_V$, $R > 0$, then the metrics ρ_U and ρ_W induced on the ball $\mathcal{B}_V(f, R)$ from \mathcal{B}_U and \mathcal{B}_W are quasi-isometric, i.e. $C^{-1}\rho_M \leq \rho_U \leq C\rho_W$ (with the constant depending on all the data specified).
 - Let $U \succ V$, and $\phi_i : (\mathcal{B}_U, \mathcal{B}_V) \rightarrow (\mathbb{C}, \mathbb{C})$ be a family of linear functionals continuous on the both spaces. Let us consider the common kernels of these functionals in the corresponding spaces: $\mathcal{L}_U \subset \mathcal{B}_U$ and $\mathcal{L}_V = \mathcal{L}_U \cap \mathcal{B}_V$. Then $\text{codim}(\mathcal{L}_U|\mathcal{B}_U) = \text{codim}(\mathcal{L}_V|\mathcal{B}_V)$.

For any $U \succ V$, let us identify any $f \in \mathcal{B}_V$ with its image $i_{U,V}f \in \mathcal{B}_U$ and span the equivalence relation generated by these identifications. Thus $f \in \mathcal{B}_U$ and $g \in \mathcal{B}_V$ are equivalent if there is a common majorant $W \succ (U, V)$ such that $i_{W,V}f = i_{W,U}g$ (then by injectivity this holds for any common majorant). The equivalence classes will be called *germs*. The space of germs is called the *inductive limit* of the Banach spaces \mathcal{B}_V and is denoted by $\mathcal{B} = \lim_{\rightarrow} \mathcal{B}_V$.

Every space \mathcal{B}_V is naturally injected into the space of germs, and will be considered as a subset of the latter. Given a subset $\mathcal{X} \subset \mathcal{B}$, the intersection $\mathcal{X}_V \equiv \mathcal{X} \cap \mathcal{B}_V$ will be called a (*Banach*) *slice* of \mathcal{X} .

Let us supply it with the *inductive limit topology*. In this topology, a set $\mathcal{X} \subset \mathcal{B}$ is claimed to be open if all its Banach slices \mathcal{X}_V are open. The axioms of topology are obviously satisfied, and the linear operations are obviously continuous (note that the product topology on $\mathcal{B} \times \mathcal{B}$ coincides with the natural inductive limit topology). Thus \mathcal{B} is a topological vector space. Since

points are obviously closed in this topology, \mathcal{B} is Hausdorff. The following lemma summarizes some useful general properties of inductive limits.

- Lemma 5.3.** (i) *In the inductive limit topology, $f_n \rightarrow f$ if and only if all the maps f_n and f belong to the same Banach slice \mathcal{B}_V and $f_n \rightarrow f$ in the intrinsic topology of \mathcal{B}_V . Any cluster point f of a set $K \subset \mathcal{B}$ is a limit of a sequence $\{f_n\} \subset K$.*
- (ii) *A set $\mathcal{X} \subset \mathcal{B}$ is open if and only if it is sequentially open.*
- (iii) *If X is a metric space and $\phi : (X, a) \rightarrow (\mathcal{QL}, g)$ is a continuous map then there is neighborhood $D \ni a$ and an element $V \in \mathbb{V}$ such that $\phi D \subset \mathcal{B}_V$.*
- (iv) *A set $\mathcal{K} \subset \mathcal{B}$ is compact if and only if it is sequentially compact. Such a set sits in some Banach space \mathcal{B}_V and bears an induced “Montel metric” which is well-defined up to a quasi-isometry.*
- (v) *A map $\phi : \mathcal{B} \rightarrow T$ to a topological space T is continuous iff every restriction $\phi|_{\mathcal{B}_V}$ is continuous.*

Remarks. 1. Any continuous curve $\gamma : (\mathbb{R}, 0) \rightarrow (\mathcal{B}, g)$ locally sits in some space \mathcal{B}_V ;

2. Given a continuous transformation $R : (\mathcal{B}, f) \rightarrow (\mathcal{T}, g)$ between two spaces of germs over \mathbb{V} and \mathbb{U} respectively, for any $V \in \mathbb{V}$ there exist an $\epsilon > 0$ and an element $U \in \mathbb{U}$ such that $R(\mathcal{B}_V(f, \epsilon)) \subset \mathcal{B}_U$.

3. The third statement of the above lemma shows that the space \mathcal{B} is not Fresche, i.e., it is not metrizable, and thus does not have a local countable base of neighborhoods. However, as we see, the sequential description of basic topological properties (cluster points, compactness, continuity etc.) is adequate in this space.

