PARABOLIC LIMITS OF RENORMALIZATION

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ABSTRACT. In this paper we give a combinatorial description of the renormlization limits of infinitely renormalizable unimodal maps with *essentially bounded* combinatorics admitting quadratic-like complex extensions. As an application we construct a natural analogue of the period-doubling fixed point. Dynamical hairiness is also proven for maps in this class. These results are proven by analyzing *parabolic towers*: sequences of maps related either by renormalization or by *parabolic renormalization*.

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1. Introduction

In this paper we extend the well-known combinatorial description of renormalization limits of unimodal maps with bounded combinatorics to renormlization limits of maps with *essentially bounded* combinatorics. This class of maps was introduced by Lyubich [L2] and their complex geometry was studied in [LY]. Roughly speaking the high renormalization periods of such maps are due to their renormalizations being small perturbations of parabolic maps. Although this leads to the creation of unbounded combinatorics, the *essential geometry* of these maps remains bounded away from zero.

Let us state our results (see §2 for background). In §3.1 we construct a countable collection of maximal tuned Mandelbrot copies $\{M_n^{(3)}\}_{n=1}^{\infty}$ that accumulate at $c=-1.75$, the root point of $M^{(3)}$. These copies have "essentially period tripling" combinatorics. Our first result produces the analog of the renormalization fixed point in the essentially period tripling situation:

Theorem 1.1. *There is a unique quadratic-like germ F such that*

$$\mathcal{R}^n(f) \to F$$

for any quadratic-like map f in the hybrid class of an infinitely renormalizable real quadratic with a tuning invariant

$$\tau(f) = (M_{n_1}^{(3)}, M_{n_2}^{(3)}, \dots, M_{n_k}^{(3)}, \dots)$$

satisfying $n_k \to \infty$ as $k \to \infty$. Any quadratic-like representative of F is hybrid equivalent to $z^2 - 1.75$ and hence has a period three parabolic orbit.

In order to state our second theorem we need to fix some notation. Let Ω denote the space of *unimodal non-renormalizable permutations*, or *shuffles*, and let $p_e(\sigma)$ be the essential period of $\sigma \in \Omega$. Let

$$\Omega_p = \{\sigma \in \Omega : p_e(\sigma) \le p\}$$

and let Ω_p^{cpt} be the compactification of Ω_p defined in §3.3. Let

$$\Sigma_p = \Pi_{-\infty}^{\infty} \Omega_p^{cpt}$$

with coordinate projections $\pi_n : \Sigma_p \to \Omega_p^{cpt}$ and let $\omega : \Sigma_p \to \Sigma_p$ be the left shift operator. Let $\Omega_{p,*}^{cpt}$ denote the space Ω_p^{cpt} with the symbol $*$ adjoined and let $\Sigma_p^* = \Pi_{-\infty}^{\infty} \Omega_{p,*}^{cpt}$. We will denote the left shift on Σ_p^* by ω as well. For any quadratic-like map f hybrid equivalent to an infinitely renormalizable real polynomial let

$$\bar{\sigma}(f) = (\dots, *, *, \sigma_0, \sigma_1, \sigma_2, \dots)$$

where σ_n is the shuffle corresponding to the n-th Mandelbrot copy in $\tau(f)$. Let $\bar{p}_e(f) = \sup_{n \ge 0} p_e(\pi_n(\bar{\sigma}(f)))$. Let $GQuad(m)$ be the space of quadratic-like germs with modulus at least m. We can now state the combinatorial classification of all limits of renormalization of an infinitely renormalizable real quadratic with essentially bounded combinatorics.

Theorem 1.2. *There is an $m > 0$ so that for any $p > 1$ there exists a continuous map*

$$h : \Sigma_p \to GQuad(m)$$

with the following property. Let f be a quadratic-like map in the hybrid class of an ∞-renormalizable real quadratic with $\bar{p}_e(f) \leq p$ and let $\bar{\sigma} \in \Sigma_p$ be a limit point of $\bar{\sigma}_n = \omega^n(\bar{\sigma}(f))$. If $\bar{\sigma}_{n_i} \to \bar{\sigma}$ then

$$\mathcal{R}^{n_i}(f) \to h(\bar{\sigma}). \tag{1.1}$$

Furthermore, if $\sigma_0 = \pi_0(\bar{\sigma}) \in \Omega_p$ then $h(\bar{\sigma})$ is renormalizable by the shuffle type σ_0 and h is a conjugacy between ω and $\mathcal{R}_{\sigma_0}$. If $\sigma_0 \notin \Omega_p$ then the inner class of $h(\bar{\sigma})$ is the root of a maximal tuned Mandelbrot copy $M(\sigma_0)$.

Let us comment on the ideas involved in the paper. Recall that the central objects of McMullen's argument [McM2] are *towers:* sequences of quadratic-like maps related by renormalization. A *forward tower* is a one-sided sequence and a *bi-infinite tower* is a two-sided infinite sequence. The question of convergence of renormalization is equivalent to the question of *combinatorial rigidity* of the corresponding limiting bi-infinite towers. However, for maps with essentially bounded combinatorics the limiting towers may contain parabolic maps and we lose the renormalization relation between levels. In this case a new relation appears: *parabolic renormalization.* That is, the maps in the limiting towers are related by either classical or parabolic renormalization. A tower which contains a parabolic renormalization is called a *parabolic tower.* Our proof of the rigidity of bi-infinite parabolic towers with definite modulus and essentially bounded combinatorics consists of first analyzing forward towers and then analyzing bi-infinite towers.

Our analysis of forward parabolic towers was motivated by the work of A. Epstein [E], which considered general holomorphic dynamical systems (with maximal domains of definition) and their geometric limits. The phenomenon studied there was the renormalization (different from the sense used in this paper) of a parabolic orbit at the ends of its Ècalle-Voronin cylinders. The phenomenon we study occurs away from the ends and as a result the forward infinite towers in this paper look in many ways like infinitely renormalizable real quadratic maps.

The combinatorial rigidity of forward parabolic towers with polynomial base map follows from the theory of *quadratic-like families* and from the combinatorial rigidity of quadratic polynomials with complex bounds and real combinatorics (see Proposition 6.1). After analyzing the Julia set of a foward tower we prove any quasiconformal conjugacy of a forward infinite parabolic tower with essentially bounded combinatorics and complex bounds is a hybrid conjugacy (see §6.4).

Then following the arguments of McMullen we prove in §6.5 the rigidity of bi-infinite towers. That is, we first prove

Theorem 1.3 (Dynamical Hairiness)**.** *The union of the Julia sets of the forward infinite subtowers of a bi-infinite tower with essentially bounded combinatorics and complex bounds is dense in the plane.*

Then we prove

Theorem 1.4. *Any quasiconformal equivalence of a bi-infinite tower with essentially bounded combinatorics and complex bounds is affine.*

Let us mention a parallel with critical circle maps. The theory of renormalization of unimodal maps is closely related to renormalization theory of critical circle maps. The rotation number ρ, more specifically its continued fraction expansion, determines the combinatorics of a circle map. If the factors in its expansion are bounded then the map has bounded combinatorics and has unbounded combinatorics otherwise. If a circle map has unbounded combinatorics then the rotation numbers of the renormalizations contain rational limit points and the corresponding limit of renormalization contain parabolic periodic points. That is, the only kind of unbounded combinatorics in the theory of critical circle maps is the essentially bounded combinatorics. DeFaria [deF] analyzed the renormalization limits of critical circle maps with bounded combinatorics and Yampolsky [Y] proves complex bounds for arbitrary combinatorics. We expect the techniques in this paper can be adapted to analyze renormalization limits of critical circle maps with arbitrary combinatorics.

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1.1. **Notation.**

- $\mathbb{H} \subset \mathbb{C}$ denotes the complex upper half-plane, $\widehat{\mathbb{C}}$ the Riemann sphere, $\mathbb{N} = \mathbb{N}_0$ the non-negative integers and $\mathbb{N}_+$ the positive integers.
- $[a, b]$ will also denote the interval $[b, a]$ if $b < a$.
- $\operatorname{diam}(U)$ denotes the euclidean diameter of $U \subset \mathbb{C}$ and $|I|$ the diameter of $I \subset \mathbb{R}$.
- $\operatorname{cl}(X)$, $\operatorname{int}(X)$ and ∂X denote the closure, interior and boundary of X in $\mathbb{R}$ if $X \subset \mathbb{R}$ and in $\mathbb{C}$ otherwise.
- $U \Subset V$ means U is compactly contained in V. Namely $\operatorname{cl}(U)$ is compact and $\operatorname{cl}(U) \subset V$.
- in a dynamical context f^n denotes f composed with itself n times.
- if V is a simply connected domain and $U \subset V$ then $\operatorname{mod}(U, V) = \sup_A \operatorname{mod}(A)$ where A is an annulus separating U from ∂V.
- $Dom(f)$ and $Range(f)$ denote the domain and range of f.
- $Comp(X)$ denotes the collection of connected components of X and $Comp(X, Y)$ denotes the components of X intersecting Y.
- $P_c(z) = z^2 + c$.

2. Background

2.1. **Quadratic-like maps.**

We will assume the reader is familiar with the theory of quasi-conformal maps and the Measurable Riemann Mapping Theorem (see [LV]).

A holomorphic map $f : U \to V$ is *quadratic-like* if U and V are topological disks in $\mathbb{C}$ with $U \Subset V$ and f is a branched double cover of U onto V. By topological disk we mean a simply connected domain in $\mathbb{C}$. A topological disk whose boundary is a Jordan curve will be called a *Jordan disk*. Unless otherwise indicated we will assume the critical point of a quadratic-like map is at the origin. A point $z \in U$ is *non-escaping* if $f^n(z)$ is defined for all $n \geq 0$. For a quadratic-like $f : U \to V$ define

- The *filled Julia set* $K(f) = \operatorname{cl}\{z \in U : z \text{ is non-escaping}\}$

- The *Julia set* $J(f) = \partial K(f)$
- The *post-critical set* $P(f) = \mathrm{cl}\{\bigcup_{n\geq 1} f^n(0)\}$

An actual quadratic polynomial can be considered quadratic-like by taking $V = \{z : |z| < R\}$ for some large R. Following [McM2], define *Quad* to be the union of all quadratic-like maps $f : U \to V$ and all quadratic polynomials $f : \mathbb{C} \to \mathbb{C}$ with a non-escaping critical point at the origin.

Impose on *Quad* the *Carathéodory topology.* That is, a sequence $f_n : U_n \to V_n$ converges to $f : U \to V$ iff $(U_n, 0)$ and $(V_n, f_n(0))$ converge to $(U, 0)$ and $(V, f(0))$, respectively, in the Carathéodory topology on pointed domains in the Riemann sphere $\widehat{\mathbb{C}}$ and f_n converges uniformly to f on compact subsets of U. We note the facts:

1. For any compact connected $U \subset V$ if $\mathrm{mod}(U, V) \geq m$ then V contains an $\epsilon(m)$-scaled neighborhood of U, where an ϵ-scaled neighborhood of a domain U is an $\epsilon \cdot \mathrm{diam}(U)$ neighborhood of U
2. If the domains are K-quasidisks then the Carathéodory convergence of pointed domains is equivalent to the Hausdorff convergence of their closures.
3. The set of K-quasidisks in $\mathbb{C}$ containing a definite neighborhood of the origin and with bounded diameter is compact in the Hausdorff topology.
4. If a sequence of pointed domains (U_n, u_n) in $\widehat{\mathbb{C}}$ converges in the Carathéodory topology to the domain (U, u), and if U_n and U are all hyperbolic Riemann surfaces, then the hyperbolic metrics on U_n converge in the C^∞ norm uniformly on compact set of U to the hyperbolic metric on U.

Define the subspaces

$$Quad(m) = \{f \in Quad : f \text{ is a polynomial or } \mathrm{mod}(U, V) \geq m\}$$

and

$$RQuad = \{f \in Quad : f(\bar{z}) = \overline{f(z)}\}.$$

The following compactness lemma is a basic tool in renormalization theory:

Lemma 2.1 ([McM1, Theorem 5.8])**.** *For any $C_0 > 0$, $C_1 < \infty$, $m > 0$, the set*

$$\{f \in Quad(m) : C_0 \leq \mathrm{diam}\, K(f) \leq C_1\}$$

is compact.

Define $GQuad(m)$ to be the quotient space of $Quad(m)$ by the relation $f \sim g$ iff $f = g$ on a neighborhood of zero. Define the set of *quadratic-like germs* to be $GQuad = \cup_m GQuad(m)$. Convergence of germs will always take place in some $GQuad(m)$. The germ of f will be denoted by $[f]$. From [McM2, Lemma 7.1] the (filled) Julia set of a quadratic-like germ is well defined and consequently if f and g are two quadratic-like representatives of a germ in $GQuad(m)$ then $f = g$ on an $\epsilon(m/2)$-scaled neighborhood of $K(f) = K(g)$. Since $K(f)$ is an upper semi-continuous function on *Quad*, if $f_k \in GQuad(m)$ converges to f then for any sequence of representatives $g_k \in Quad(m)$ it follows g_k converges to g on a definite neighborhood of $K(f)$. Given $f \in GQuad$ let

$$\mathrm{mod}(f) = \sup \mathrm{mod}(U, V)$$

where the supremum is taken over all quadratic-like representations of f.

Let $f \in Quad$. For a given $x \neq 0$ let $x' = f^{-1}(f(x)) \setminus \{x\}$. If $x = 0$ let $x' = 0$. There are two fixed points α and β of f counted with multiplicity and labeled so that $J(f) \setminus \{\beta\}$ is connected. The only case when $\alpha = \beta$ is when $I(f) = 1/4$. We say $f \in Quad$ is *normalized* if $\beta(f) = 1$. We normalize a germ by normalizing any quadratic-like representative.

A *quasi-conformal equivalence* ϕ between quadratic-like maps f and g is a quasiconformal map from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ such that $\phi \circ f = g \circ \phi$. A quasi-conformal equivalence is a *hybrid* equivalence if $\bar{\partial}\phi|_{K(f)} = 0$ as a distribution.

Proposition 2.2 (Straightening,[DH2])**.** *Any quadratic-like map f is hybrid equivalent to a quadratic polynomial. If $K(f)$ is connected the polynomial is unique up to affine conjugacy. Moreover, if $f \in Quad(m)$ then the equivalence can be chosen to be a conjugacy on an $\epsilon(m)$-scaled neighborhood of $K(f)$ and with dilatation bounded above by $K(m) < \infty$.*

The *inner class* of a map $f \in Quad$, denoted $I(f)$, is the unique c value such that f is hybrid equivalent to P_c. The inner class of a germ $I([f])$ is the inner class of any quadratic-like representative. The *Mandelbrot set*, M, is the set of $c \in \mathbb{C}$ such that the Julia set of $z^2 + c$ is connected. Let $\mathcal{H}(c) = I^{-1}(c)$ for $c \in M$. Note $\mathcal{H}(c) \subset Quad$.

Proposition 2.3 ([DH2],[McM2, Proposition 4.7])**.** *$I : Quad \to M$ is continuous.*

2.2. **Renormalization.** A parameter value $c \in \mathbb{C}$ is called *super-stable* if 0 is periodic under P_c. To each super-stable $c \neq 0$ there is associated a homeomorphic copy of M containing c called the *Mandelbrot set tuned by* c, or, briefly, an *M-copy*, and denoted by $c \star M$. The *root* of $c \star M$ is the point corresponding to 1/4 and the *center* is the point c. For every copy $c \star M$ there is a $p > 1$ such that for any $c' \in c \star M$, except possibly the root, and any $f \in \mathcal{H}(c')$ there is a domain $U \ni 0$ such that $f^p|_U \in Quad$. The map $f^p|_U$ is called a (complex) *pre-renormalization* of f and f is said to be *renormalizable* of period p. This pre-renormalization is always *simple*, meaning the iterates of $J(f^p|_U)$ under f are either are disjoint or intersect only along the orbit of $\beta(f^p|_U)$. The *period* of the copy, $p(c \star M)$, is the maximal such p and we say $c \star M$ is *maximal* if there is only one such p. We say $c \star M$ is *real* if c is real. The only real maximal M-copy for which the root point is not renormalizable is the period two copy $M^{(2)}$. We will denote the real period three copy by $M^{(3)}$. Define $\mathcal{H}(c \star M)$ to be the set of renormalizable $f \in I^{-1}(c \star M)$. In Fig. 1 we have drawn the Mandelbrot set highlighting $M^{(2)}$ and $M^{(3)}$. The root points are $c = -0.75$ and $c = -1.75$, respectively.

Let $c \star M$ be a maximal M-copy with period p and suppose $f \in \mathcal{H}(c \star M)$. If $f^p|_U$ and $f^p|_{U'}$ are two pre-renormalizations then $[f^p|_U] = [f^p|_{U'}]$. Hence we can define the *renormalization* $\mathcal{R}(f)$ to be the normalized quadratic-like germ of any pre-renormalization of period p. We define the renormalization of a germ $\mathcal{R}([f])$ to be the renormalization of a quadratic-like representative. A map $f \in Quad$ is *infinitely renormalizable* if $\mathcal{R}^n(f)$ is defined for all $n \geq 0$, or, equivalently, if $I(f)$ is contained in infinitely many M-copies. The tuning invariant of an infinitely renormalizable map $f \in Quad$ is

$$\tau(f) = (M_0, M_1, M_2, \dots)$$

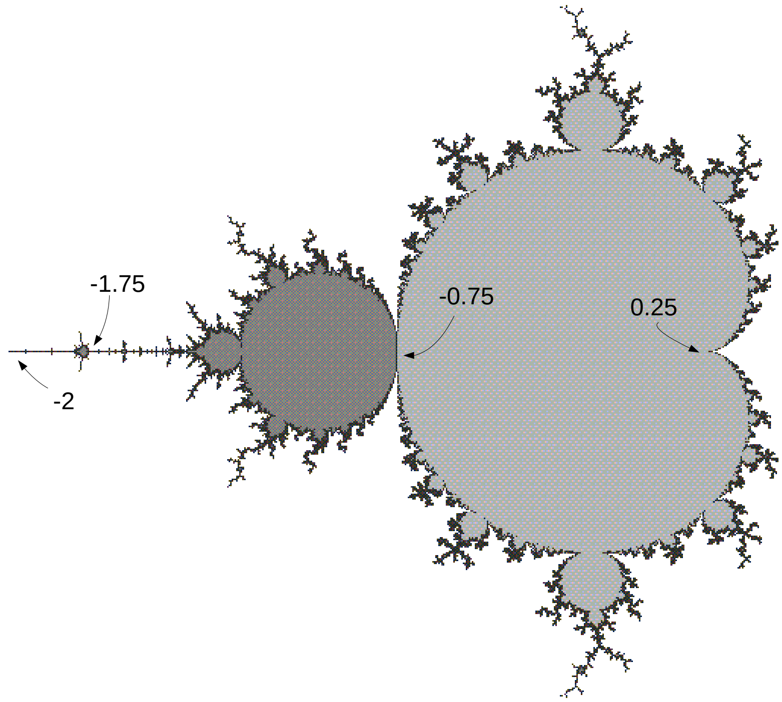

FIGURE 1. The Mandelbrot set.

where M_n is the maximal M-copy containing $I(\mathcal{R}^n(f))$. We say f, even if it is only finitely renormalizable, has *real combinatorics* if all M-copies in $\tau(f)$ are real. See [D2] for a more complete description of tuning.

Let us turn to renormalization in real dynamics. Let $I \subset \mathbb{R}$ be a closed interval. A continuous map $f : I \to I$ is *unimodal* if $f(\partial I) \subset \partial I$ and there is a unique extremum c of $f|_I$. For $f \in RQuad$ let $B(f) = [\beta, \beta']$, and $A(f) = [\alpha, \alpha'] \subset B(f)$. Note that $K(f) \cap \mathbb{R} = B(f)$. The next lemma follows from Lemma 2.1 and the continuity of β and β'. We say A is C-*commensurable* to B if $C^{-1} \le A/B \le C$.

Lemma 2.4. *For $m > 0$, $|\beta(f)|$ and $|B(f)|$ are $C(m)$-commensurable to* $\operatorname{diam} K(f)$ *for any $f \in RQuad(m)$.*

Any $f \in RQuad$ is unimodal on $B(f)$ (and this interval is maximal). We say a unimodal map $f|_{[a,b]}$ with $a < b$ is *positively oriented* if $f(b) = b$. The quadratic family P_c is positively oriented. A unimodal map $f : I \to I$ is *real-renormalizable* if there is an interval $I' \ni c$ and an $n > 1$ such that $f^n|_{I'}$ is unimodal. Unlike complex renormalization, we can canonically define real-renormalization as acting on unimodal maps as follows. Define the real pre-renormalization f_1 of a unimodal map f as $f^n|_{I'}$ where n is minimal and I is maximal and define the real-renormalization $\mathcal{R}(f)$ as f_1 conjugated by $x \mapsto x/\beta(f_1)$ where $\beta(f)$ is the boundary fixed point of f.