4. Note that the Banach slices \mathcal{B}_V are dense in the space of germs \mathcal{B} , so that their intrinsic topology is not induced from \mathcal{B} . However, by compactness, the intrinsic topology on the Banach balls $\mathcal{B}_V(f, R)$ is induced from \mathcal{B} .

5. If \mathbb{V} is directed then the space of germ is clearly a linear vector space.

Let us define a *sublimit* of the directed family \mathcal{B}_V , $V \in \mathbb{V}$ as the inductive limit of Banach spaces \mathcal{B}_V corresponding to a directed subset $\mathbb{U} \subset \mathbb{V}$ (which is not necessarily exhausting).

All linear operators $A : \mathcal{B} \rightarrow \mathcal{T}$ between spaces of germs are assumed to be continuous. Let us supply this space with the following convergence: A sequence of linear operators $A_n : \mathcal{B} \rightarrow \mathcal{T}$ converges to an operator A if for any $V \in \mathbb{V}$ there is a $U \in \mathbb{U}$ such that for all sufficiently big n , $A_n(\mathcal{B}_V) \subset \mathcal{T}_U$ and $A_n|_{\mathcal{B}_V} \rightarrow A|_{\mathcal{B}_V}$ in the uniform operator topology.

5.3. Analytic maps. We will give most of the definitions in the complex analytic category automatically accepting the corresponding smooth notions. So the Banach spaces under consideration are assumed to be over \mathbb{C} .

Let us consider an inductive limit \mathcal{B} over \mathbb{V} . By definition, a function $\phi : \mathcal{B} \rightarrow \mathbb{C}$ is complex analytic if all the restrictions $\phi|_{\mathcal{T}_V}$ are complex analytic in the Banach sense.

Let us have a continuous map $R : \mathcal{V} \rightarrow \mathcal{B}'$, where \mathcal{V} is an open subset of \mathcal{T} and \mathcal{B}' is an inductive limit space over \mathbb{V}' . It is called *differential* at a point $f \in \mathcal{B}$ if there is a real linear operator $A \equiv DR(f) : \mathcal{B} \rightarrow \mathcal{B}'$ such that

$$R(f + h) - R(x) = Ah + \omega(h),$$

where $\|\omega(h)\| = o(\|h\|)$ in the induced metric (note that this makes sense as by Lemma 5.3, $\omega(h)$ locally sits in the ball of a Banach slice \mathcal{B}'_V which bears the well-defined, up to quasi-isometry, induced metric).

As usual, a map $R : \mathcal{V} \rightarrow \mathcal{B}'$ is called *smooth* if it is differentiable at every point $f \in \mathcal{V}$ and the differential $DR(f)$ depends continuously on f (which amounts to differentiability of all Banach restrictions). A map $R : \mathcal{V} \rightarrow \mathcal{B}'$ is called *analytic* if it is smooth, and the differentials $DR(f)$ are linear over \mathbb{C} .

5.4. Complex structures. Let us have a family of Banach spaces \mathcal{B}_V labeled by elements V of some set \mathbb{V} , and open sets $\mathcal{U}_V \subset \mathcal{B}_V$. Let us have a set \mathcal{QL} and a family of injections $j_V : \mathcal{U}_V \rightarrow \mathcal{QL}$. The images $\mathcal{S}_V \equiv j_V \mathcal{U}_V$ are called Banach slices in \mathcal{QL} . The images $j_V \mathcal{V}_V \subset \mathcal{S}_V$ of open sets $\mathcal{V}_V \subset \mathcal{U}_V$ are called *Banach neighborhoods*. We assume the following properties (compare with C1 and C2):

- P1: *countable base and compactness.* There is a countable family of slices \mathcal{S}_i with the following property: Any $f \in \mathcal{QL}$ has a Banach neighborhood \mathcal{V}_V compactly contained in some \mathcal{S}_i .
- P2: *analyticity.* If some Banach neighborhood $\mathcal{V}_V \subset \mathcal{S}_V$ is also contained in another slice \mathcal{S}_U , then the transit map $j_{U,V} = j_U^{-1} \circ j_V : \mathcal{V}_V \rightarrow \mathcal{S}_U$ is analytic.
- P3: *density.* The differential $Dj_{U,V}(f)$ of the above transit map has a dense image in \mathcal{B}_U .