Suppose $f \in RQuad$ is real-renormalizable and positively oriented. Let f_1 be a pre-renormalization and let $\sigma(f)$ be the permutation induced on the orbit of $B(f_1)$ labeled from left to right. Any permutation that can be so realized is called a *unimodal non-renormalizable permutation*, or a *shuffle*. The permutation on two symbols we will denote by $\sigma^{(2)}$. If $\sigma(f) = \sigma^{(2)}$ we say f is *immediately renormalizable*. The map $c \star M \mapsto \sigma(P_c)$ from the set of real maximal M-copies to the set of shuffles is a bijection. We will denote the shuffle corresponding to $c \star M$ by $\sigma(c \star M)$ and the real maximal M-copy corresponding to σ by $M(\sigma)$. We will occasionally use the notation $\mathcal{R}_\sigma$ to denote the complex renormalization operator acting on $\mathcal{H}(M(\sigma))$ and on its germs. If $g \in \mathcal{H}(M(\sigma))$ then define $\sigma(g) = \sigma$. For an infinitely renormalizable $f \in Quad$ with real combinatorics define

$$\bar{\sigma}(f) = (\sigma(M_0), \sigma(M_1), \sigma(M_2), \dots)$$

where $\tau(f) = (M_0, M_1, M_2, \dots)$.

An ∞-renormalizable map $f \in Quad$ has *complex bounds* if there is some $m > 0$ such that the domain U_k and range V_k of the k-th complex pre-renormalization f_k can be chosen to satisfy $\text{mod}(U_k, V_k) \geq m$ for all $k \geq 1$. The following theorem establishes combinatorial rigidity of infinitely renormalizable maps with real combinatorics and complex bounds.

Theorem 2.5 ([L3]). *If P_c and $P_{c'}$ are two ∞-renormalizable quadratics with complex bounds and the same real combinatorics then $c = c'$.*

Complex bounds are proven to exist for real quadratics:

Theorem 2.6 ([LY, L2, S, LS]). *Real infinitely renormalizable quadratics have complex bounds. That is, if $f_k : U_k \to V_k$ is a complex renormalization of an ∞-renormalizable real quadratic f then $[f_k] \in GQuad(m)$ for some $m > 0$ independent of k. Moreover, U_k and V_k can be chosen to be K-quasidisks,*

$$\text{diam}(V_k) \leq C \cdot |B(f_k)|,$$

and, if $\sigma(\mathcal{R}^{k-1}(f)) \neq \sigma^{(2)}$ then the unbranched condition holds:

$$P(f) \cap V_k = P(f_k).$$

The values m, C and K are independent of f.

When we make an additional assumption on the combinatorics we obtain the unbranched condition on all levels.

Lemma 2.7. *Let $\epsilon > 0$. Suppose f is an infinitely renormalizable real quadratic with $I(\mathcal{R}^k(f)) \geq -2 + \epsilon$ for all $k \geq 0$. Then there is an $m > 0$ such that the domain U_k and range V_k of the k-th pre-renormalization can be chosen to satisfy*

- $\text{mod}(U_k, V_k) \geq m$
- *U_k and V_k are K-quasidisks*
- $\text{diam}(V_k) \leq C \cdot |B(f_k)|$
- $P(f) \cap V_k = P(f_k)$

for all $k \geq 1$. The constants m and K then depend on ϵ.

Proof. If $\sigma(\mathcal{R}^{k-1}(f)) \neq \sigma^{(2)}$ then let U_k and V_k be from Theorem 2.6. So assume $\sigma(\mathcal{R}^{k-1}(f)) = \sigma^{(2)}$. Let $h : U'_{k-1} \to V'_{k-1}$ and $h_1 : U'_k \to V'_k$ be the $(k-1)$-st and k-th pre-normalization from

Theorem 2.6 rescaled so that $\mathrm{diam}(K(h)) = 1$. Let $E = P(h) \setminus P(h_1)$. From the following lemma, Proposition 2.3 and the assumption $I(\mathcal{R}^k(f)) \geq -2 + \epsilon$ we obtain

$$\mathrm{dist}(E, B(h_1)) = |h^3(0) - \alpha(h)| \geq C(\epsilon, m) > 0.$$

From a construction of Sands, V_k' can be chosen to be the union of a euclidean disk centered at 0 of radius $|\beta(h_1)|$ and two small euclidean disks centered at $\pm\beta(h_1)$ of radius $\epsilon' > 0$. The modulus $\mathrm{mod}(U_k', V_k')$ is bounded below by a function $m'(\epsilon') > 0$. Choose $\epsilon' < C(\epsilon, m)$. ☐

2.3. **Generalized quadratic-like maps.** A holomorphic map f is *generalized quadratic-like* if $Range(f) = V$ is a topological disk, each $U \in Comp(Dom(f))$ is a topological disk compactly contained in V and $f|_U$ is a conformal isomorphism except for a distinguished component U_0, the *central component*, where $f|_{U_0}$ is a branched double cover onto V. We will consider only generalized quadratic-like maps whose domain has of finitely many components. Define the *filled Julia set*, $K(f)$, the *Julia set*, $J(f)$, and the *post-critical set*, $P(f)$, as for quadratic-like maps.

Let Gen be the union of $Quad$ and the space of generalized quadratic-like maps with a non-escaping critical point at the origin. Let $RGen$ be the space of real-symmetric maps in Gen with real-symmetric domains. Define

$$Gen(m) = \{f \in Gen : \mathrm{mod}(Dom(f), Range(f)) \geq m\}.$$

Impose on Gen the Carathéodory topology as follows. For a given $f \in Gen$ let $f(0)$ be the basepoint of $Range(f)$ and let $u_f = f^{-1}(f(0))$ be the basepoints of $Comp(Dom(f))$. A sequence $f_n \in Gen$ converges to f iff

- u_{f_n} converges in the Hausdorff topology to u_f
- if X is any Hausdorff limit point of $\widehat{\mathbb{C}} \setminus Dom(f_n)$ then

$$Dom(f) = Comp(\widehat{\mathbb{C}} \setminus X, u_f)$$

- $f_n \to f$ on compact subsets of $Dom(f)$.

The space of generalized quadratic-like germs is the quotient space of Gen by the relation $f \sim g$ iff $f = g$ on a neighborhood of $u_f = u_g$.

Define the *geometry* of $f \in Gen$ as

$$\mathrm{geo}(f) = \inf_{U \in Comp(Dom(f))} \frac{\mathrm{diam}(K(f) \cap U)}{\mathrm{diam}\, K(f)}.$$

The following lemma is a direct generalization of Lemma 2.1.

Lemma 2.8. *For a given $m > 0$, $C_0 > 0$ and C_1 the set*

$$\{f \in Gen(m) : \mathrm{geo}(f) \geq C_0 \text{ and } C_0 \leq \mathrm{diam}\, K(f) \leq C_1\}$$

is compact.

Suppose $f : \cup U_j \to V$ is a generalized quadratic-like map with critical point at the origin and suppose $U \subset \mathbb{C}$ is open. Define the open sets D_0 and D_+ by

$$D_{0/+} = \{z : f^n(z) \in U \text{ for some } n \in \mathbb{N}_{0/+}\}$$

and the maps $L_{0/+}(f, U) = L_{0/+} : D_{0/+} \to U$ by $L_{0/+}(z) = f^n(z)$ for the minimal *landing time* $\text{ltime}(z) = n \in \mathbb{N}_{0/+}$ such that $f^n(z) \in U$. We call L_0 the *first landing map* and L_+ the *strict first landing map.*

Define the *first return map to* U, $R(f, U) = R$ by

$$R = L_+|_{D_+ \cap U}.$$

Let $R = R(f, U_0)$ and suppose $0 \in Dom(R)$. Define the *generalized renormalization* of f as R restricted to $Comp(Dom(R), P(f) \cup \{0\})$.

Lemma 2.9. *Let* $\lambda > 0$*,* $m > 0$ *and* $r \in \mathbb{N}$*. Suppose* $f \in Gen(m)$ *satisfies* $\text{geo}(f) \geq \lambda$ *and suppose* $g \in Gen$ *is a restriction of* $R(f, U_0)$ *such that*

$$Dom(g) = Comp(Dom(R), u_g) \text{ and } \sup_{z \in Dom(g)} \text{ltime}(z) \leq r.$$

Then there exists $C(\lambda, m, r) > 0$ *such that* $\text{geo}(g) \geq C$*.*

Proof. Assume $\operatorname{diam} K(f) = 1$. Let $\cup_j U_j = Dom(f)$ and let $K_j = U_j \cap K(f)$. Since $\operatorname{diam} K_j \geq \lambda$ and $\text{mod}(K_j, U_j) \geq m$ it follows that U_j contains an $\epsilon(\lambda, m)$ neighborhood of K_j. For each $z \in u_g$ let $U_{z,0}, U_{z,1}, \ldots, U_{z,k} = U_0$ be the pull back of U_0 along the orbit $z, f(z), \ldots, f^k(z) = g(z)$. Assume $f(z) \in \cup_{j \neq 0} U_j$. From the Koebe Distortion Theorem and the fact that $k \leq r$ it follows $U_{z,1}$ contains a definite neighborhood of $f(z)$ and that $f^{k-1} : U_{z,1} \to U_0$ has bounded distortion. Hence each $U_{z,0}$ contains a definite neighborhood of z by Lemma 2.8. The lemma follows by pulling $\cup_{z \in u_g} U_{z,0}$ back to each $U_{z,1}$ by a map with bounded distortion and bounded derivative and then to $U_{z,0}$. □

Let $L_0 = L_0(f, U)$ and suppose $0 \in Dom(L_0)$. Define the *first through map*, $T = T(f, U)$, of f by

$$T = f \circ L_0.$$

We shall analyze the geometry of certain first through maps in §2.5.

2.4. **Families of generalized quadratic-like maps.** In this section we summarize the theory of holomorphic families of generalized quadratic-like maps. For further details see [L4]. Let $D \subset \mathbb{C}$ be a Jordan disk and fix $* \in D$. Let π_1 and π_2 be the coordinate projections of $\mathbb{C}^2$ to the first and second coordinates. Given a set $\mathbb{X} \subset \mathbb{C}^2$ let $X_\lambda = \pi_2(\mathbb{X} \cap \pi_1^{-1}(\lambda))$. An open set $\mathbb{X} \subset \mathbb{C}^2$ is a *Jordan bidisk* over D if $\pi_1(\mathbb{X}) = D$ and X_λ is a Jordan disk for all $\lambda \in D$. We say $\mathbb{X}$ admits an extension to the boundary if $\text{cl}(\mathbb{X})$ is homeomorphic over $\text{cl}(D)$ to $\text{cl}(D) \times \text{cl}(\mathbb{D})$. A section $\Psi : \text{cl}(D) \to \text{cl}(\mathbb{X})$ is a *trivial section* if there is a fiber-preserving homeomorphism $h : \text{cl}(\mathbb{X}) \to \text{cl}(D) \times \text{cl}(\mathbb{D})$ such that $(h \circ \Psi)(\lambda) = (\lambda, 0)$. Given a Jordan bidisk $\mathbb{X}$ which admits an extension to the boundary we define the *frame* $\delta\mathbb{X}$ as the torus $\cup_{\lambda \in \partial D} \cup_{z \in \partial X_\lambda} (\lambda, z)$. A section $\Phi : D \to \mathbb{X}$ is *proper* if it admits a continuous extension to ∂D and $\Phi(\partial D) \subset \delta\mathbb{X}$. Let Φ be a proper section and let Ψ be a trivial section. Let $\phi = \pi_2 \circ \Phi$ and $\psi = \pi_2 \circ \Psi$. Define the *winding number* of Φ to be the winding number of the curve $(\phi - \psi)|_{\partial D}$ around the origin.

Lemma 2.10 (Argument Principle)**.** *Let* $\mathbb{X}$ *be a Jordan bidisk over* D *that admits an extension to the boundary. Let* $\Phi : D \to \mathbb{X}$ *be a proper section and let* $\Psi : \text{cl}(D) \to \text{cl}(\mathbb{X})$ *be a continuous section, holomorphic on* D*. Let* $\phi = \pi_2 \circ \Phi$ *and* $\psi = \pi_2 \circ \Psi$*. Suppose there are no solutions to*

$\phi = \psi$ on ∂D. Then the number of solutions to $\phi = \psi$ counted with multiplicity is equal to the winding number of Φ.

Let $\cup_j \mathbb{U}_j$ be a pairwise disjoint collection of Jordan bidisks over D with $0 \in U_\lambda = U_{0,\lambda}$. Let $\mathbb{V}$ be a Jordan bidisk over D such that each $U_{j,\lambda}$ is compactly contained in V_λ. Let

$$\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}$$

be a fiber-preserving holomorphic map such that each fiber map $f_\lambda : \cup_j U_{j,\lambda} \to V_\lambda$ is a generalized quadratic-like map with critical point at the origin and which on each branch $f_\lambda|_{U_{j,\lambda}}$ admits a holomorphic extension to a neighborhood of $U_{j,\lambda}$. Let $\mathbf{h}$ be a holomorphic motion

$$h_\lambda : (\partial V_*, \cup_j \partial U_{j,*}) \to (\partial V_\lambda, \cup_j \partial U_{j,\lambda})$$

over D with basepoint $* \in D$ which respects the dynamics. We say $(\mathbf{f}, \mathbf{h})$ is a *holomorphic family of generalized quadratic-like maps over D*. When $\cup \mathbb{U}_j$ consists of only one bidisk then the family is a *DH quadratic-like family*. A family is *proper* if

1. $\mathbb{V}$ admits an extension to the boundary
2. for each $z \in \cup_j \partial U_{j,*}$ the section $\lambda \mapsto (\lambda, h_\lambda(z))$ extends continuously to ∂D and is a trivial section
3. the critical-value section $\Phi(\lambda) = (\lambda, f_\lambda(0))$ is proper.

The *winding number* of a proper family is the winding number of the critical value section.

Theorem 2.11 ([DH2]). *If $(\mathbf{f}, \mathbf{h})$ is a proper DH quadratic-like family over D with winding number 1 then*

$$M(\mathbf{f}, \mathbf{h}) = \{\lambda \in D : J(f_\lambda) \text{ is connected}\}$$

is homeomorphic to the standard Mandelbrot set M. The homeomorphism is given by the inner class map $\lambda \mapsto I(f_\lambda)$.

We finish this section with the renormalization of a family. Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}, \mathbf{h})$ be a proper holomorphic family of generalized quadratic-like maps over D with winding number 1. If $0 \in R(f_\lambda, U_{0,\lambda})$ let $\bar{i}_\lambda$ be the *return itinerary* of f_λ: the (possibly empty) sequence of indices of off-critical pieces $\{U_{j,\lambda}\}$ through which the critical point passes before returning to $U_{0,\lambda}$. For such an f_λ we can define a holomorphic motion $\mathbf{h}'$ of the boundaries of the domain and range of the return map to $U_{0,\lambda}$ by pulling back the holomorphic motion $\mathbf{h}$ by f_λ. The motion $\mathbf{h}'$ has basepoint λ and is defined over the neighborhood of λ having the itinerary $\bar{i}_\lambda$.

Lemma 2.12 ([L4, Lemma 3.6]). *Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}_j, \mathbf{h})$ be a proper generalized quadratic-like family over D with winding number 1. Let $* \in D$ be the basepoint and let $g_* = R(f_*, U_{0,*})$. Suppose $0 \in Dom(g_*)$. Then the set*

$$D' = \{\lambda \in D : \bar{i}_\lambda = \bar{i}_*\}$$

is a Jordan disk and the family of first return maps $(\mathbf{g}, \mathbf{h}')$ over D' is proper and has winding number 1.

2.5. **Parabolic periodic points.** The limits of maps with unbounded but essentially bounded combinatorics are maps with parabolic periodic points. This section reviews the local theory near parabolic orbits and their perturbations. The main results are the existence and continuity of Fatou coordinates. These results were proven in [DH1] and [La] for perturbations lying in an analytic family and later generalized in [Sh]. Our presentation is based on [Sh].

Throughout this section we give the space of holomorphic maps the "compact-open topology with domains". A basis for this topology is given by the sets

$$\mathcal{N}(f, K, \epsilon) = \{g : |g(z) - f(z)| < \epsilon \text{ for } z \in K\}$$

where $K \subset Dom(f)$ is compact and $\epsilon > 0$. If a sequence of quadratic-like maps converges to $f : U \to \mathbb{C}$ in the Carathéodory topology then it also converges to $f : U \to \mathbb{C}$ in this topology.

Let $\mathcal{P}_0$ be the space of holomorphic maps f_0 with a fixed point ξ_0 that is *parabolic* and *non-degenerate*: $f_0'(\xi_0) = 1$ and $f_0''(\xi_0) \neq 0$. For example, choose any quadratic-like map f_0 hybrid equivalent to $z^2 + 1/4$. Choose a neighborhood $N \ni \xi_0$ so that $f_0|_N$ is a diffeomorphism and maps N onto a neighborhood $N' \ni \xi_0$.

Proposition 2.13 (Fatou coordinates)**.** *Let $f_0 \in \mathcal{P}_0$ and choose N and N' as above. Then there exist topological disks $D_\pm \Subset N \cap N'$, whose union forms a punctured neighborhood of ξ_0 and which satisfy*

$$f_0^{\pm 1}(\mathrm{cl}(D_\pm)) \subset D_\pm \cup \{\xi_0\} \text{ and } \cap_{n \geq 0} f_0^{\pm n}(\mathrm{cl}(D_\pm)) = \{\xi_0\}.$$

Moreover, there exist univalent maps $\Phi_\pm : D_\pm \to \mathbb{C}$ such that

1. *$\Phi_\pm$ are unique up to post-composing with a translation*
2. *$Range(\Phi_+)$ and $Range(\Phi_-)$ contain a right and left half-plane, respectively*
3. *$\Phi_\pm(f_0(z)) = \Phi_\pm(z) + 1$*

The disks $D_\pm$ are called *incoming* and *outgoing petals* and the maps $\Phi_\pm$ are called the *Fatou coordinates.* The Fatou coordinates induce conformal isomorphisms between the *Ècalle-Voronin cylinders* $\mathcal{C}_\pm = D_\pm/f_0$ and $\mathbb{C}/\mathbb{Z}$. Let $\pi_\pm$ denote the projection of $D_\pm$ to $\mathcal{C}_\pm$ and extend π_+ to the attracting basin of ξ_0 by $\pi_+(z) = \pi_+(f_0^n(z))$ for a large enough n.

A *transit map* $g : \mathcal{C}_+ \to \mathcal{C}_-$ is a conformal isomorphism which respects the ends $\pm\infty$. A holomorphic map $\widetilde{g} : U \to \mathbb{C}$ is a *local lift* of a transit map g if $\mathrm{cl}(U) \subset D_+$, $Range(\widetilde{g}) \subset D_-$, and

$$g \circ \pi_+ = \pi_- \circ \widetilde{g}.$$

When written in Fatou coordinates, $\widetilde{g}$ is a translation T_a by a complex number a. The quantity $\bar{a} = a \mod \mathbb{Z}$, called the *phase*, depends only on g (and the normalization of Fatou coordinates) and uniquely specifies g. We will use the notation $g_{\bar{a}}$ to denote the transit map with phase $\bar{a}$.

To simplify future notation, let $\Phi = \Phi_+$ and $\phi = \Phi_-^{-1}$. Also, we shall freely use the notation Φ_n, Φ_f, $\mathcal{C}_{n,\pm}$, etc to indicate a dependence on an index n or map f.

We now consider perturbations of $f_0 \in \mathcal{P}_0$. Since ξ_0 is a non-degenerate parabolic fixed point the generic perturbation will cause it to bifurcate into two nearby fixed points ξ_f and ξ_f' with multipliers λ_f and λ_f', respectively. Let N be the neighborhood of ξ_0 chosen for Proposition 2.13

and let $\mathcal{P}$ be the space of holomorphic maps which are diffeomorphisms of N. Let $\mathcal{P}_1$ be the set of $f \in \mathcal{P}$ with exactly two fixed points ξ_f and ξ'_f in N satisfying

$$\arg(1-\lambda_f), \arg(1-\lambda'_f) \in [\pi/4, 3\pi/4] \cup [-3\pi/4, -\pi/4]. \tag{2.1}$$

Theorem 2.14 (Douady coordinates)**.** *Let $f_0 \in \mathcal{P}_0$. There is a neighborhood $\mathcal{N}$ of f_0 such that if $f \in (\mathcal{N} \cap \mathcal{P}_1)$ then there exist univalent maps $\Phi_f = \Phi_{f,+}$ and $\phi_f = (\Phi_{f,-})^{-1}$, unique up to translation, and a constant $a_f \in \mathbb{C}$ satisfying*

1. *$\Phi_f(f(z)) = \Phi_f(z) + 1$ and $\phi_f(w+1) = f(\phi_f(w))$ where defined*
2. *$\mathcal{C}_{f,+} = Dom(\Phi_f)/f$ and $\mathcal{C}_{f,-} = Range(\phi_f)/f$ are conformally cylinders and one can choose fundamental domains $S_{f,\pm}$ to depend on $f \in \mathcal{P}_1$ continuously in the Hausdorff topology.*
3. *(see Fig. 2) for $z \in S_{f,+}$ there is an $n > 0$ such that $f^n(z) \in S_{f,-}$ and for n minimal*

$$f^n(z) = (\phi_f \circ T_{a_f+n} \circ \Phi_f)(z). \tag{2.2}$$

If we fix points $z_\pm \in D_\pm$ and normalize $\Phi_{f,\pm}$ by $\Phi_{f,\pm}(z_\pm) = 0$ then $\Phi_{f,\pm}$ depend continuously on $f \in \mathcal{N} \cap (\mathcal{P}_0 \cup \mathcal{P}_1)$.

Suppose $f_0 \in \mathcal{P}_0$ and $f \in \mathcal{P}_1 \cap \mathcal{N}$ where $\mathcal{N}$ is from Theorem 2.14. The discontinuous map from $S_{f,+}$ to $S_{f,-}$ defined by equation 2.2 projects to a transit map $g_f : \mathcal{C}_{f,+} \to \mathcal{C}_{f,-}$ with phase $\bar{a}_f = a_f \mod \mathbb{Z}$. This map describes how a long orbit of f "passes though the gate" between ξ_f and ξ'_f. The following lemma relates the convergence of $\bar{a}_f$ to the convergence of local lifts.

Lemma 2.15. *Let $f_k \in \mathcal{P}_1$ converge to $f_0 \in \mathcal{P}_0$ and suppose $\bar{a}_{f_k} \to \bar{a}$. Then for any local lift $\widetilde{g}$ of $g_{\bar{a}}$ there exists a sequence n_k such that*

$$f_k^{n_k} \to \widetilde{g}$$

uniformly as $k \to \infty$.

Proof. Let $K = \mathrm{cl}(Dom(\widetilde{g}))$ and define $a \in \mathbb{C}$ by $\widetilde{g} = \phi \circ T_a \circ \Phi$. Let K_1 be a compact set in $\mathbb{C}$ containing $\Phi(K)$ in its interior and let K_2 be a compact set in $Dom(\phi)$ containing $T_a(K_1)$ in its interior. Let a_k be the constant a_{f_k} in Proposition 2.14. Since $\bar{a}_{f_k} \to \bar{a}$ there exists a sequence n_k so that $a_k + n_k \to a$.

For k large enough $K \subset Dom(\Phi_{f_k})$, $\Phi_{f_k}(K) \subset K_1$, $T_{a_k+n_k}(K_1) \subset K_2$ and $K_2 \subset Dom(\phi_{f_k})$. The lemma follows from equation (2.2) and since $\Phi_{f_k}, T_{a_k+n_k}, \phi_{f_k}$ converge to Φ, T_a, ϕ uniformly on K, K_1, K_2, respectively. □

The following lemma gives a simple condition under which perturbed Fatou coordinates exist.

Lemma 2.16. *Suppose f_n is a sequence of quadratic-like maps converging in the Carathéodory topology to a quadratic-like map $f \in \mathcal{P}_0$. Suppose the fixed points of f_n are repelling. Then $f_n \in \mathcal{P}_1$ for n large enough.*

Proof. Using the holomorphic index (see [M1]) one can prove that

$$\frac{1}{1-\lambda_{f_n}} + \frac{1}{1-\lambda'_{f_n}}$$

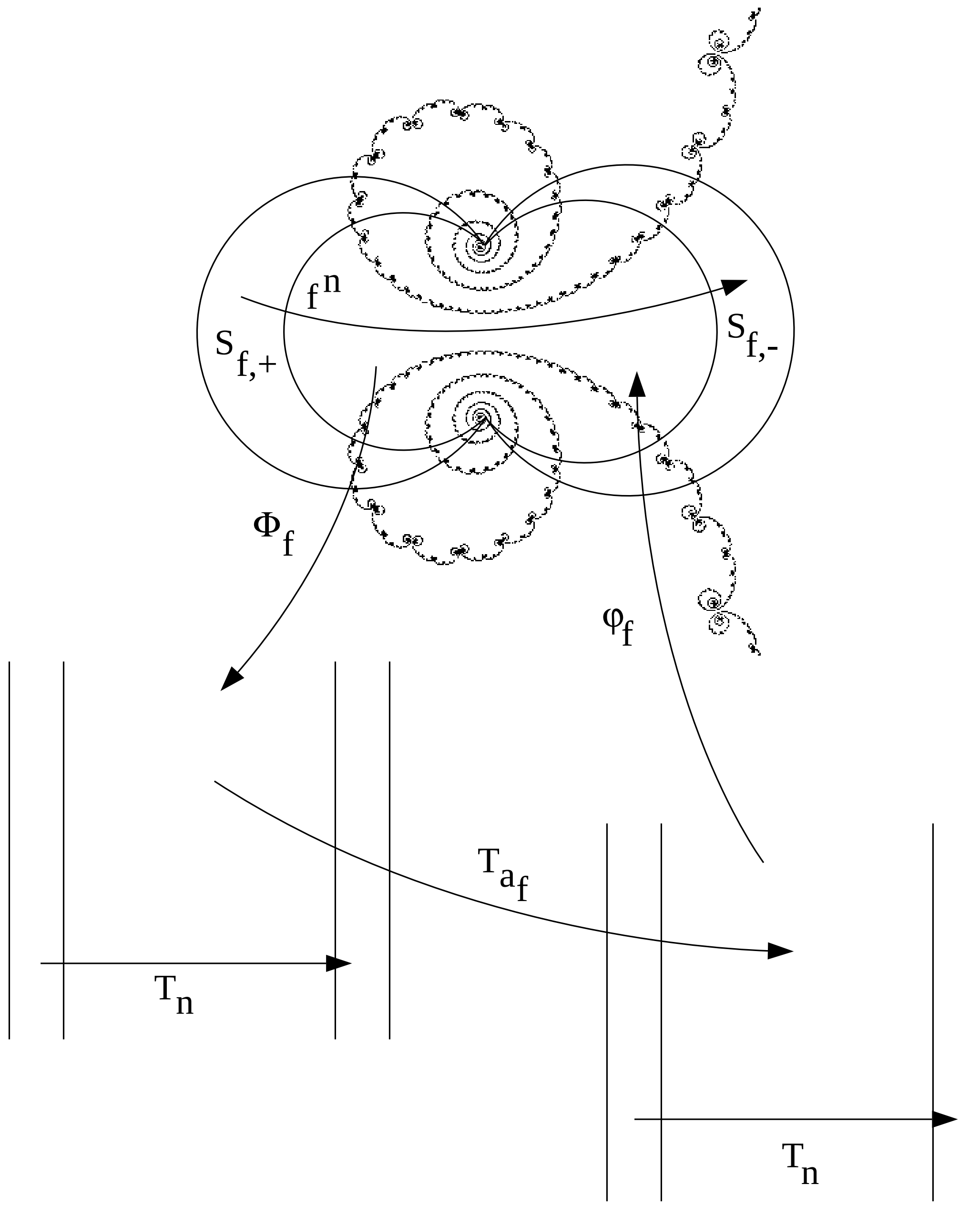

FIGURE 2. Perturbed Fatou coordinates.

converges as $f_n \to f$. Since λ, $\lambda' \in \mathbb{C} \setminus \mathbb{D}$ it follows

$$|\arg(1 - \lambda_{f_n})| \to \pi/2 \text{ and } |\arg(1 - \lambda'_{f_n})| \to \pi/2$$

as $n \to \infty$. In particular, $f_n \in \mathcal{P}_1$ for n large. □

For any $z \in D_+ \cap D_-$ define the Ècalle-Voronin transformation $\mathcal{E}$ by

$$\mathcal{E}(\pi_-(z)) = \pi_+(z).$$

One can show that $\mathcal{E}$ extends holomorphically to the two ends of $\mathcal{C}_-$ by using the Fatou coordinates and the standard isomorphism $\pi(z) = \exp(2\pi i z)$ of $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C} \setminus 0$. The following lemma is useful for controlling the dynamics near the ends of the Ècalle-Voronin cylinders.

Lemma 2.17. *Suppose $f_0 \in \mathcal{H}(1/4)$ and $g : \mathcal{C}_+ \to \mathcal{C}_-$ is a transit map such that the critical point of f_0 escapes $K(f_0)$ under iterates of f_0 and local lifts of g. Then*

$$|(g \circ \mathcal{E})'(\pm\infty)| > 1.$$

Proof. We will prove the lemma with the critical point escaping after just one iterate of a local lift of g. Assume the critical point of f is at the origin. Let $R = g \circ \mathcal{E}$ and $J_- = \pi_-(J(f_0))$. Let $V_{\pm\infty}$ denote the connected components of $(\mathcal{C}_- \setminus J_-) \cup \{\pm\infty\}$ containing $\pm\infty$ and let $U_{\pm\infty} = g^{-1}(V_{\pm\infty})$ (see Fig. 3).

Note that $\mathcal{E}$ can be extended to $V_{\pm\infty}$ as a branched cover. The set of critical points is the backward orbit of 0 and the only critical value is $\pi_+(0)$.

Since $\pi_+(0) \notin U_{\pm\infty}$ and each $U_{\pm\infty}$ is simply connected there is a branch of $\mathcal{E}^{-1}$ defined on $U_{\pm\infty}$ preserving $\pm\infty$. Composing $\mathcal{E}^{-1} \circ g^{-1}$ we have constructed a branch of R^{-1} which maps each $V_{\pm\infty}$ strictly inside itself and fixes $\pm\infty$. The lemma follows from the Schwarz lemma. ☐

We close this section with a lemma on the geometry of some particular first through maps. Let $m > 0$ and let

$$X = \{f \in Quad(m) : f \in \mathcal{H}(1/4) \text{ and } \operatorname{diam} K(f) = 1\}.$$

From Lemma 2.1, X is compact (in the Carathéodory topology). For each $f \in X$ choose a neighborhood $N \ni \beta(f)$ on which f is a diffeomorphism and let $\mathcal{N}_1, \ldots, \mathcal{N}_k$ be a finite cover of X by the neighborhoods from Theorem 2.14. In order to preserve certain compactness properties, we will need $\mathcal{N}_i$ to be closed neighborhoods. By rescaling we can extend the neighborhoods $\mathcal{N}_i$ to be a finite cover of $\{f \in Quad(m) : f \in \mathcal{H}(1/4)\}$. Note the coordinates do not necessarily agree on the overlaps $\mathcal{N}_i \cap \mathcal{N}_j$.

Now suppose $f \in Gen(m)$ is a generalized quadratic-like map such that the critical point escapes the central component U_0. Suppose $f \in (\mathcal{N}_i \cap \mathcal{P}_1)$ for some $1 \leq i \leq k$ and suppose the critical point of f passes once through the gate before landing in the off-critical pieces. In this case we say f has a *saddle-node cascade.* Let $T = T(f, \cup_{j \neq 0} U_j)$ be the first through map of f and define the *modified landing time* $l(z)$ of $z \in Dom(T)$ as follows. For each $\mathcal{N}_i \ni f$, there is a choice of fundamental domains $S_{f,\pm}$. Write $T(z)$ as a composition of f, $(f|_{N_i})^{-1}$ and of the discontinuous map $\widetilde{g}_f : S_{f,+} \to S_{f,-}$ defined in equation (2.2). Define $l_i(z)$ as the minimal number of maps in this composition and define $l(z) = \max_i l_i(z)$.

Lemma 2.18. *Let $f_k \in Gen(m)$ be a sequence of maps with saddle-node cascades such that the modified landing times $l(0)$ are bounded. Suppose $f_k \to f$ and $K(f|_{U_0})$ is connected. Then $f|_{U_0} \in \mathcal{H}(1/4)$.*

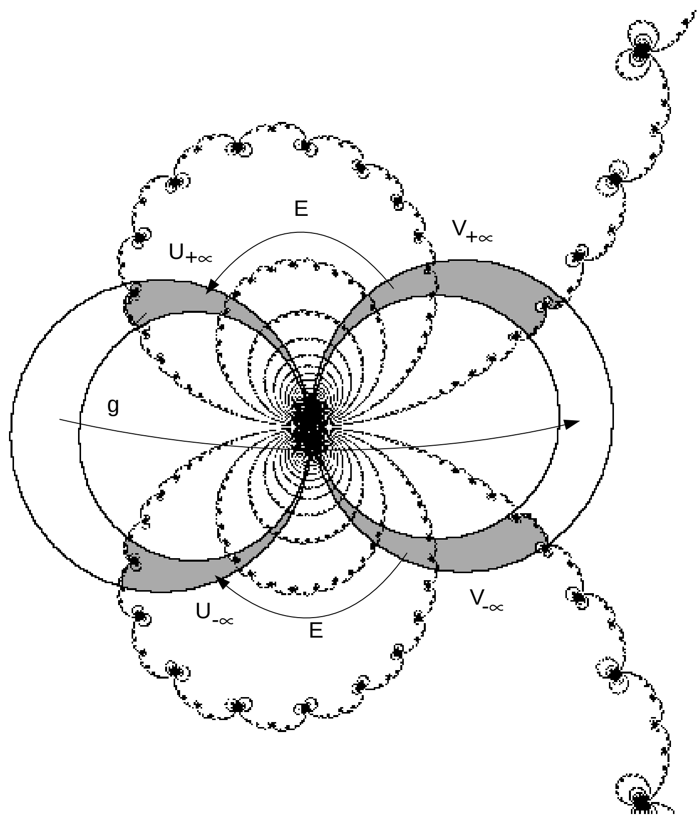

FIGURE 3. A blow-up of the Julia set of $f_0 = z^2 + 1/4$ with pre-images by f_0, g and $\mathcal{E}$ highlighting the sets $U_{\pm\infty}$ and $V_{\pm\infty}$.

Proof. Fix some neighborhood $\mathcal{N}_i$ containing f_k and f. Let n_k be the transit time defined by equation 2.2 for the orbit of the origin. If $f|_{U_0} \notin \mathcal{H}(1/4)$ then n_k is bounded from above. But then the origin escapes $U_{k,0}$ under a bounded number of iterates of f_k, which is a contradiction. □

Lemma 2.19. *Let $\lambda > 0$, $m > 0$ and $r \in \mathbb{N}$. Suppose $f \in Gen(m)$ has a saddle-node cascade and $\text{geo}(f) \geq \lambda$. Then the phase $\bar{a}_f$ of the induced transit map lies in a pre-compact subset of $\mathbb{C}/\mathbb{Z}$. Suppose $g \in Gen$ is a restriction of the first through map T such that $Dom(g) = Comp(Dom(T), u_g)$ and for $z \in u_g$,*

$$l(z) \leq r.$$

Then there exists $C(\lambda, m, r) > 0$ *such that* $\operatorname{geo}(g) \geq C$.

Proof. Let us prove the first statement. Suppose $f \in \mathcal{N}_i$. Let $c_1 = f^{r_1}(0)$ be the first moment when the orbit of 0 lands in $S_{f,+}$. We can assume r_1 is uniform over the neighborhood $\mathcal{N}_i$. Then c_1 lies in a pre-compact subset of $\mathcal{C}_{f,+}$. Let $c_2 = \widetilde{g}_f(c_1)$. Since $f^n(c_2) \in \cup_{j \neq 0} U_j$ for some $n \leq r$, it follows c_2 lies in a pre-compact subset of $\mathcal{C}_{f,-}$. Hence the phase $\bar{a}_f$, measured in the coordinates from $\mathcal{N}_i$, lies in a pre-compact subset of $\mathbb{C}/\mathbb{Z}$.

The bound on the geometry is clear since the perturbed Fatou coordinates converge and the transit maps g_f lie in a pre-compact subset and the number of iterates of g_f, f and $(f|_{N_i})^{-1}$ is bounded. □

3. Combinatorics

3.1. **Essentially period tripling.** An ∞-renormalizable map f has *bounded combinatorics* if $\tau(f)$ contains a finite number of distinct maximal tuned Mandelbrot sets, or, equivalently, if $\bar{\sigma}(f)$ contains a finite number of distinct shuffles. In this section we construct an infinite set of maximal tuned Mandelbrot sets with bounded essential period. Hence any map whose tuning invariant is chosen from these Mandelbrot sets will have essentially bounded combinatorics. On the other hand if the tuning invariant contains an infinite number of distinct Mandelbrot sets then the map will not have bounded combinatorics. The simplest way to construct such a collection of maximal tuned Mandelbrot sets is by perturbing in a particular way the map $z^2 - 1.75$, the root point of the period three tuned copy. For this reason we say these copies are essentially period tripling.

Let $f(x) = x^2 - 1.75$ and let ξ be the parabolic periodic orbit of period three. Recall $A = A(f) = [\alpha, \alpha']$. Let g be the first return map of f on A (see Fig. 4). Let I^1 and I^1_1 be the two indicated intervals satisfying $g|_{I^1} = f^3$ and $g|_{I^1_1} = f^2$.

Fix a small $\epsilon > 0$ and consider $c \in (-1.75, -1.75 + \epsilon)$. The periodic point ξ bifurcates and the orbit of the critical point under f_c^3 now escapes the interval I^1. Let c_n be the parameter value (see Fig. 5) so that for $f = f_{c_n}$,

- $f^{3i}(0) \in I^1$ for $i = 1, \ldots, n-1$,
- $f^{3n}(0) \in I^1_1$,
- $f^{3n+2}(0) = 0$.

In the next section we will justify the claim that c_n exists and is the center of a maximal tuned Mandelbrot set, denoted $M_n^{(3)}$. Equivalently, if we let $\sigma_n^{(3)}$ be the permutation induced on the orbit by f_{c_n} of the origin labeled from left to right, then $\sigma_n^{(3)}$ is a shuffle. In Fig. 6 we have drawn the period three tuned Mandelbrot set and a few of the $M_n^{(3)}$ accumulating at its root point. Any map f_c for $c \in M_n^{(3)}$ will be renormalizable with essential period $p_e(f_c) = 5$.

In Fig. 7 we have drawn the filled Julia sets for $z^2 - 1.75$ and for $z^2 - c_n$ for some c_n with n large. Fig. 8 shows two blow-ups of the Julia set of $f = z^2 - c_n$. The "ghost" boundary of the basin of ξ is visible in the left picture and the pre-images of this ghost boundary nest down to $J(\mathcal{R}(f))$ in the right picture.

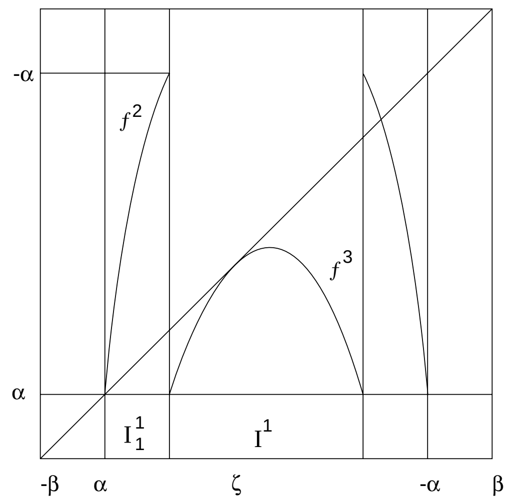

FIGURE 4. The first return map for $x^2 - 1.75$.

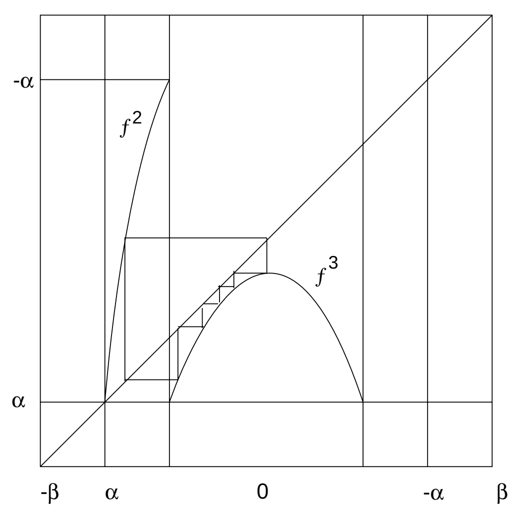

FIGURE 5. The first return map for $x^2 + c_5$ and the orbit of the origin.

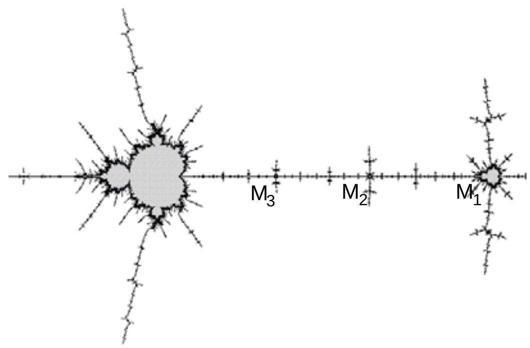

FIGURE 6. The Mandelbrot set near the real period three tuned copy.

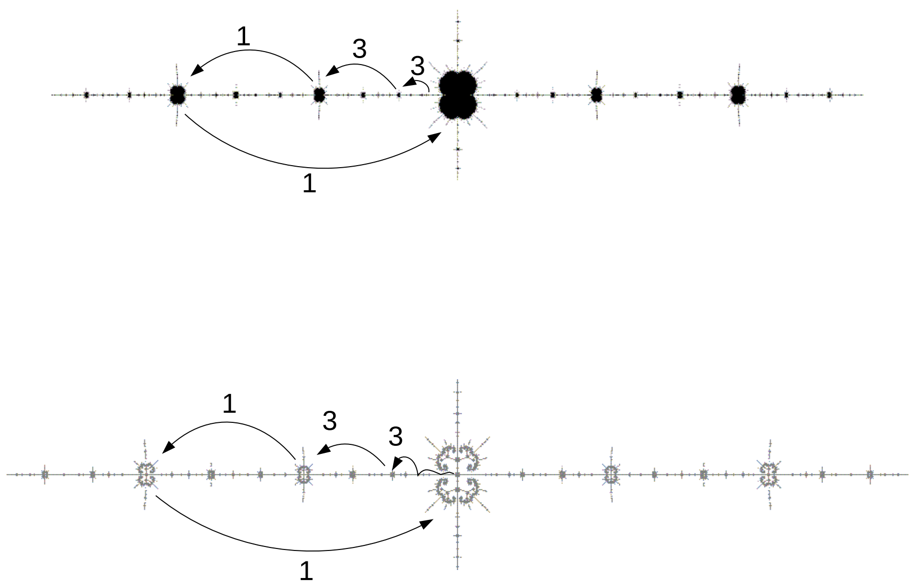

FIGURE 7. The filled Julia set of the map $z^2 - 1.75$ (above) and of $f = z^2 - c_n$ for some large n.