We endow \mathcal{QL} with the finest topology which makes all the injections j_V continuous by declaring a set $\mathcal{V} \subset \mathcal{QL}$ open if and only if all its Banach slices $j_V^{-1} \mathcal{V}$ are open. Lemma 5.3 should be minor modified in this more general situation:

Lemma 5.4. *In \mathcal{QL} , $f_n \rightarrow f$ if and only if the sequence $\{f_n\}$ sits in a finite union of the Banach slices, and the corresponding subsequences converge to f in the Banach metric. All other statements of Lemma 5.3 are valid in \mathcal{QL} as well with the modification that a single Banach slice in (iii) and (iv) should be replaced with a finite union of Banach slices.*

We say that a topological space \mathcal{QL} as above is endowed with *complex analytic structure* modeled on the family of Banach spaces. A subset $\mathcal{QL}^\#$ will be called a *slice* of \mathcal{QL} if it is a union of some family of Banach neighborhoods $j_V \mathcal{V}_V$. It naturally inherits from \mathcal{QL} complex analytic structure.

By definition, a smooth curve in \mathcal{QL} locally sits in some Banach slice and is smooth there. Since the transit maps between the Banach slices are analytic, this notion is well-defined. Moreover, the tangency of two smooth curves γ_1 and γ_2 through f is well-defined via the local Banach charts as well. Thus we can define a tangent vector to \mathcal{QL} at f as a class of tangent curves. This is generally not a linear space but rather a union of Banach spaces $T_f \mathcal{S}_V \approx \mathcal{B}_V$ (the space of smooth curves via f lying in \mathcal{S}_V).

Let us call a point $f \in \mathcal{QL}$ *regular* if any two Banach neighborhoods $\mathcal{U} \subset \mathcal{S}_U$ and $\mathcal{V} \subset \mathcal{S}_V$ around f are contained in a common slice \mathcal{S}_W . At such a point the tangent space $T_f \mathcal{QL}$ is a linear space identified with the inductive limit of the Banach spaces,

$$T_f \mathcal{QL} = \lim_{\rightarrow U: f \in \mathcal{S}_U} T_f \mathcal{S}_U.$$

A map $R : \mathcal{QL}^1 \rightarrow \mathcal{QL}^2$ is called *analytic* if it locally transfers any Banach slice \mathcal{S}_U to some slice \mathcal{S}_V , and its Banach restriction $j_V^{-1} R \circ j_U$ is analytic. An analytic map has a well-defined

differential $DR(f) : T_f \mathcal{QL}^1 \rightarrow T_{Rf} \mathcal{QL}^2$ continuously depending on f whose Banach restrictions are linear.

An analytic map is called *immersion* if it has an injective differential. The image \mathcal{X} of an injective immersion $i : \mathcal{M} \rightarrow \mathcal{QL}$ is called an *immersed submanifold*. It is called an (*embedded*) *submanifold* if additionally i is a homeomorphism onto \mathcal{X} supplied with the induced topology. For example, if there is an analytic projection $\pi : \mathcal{QL} \rightarrow \mathcal{M}$ such that $\pi \circ i = \text{id}$ then \mathcal{X} is a submanifold in \mathcal{M} . (Note that in this case π is a submersion at every point of \mathcal{X} .) By definition, the dimension of \mathcal{X} is equal to the dimension of \mathcal{M} .

If $i : (\mathcal{M}, m) \rightarrow (\mathcal{X}, f) \subset (\mathcal{QL}, f)$ is an immersion, then the tangent space $T_f \mathcal{X}$ is defined as the image of the differential $Di(m)$. If the points m and f are regular then $T_f \mathcal{X}$ is a linear subspace in $T_f \mathcal{QL}$, so that we have a well-defined notion of codimension of \mathcal{X} at f . Moreover, if \mathcal{M} is a Banach manifold (in particular, a finite dimensional manifold) then \mathcal{X} locally sits in a Banach slice of \mathcal{QL} .

Lemma 5.5. *Let $\mathcal{N} \subset \mathcal{M}$ be a connected submanifold in \mathcal{M} . Then $\text{codim}_g \mathcal{N}$ is constant.*

As usual, two submanifolds \mathcal{X} and \mathcal{Y} in \mathcal{M} are called *transversal* at a point $g \in \mathcal{X} \cap \mathcal{Y}$ if $T_g \mathcal{X} \oplus T_g \mathcal{Y} = T_g \mathcal{M}$.

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¹Specific references to this paper are given according to its revised version available via Internet on www.math.sunysb.edu/~mlyubich/universe.ps.gz.

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