3.2. **Essentially bounded combinatorics.** In this section we describe the *return type sequence* of a given shuffle σ and define the *essential period* $p_e(\sigma)$.

Suppose $f \in Quad$ is renormalizable, has real combinatorics, and $\sigma(f) \neq \sigma^{(2)}$. In real contexts we will assume $f \in RQuad$. Define the complex principal nest $V^0 \supset V^1 \supset V^2 \supset \dots$ of f as follows. Choose a straightening of f to a polynomial f_c and pull the equipotential and external ray foliations of f_c back to f. Cut the domain D bounded by a fixed equipotential level by the closure of the rays that land at α and at α'. The resulting set of connected components is called the *initial Yoccoz puzzle*. Let V^0 be the component containing 0 and let

$$V^m = Comp(Dom(R(f, V^{m-1})), 0).$$

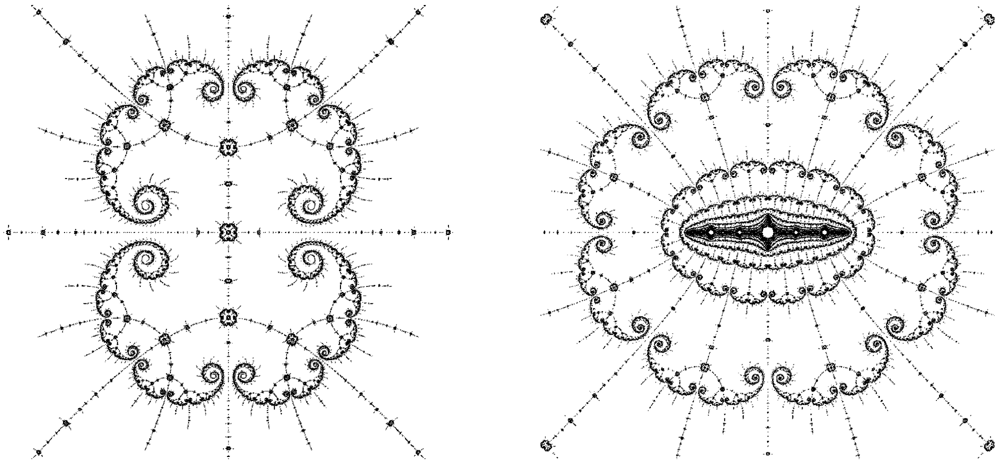

FIGURE 8. Blow-ups of $J(f)$ near the origin.

For $m \geq 0$ let $I^m = V^m \cap \mathbb{R}$.

For $m \geq 1$, let $g_m : \cup_i V_i^m \to V^{m-1}$ be the *generalized renormalization* of f on V^{m-1}. We will also denote the restriction to the real line $g_m : \cup_i I_i^m \to I^{m-1}$ by g_m. Number the intervals $\cup_i I_i^m$ (and domains V_i^m) from left to right and so that $0 \in I_0^m = I^{m+1}$. See Fig. 5 for an example of the first two levels of the real principal nest and the graph of g_1.

Lemma 3.1 ([L3]). *Let $m > 0$, $n > 0$ and let $f \in Quad(m)$. Suppose f has real combinatorics, is not immediately renormalizable, and the return time of any $z \in Dom(g_1)$ through the initial Yoccoz puzzle until the first return to V^0 is bounded above by n. Then* $\operatorname{mod}(Dom(g_1), Range(g_1)) \geq m_0(m,n) > 0$, $\operatorname{geo}(g_1) \geq C(m,n) > 0$ *and* $\operatorname{diam} K(g_1)/\operatorname{diam} K(f) \geq C'(m,n) > 0$.

The *return type* of g_m is defined as follows (see [L6] for details). Let $g \in RGen$ have finite type and let $\cup_i I_i = Dom(g) \cap \mathbb{R}$ numbered from left to right with $0 \in I_0$. Let (Γ, ϵ) be the free ordered signed semigroup generated by $\{I_i\}$ where $\epsilon : \{I_i\} \to \{\pm 1\}$ is the sign function defined for $i \neq 0$ by $\epsilon(I_i) = +1$ iff $g|_{I_i}$ is orientation preserving and for $i = 0$ by $\epsilon(I_0) = +1$ iff 0 is a local minimum of g. Let $h \in RGen$ be a restriction of $R(g, I_0)$ to finitely many components of its domain and let $\cup_j J_j = Dom(h) \cap \mathbb{R}$. Let (Γ', ϵ') be the corresponding signed semigroup for h. Let $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ be the homomorphism generated by assigning to each J_j the word $I_{i_1} I_{i_2} \cdots I_{i_n}$ where I_{i_k} is the interval containing $g^k(J_j)$ and n is the return time of J_j to I_0. The homomorphism $\chi : \Gamma' \to \Gamma$ is the *return type* of h.

A homomorphism $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ between free ordered signed semigroups is called *unimodal* if the image of every generator is a word ending with the central interval and if the map is strictly monotone on the intervals to the right and left of center and has an extremum at the center. We say a unimodal χ is *admissible* if

$$\epsilon'(I'_j) = \operatorname{sgn}(j)\epsilon(\chi(I'_j)) \text{ for } j \neq 0 \text{ and } \epsilon'(I'_j) = \epsilon(\chi(I'_j)) \text{ for } j = 0.$$

Let us describe the initial combinatorics of f. Let (Γ_0, ϵ_0) be the signed semigroup generated by $+I^0$ and two intervals $-I_{-1}^0$ and $+I_1^0$. We say a unimodal homomorphism $\chi : (\Gamma, \epsilon) \to (\Gamma_0, \epsilon_0)$

is *zero-admissible* if it is admissible and additionally for each I_i there is a $p_i \geq 0$ with $p_0 \geq 1$ and such that

$$\chi(I_i) = I_{-1} I_1^{p_i} I_0.$$

The initial combinatorics of f is described by the homomorphism assigning to each I_i^1 its itinerary by f through the intervals I^0 and the connected components of $B(f) \setminus I^0$. In general if h_1 is any restriction of the first return map to V^0 then the return type of h_1 is the homomorphism mapping to any interval in its domain its itinerary through the above intervals. Note that if f has negative orientation then repeat the construction with all signs reversed.

The combinatorics of f up to level m is described by the sequence S_m of admissible unimodal homomorphisms

$$\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0$$

where χ_m is the return type of g_m and χ_1 is zero-admissible. Each S_m is *irreducible*, meaning the orbit of the critical point enters every interval I_i^m. Since f is renormalizable there exists an m' such that Γ_m is the semigroup with one generator for all $m \geq m'$. Let $S(\sigma) = S_{m'}$ for the smallest such value of m'. Then the shuffle $\sigma(f)$ is uniquely specified by $S_{m'}$. Moreover, we have the following

Theorem 3.2 ([L6]). *Let S be an irreducible finite sequence of admissible unimodal homomorphisms:*

$$\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0.$$

Suppose Γ_m is the only semigroup with one generator, Γ_0 is as above and χ_1 is zero-admissible. Then there is a unique shuffle σ such that $S(\sigma) = S$.

We can now justify our construction of the essentially period tripling shuffles $\sigma_n^{(3)}$ from §3.1. Consider the following signed semigroups generated by the specified intervals

$$\begin{aligned} \Gamma &= \langle +I_{-1}, -I_0 \rangle \\ \Gamma' &= \langle -I_0 \rangle \end{aligned} \tag{3.1}$$

and consider the following homomorphisms

$$\begin{array}{lll} \chi_0 : \Gamma \to \Gamma_0 & \text{generated by} & I_{-1} \mapsto I_{-1} I_0 \text{ and } I_0 \mapsto I_{-1} I_1 I_0 \\ \chi : \Gamma \to \Gamma & \text{generated by} & I_{-1} \mapsto I_{-1} I_0 \text{ and } I_0 \mapsto I_0 \\ \chi' : \Gamma' \to \Gamma & \text{generated by} & I_0 \mapsto I_{-1} I_0. \end{array} \tag{3.2}$$

Then the sequence corresponding to the essentially period tripling combintorics $\sigma_n^{(3)}$ is

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0$$

where χ is repeated $n-1$ times.

A level $m > 0$ is called *non-central* iff

$$g_m(0) \in V^{m-1} \setminus V^m.$$

Let $m(0) = 0$ and let $0 < m(1) < m(2) < \cdots < m(\kappa)$ enumerate the non-central levels, if any exist, and let $h_k \equiv g_{m(k)+1}$, $k = 0, \ldots, \kappa$.

The nest of intervals (or the corresponding nest of pieces V^m)

$$I^{m(k)+1} \supset I^{m(k)+2} \supset \ldots \supset I^{m(k+1)} \tag{3.3}$$

is called a *central cascade*. The *length* l_k of the cascade is defined as $m(k+1) - m(k)$. Note that a cascade of length 1 corresponds to a non-central return to level $m(k)$.

A cascade 3.3 is called *saddle-node* if $0 \notin h_k I^{m(k)+1}$. Otherwise it is called *Ulam-Neumann*. For a long saddle-node cascade the map h_k is combinatorially close to $z \mapsto z^2 + 1/4$. For a long Ulam-Neumann cascade it is close to $z \mapsto z^2 - 2$.

The next lemma shows that for a long saddle-node cascade, the map $h_k : I^{m(k)+1} \to I^{m(k)}$ is a small perturbation of a map with a parabolic fixed point.

Lemma 3.3 ([L2]). *Let $h_k : U_k \to V_k$ be a sequence of real-symmetric quadratic-like maps with* $\operatorname{mod}(h_k) \geq \epsilon > 0$ *having saddle-node cascades of length $l_k \to \infty$. Then any limit point of this sequence in the Carathéodory topology $f : U \to V$ is hybrid eqivalent to $z \mapsto z^2 + 1/4$, and thus has a parabolic fixed point.*

Proof. It takes l_k iterates for the critical point to escape U_k under iterates of h_k. Hence the critical point does not escape U under iterates of f. By the kneeding theory [MT] f has on the real line topological type of $z^2 + c$ with $-2 \leq c \leq 1/4$. Since small perturbations of f have escaping critical point, the choice for c boils down to only two boundary parameter values, 1/4 and -2. Since the cascades of h_k are of saddle-node type, $c = 1/4$. □

Since both fixed points of such a sequence h_k are repelling, it follows from Lemma 2.16 that for k large enough h_k has perturbed Fatou coordinates and so h_k has a saddle-node cascade in the sense described in §2.5.

Let $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$ and let $h_k x \in I^j \setminus I^{j+1}$. Set

$$d(x) = \min\{j - m(k), m(k+1) - j\}.$$

This parameter shows how deep the orbit of x lands inside the cascade. Let us now define d_k as the maximum of $d(x)$ over all $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$. Given a saddle-node cascade (3.3), let us call all levels $m(k) + d_k < l < m(k+1) - d_k$ *neglectable*.

Let f be renormalizable and f_1 a pre-renormalization of f. Define the *essential period* $p_e = p_e(f)$ as follows. Let p be the period of the periodic interval $J = B(f_1)$, and set $J_k = f^k J$, for $0 \leq k \leq p-1$. Let us remove from the orbit $\{J_k\}_{k=0}^{p-1}$ all intervals whose first landing to some $I^{m(k)}$ belongs to a neglectable level, to obtain a sequence of intervals $\{J_{n_i}\}_{i=1}^{m}$. The essential period is the number of intervals which are left, $p_e(f) = m$. Note the essential period of a shuffle is well-defined and in this way we can define $p_e(f)$ for any $f \in Quad$ with real combinatorics.

Let us give some examples of combinatorial types involving long saddle-node cascades with neglectable levels. Let Γ, Γ', χ, χ' and χ_0 be from 3.1 and 3.2.

Example 3.1 (Goes Through Twice). Let $\chi_2 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_0$ and $I_{-1} \mapsto I_{-1}^2 I_0$. Then any sequence of the form

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_2} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0$$

will correspond to a shuffle where the critical orbit moves up through the cascade until the top, returns to the level of χ_2, moves up through the cascade again and then returns to the renormalization interval. If the total number of levels in the sequence is m then the number of neglectable levels will be roughly $m - 2\min(d, m-d)$ where d is the level of χ_2.

Example 3.2 (Two Cascades). As a second example imagine perturbing the right-hand picture in Fig. 8 so that the renormalization becomes hybrid equivalent to $z^2 + \frac{1}{4}$. Now any further perturbation will cause the parabolic orbit to bifurcate and we can create another long cascade. More specifically, let $\chi_3 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_{-1}I_0$ and $I_{-1} \mapsto I_{-1}^2 I_0$ and consider a sequence of the form

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_3} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0.$$

Since χ_3 has a non-central return the two long sequences of χ form two seperate saddle-node cascades, each with a long sequence of neglectable levels.

3.3. **Parabolic shuffles.** Let Ω_p be the space of shuffles σ satisfying $p_e(\sigma) \leq p$. In this section we construct a compactification Ω_p^{cpt} of Ω_p which will form the elements of our combinatorial description of renormalization limits.

Suppose $f \in RQuad$ is renormalizable and let

$$\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0$$

be its sequence of return types. Let l be a neglectable level and let $\chi_l : (\Gamma_l, \epsilon_l) \to (\Gamma_{l-1}, \epsilon_{l-1})$ be the return type of g_l. It is clear that if both level $l-1$ and $l+1$ are neglectable then (Γ_l, ϵ_l) and $(\Gamma_{l-1}, \epsilon_{l-1})$ are generated by configurations of the form

$$\pm I_{-p}, \pm I_{-p+1}, \ldots, \pm I_{-1}, -I_0$$

or by

$$+I_0, \pm I_1, \ldots, \pm I_{p-1}, \pm I_p$$

for some $p \geq 1$. We claim that $(\Gamma_1, \epsilon_l) \cong (\Gamma_{l-1}, \epsilon_{l-1})$ and that χ_l is defined by $I_i \mapsto I_i I_0$ for $i \neq 0$ and $I_0 \mapsto I_0$. First it is clear $I_0 \mapsto I_0$. Now if $\chi_l(I_i)$ contained more than one off-critical interval then l would not be a neglectable level. Since χ_l is unimodal it follows Γ_{l-1} contains at least as many intervals as Γ_l. Since the return type sequence is irreducible Γ_{l-1} contains exactly the same number of intervals as Γ_l. Hence $I_i \mapsto I_i I_0$. The claim that the signs agree follows from the condition that χ_l be admissible.

Hence we can "insert" another neglectable level into S before l to obtain another irreducible sequence S' of return types:

$$\Gamma_m \xrightarrow{\chi_m} \cdots \Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1} \cong \Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1} \cdots \xrightarrow{\chi_1} \Gamma_0.$$

From Theorem 3.2 there is a unique shuffle σ' such that $S(\sigma') = S'$.

We say two shuffles σ and σ' in Ω_p are *essentially equivalent* if one can insert a finite number of neglectable levels into σ and σ' and obtain equal shuffles. Let Ξ be the partition of Ω_p into essentially-equivalent equivalence classes. Let $U \in \Xi$ be a non-trivial equivalence class. Then there is an $n = n_U > 0$ such that for any $\sigma \in U$ the return type sequence $S(\sigma)$ has exactly n different cascades $S_1, S_2, \ldots, S_n$, canonically ordered, containing neglectable levels. Let l_k,

$k = 1, \ldots, n$, denote the number of neglectable levels in the cascade S_k. The map $\theta_U : U \to \mathbb{N}_+^n$ given by $\sigma \mapsto (l_1, l_2, \ldots, l_n)$ is a homeomorphism. Let

$$\overline{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{+\infty\}$$

be the one-point compactification of $\mathbb{N}$. Define $U^{cpt} \supset U$ as the unique space such that θ_U extends to a homeomorphism $\theta_U : U^{cpt} \to \overline{\mathbb{N}}_+^n$. Define $\Omega_p^{cpt} \supset \Omega_p$ as the union of the trivial classes of Ξ and of the spaces U^{cpt} for non-trivial $U \in \Xi$. An element of $\Omega_p^{cpt} \setminus \Omega_p$ is called an *end* and can be represented by a "sequence" of return types where infinitely long sequences of neglectable levels are allowed:

$$\Gamma_m \overset{\chi_m}{\to} \cdots \overset{\chi_{l+2}}{\to} \Gamma_{l+1} \overset{\chi_{l+1}}{\to} (\Gamma_l \overset{\chi_l}{\to} \Gamma_{l-1})^\infty \overset{\chi_{l-1}}{\to} \Gamma_{l-2} \overset{\chi_{l-3}}{\to} \cdots \overset{\chi_1}{\to} \Gamma_0.$$

The following lemma is evident from the definition of essential period and Ω_p^{cpt}.

Lemma 3.4. *For any $p > 1$ the space Ω_p^{cpt} is metrizable and compact.*

Let $\mathcal{M}_p = \{M(\sigma)\}_{\sigma \in \Omega_p}$ be the collection of M-copies corresponding to Ω_p and let $\mathcal{C}_p = \{c(\sigma)\}_{\sigma \in \Omega_p}$ be the corresponding collection of centers. We now describe the topology of $\mathcal{C}_p$ and how $\mathrm{cl}(\mathcal{C}_p)$ compares to Ω_p^{cpt}. For any $U \in \Xi$ with $n = n_U \geq 1$ let $\mathcal{C}_U \subset \mathcal{C}_p$ denote the collection of centers of $\{M(\sigma)\}_{\sigma \in U}$. Since Ξ is a finite partition it suffices to describe the topology of the sets $\mathcal{C}_U$. We claim for each non-trivial $U \in \Xi$ there is a homeomorphism of $\mathbb{R}$ which maps $\mathcal{C}_U$ to the image of the function $F : \mathbb{N}_+^n \to \mathbb{R}$ given by

$$F(x_1, x_2, \ldots, x_n) = 2^{-x_1} + 2^{-x_1 x_2 - 1} + \cdots + 2^{-x_1 x_2 \cdots x_n - n + 1}$$

where $n = n_U$ (see Fig. 9).

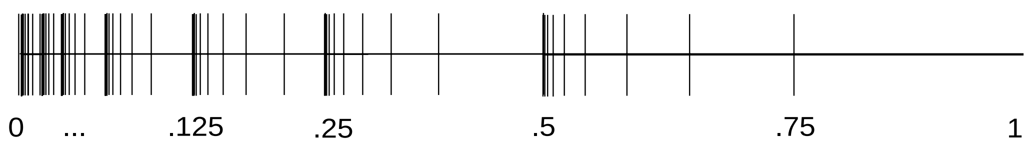

FIGURE 9. The image of F for $n = 2$.

To be more specific the limit points of $\mathcal{C}_U$ are root points of the M-copies obtained by "truncating" the return type sequences of $\sigma \in U$ at the neglectable levels. Let us describe how to truncate a return type sequence

$$\Gamma_m \overset{\chi_m}{\to} \Gamma_{m-1} \overset{\chi_{m-1}}{\to} \cdots \overset{\chi_1}{\to} \Gamma_0$$

at a level l. Let (Γ_T, ϵ_T) be the semigroup generated by I_0 with $\epsilon_T(I_0) = \epsilon_l(I_0^l)$ and let χ_T be the homomorphism defined by $I_0 \mapsto \chi_l(I_0^l)$. Let S' be the sequence

$$\Gamma_T \overset{\chi_T}{\to} \Gamma_{l-1} \overset{\chi_{l-1}}{\to} \cdots \overset{\chi_2}{\to} \Gamma_1 \overset{\chi_1}{\to} \Gamma_0.$$

One can check that S' is a sequence of admissible unimodal return types. If S' is not irreducible then simply remove all intervals I_i^m not in the combinatorial orbit of the critical point and shorten the sequence if necessary. We obtain a unique shuffle $\sigma' = \lfloor \sigma \rfloor_l$, the shuffle σ truncated at level l.

Let $U \in \Xi$ satisfy $n = n_U \geq 1$. Any shuffle $\sigma \in U$ has n cascades with neglectable levels of lengths $x_1, \dots, x_n$ respectively. As $x_1 \to \infty$, the corresponding centers accumulate at the root of the tuned M-copy corresponding to any $\sigma \in U$ truncated at the first neglectable level. If we fix x_1 and let $x_2 \to \infty$ the corresponding centers accumulate at the root of the M-copy corresponding to truncating at the second cascade of neglectable levels. In general if we fix the lengths of the first k sequences of neglectable levels and let the length of the $k+1$-st sequence grow the centers converge to the root of the M-copy corresponding to truncating at the $(k+1)$-st neglectable sequence.

Given an end $\tau \in \Omega_p^{cpt}$ let

$$c(\tau) = root(\lfloor \sigma \rfloor_l)$$

where $\sigma \in \Omega_p$ is in a sufficiently small neighborhood of τ, l is a neglectable level of σ which belongs to the first infinitely long cascade of τ, and $root(\sigma)$ is the root of the M-copy $M(\sigma)$. The map $c : \Omega_p^{cpt} \to \mathbb{R}$ is continuous and its image is $\mathrm{cl}(\mathcal{C}_p)$.

We return to our examples. Choose a large p so that the shuffles from Example 3.1 and Example 3.2 are contained in Ω_p.

First consider the essentially period tripling shuffles $\sigma_n^{(3)}$. Then $c(\sigma_n^{(3)}) \to root(\sigma^{(3)})$ where $\sigma^{(3)}$ is the period tripling shuffle. Moreover, $\sigma_n^{(3)}$ converges to an end $\tau_1 \in \Omega_p^{cpt}$.

Now consider the shuffles $\sigma_{m,d}$ from Example 3.1 (Goes Through Twice). First fix $d > 1$ and let $m \to \infty$. Then $c(\sigma_{m,d}) \to root(\lfloor \sigma_{m,d} \rfloor_l) \neq root(\sigma^{(3)})$ where l is any negectable level and, in much the same spirit as essential period tripling, $\sigma_{m,d}$ converges in Ω_p^{cpt} to an end. Now fix $m - d > 1$ and let $m \to \infty$. Then $c(\sigma_{m,d}) \to root(\sigma^{(3)})$ and $\sigma_{m,d}$ converges to an end $\tau_2 \in \Omega_p^{cpt}$.

Finally consider the shuffles σ_{l_1,l_2} from Example 3.2 (Two Cascades). Fix $l_1 > 1$ and let $l_2 \to \infty$. Then $c(\sigma_{l_1,l_2}) \to r_{l_1} = root(\lfloor \sigma_{l_1,l_2} \rfloor_l)$ where l is any negectable level in the second cascade. The sequence $r_{l_1} \to root(\sigma^{(3)})$ as $l_1 \to \infty$. Moreover, for any sequence of l_2 if we let $l_1 \to \infty$ then $c(\sigma_{l_1,l_2}) \to root(\sigma^{(3)})$. Now consider the limits of σ_{l_1,l_2} in Ω_p^{cpt}. If we fix l_2 and let $l_1 \to \infty$ the shuffles will converge to an end τ_{∞,l_2}.

This completes our description of the topology of $\mathcal{M}_p$ and how $\mathrm{cl}(\mathcal{C}_p)$ compares to Ω_p^{cpt}.

4. Parabolic Renormalization

Let $c \star M$ be a maximal tuned Mandelbrot set with root c' and suppose $f \in \mathcal{H}(c')$ is renormalizable and let f_0 be a pre-renormalization. Let $\xi = \beta(f_0)$. Choose incoming and outgoing petals $D_\pm$ around the parabolic point ξ and let $\mathcal{C}_\pm$ denote the respective Ècalle-Voronin cylinders and $\pi_\pm$ the projections with π_+ extended to $B = \mathrm{int}(K(f_0))$. Fix a transit map $g : \mathcal{C}_+ \to \mathcal{C}_-$ satisfying

$$g(\pi_+(0)) \notin \pi_-(K(f_0)).$$

Given a collection $\{f_\alpha\}$ of homolorphic maps let $\langle f_\alpha \rangle$ denote the set of restrictions of all finite compositions of $\{f_\alpha\}$. Let

$$\mathcal{F}(f, g) = \langle f \cup \{\text{all local lifts of } g \text{ to } D_\pm\} \rangle.$$

Note that $\mathcal{F}(f, g)$ is independent of the choice of petals $D_\pm$. A collection $\mathcal{F}$ of holomorphic maps closed under composition and restriction is called a *conformal dynamical system*. Define the orbit of a point $z \in \mathbb{C}$ as

$$\mathrm{orb}(z) = \mathrm{orb}(\mathcal{F}, z) = \bigcup_{h \in \mathcal{F}} h(z).$$

We say $\mathcal{F}$ is contained in any *geometric limit* of a sequence $\mathcal{F}_n$ if for any $f \in \mathcal{F}$ there are $f_n \in \mathcal{F}_n$ such that $f_n \to f$ on compact sets.

We say the pair (f, g) is *parabolic renormalizable* if there is a neighborhood $U \ni 0$ and a map $h \in (\mathcal{F}(f, g) \setminus \langle f \rangle)$ such that

$$h|_U \in Quad.$$

We call such an $h|_U$ a *parabolic pre-renormalization* of (f, g) and we call the germ of a normalized pre-renormalization a *parabolic renormalization* of (f, g). In the next section we will show that the domain U of the pre-renormalization can be canonically chosen.

4.1. **Essentially period tripling.** In this section we describe a construction from [DD] for finding a canonical representation of the parabolic renormalization in the essentially period tripling case. For simplicity we will state the construction for the quadratic map $P_{-1.75}$. However, it is clear how to generalize this construction to any map $f \in \mathcal{H}(-1.75)$.

Recall from §3.1 the sequence of maximal tuned Mandelbrot sets $c_n \star M$ with essentially period tripling combinatorics accumulate at the root of the period three tuned copy, $c = -1.75$. Let $f = P_{-1.75}$ and choose f_0 and $D_\pm$ as above. Let $B = \mathrm{int}(K(f_0))$ and let $f_n = P_{c_n}$. Choose n_0 sufficiently large and choose

$$U_- \in \bigcap_{n \geq n_0} Comp(\mathrm{int}(K(f)) \setminus B, P(f_n))$$

such that $U_- \Subset D_-$. Let t be the landing time of U_- to B under f.

Let

$$\mathcal{D}_f = \{g : \mathcal{C}_+ \to \mathcal{C}_- | g \text{ is a transit map and } g(\pi_+(0)) \in \pi_-(U_-)\}.$$

The phase map gives a conformal isomorphism of $\mathcal{D}_f$ to a disk $D_f \Subset \mathbb{C}/\mathbb{Z}$. Note that D_f is a Jordan domain. Choose a branch of π_-^{-1} so that $Range(\pi_-^{-1}) \supset U_-$. For $g \in \mathcal{D}_f$ let W_g be the connected component of $(\pi_+^{-1} \circ g^{-1} \circ \pi_-)(U_-)$ containing 0. Since $\pi_-(U_-)$ is a topological disk, it follows the map $R_{\bar{a}} : W_g \to B$ given by

$$R_{\bar{a}} = f^t \circ \pi_-^{-1} \circ g_{\bar{a}} \circ \pi_+$$

is quadratic-like with possibly disconnected Julia set. If $J(R_{\bar{a}})$ is connected then we have constructed a parabolic pre-renormalization of (f, g).

Fix any $* \in D_f$. Define the holomorphic motion

$$h_{\bar{a}} : (\partial B, \partial W_{g_*}) \to (\partial B, \partial W_{g_{\bar{a}}})$$

on ∂B by the identity and locally on $\partial W_{g_{\bar{a}}}$ by pulling back under $R_{\bar{a}}$. Let $\mathbb{V} = \{(\bar{a}, z) : \bar{a} \in D_f, z \in B\}$ and $\mathbb{U} = \{(\bar{a}, z) : \bar{a} \in D_f, z \in W_{g_{\bar{a}}}\}$. Let $\mathbf{f} : \mathbb{U} \to \mathbb{V}$ be defined by

$$\mathbf{R}(\bar{a}, z) = (\bar{a}, R_{\bar{a}}(z)).$$

Lemma 4.1. *The family $(\mathbf{R}, \mathbf{h})$ is a proper DH quadratic-like family with winding number 1.*

Proof. The map f^t is a conformal isomorphism of a neighborhood of U_- onto a neighborhood of B. There is a branch of π_-^{-1} such that the map $(\pi_-^{-1} \circ g_{\bar{a}} \circ \pi_+)(0)$ is a conformal isomorphism of a neighborhood of D_f onto a neighborhood of U_-. The lemma follows. □

The following lemma states that the renormalization operators $\mathcal{R}_{\sigma_n^{(3)}}$ converge to essentially period tripling parabolic renormalization.

Lemma 4.2. *Let $f \in \mathcal{H}(-1.75)$. Suppose $f_k \in Quad$ is a sequence of renormalizable maps with $f_k \to f$ and $\sigma(f_k) \to \tau$. Let $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$ be the induced transit maps with phases $\bar{a}_k$. Then*

1. *$\{\bar{a}_k\}$ is pre-compact*
2. *if $\bar{a}_{k_j} \to \bar{a}$ is a convergent subsequence then $J(R_{\bar{a}})$ is connected and*
$$[h_{k_j}] \to [R_{\bar{a}}]$$
where h_k is a pre-renormalization of f_k
3. *$\mathcal{F}(f, g_{\bar{a}})$ is contained in any geometric limit of $\langle f_{k_j} \rangle$*

Proof. Let h_k be a pre-renormalization of f_k. Since $\sigma(f_k) \to \tau$ and $f_k \to f$ we can write
$$h_k = f_k^{N_1} \circ \widetilde{g}_k \circ f_k^{N_2} \tag{4.1}$$
on some neighborhood of the origin for some fixed N_1, N_2 and some choice of local lift $\widetilde{g}_k$ of the induced transit maps $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$. The first claim is that h_k can be chosen in $Quad(m')$ for some $m' > 0$. Let V' be an ϵ-neighborhood of the central basin B of f for some small $\epsilon > 0$. Choose ϵ small enough and N_1 and N_2 large enough so that for large k the right-hand side of (4.1) can be used to define a pre-renormalization h_k with range V'. Let $U'_k = h_k^{-1}(V')$. By taking k larger still we can assume U'_k is contained in an $\epsilon/2$ neighborhood of B. It follows there is an $m' > 0$ so that $\mathrm{mod}(U'_k, V') \geq m'$. Moreover, $\mathrm{diam}(U'_k) \geq C > 0$ for some C independent of k. Hence (4.1) holds on a definite neighborhood of the origin.

From the convergence of Fatou coordiantes and the convergence of f_k it follows that $\{\bar{a}_k\}$ is pre-compact. Let $\bar{a}_{k_j} \to \bar{a}$ be a convergent subsequence. Then h_{k_j} converges on a definite neighborhood of the origin to the map $f^{N_1} \circ \widetilde{g}_{\bar{a}} \circ f^{N_2}$ for an appropriate local lift $\widetilde{g}_{\bar{a}}$ of $g_{\bar{a}}$. Since the origin is non-escaping under all h_k it follows $J(R_{\bar{a}})$ is connected. The last statement follows from the fact that $f_k \to f$ and Lemma 2.15. □

Moreover, the proof of the previous lemma can be modified to prove the following

Lemma 4.3. *Suppose $f \in \mathcal{H}(-1.75)$ and $f_k \in \mathcal{H}(-1.75)$ satisfy $f_k \to f$. Suppose $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$ is a transit map with phase $\bar{a}$ such that $R_k = R_{\bar{a}_k}$ is defined. Then*

1. *$\{\bar{a}_k\}$ is pre-compact*
2. *if $\bar{a}_{k_j} \to \bar{a}$ is a convergent subsequence then*
$$R_{k_j} \to R_{\bar{a}}.$$
3. *$\mathcal{F}(f, g_{\bar{a}})$ is contained in any geometric limit of $\mathcal{F}(f_{k_j}, g_{k_j})$*

We finish this section with two useful properties of parabolic renormalization. The first property is that open sets intersecting the Julia set of the parabolic pre-renormalization iterate under $\mathcal{F}(f,g)$ to open sets intersecting $J(f)$.

Lemma 4.4. *Let $f \in \mathcal{H}(-1.75)$ and $g : \mathcal{C}_+ \to \mathcal{C}_-$ be a transit map with phase $\bar{a}$ such that $J(R_{\bar{a}})$ is connected. Suppose U is an open set satisfying*

$$U \cap J(R_{\bar{a}}) \neq \emptyset.$$

Then there is an $h \in \mathcal{F}(f,g)$ such that $U \cap Dom(h) \neq \emptyset$ and

$$h(U) \supset J(f).$$

Proof. From the construction of $R_{\bar{a}}$ it is clear that there is an $h \in \mathcal{F}(f,g)$ such that h is a quadratic-like extension of $R_{\bar{a}}$ to a small neighborhood of $B = Range(R_{\bar{a}})$. It follows that there an $m \geq 0$ such that $h^m(U) \cap \partial B \neq \emptyset$. But $\partial B \subset J(f)$. Iterating f further covers all of $J(f)$. □

The second property is that no quadratic-like representative of $[R_{\bar{a}}]$ can have too large a domain.

Lemma 4.5. *Let $f \in \mathcal{H}(-1.75)$ and $g : \mathcal{C}_+ \to \mathcal{C}_-$ be a transit map with phase $\bar{a}$ such that $R_{\bar{a}}$ is defined. If $(\widetilde{f} : U \to V) \in Quad$ satisfies $[\widetilde{f}] = [R_{\bar{a}}]$ then*

$$U \subset Range(R_{\bar{a}}).$$

Proof. Let f_1 be a pre-renormalization of f and let $B = Range(R_{\bar{a}})$. Suppose $U \cap \partial B \neq \emptyset$ and let U' be the connected component of $U \cap B$ containing 0. Since f_1-preimages of 0 accumulate on $J(f_1) = \partial B$ there exists an $n > 0$ and $z_0 \in U'$ such that $f_1^n(z_0) = 0$. Since $[\widetilde{f}] = [R_{\bar{a}}]$ it follows $\widetilde{f}$ has a critical point at z_0, which is a contradiction. □

4.2. **Generalized parabolic renormalization.**

In this section we modify the construction of parabolic renormalization to act on generalized quadratic-like maps.

Let $f : \cup_j U_j \to V$ be a generalized quadratic-like map with $f_0 = f|_{U_0} \in \mathcal{H}(1/4)$. Let $\xi = \beta(f_0)$. Choose incoming and outgoing petals $D_\pm$ around the parabolic point ξ and let $\mathcal{C}_\pm$ denote the respective Ècalle-Voronin cylinders and $\pi_\pm$ the projections with π_+ extended to $B = \text{int}(K(f_0))$.

For a given $g : \mathcal{C}_- \to \mathcal{C}_+$ let L_0 be the first landing map under $\mathcal{F}(f,g)$ to $\cup_{j\neq 0} U_j$. Note that if C is a connected component of $Dom(L_0)$ then there is an $h \in \mathcal{F}(f,g)$ such that $C \Subset Dom(h)$ and $h(z) = L_0(z)$ for all $z \in C$. Let T be the *first through map*

$$T = f \circ L_0. \tag{4.2}$$

Note that T is not a generalized quadratic-like map. However, T has at most one critical value, and, with a slight abuse of notation we will treat T as a generalized quadratic-like map.

Let $X \subset \mathbb{C}/\mathbb{Z}$ be the set of phases $\bar{a}$ such that for $g = g_{\bar{a}}$,

$$0 \in Dom(L_0).$$

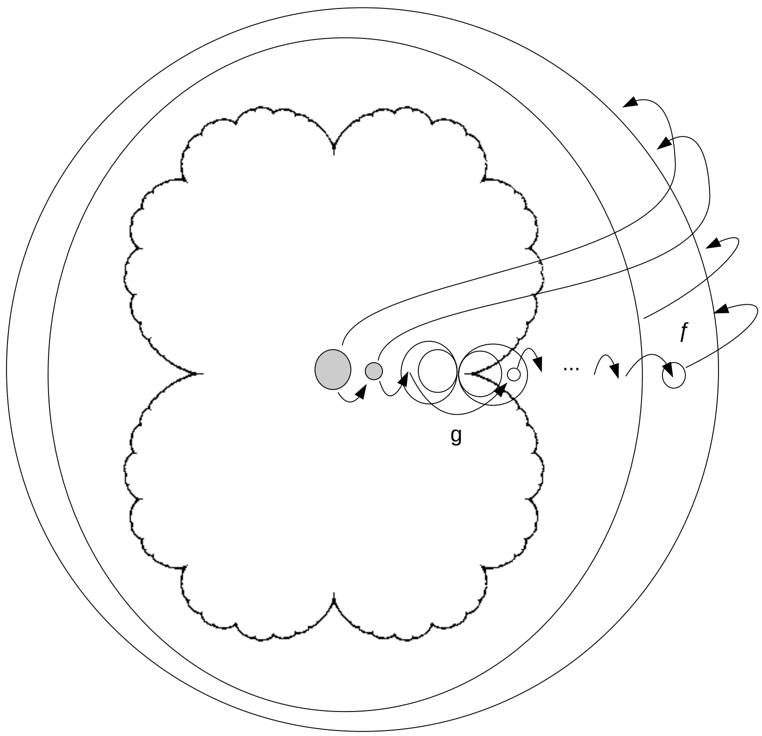

FIGURE 10. The first through map T with some components of the domain shaded.

It is clear that X is a countable pairwise disjoint collection of Jordan disks. Let D be a connected component of X. Let $\mathbf{T}$ denote the family over D of first through maps T. The construction of the holomorphic motion $\mathbf{h}$ described before Lemma 4.1 carries over unchanged to this situation. Moreover, one can modify the proof of Lemma 4.1 to prove

Lemma 4.6. *For any connected component D of X, the family $(\mathbf{T}, \mathbf{h})$ over D is a proper generalized quadratic-like family with winding number 1.*

The following two lemmas are generalizations of Lemma 4.2 and Lemma 4.3, respectively. We omit the proofs.

Lemma 4.7. *Suppose $f \in Gen$ satisfies $f|_{U_0} \in \mathcal{H}(1/4)$ and $f_k \in Gen$ is a sequence converging to f. Let T_k be the first through map for f_k and suppose that $T_k \in Gen$. Let g_k be the induced transit maps of f_k with phase $\bar{a}_k$. Suppose $\bar{a}_{k_j} \to \bar{a}$ is a convergent subsequence and suppose $0 \in Dom(T_{\bar{a}})$ where $T_{\bar{a}}$ is the first through map for $\mathcal{F}(f, g_{\bar{a}})$, Then*

$$T_{k_j} \to T_{\bar{a}}$$

Moreover, $\mathcal{F}(f, g_{\bar{a}})$ is contained in any geometric limit of $\langle f_{k_j} \rangle$.

Lemma 4.8. *Suppose $f \in Gen$ and $f_k \in Gen$ satisfy $f|_{U_0} \in \mathcal{H}(1/4)$, $f_k|_{U_{k,0}} \in \mathcal{H}(1/4)$, and $f_k \to f$. Let g_k be a transit map of f_k with phases $\bar{a}_k$. Let T_k be the first through map for $\mathcal{F}(f_k, g_k)$ and suppose $T_k \in Gen$. Suppose $\bar{a}_{k_j} \to \bar{a}$ is a convergent subsequence and suppose*

$0 \in Dom(T_{\bar{a}})$ where $T_{\bar{a}}$ is the first through map for $\mathcal{F}(f, g_{\bar{a}})$, Then

$$T_{k_j} \to T_{\bar{a}}$$

Moreover, $\mathcal{F}(f, g_{\bar{a}})$ is contained in any geometric limit of $\mathcal{F}(f_{k_j}, g_{k_j})$.

5. Towers

Let $S \subset \mathbb{Z}$ be a set of consecutive integers containing $\mathbb{N}_0$ and let f_n be a sequence of maps in Gen indexed by $n \in S$. Let $U_{n,0}$ be the central component of f_n. Let

$$S_{\mathcal{C}} = \{n \in S : f_n|_{U_{n,0}} \in \mathcal{H}(1/4)\}$$

and let

$$S_Q = \{n \in S : f_n \in Quad\}.$$

For $n \in S_{\mathcal{C}}$ let $g_n : \mathcal{C}_{n,+} \to \mathcal{C}_{n,-}$ be a transit map between the Ècalle-Voronin cylinders $\mathcal{C}_{n,\pm}$ of f_n. The collection of maps

$$\mathcal{T} = \{f_n : n \in S\} \cup \{g_n : n \in S_{\mathcal{C}}\}$$

is called a *tower* iff for each pair $n, n+1 \in S$ one of the following conditions hold:

- T1: $n \in S_Q$, f_n is immediately renormalizable and $[f_{n+1}] = [h]$ where h is a pre-renormalization of f_n of minimal period
- T2: $n \in S_Q$, f_n is not immediately renormalizable and $[f_{n+1}] = [h]$ where h is a restriction of the first return map to the initial central puzzle piece of f_n
- T3: $n \notin (S_Q \cup S_{\mathcal{C}})$ and $[f_{n+1}] = [h]$ where h is a restriction of the first return map $R(f_n, U_{n,0})$ or first through map $T(f_n, \cup_{j \neq 0} U_{n,j})$
- T4: $n \in S_{\mathcal{C}}$ and $[f_{n+1}] = [h]$ where h is a restriction of the first through map of the pair (f_n, g_n).

We shall often identify g_n with the set of local lifts of g_n for some choice of incoming and outgoing petals $D_{n,\pm}$. If $S_{\mathcal{C}} \neq \emptyset$ then $\mathcal{T}$ is a *parabolic tower*.

Let Tow be the space of towers with the following topology: a sequence $\mathcal{T}_m = \{f_{m,n}, g_{m,n}\}$ converges to $\mathcal{T} = \{f_n, g_n\}$ iff

- $S_m \to S$ and $S_{m,\mathcal{C}} \to S'_{\mathcal{C}} \subset S_{\mathcal{C}}$
- if $n \in S \setminus S_{\mathcal{C}}$ then $f_{m,n} \to f_n$
- if $n \in S'_{\mathcal{C}}$ then $f_{m,n} \to f_n$ and $g_{m,n} \to g_n$
- if $n \in S_{\mathcal{C}} \setminus S'_{\mathcal{C}}$ then $f_{m,n}|_{U_{m,n,0}}$ has both fixed points repelling, $f_{m,n} \to f_n$ and $h_{m,n} \to g_n$ where $h_{m,n}$ is the induced transit map on the perturbed Ècalle-Voronin cylinders.

If $S = \mathbb{Z}$ then $\mathcal{T}$ is a *bi-infinite tower* and otherwise $\mathcal{T}$ is a *forward tower*. The map $f_{min(S)}$ in a forward tower is called the *base map*. Define $Dom(\mathcal{T})$ and $Range(\mathcal{T})$ to be the domain and range of the base map.

Let $\mathcal{T}$ be a forward tower and let f_m be the base map of $\mathcal{T}$. Let

$$\mathcal{F}(\mathcal{T}) = \langle \mathcal{T} \setminus \{f_n : n > m\} \rangle$$

where recall $\langle f_\alpha \rangle$ denotes the set of restrictions of all finite compositions of $\{f_\alpha\}$. Define the orbit of $z \in Dom(\mathcal{T})$ by

$$\text{orb}(z) = \text{orb}(\mathcal{F}(\mathcal{T}), z).$$

Note that if $Range(f_n) \subset Range(f_m)$ and $z \in Dom(f_n)$ then $f_n(z) \in \mathrm{orb}(z)$. We say $\mathrm{orb}(z)$ *escapes* if $\mathrm{orb}(z) \cap (Range(\mathcal{T}) \setminus Dom(\mathcal{T})) \neq \emptyset$. Define the *filled Julia set*, $K(\mathcal{T})$, the *Julia set*, $J(\mathcal{T})$, and the *post-critical set*, $P(\mathcal{T})$, as for quadratic-like maps. For a bi-infinite tower $\mathcal{T}$ define the post-critical set

$$P(\mathcal{T}) = \mathrm{cl} \bigcup_{S' \subset S} P(\mathcal{T}|_{S'})$$

where $\mathcal{T}|_{S'}$ ranges over forward subtowers of $\mathcal{T}$ and where the closure is taken as a subset of $\widehat{\mathbb{C}}$.

Two towers $\mathcal{T}$ and $\mathcal{T}'$ with $S(\mathcal{T}) = S(\mathcal{T}')$ are *quasi-conformally equivalent* if there is a quasi-conformal map ϕ such that

1. ϕ is a quasi-conformal conjugacy of f_n and f_n' on a neighborhood of $K(f_n)$ to a neighborhood of $K(f_n')$ for all $n \in S$,
2. ϕ induces a quasi-conformal conjugacy of the transit maps g_n and g_n' for $n \in S_{\mathcal{C}}$.

A quasi-conformal equivalence ϕ between two forward towers is a *hybrid* equivalence if $\bar{\partial}\phi|_{K(\mathcal{T})} \equiv 0$ and is a *holomorphic* equivalence if ϕ is holomorphic. The following proposition is the analogue of Proposition 2.2 for towers.

Proposition 5.1 (Straightening)**.** *Let $\mathcal{T}$ be a forward tower such that its base map is quadratic-like. Then $\mathcal{T}$ is hybird equivalent to a tower with a quadratic base map.*

Proof. Let f_m be the base map of $\mathcal{T}$. From Proposition 2.2 there is a hybrid equivalence ϕ between f_m and a unique polynomial of the form $z^2 + c$. Let $u(z)$ be the complex dilatation of ϕ and let $\mu = u(z)d\bar{z}/dz$ be the corresponding Beltrami differential. Since ϕ is quasi-conformal there is a $k < 1$ such that $\|u(z)\|_\infty \leq k$. Let $U \supset K(f_m)$ be the domain on which ϕ is a conjugacy.

Define the Beltrami differential μ' by

$$\mu'|_{K(\mathcal{T})} \equiv 0$$

and if $z \in (U \setminus K(\mathcal{T}))$ by

$$\mu'|_{U'} = h^*(\mu)$$

where $h \in \mathcal{F}(\mathcal{T})$ and $U' \ni z$ satisfy $h(U') \subset (U \setminus K(f_m))$. There are restrictions f_n' of f_n such that $[f_n'] = [f_n]$ and μ' is invariant under the foward tower $\mathcal{T}' = \{f_n', g_n\}$.

Write $\mu'(z) = u'(z)d\bar{z}/dz$. Since all maps in $\mathcal{T}'$ are holomorphic $\|u'(z)\|_\infty \leq k < 1$. Let ϕ_1 be the solution to the Beltrami equation

$$\bar{\partial}\phi_1 = u' \cdot \partial\phi_1$$

and let

$$\mathcal{T}'' = \{\phi_1 \circ h \circ \phi_1^{-1} : h \in \mathcal{T}'\}.$$

We claim $\mathcal{T}''$ is again a forward tower and that ϕ_1 is a hybrid equivalence between $\mathcal{T}$ and $\mathcal{T}''$. Let $n \in S_{\mathcal{C}}$ and $g_n \in \mathcal{T}'$. Let $g_n'' = \phi_1 \circ g_n \circ \phi_1^{-1}$ and $f_n'' = \phi_1 \circ f_n' \circ \phi_1^{-1}$. Since ϕ_1 conjugates forward and backward orbits of f_n to orbits of f_n'', it follows that g_n'' is a map on the Ècalle-Voronin cylinders of f_n''. Since ϕ_1 is a homeomorphism, it is evident that g_n'' is a homeomorphism. Moreover, μ' is invariant under g_n, and so g_n'' is conformal. That is, the conjugate of a transit map in $\mathcal{T}$ is a transit map in $\mathcal{T}''$. The other properties of a tower are clear.

The base map of $\mathcal{T}''$ is holomorphically equivalent to a polynomial. Hence $\mathcal{T}''$ is holomorphically equivalent to a tower with a polynomial base map. □

6. Limiting Towers

In this paper we study the parabolic towers that are limits of certain McMullen towers. To be precise we make the following definition. For a given $\kappa > 0$ let $Tow(\kappa)$ be the closure of the set $Tow_0(\kappa)$ of $\mathcal{T} \in Tow$ satisfying

1. $S_{\mathcal{C}} = \emptyset$
2. $f_0 \in Quad$ is normalized
3. $f_n \in Gen(1/\kappa)$ for all $n \in S$
4. if $n \in S_Q$ then f_n has real combinatorics and $p_e(f_n) \leq \kappa$
5. if $n \notin S_Q$ then $[f_n] = [h_n]$ where h_n is either the generalized renormalization of f_{n-1} or the first through map T of f_{n-1} restricted to $Comp(Dom(T), P(f_{n-1}))$
6. if f_n is a first through map then f_{n-1} has a saddle-node cascade in the sense described in §2.5
7. quadratic-like levels are at most κ apart: if $n, m \in S_Q$ are adjacent quadratic-like levels then $|m - n| \leq \kappa$.
8. $V_n = Range(f_n)$ is a κ-quasidisk
9. $\operatorname{diam} V_n \leq \kappa \operatorname{diam} K(f_n)$
10. Unbranched Property: $V_n \cap P(f_m) = P(f_n)$ for $n \geq m$

We will refer to towers in $Tow(\kappa)$ as towers with *essentially bounded combinatorics and complex bounds*. Over the next several sections we will analyze the basic properties of towers in $Tow(\kappa)$.

The combinatorics of a tower $\mathcal{T} \in Tow_0(\kappa)$ is the sequence $\bar{\sigma}(\mathcal{T})$ indexed by $n \in S_Q$ of shuffles $\sigma_n = \sigma(f_n)$. Recall $\Omega^{cpt}(\kappa)$ is the compactification of the space $\Omega(\kappa)$ of shuffles σ with $p_e(\sigma) \leq \kappa$. Suppose $\mathcal{T}_m \in Tow_0(\kappa)$ is a sequence of towers converging to the parabolic tower $\mathcal{T} \in Tow(\kappa)$. The combinatorics of $\mathcal{T}$ is the sequence $\bar{\sigma}(\mathcal{T})$ indexed by $n \in S_Q$ of shuffles and ends given by $\lim_{m\to\infty} \sigma_{m,n}$. Clearly the combinatorics of a tower is invariant under hyrbid equivalence.

Two towers $\mathcal{T} = \{f_n, g_n\}$ and $\mathcal{T}' = \{f'_n, g'_n\}$ are *combinatorially equivalent* if $S(\mathcal{T}) = S(\mathcal{T}')$ and $\bar{\sigma}(\mathcal{T}) = \bar{\sigma}(\mathcal{T}')$.

Proposition 6.1 (Forward Combinatorial Rigidity)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be forward towers hyrbid equivalent to towers in $Tow(\kappa)$. Let f_m and f'_m be the respective base maps. Suppose $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ and $[f_m] = [f'_m]$. Then $[f_n] = [f'_n]$ for all $n \in S$ and $g_n = g'_n$ for $n \in S_{\mathcal{C}}$.*

Proof. First, it is clear that $[f_n] = [f'_n]$ for $m \leq n \leq \min\{S_{\mathcal{C}}\}$ where $\min\{\emptyset\} = \infty$. Now suppose by induction that $n \in S_{\mathcal{C}}$ and $[f_n] = [f'_n]$. We claim $g_n = g'_n$. Let $L_{\bar{a}}$ be the first landing map of $\mathcal{F}(f_n, g_{\bar{a}})$ to the off-critical pieces of f_n and let

$$X = \{\bar{a} : 0 \in Dom(L_{\bar{a}})\}.$$

Let $(\mathbf{T}, \mathbf{h})$ be the holomorphic family over the component $D \subset X$ containing g_n of generalized quadratic-like maps constructed in §4.2. Construct the sequence of families of first return maps as described in §2.4 until the next level where $f_n|_{U_{n,0}} \in Quad$. Similarly construct the families containing g'_n using f'_n. Since $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ it follows from Theorem 2.5, uniqueness of root points and Theorem 2.11 that $g_n = g'_n$. □

Combining this result with straightening we have the following

Corollary 6.2. *Any two combinatorially equivalent forward towers $\mathcal{T}, \mathcal{T}' \in Tow(\kappa)$ are hybrid equivalent.*

Proof. Straighten $\mathcal{T}$ and $\mathcal{T}'$ to the towers $\mathcal{T}_1$ and $\mathcal{T}_2$ with quadratic base maps. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are combinatorially equivalent it follows from Theorem 2.5 and the uniqueness of root points that the base maps are equal. Hence by Proposition 6.1 $\mathcal{T}_1$ and $\mathcal{T}_2$ are hybrid equivalent. □

We now prove compactness:

Proposition 6.3. *For any $\kappa > 0$ the space $Tow(\kappa)$ is compact.*

Proof. Let $\mathcal{T}_m = \{f_{m,n}, g_{m,n}\}$ be a sequence in $Tow(\kappa)$. By selecting a subsequence we may assume the index set $S(\mathcal{T}_m)$ converges to some index set S. If $f_{m,n}$ is a first through map then the modified landing times $l(z)$ are bounded for all $z \in u_{f_{m,n}}$ since the essential period is bounded. By Lemma 2.9, Lemma 2.19 and Lemma 3.1, there exists a function $C(\kappa)$ such that for all $f_{m,n}$,

$$\operatorname{geo}(f_{m,n}) \geq C(\kappa) > 0.$$

From Lemma 2.8 we can select a subsequence $\mathcal{T}_{m_k}$ so that $f_{m_k,n}$ converges on all levels $n \in S$ to some generalized quadratic-like maps f_n. Let $S_{\mathcal{C}} \subset S$ be the levels with $f_n|_{U_{n,0}} \in \mathcal{H}(1/4)$. From Lemma 2.19 we can choose a subsequence so that the transit maps on each level $n \in S_{\mathcal{C}}$ converge. From Lemma 4.7 and Lemma 4.8 the limiting collection of maps will form a tower. The other properties of a tower are clear. □

Lemma 6.4. *Let $\mathcal{T} \in Tow(\kappa)$. Then $\operatorname{diam} K(f_n) \to 0$ as $n \to \infty$. If $\mathcal{T}$ is a bi-infinite tower then $\operatorname{diam} K(f_n) \to \infty$ as $n \to -\infty$.*

Proof. Let us prove the first statement. We can assume there are an infinite number of levels $n \to \infty$ where f_n is not immediately renormalizable, for otherwise $\mathcal{T}$ is eventually a McMullen tower with period-doubling combinatorics and the result follows. Choose a subsequence f_{n_k}, $n_k \to \infty$, of generalized quadratic-like with at least one off-critical piece.

Suppose by contradiction that $\operatorname{diam} K(f_{n_k}) \geq \epsilon > 0$. Let $\cup_j U_{k,j} = Dom(f_{n_k})$ and $K_{k,j} = K(f_{n_k}) \cap U_{k,j}$. We may assume $K_{k+1,j} \subset K_{k,0}$ by selecting levels of first return.

Then since $\operatorname{geo}(f_{n_k}) \geq C(\kappa) > 0$ and $\operatorname{mod}(K_{k,j}, U_{k,j}) \geq 1/\kappa$ it follows that $U_{k,j}$ contains a definite neighborhood of $K_{k,j}$. Hence there is eventually some $j_1, j_2 \neq 0$ and $k_2 > k_1$ with $K_{k_2,j_2} \cap U_{k_1,j_1} \neq \emptyset$. But this is a contradiction since $K_{k_2,j_2} \subset K_{k_1,0}$ and $K_{k_1,0} \cap U_{k_1,j_1} = \emptyset$.

The second statement is analogous. □

Proposition 6.5 ([McM2, Corollary 5.12])**.** *The postcritical set $P(\mathcal{T})$ varies continuously with $\mathcal{T} \in Tow(\kappa)$.*

Proof. Let $\mathcal{T}_m$ be a sequence of towers in $Tow(\kappa)$ converging to a tower $\mathcal{T}$. Assume $\mathcal{T}$ is a forward tower. If $z \in \mathrm{orb}(\mathcal{T}, 0)$ then $d(z, P(\mathcal{T}_m)) \to 0$ as $m \to \infty$ since $\mathcal{F}(\mathcal{T})$ is contained in any geometric limit of $\mathcal{T}_m$. Hence $P(\mathcal{T}) \subset \liminf_m P(\mathcal{T}_m)$. We must show $\limsup_m P(\mathcal{T}_m) \subset P(\mathcal{T})$.

For $n \in S_Q$ let $K_n(0) = K(f_n)$ and let $K_n(i)$ enumerate the orbit of $K(f_n)$ by $\mathcal{T}$. That is,

$$\cup_i K_n(i) = \{h(z) : z \in K(f_n), h \in \mathcal{F}(\mathcal{T})\}.$$

Let $\delta_n = \sup_i \operatorname{diam} K_n(i)$. The arguments proving $\operatorname{diam} K_n(0) \to 0$ can be adapted to prove $\delta_n \to 0$. Let $\epsilon > 0$ and let N be large enough so that $\delta_N < \epsilon$. Let

$$\cup_i K_{m,n}(i) = \{h(z) : z \in K(f_{m,n}), h \in \mathcal{F}(\mathcal{T}_m)\}.$$

Since $\mathcal{T}_m \to \mathcal{T}$ it follows that for $m > N$ large enough $\cup_i K_{m,n}(i)$ is contained in an ϵ-neighborhood of $\cup_i K_n(i)$. Hence $P(\mathcal{T}_m)$ is contained in a 2ϵ-neighborhood of $P(\mathcal{T})$.

Now suppose $\mathcal{T}$ is a bi-infinite tower. From the continuity of $P(\mathcal{T})$ for forward towers and the unbranched property for $Tow_0(\kappa)$ it follows that $P(\mathcal{T}|_{S_n}) = V_n \cap P(\mathcal{T})$ where $S_n \subset S$ is any index set of a forward tower. Since V_n contains an $\epsilon(\kappa)$-scaled neighborhood of $K(f_n)$ and $\operatorname{diam} K(f_n) \to \infty$ as $n \to -\infty$, it follows that $P(\mathcal{T}) = \{\infty\} \cup_{S' \subset S} P(\mathcal{T}|_{S'})$. ☐

6.1. **Expansion of the hyperbolic metric.** One of the central ideas in McMullen's arguments is that maps in a tower expand the hyperbolic metric on the complement of the post-critical set. In this section we prove similar propositions.

Lemma 6.6. *There are continuous increasing functions $C_1(s)$ and $C_2(s)$ such that if $f : X \hookrightarrow Y$ is an inclusion between two hyperbolic Riemann surfaces and $x \in X$ then, letting $s = d(x, Y \setminus X)$,*

$$0 < C_1(s) \le \|Df(x)\| \le C_2(s) < 1.$$

Moreover, $C_2(s) \to 0$ as $s \to 0$.

Proof. The inequality $\|Df(x)\| \le C_2(s) < 1$ and the properties of $C_2(s)$ are found in [McM2]. Lift f to the universal cover $\pi : \mathbb{D} \to Y$ and normalize so that $x = f(x) = 0$. The inclusion $B_s \equiv \{z : d_{\mathbb{D}}(0, z) < s\} \hookrightarrow \mathbb{D}$ factors through f and so $\|Df(0)\| \ge 1/r(s)$ where $r(s)$ is the radius of B_s measured in the euclidean metric. ☐

The following Proposition states when maps in a forward tower $\mathcal{T} \in Tow(\kappa)$ expand the hyperbolic metric on $Range(\mathcal{T}) \setminus P(\mathcal{T})$ and gives an estimate on the amount of expansion and the variation of expansion.

Recall if the base map of $\mathcal{T}$ is $f_m : U_m \to V_m$ then $Range(\mathcal{T}) = V_m$ and $Dom(\mathcal{T}) = U_m$. We will use the notation ρ_m, $\|\cdot\|_m$, $d_m(\cdot,\cdot)$ and $\ell_m(\cdot)$ to denote the hyperbolic metric, norm, distance and length on $Range(\mathcal{T}) \setminus P(\mathcal{T})$.

Proposition 6.7. *Let $\mathcal{T} \in Tow(\kappa)$ be a forward tower with base map $f_m : U_m \to V_m$. Suppose that $h \in \mathcal{F}(\mathcal{T})$ and let $Q_h = h^{-1}(P(\mathcal{T}))$. Then*

$$\|Dh(z)\|_m > 1$$

for any $z \in (Dom(h) \setminus Q_h)$. Moreover, if $(Q_h \setminus P(\mathcal{T})) \ne \emptyset$ then

$$C_2^{-1}(s_2) \le \|Dh(z)\|_m \le C_1^{-1}(s_1)$$

where $s_1 = d_m(z, Q_h \cup \partial Dom(h))$ *and* $s_2 = d_m(z, Q_h)$. *Finally, if* γ *is a path in* $Dom(h) \setminus Q_h$ *with endpoints* z_1 *and* z_2, *then*

$$\|Dh(z_2)\|_m^{1/\alpha} \leq \|Dh(z_1)\|_m \leq \|Dh(z_2)\|_m^{\alpha}$$

where $\alpha = exp(M\ell_m(h(\gamma)))$ *for a universal* $M > 0$.

Proof. We apply McMullen's argument to the approximations of h. Let $\mathcal{T}_j \in Tow_0(\kappa)$ converge to $\mathcal{T}$. We can assume $S_j = S$ and $m = 0$. Let ρ_0 be the hyperbolic metric on $V_0 \setminus P(\mathcal{T})$ and let $\rho_{j,0}$ be the hyperbolic metric on $V_{j,0} \setminus P(f_{j,0})$.

Since

$$f_0 : (U_0 \setminus f_0^{-1}(P(\mathcal{T})) \to (V_0 \setminus P(\mathcal{T}))$$

is a covering map and the inclusion

$$\imath : (U_0 \setminus f_0^{-1}(P(\mathcal{T}))) \hookrightarrow (V_0 \setminus P(\mathcal{T}))$$

is a contraction by the Schwarz Lemma, we see f_0 expands ρ_0. That is, $\|Df_0(z)\|_0 > 1$ for $z \in (U_0 \setminus f_0^{-1}(P(\mathcal{T})))$. Similarly $f_{j,0}$ expands $\rho_{j,0}$.

Suppose $h \in \mathcal{F}(\mathcal{T})$. Let $z \in (Dom(h) \setminus h^{-1}(P(\mathcal{T}))$. Choose compact sets $K_1 \subset (Dom(h) \setminus P(\mathcal{T}))$ and $K_2 \subset (Range(h) \setminus P(\mathcal{T}))$ which contain neighborhoods of z and $h(z)$, respectively. Since the domains $\mathrm{cl}(V_{j,0})$ converge in the Hausdorff topology to $\mathrm{cl}(V_0)$ and the post-critical sets $P(f_{j,0})$ converge to $P(\mathcal{T})$, the hyperbolic metrics $\rho_{j,0}$ converge uniformly on K_1 and K_2 to ρ_0. For large enough j we have $Range(h) \subset V_{j,0}$ since $Range(h) \subset V_0$. Hence there are iterates t_j such that

$$f_{j,0}^{t_j} \to h$$

uniformly on K_1 in the C^1 topology as $j \to \infty$. Thus maps arbitrarily close to h expand metrics arbitrarily close to ρ_0. Hence h is non-contracting: $\|Dh(z)\|_0 \geq 1$. To prove h is expanding, it suffices to assume h is a local lift of a transit map. For this we use induction on $n \in S_{\mathcal{C}}$. First the base case. Let $n = \min S_{\mathcal{C}}$ and let h' be another local lift of g_n such that

$$h = f_0 \circ h'.$$

Since f_0 is expanding and $\widetilde{h}$ is non-contracting it follows h is expanding and the base case holds. Now suppose by induction that local lifts of $g_{n_1}, \ldots, g_{n_k}$ expand ρ_0 for the first k levels in $S_{\mathcal{C}}$. Let h be a local lift of $g_{n_{k+1}}$ where n_{k+1} is the next level in $S_{\mathcal{C}}$ after n_k. There is a restriction f of $f_{n_{k+1}}$ so that $f \in \mathcal{F}(\mathcal{T})$ and we can assume the attracting and repelling petals $D_{\pm}$ were chosen to lie in $Dom(f)$. But then like before there is another local lift h' so that $h = f \circ h'$ and we again see h must be expanding.

Now we estimate how much h expands ρ_0. Choose $z \in (Dom(h) \setminus Q_h)$ and let K_1 and K_2 be closed neighborhoods of z and $h(z)$ as above. Then just as above for large j, we can find iterates t_j such that $f_{j,0}^{t_j} \to h$ uniformly on K_1 as $j \to \infty$. Let $V_j^{-n} = f_{j,0}^{-n}(V_{j,0})$ and $P_j^{-n} = f_{j,0}^{-n}(P(f_{j,0}))$. Since

$$f_{j,0}^{t_j} : V_j^{-t_j} \setminus P_j^{-t_j} \to V_{j,0} \setminus P(f_{j,0})$$

is a local isometry we can apply Lemma 6.6 to the inclusion

$$\imath : V_j^{-t_j} \setminus P_j^{-t_j} \hookrightarrow V_{j,0} \setminus P(f_{j,0})$$

to get the inequalities

$$C_2^{-1}(s) \le \|Df_{j,0}^{t_j}(z)\|_{\rho_{j,0}} \le C_1^{-1}(s)$$

where

$$s = d_{\rho_{j,0}}(z, P_j^{-t_j} \cup \partial V_j^{-t_j}).$$

Since C_1 and C_2 are increasing,

$$C_2^{-1}(s_2') \le \|Df_{j,0}^{t_j}(z)\|_{\rho_{j,0}} \le C_1^{-1}(s_1')$$

where

$$s_1' = d_{\rho_{j,0}}(z, P_j^{-t_j} \cup \partial K_1) \text{ and } s_2' = d_{\rho_{j,0}}(z, P_j^{-t_j} \cap K_1).$$

But $f_{j,0}^{t_j} \to h$ uniformly on K_1, $(P_j^{-t_j} \cap K_1) \to (Q_h \cap K_1)$ and $\rho_{j,0} \to \rho_0$ uniformly on K_1 and K_2 as $j \to \infty$. Thus the second statement of the Proposition follows if we let K_1 range over larger and larger compact sets in $Dom(h) \setminus P(\mathcal{T})$.

To conclude let us prove the last statement about the variation of expansion. From [McM1, Cor 2.27] the variation in $\|Df_{j,0}^{t_j}(z)\|_0$ is controlled by the distance between z_1 and z_2 measured in the hyperbolic metric on $V_j^{-t_j} \setminus P_j^{-t_j}$. Since $f_{j,0}^{t_j}$ is a covering map, this distance is bounded above by the length of $f_{j,0}^{t_j}(\gamma)$ measure on $V_{j,0} \setminus P(f_{j,0})$. As $j \to \infty$ this length converges to $\ell_0(h(\gamma))$. The statement follows. □

The following corollary can be used to control the expansion of the hyperbolic metric on one level with bounds from a deeper level.

Corollary 6.8. *Let $\mathcal{T} \in Tow(\kappa)$ be a forward tower with base map $f_m : U_m \to V_m$. Suppose $n \in S_Q$ is a level such that $V_n \subset V_m$ and let $\mathcal{T}'$ be the tower $\mathcal{T}$ restricted to the levels $n' \ge n$. Let $h \in \mathcal{F}(\mathcal{T}')$ and let $Q_h = h^{-1}(P(\mathcal{T}'))$. Then if $(Q_h \setminus P(\mathcal{T}')) \ne \emptyset$ and $z \in Dom(h) \setminus Q_h$,*

$$C_2^{-1}(s_2) \le \|Dh(z)\|_m$$

where $s_2 = d_n(z, Q_h)$.

Proof. We may assume $m = 0$. Since $V_n \subset V_0$ and $P(\mathcal{T}') = P(\mathcal{T}) \cap V_n$ we see

$$(V_n \setminus P(\mathcal{T}')) \subset (V_0 \setminus P(\mathcal{T}))$$

and so

$$d_0(z, Q_h) \le d_n(z, Q_h).$$

Since the function C_2 in Proposition 6.7 is increasing,

$$C_2^{-1}(d_n(z, Q_h)) \le C_2^{-1}(d_0(z, Q_h)).$$

Finally, since $Range(h) \subset V_n$ and $V_n \cap P(\mathcal{T}) = P(\mathcal{T}')$,

$$h^{-1}(P(\mathcal{T}')) = h^{-1}(P(\mathcal{T})).$$

Since $h \in \mathcal{F}(\mathcal{T})$ it follows from Proposition 6.7 that

$$C_2^{-1}(d_0(z, Q_h)) \le \|Dh(z)\|_0.$$

□

In order to apply this corollary we need to get a bound on $s_2 = d_n(z, Q_h)$. This is done by compactness:

Lemma 6.9. *Let $\mathcal{T} \in Tow(\kappa)$, $n \in S_Q$ and $z \in f_n^{-1}(V_n \setminus U_n)$. Then $d_n(z, Q_{f_n}) \leq C(\kappa)$.*

Proof. By shifting we may assume $n = 0$. Since U_0 and V_0 are κ-quasidisks, the set $V_0' = \mathrm{cl}(f_0^{-1}(V_0 \setminus U_0))$ varies continuously with $\mathcal{T} \in Tow(\kappa)$. Since $P(\mathcal{T})$ varies continously the hyperbolic metric ρ_0 and the set Q_{f_0} vary continuously. Therefore the function F on $Tow(\kappa)$ given by

$$F(\mathcal{T}) = \sup_{z \in V_0'} d_0(z, Q_{f_0})$$

is continuous. Since $Tow(\kappa)$ is compact by Lemma 6.3, there is a $C(\kappa)$ such that $F(\mathcal{T}) \leq C$. ☐

6.2. **Equivalent definitions of the Julia set.** There are several equivalent definitions of the Julia set of a rational map. In this section we present the analogous result for forward towers.

Fix a forward tower $\mathcal{T} \in Tow(\kappa)$. The full orbit of a point under $\mathcal{T}$, much like the full orbit of a point very near the origin in the Feigenbaum map, can be disected to reveal much more structure. For forward towers this can be done by iterating deeper maps when possible.

By shifting we may assume $S = \mathbb{N}_0$. By restricting each $f_n \in \mathcal{T}$ construct a tower $\mathcal{T}' = \{f_n', g_n'\}$ such that

1. $[f_n] = [f_n']$ and $g_n = g_n'$
2. $V_{n+1}' \subset U_{n,0}'$ for each $n \in S$ except $V_{n+1}' = V_n'$ for each $n \in S$ such that f_{n+1} is a first through map
3. $U_0 = U_0'$.

Note that $\mathcal{T}'$ may no longer be a tower in $Tow(\kappa)$ but that $\mathcal{F}(\mathcal{T}') = \mathcal{F}(\mathcal{T})$. For any non-zero $z \in U_0$ define the *depth* of z to be

$$\mathrm{depth}(z) = \max\{n \in S : z \in U_{n,0}'\}.$$

For a point $z \in U_0$ we say a (possibly finite) sequence $(z_0, z_1, z_2, ...)$ is a *sub-orbit* of z (in $\mathcal{T}'$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i \in V_0 \setminus U_0$ then z_{i+1} is not defined
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in Dom(\widetilde{g}_n)$ then $z_{i+1} = \widetilde{g}_n(z_i)$ for some local lift $\widetilde{g}_n \in \mathcal{T}$
- otherwise $z_{i+1} = f'_{\mathrm{depth}(z_i)}(z_i)$

Note any sub-orbit of z is a subset of $\mathrm{orb}(z)$ and $\mathrm{orb}(z)$ escapes iff there exists a sub-orbit $z_0, \ldots, z_N$ such that $z_N \in V_0 \setminus U_0$.

A point $z \in U_0$ is called *periodic* (in $\mathcal{T}$) if there exists $h \in \mathcal{F}(\mathcal{T})$ such that $h(z) = z$. Equivalently, $z \neq 0$ is periodic iff there is an $x \in \mathrm{orb}(z)$ such that $z \in \mathrm{orb}(x)$ and a sub-orbit $x_0, x_1 = h_1(x), ...x_N = h_N(x)$ of x such that $x_0 = x_N$ and $x_0 \neq x_i$ for $0 < i < N$. The *multiplier*, λ, of the periodic orbit through z is defined to be $Dh_N(x)$. The multiplier does not depend on the sub-orbit. A periodic orbit is called *superattracting, attracting, repelling, neutral* if λ satisfies $\lambda = 0, |\lambda| < 1, |\lambda| > 1, |\lambda| = 1$, respectively.

Lemma 6.10. *Let $\mathcal{T} \in Tow(\kappa)$. The only non-repelling periodic orbits in $\mathcal{T}$ are the orbits through the parabolic points of f_n for $n \in S_{\mathcal{C}}$.*

Proof. Let $z_0, ..., z_N$ be the periodic orbit. Since the only non-repelling periodic orbits in $P(\mathcal{T})$ are the orbits through the parabolic points, we can assume the orbit is disjoint from $P(\mathcal{T})$. By Proposition 6.7,

$$\|Dh_N(z)\|_0 > 1$$

But then

$$|\lambda| = |Dh_N(z)| > 1$$

in the euclidean metric as well. □

For a given level $n \in S_{\mathcal{C}}$ let $B_n = K(f_n|_{U_{n,0}})$ be the *central basin* of level n. A connected compact set $K \subset U_0$ is *iterable* if $K \cap \partial B_n = \emptyset$ for all central basins B_n. Mimicing the definition of sub-orbits of points, we say a (possibly finite) sequence of compact sets $(K_0, K_1, K_2, \dots)$ is a *sub-orbit* of K (in $\mathcal{T}'$) if the following conditions are satisfied:

- $K_0 = K$
- all K_i are iterable except possibly the last one, if it exists
- if $K_i \subset Dom(\widetilde{g}_n)$ then $K_{i+1} = \widetilde{g}_n(K_i)$ for some local lift $\widetilde{g}_n \in \mathcal{T}$
- otherwise $K_{i+1} = f'_d(K_i)$ where $d = \min_{z \in K_i} \operatorname{depth}(z)$.

Now that we have said what it means to iterate an iterable compact set, we can prove the following

Proposition 6.11. *Suppose $\mathcal{T} \in Tow(\kappa)$ and let $y \in J(\mathcal{T})$. The following are two equivalent definitions of the Julia set:*

1. $J(\mathcal{T}) = \mathrm{cl}\{z \in Dom(\mathcal{T}) : z \text{ is a repelling periodic point}\}$
2. $J(\mathcal{T}) = \mathrm{cl}\{z \in Dom(\mathcal{T}) : z \text{ is a pre-image of } y\}$

Proof. By shifting we may assume $S = \mathbb{N}_0$. We may also assume $S_{\mathcal{C}} \neq \emptyset$. Let $z \in \partial K(\mathcal{T})$ and let W be a connected neighborhood of z. We can assume $W \subset \mathrm{int}(K(f_0))$. Let $K = \mathrm{cl}(W)$. We can form the suborbit $K_i = h_i(K)$ from K until the first moment when K_i is not iterable. Such a moment must exist since the orbit of $z \in K$ never escapes but the orbit of some other point in K does escape.

Case 1: Suppose $\mathrm{int}(K_i) \cap \partial B_n \neq \emptyset$ for some $n \in S$. Then by arguing as in Lemma 4.4 there is a open set $W' \subset K_i$ and composition $h \in \mathcal{F}(\mathcal{T})$ defined on W' such that $h(W') \cap J(f_0) \neq \emptyset$. There is then an open set $W'' \subset h(W')$ and an $N \geq 0$ such that $K(f_0) \subset f_0^N(W'')$. Since $W \subset K(f_0)$ there exists a point $z_0 \in W$ such that

$$(f_0^N \circ h \circ h_i)(z_0) = z_0.$$

By Lemma 6.10, if we chose W to be small enough, z_0 must be repelling.

Case 2: If K_i is not iterable because $K_i \cap (V_0 \setminus U_0) \neq \emptyset$, then by perhaps choosing a smaller neighborhood W and iterating f_0 more, we can assume that the moment when K_i is not iterable is because $\mathrm{int}(K_i) \cap \partial B_n \neq \emptyset$ for $n = \min S_{\mathcal{C}}$ and we can argue as in case 1.

Case 3: Suppose $\text{int}(K_i) \cap \partial B_n = \emptyset$ for some $n \in S_{\mathcal{C}}$ but that $\partial K_i \cap \partial B_n \neq \emptyset$. Then by choosing a slightly smaller neighborhood W we can assume K_i is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time K_i is not iterable K_i falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood W' of z we see that the orbit of W' is defined for all iterates. But this is impossible since then W' never escapes, contradicting the fact that $z \in \partial K(\mathcal{T})$. Thus after a finite number of restrictions, the non-iterable set K_i must fall into the cases considered above. Thus

$$J(\mathcal{T}) \subset \text{cl}\{z \in U_0 : z \text{ is a repelling periodic point}\}.$$

Let $z \in K(\mathcal{T})$ and let W be a connected neighborhood of z. Suppose W contains a repelling periodic point z_0. Again let $K = \text{cl}(W)$ and start forming the sub-orbit $K_i = h_i(K)$ through K. Claim there is a moment when K_i is not iterable. For otherwise the maps h_i form a normal family on W and that contradicts the fact that W contains a repelling periodic point. Thus there is a non-iterable iterate K_i.

Case 1: Just as case 1 above, there is a open set $W' \subset K_i$ and composition $h \in \mathcal{F}(\mathcal{T})$ defined on W' such that $h(W') \cap J(f_0) \neq \emptyset$. But then there is a point in $h(W')$ that escapes and thus there is a point in W that escapes as well.

Case 2: If K_i is not iterable because $K_i \cap (V_0 \setminus U_0) \neq \emptyset$, then we have found a point in W that escapes.

Case 3: Suppose $\text{int}(K_i) \cap \partial B_n = \emptyset$ but that $\partial K_i \cap \partial B_n \neq \emptyset$. Then by choosing a slightly smaller neighborhood W that still contains the repelling periodic point z_0, we can assume K_i is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time K_i is not iterable K_i falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood W' containing z_0 we see that the orbit of W' is defined for all iterates. But this is impossible since the iterates of W' cannot form a normal family. Thus after a finite number of restrictions, the non-iterable set K_i must fall into the cases considered above. Thus

$$J(\mathcal{T}) \supset \text{cl}\{z \in U_0 : z \text{ is a repelling periodic point}\}.$$

To prove the second statement, notice that the argument proving the first also proves that if $y \in J(\mathcal{T})$ then any point in U_0 has a pre-image arbitrarily close to y. That is,

$$J(\mathcal{T}) \subset \text{cl}\{z \in U_0 : \text{ there is an } h \text{ such that } h(z) = y\}.$$

The reverse inclusion follows from the fact that $J(\mathcal{T})$ is closed and backward invariant and that $y \in J(\mathcal{T})$. □

6.3. **The interior of the filled Julia set.** An infinitely renormalizable quadratic-like map $f \in RQuad$ has a filled Julia set with empty interior. The same statement holds for forward towers:

Proposition 6.12. *For any $\mathcal{T} \in Tow(\kappa)$,*

$$\text{int}(K(\mathcal{T})) = \emptyset.$$

As a corallary we have

Proposition 6.13. *The Julia set $J(\mathcal{T})$ varies continuously with $\mathcal{T} \in Tow(\kappa)$.*

Proof. □

The proof of Proposition 6.12 is broken into propositions Proposition 6.16 and Proposition 6.17 and will occupy the rest of this section.

By shifting we may assume $S = \mathbb{N}_0$. Suppose by contradiction that

$$\mathcal{O} = Comp(\text{int}(K(\mathcal{T})))$$

is non-empty. Let $U \in \mathcal{O}$ and $z \in U$. Let $K \subset U$ be a compact and connected neighborhood of z. Recall B_n are the central basins of $\mathcal{T}$. Since $\partial B_n \subset J(\mathcal{T})$ for all $n \in S_{\mathcal{C}}$ it follows that K is iterable. Since $J(\mathcal{T})$ is backward invariant we see that all the iterates of K are iterable as well. Thus the orbit of K is well defined and contains the orbit of z and so, letting K range over larger and larger compact subset of U, we can define the orbit of U, $\text{orb}(U)$, to be components containing the orbit of K.

A component $U \in \mathcal{O}$ is called *periodic* if $U' \in \text{orb}(U)$ implies $U \in \text{orb}(U')$. A component $U \in \mathcal{O}$ is called *pre-periodic* if U is not itself periodic but there is a peridic component in $\text{orb}(U)$.

The classification of periodic components is based on the following

Proposition 6.14. [L1, M1] *Let $h : U \to U$ be an analytic transform of a hyperbolic Riemann surface U. The we have one of the following possibilities:*

1. *h has an attracting or superattracting fixed point in U to which all orbits converge*
2. *all orbits tend to infinity*
3. *h is conformally conjugate to an irrational rotation of the disk, the punctured disk or an annulus*
4. *h is a conformal homeomorphism of finite order*

The following proposition expands on case 2) above

Proposition 6.15. [L1, M1] *Let U be a hyperbolic domain on the sphere, and $h : U \to U$ an analytic transform continuous up to the boundary. Suppose that the set of fixed points of h on ∂U is totally disconnected. Then in case 2) of Proposition 6.14 there is a fixed point $\alpha \in \partial U$ such that $h_m(z) \to \alpha$ for every $z \in U$.*

We shall use these two propositions to prove

Proposition 6.16. *No $U \in \mathcal{O}$ is periodic or pre-periodic.*

Proof. Suppose $U \in \mathcal{O}$ is a periodic component. Suppose $\text{cl}(U)$ is iterable and all iterates of $\text{cl}(U)$ are interable. Then since U is periodic there exists a univalent map $h \in \mathcal{F}(\mathcal{T})$ defined on a neighborhood of $\text{cl}(U)$ such that $h(\text{cl}(U)) = \text{cl}(U)$. Let us examine the possiblities from Proposition 6.14.

Since $\text{cl}(U)$ is disjoint from $P(\mathcal{T})$, Lemma 6.10 implies any periodic point in $\text{cl}(U)$ must be repelling. Thus there cannot be an superattracting or attracting orbits. Suppose all iterates tend to ∂U. Now the set of points on ∂U fixed by h are isolated, since otherwise h would be the identity on an open set and that would contradict Proposition 6.7. Applying Proposition 6.15 again contradicts Lemma 6.10.

The other possibilities in Proposition 6.14 are ruled out because h expands the hyperbolic metric on $U_0 \setminus P(\mathcal{T})$ and any map conjugate to a rotation will have high iterates arbitrarily close to the identity.

Now suppose there is a component U' from $\mathrm{orb}(U)$ such that $\mathrm{cl}(U')$ is not iterable. To simplify the exposition we will assume $\mathcal{T}$ is a real-symmetric tower. However, this is not essential. Since U is periodic we may assume $U = U'$. Since $U \subset K(f_0)$ there must be an $n \in S_{\mathcal{C}}$ such that $\mathrm{cl}(U) \cap \partial(B_n) \neq \emptyset$. Since $U \cap J(\mathcal{T}) = \emptyset$ it follows that $U \subset B_n$ and if $n' \in S_{\mathcal{C}}$ is the next parabolic level after n then $\mathrm{cl}(U) \cap B_{n'} = \emptyset$.

Let $K = \mathrm{cl}(U)$, $f = f_n|_{U_{n,0}}$ and $\xi = \beta(f)$. Since B_n and ∂B_n are invariant by f, it follows $K_k = f^k(K) \subset \mathrm{cl}(B_n) \setminus B_{n'}$ and $\partial K_k \cap \partial B_n \neq \emptyset$ for all $k \geq 0$. Let

$$\mathcal{B} = Comp(B_n \setminus \left(\bigcup_{k \geq 0}^{\infty} f^{-k}(\mathbb{R}) \right))$$

be the collection of components of the partition pictured in Fig. 11.

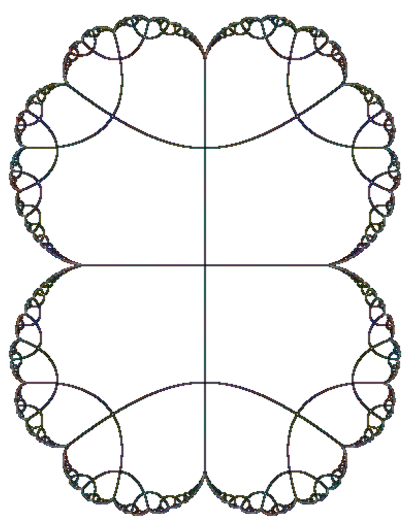

FIGURE 11. The tiling of B_n.

First we claim that $U \cap \mathbb{R} = \emptyset$. Let $n' \in S_Q$ be the largest quadratic-like level before n. Let $B_{n''}$ be the central basin of the first level $n'' \in S_{\mathcal{C}}$ after n'. Then the $f_{n'}$ pre-images of $B_{n''}$ cover a dense subset of $\mathbb{R} \cap K(\mathcal{T}_{n'})$ where $\mathcal{T}_{n'} \subset \mathcal{T}$ is the forward tower with levels $m \geq n'$. It follows that the pre-images by $\mathcal{F}(\mathcal{T})$ cover a dense subset of $\mathbb{R} \cap B_n$ and accumulate at ξ. Since $\partial B_{n''} \subset J(\mathcal{T})$ the claim is established. Since U is periodic under $\mathcal{T}$, we can assume $U \subset A$ where $A \in \mathcal{B}$ satisfies $\xi \in \partial A$. Without loss of generality assume $A \subset \mathbb{H}$.

Let $\gamma = \partial A$. Let

$$\gamma_1 = \bigcup_{\widetilde{g}_n} \widetilde{g}_n^{-1}(\gamma).$$

Since g_n is a real translation, $\gamma_1 \subset \mathbb{H}$. Let

$$\gamma_2 = \bigcup_{k \geq 0} (f^{-1})^k(\gamma_1)$$

where the branch of f^{-1} is chosen so that $f^{-1}(\mathbb{H} \cap B_n) \subset \mathbb{H} \cap B_n$ (see Fig. 12). It follows

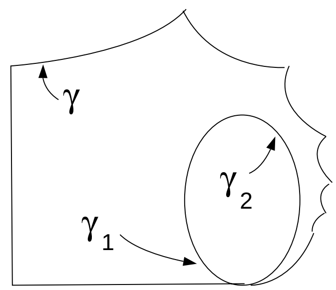

FIGURE 12. The curves γ, γ_1 and γ_2.

from Lemma 4.4 that U is contained in the domain A_1 bounded by γ_2. Continue this process. That is, the pre-image of γ_2 by $\widetilde{g}_n$ is contained in A_1 and pulling back by f^{-1} we see that U is contained in a domain $A_2 \subset A_1$. By Lemma 2.17,

$$\bigcap_{m \geq 1}^{\infty} A_m = \emptyset$$

and so a non-iterable periodic component U cannot exist. ☐

A component $U \in \mathcal{O}$ that is neither periodic nor pre-periodic is called *wandering*.

Proposition 6.17. *No $U \in \mathcal{O}$ is wandering.*

Proof. Suppose $U \in \mathcal{O}$ is wandering. Let $K \subset U$ be compact and connected. Then K is iterable and all iterates of K are iterable. Fix an $z \in \text{int}(K)$. Since each map h from the orbit of K is defined on a neighborhood of K and since $Q_h = h^{-1}(P(\mathcal{T})) \subset J(\mathcal{T})$, it follows from Proposition 6.7 that

$$\sup_h \|Dh(z)\|_0 < \infty. \tag{6.1}$$

Suppose there is an $\epsilon > 0$ such that there are an infinite number of iterates h_n satisfying

$$d(h_n(z), P(\mathcal{T})) > \epsilon$$

where the distance is just the euclidean distance. Order the h_n to match the ordering on the orbit. That is, if $n < m$ then $h_m(z) \in \text{orb}(h_n(z))$. Since each $h_n(z)$ lies in a compact subset of the hyperbolic surface $V_0 \setminus P(\mathcal{T})$,

$$d_0(h_n(z), Q_{f_0}) \leq C_\epsilon,$$

and so from Proposition 6.7,

$$\|Df_0(h_n(z))\|_0 \geq C > 1. \tag{6.2}$$

But then

$$\|Dh_{n+1}(z)\|_0 \geq \|D(f_0 \circ h_n)(z)\|_0 = \|Df_0(h_n(z))\|_0 \cdot \|Dh_n(z)\|_0 \geq C\|Dh_n(z)\|_0 \tag{6.3}$$

which as $n \to \infty$ contradicts equation 6.1.

So we can assume

$$\limsup_h d(h(z), P(\mathcal{T})) = 0.$$

Let $\mathcal{T}_n = \mathcal{T}|_{S_n}$ be the restriction of $\mathcal{T}$ to levels $m \geq n$. Let $\mathcal{K}_n$ be the collection of little filled Julia sets $\mathcal{K}_n = \mathrm{orb}(\mathcal{T}, K(\mathcal{T}_n))$.

From Lemma 2.17 we see $\mathrm{orb}(z)$ must accumulate on some $z' \notin \xi_0$ where ξ_0 is the parabolic orbit of f_0. But then z' is contained in a little filled Julia set in $\mathcal{K}_1$. By iterating forward we can assume $z' \in K(\mathcal{T}_1)$. It follows that there is a $y_1 \in \mathrm{orb}(z)$ such that $y_1 \in K(\mathcal{T}_1)$. Now again there is an accumulation point $\mathrm{orb}(y_1)$ disjoint from ξ_1, the parabolic orbit of f_1, and, repeating the whole argument inductively, there is a sequence of iterates $y_n \in K(\mathcal{T}_n)$.

Each y_n has a moment $x_n \in \mathrm{orb}(z)$ when $\mathrm{orb}(z)$ enters the collection of little filled Julia sets $\mathcal{K}_n$. Once $\mathrm{orb}(z)$ enters $\mathcal{K}_n$ it never leaves. It can happen that different y_n have the same moment x_n. However, since

$$\bigcap_{n \geq 0} K(\mathcal{T}_n) = \{0\}$$

there must be an infinite number of distinct entry moments x_{n_i}.

Let $z_n \in \mathrm{orb}(z)$ satisfy $f_0(z_n) = x_n$. Thus the relation between the points z, x_n, y_n and z_n is given by: $z_n \in \mathrm{orb}(z)$, $x_n = f_0(z_n)$ is the time $\mathrm{orb}(z)$ enters $\mathcal{K}_n$ and $y_n \in \mathrm{orb}(x_n)$ is the first time x_n enters $K(\mathcal{T}_n)$. Claim

$$d_0(z_n, Q_{f_0}) \leq C'.$$

Let K'_n be the component of $f_0^{-1}(\mathcal{K}_n) \setminus \mathcal{K}_n$ containing z_n. The set K'_n is called a *companion* filled Julia set of level n. Since $Q_{f_0} \cap K'_n \neq \emptyset$, it is enough to show

$$\mathrm{diam}_0(K'_n) \leq C'.$$

Consider the sets U'_n and V'_n containing K'_n which are pull-backs of $Dom(f_n)$ and V_n by the map sending z_n to y_n. By the unbranched property this pull-back is univalent. Since $\mathrm{mod}(Dom(f_n), V_n) \geq 1/\kappa$, we have $\mathrm{mod}(U'_n, V'_n) \geq 1/\kappa$ and so, from [McM1, Theorem 2.4], the diameter D_n of U'_n in the hyperbolic metric on V'_n is bounded. But $V'_n \subset (V_0 \setminus P(\mathcal{T}))$. Thus

$$\mathrm{diam}_0(K'_n) \leq \mathrm{diam}_0(U'_n) \leq D_n \leq C(\kappa)$$

and the claim is established.

But then equations 6.2 and 6.3 hold along the sequence $z_{n_i} = h_{n_i}(z)$, and we again get a contradiction to 6.1. □

6.4. **Line fields and forward towers.** A *line field* is a measurable Beltrami differential with $|u(z)| = 1$ on a set of positive measure and $|u(z)| = 0$ otherwise. A line field is *invariant* under $\mathcal{T}$ iff for every $h \in \mathcal{T}$, Dh maps the line at x to the line at $h(x)$ for almost every $x \in Dom(h)$. Using Proposition 6.11 and Proposition 6.12 we can rephrase Proposition 6.1 in terms of invariant line fields.

Before doing so, we need the following

Lemma 6.18. [L1] *Let $\mathcal{T} \in Tow(\kappa)$. The group G of homeomorphisms of $J(\mathcal{T})$ that commute with all maps $h \in \mathcal{T}$ is totally disconnected.*

Proof. Let $\phi \in G$ be a map in the connected component of the identity. Suppose z_0 is a repelling periodic point with $h(z_0) = z_0$ for some $h \in \mathcal{F}(\mathcal{T})$. Since the solutions to $h(z) = z$ are isolated ϕ must fix z_0. The lemma follows from density of repelling cycles: Proposition 6.11. □

We now prove the following version of forward tower rigidity:

Proposition 6.19 (No Line Fields for Forward Towers)**.** *No forward tower $\mathcal{T}$ hybrid equivalent to a tower in $Tow(\kappa)$ supports an invariant line field on its filled Julia set.*

Proof. By Proposition 5.1 it suffices to consider a forward tower $\mathcal{T}$ having a base map of the form $z^2 + c_0$ and by shifting we may assume $S = \mathbb{N}_0$. Since $\mathcal{T}$ is hybrid equivalent to a tower in $Tow(\kappa)$ it follows from Proposition 6.12 that $K(\mathcal{T}) = J(\mathcal{T})$. Suppose by contradiction that $\mathcal{T}$ did admit an invariant line field

$$\mu = u(z)d\bar{z}/dz$$

supported on $J(\mathcal{T})$. For any $w \in \mathbb{D}$ consider the invariant Beltrami differential

$$\mu_w = w \cdot u(z)d\bar{z}/dz$$

on $\widehat{\mathbb{C}}$. Let ϕ_w be a solution to the corresponding Beltrami equation normalized so that the map

$$f_{w,0} = \phi_w \circ f_0 \circ \phi_w^{-1}$$

is again a rational map of the form $z^2 + c_w$ for some $c_w \in \mathbb{C}$. Let $\mathcal{T}_w$ be the tower

$$\mathcal{T}_w = \{\phi_w \circ h \circ \phi_w^{-1} : h \in \mathcal{T}\}.$$

From Proposition 2.5 and the uniqueness of root points, $c_w = c_0$ for all $w \in \mathbb{D}$. Proposition 6.1 implies $f_{w,n} = f_n$ for all $n \in S$ and $g_{w,n} = g_n$ for all $n \in S_{\mathcal{C}}$ and $w \in \mathbb{D}$. So ϕ_w is a holomorphic family of quasi-conformal maps with $\phi_0 = id$ and ϕ_w mapping $J(\mathcal{T})$ homeomorphically to itself commuting with the dynamics of $\mathcal{T}$. From Lemma 6.18 $\phi_w|_{J(\mathcal{T})} = id$. But then the complex dilatation of ϕ_w is zero at all points of Lebesgue density of $J(\mathcal{T})$ and so μ is not supported on $J(\mathcal{T})$, a contradiction. □

6.5. **Line fields and bi-infinite towers.** In this section we move from studying properties of forward towers to studying bi-infinite towers. The plan of attack again follows [McM2].

Let $Tow^\infty(\kappa)$ denote the set of bi-infinite towers in $Tow(\kappa)$. Given $\mathcal{T} \in Tow^\infty(\kappa)$ define $S_{\mathcal{N}} \subset S_Q$ as follows. Let $S_{\mathcal{N},0} = \{0\}$. Then inductively let $S_{\mathcal{N},n+1} = S_{\mathcal{N},n} \cup \{m_{n+1}\}$ where $m_{n+1} = \max\{m \in S_Q | m < m', U_m \supset V_{m'}\}$ and $m' = \min S_{\mathcal{N},n}$. Define $S_{\mathcal{N}} = \cup_{n\to\infty} S_{\mathcal{N},n}$. That is, $S_{\mathcal{N}}$ is the minimal set of nested levels approaching $-\infty$. From Lemma 6.4 we see $S_{\mathcal{N}}$ is unbounded below.

Define the depth of a non-zero point $z \in \mathbb{C}$ by

$$\text{depth}(z) = \max\{m \in S_{\mathcal{N}} : z \in U_m\}.$$

For a point $z \in \mathbb{C}$ we say a (possibly finite) sequence $(z_0, z_1, z_2, ...)$ is a *sub-orbit* of z (in $\mathcal{T}$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in Dom(\widetilde{g}_n)$ then $z_{i+1} = \widetilde{g}_n(z_i)$ for some $\widetilde{g}_n \in \mathcal{T}$
- otherwise $z_{i+1} = f_{\text{depth}(z_i)}(z_i)$

Let $\rho_{-\infty}$ be the hyperbolic metric on $\mathbb{C}\backslash P(\mathcal{T})$ and as in §6.1 let ρ_n be the hyperbolic metric on $V_n \setminus P(\mathcal{T}|_{S_n})$. From Lemma 6.4 and the unbranched property the metrics ρ_n converge uniformly on compact sets to $\rho_{-\infty}$. Using the expansion from §6.1, we now prove

Theorem 1.3. *For any $\mathcal{T} \in Tow^{-\infty}(\kappa)$*

$$\lim_{n\to-\infty} J(\mathcal{T}|_{S_n}) = \widehat{\mathbb{C}}$$

in the Hausdorff topology.

Proof. Let $\mathcal{T}_n = \mathcal{T}|_{S_n}$. Let $z \notin \cup_{s\leq 0} J(\mathcal{T}_s)$. Without loss of generality we may assume $z \in U_0$. Then $\text{orb}(\mathcal{T}_s, z)$ escapes U_s for any $s \in S_{\mathcal{N}}$. Let $z_s = h_s(z)$ be the orbit point just before the first moment of escape on level s. That is, $f_s(z_s) \in V_s \setminus U_s$ and if $z' \in \text{orb}(z)$ also satisfies $f_s(z') \in V_s \setminus U_s$ then $z' \in \text{orb}(z_s)$. For a given $s \in S_{\mathcal{N}}$ let γ_s' be a hyperbolic geodesic in $V_s \setminus P(\mathcal{T}_s)$ connecting z_s with $J(\mathcal{T}_s)$. From Lemma 6.9, there is a C independent of s such that $\ell_s(\gamma_s') \leq C$. Fix a small $\epsilon > 0$ and let A be an ϵ-scaled neighborhood of $P(\mathcal{T}_s)$. Then h_s has an extension $h \in \mathcal{F}(\mathcal{T})$ that is a covering map onto $V_s \setminus A$. Let γ_s be the connected component of $h^{-1}(\gamma_s')$ containing z.

We now argue $\ell_s(\gamma_s)$ shrinks as $s \to -\infty$. The proposition would follow since ρ_s converges to $\rho_{-\infty}$ near z and since Julia sets are backward invariant. Fix an $s \in S_{\mathcal{N}}$ and let $N_s = |\{s, \ldots, 0\} \cap S_{\mathcal{N}}|$ be the minimal number of moments when the orbit of z escapes a nested level. It follows from Lemma 6.9 and Corollary 6.8 that there is a $C > 1$ such that

$$C \leq \|Df_t(z_t)\|_s$$

for any $t \in \{s, \ldots, 0\} \cap S_{\mathcal{N}}$. Hence

$$C^{N_s} \leq \|Dh_s(z)\|_s. \tag{6.4}$$

Hence the derivative at the endpoint z grows exponentially in N_s. From Proposition 6.7, there exists a $C > 1$ such that equation 6.4 holds along γ_s and hence the length of γ_s shrinks as $s \to -\infty$. ☐

A measurable line field μ on an open set U is called *univalent* if there is a univalent map $h : U \to \mathbb{C}$ such that $\mu = h^*(d\bar{z}/dz)$. The main statement in this section is the following extension of Proposition 6.19.

Theorem 6.20 (No Line Fields for Bi-infinite Towers)**.** *Let $\mathcal{T} \in Tow_\infty(\kappa)$. There does not exist a measurable line field μ in the plane such that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n)$, $n \in S_\mathcal{N}$.*

Proof. Suppose to the contrary that $\mu = u(z)d\bar{z}/dz$ is a measurable invariant line field which is non-zero on a set, B, of positive measure. Let $z \in B$ be a point of almost continuity of u and satisfying $|u(z)| = 1$. That is, for each $\epsilon > 0$, the chance of randomly choosing a point y a distance r from z that satisfies $|u(y) - u(z)| > \epsilon$ tends to 0 as r tends to 0:

$$\lim_{r \to 0} \frac{\operatorname{area}(\{y \in B(z,r) \ : |u(y) - u(z)| > \epsilon\})}{\operatorname{area} B(z,r)} = 0$$

where $B(z,r)$ is the euclidean ball of radius r centered at z. By Proposition 6.19, we can assume $z \notin K(\mathcal{T}_n)$ for any n. Let z_n be an infinite sub-orbit from z and for each $s \in S_\mathcal{N}$ let $z_{n_s} = h_{n_s}(z)$ denote the moments in the sub-orbit when z_{n_s+1} first satisfies $z_{n_s+1} \in V_s \setminus U_s$.

For a given $s \in S_\mathcal{N}$ let $\mathcal{T}_s$ denote $\mathcal{T}$ shifted so that level s is moved to level 0 and let w_s and u_s denote z_{n_s} and u shifted by s. That is, if $|B(f_s)| = \alpha_s$, then $w_s = \alpha_s^{-1} z_{n_s}$ and $u_s(z) = u(\alpha_s z)$. Then since $Tow^\infty(\kappa)$ is compact the sequence $\mathcal{T}_s$ has a subsequence which as $s \to -\infty$ converges to some $\mathcal{T}' \in Tow^\infty(\kappa)$. By choosing a further subsequence we may assume w_s converges to a $w \in \operatorname{cl}((f_0')^{-1}(V_0' \setminus U_0'))$ and, from [McM2], μ_s converges weak*, and hence pointwise almost everywhere, to a measurable line field μ' invariant by $\mathcal{T}'$ in the sense that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n')$, $n \in S_\mathcal{N}(\mathcal{T}')$.

Let D be a small disk around w in $V_0' \setminus P(\mathcal{T}_0')$. The hyperbolic diameter of D in $V_0' \setminus P(\mathcal{T}_0')$ is close to that of $D_s = \alpha_s^{-1}(D)$ in the metric on $V_s \setminus P(\mathcal{T}_s)$ for s near $-\infty$. Since D_s is disjoint from $P(\mathcal{T}_s)$, there is, by the argument given in Proposition 1.3, a univalent pullback D_s' of D_s by the map h_{n_s}. By equation 6.4 and the variation of expansion in Proposition 6.7, we see D_s' is a sequence of open sets containing z such that in the euclidean metric $\operatorname{diam}(D_s') \to 0$ and $B(z, C \operatorname{diam}(D_s')) \subset D_s'$ as $s \to -\infty$ for some constant C. Therefore from [McM1, Theorem 5.16] we can choose μ' to be univalent on D.

By Proposition 1.3, there is an $s \in S_\mathcal{N}(\mathcal{T}')$ such that $J(\mathcal{T}_s') \cap D \neq \emptyset$. By invariance, if $Dh(z) \neq 0$ and μ' is locally univalent around z then μ' agrees almost everywhere with a locally univalent line field around $h(z)$ for any composition $h \in \mathcal{F}(\mathcal{T}_s')$. From Proposition 6.11, the orbit of D by $\mathcal{T}_s'$ covers all of V_s'. So μ' agrees almost everywhere with a line field that is locally univalent on the set $V_s' \setminus P(\mathcal{T}_s')$. Since f_s' is injective on $P(\mathcal{T}_s')$ every point in $P(\mathcal{T}_s')$ except $f_s'(0)$ has an f_s' pre-image around which μ' agrees (a.e.) with a locally univalent line field. Hence μ' agrees (a.e.) with a locally univalent line field around $(f_s')^2(0)$ and 0, which is a contradiction, since then we obtain contradictory behavior of μ' around $f_s'(0)$. □

As a corollary we obtain

Theorem 1.4. *If $\mathcal{T}, \mathcal{T}' \in Tow^\infty(\kappa)$ are normalized combinatorially equivalent towers then $[f_n] = [f_n']$ for all $n \in S$ and $g_n = g_n'$ for $n \in S_\mathcal{C}$.*

Proof. Let $S_\mathcal{N}$ be the set of nested levels of $\mathcal{T}$ as constructed above. For each $n \in S_\mathcal{N}$, let ϕ_n be a hybrid equivalence between $\mathcal{T}_n$ and $\mathcal{T}_n'$ coming from straightening (see Corollary 6.2). The dilatation of ϕ_n is bounded above by a constant depending only on κ and ϕ_n fixes 0 and ∞ and maps $\beta(f_0)$ to $\beta(f_0')$. Thus we can pass to a convergent subsequence $\phi_{n_k} \to \phi$ as $n \to -\infty$.

Since ϕ_n restricts to a quasi-conformal equivalence of f_s and f'_s for $s > n$, $s \in S_{\mathcal{N}}$, on a definite neighborhood of $K(f_s)$, it follows that ϕ is a quasi-conformal equivalence. Let μ be the line field defined by ϕ and μ_n the line field defined by ϕ_n. Since $h_*(\mu_n) = \mu_n$ for all $h \in \mathcal{F}(\mathcal{T}_n)$ it follows that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n)$, $n \in S_{\mathcal{N}}$.

From Theorem 6.20, $\mu = 0$ and so ϕ is conformal. Since $\lim_{n\to-\infty} U_n = \widehat{\mathbb{C}}$, ϕ is linear and since $\mathcal{T}$ and $\mathcal{T}'$ are normalized ϕ is the identity. □

7. Proof of Theorem 1.1 and Theorem 1.2

Let $p > 1$. Let f be an ∞-renormalizable real quadratic polynomial with $\bar{p}_e(f) \leq p$. The first step in the proof of Theorem 1.2 is to construct a tower $\mathcal{T} \in Tow(\kappa)$ from f.

It follows from Lemma 2.7 that there is a $\kappa > 0$ depending only on p, and a forward tower $\mathcal{T} = \{f_n\} \in Tow_0(\kappa)$ with the following property. For $n \in S_Q$ let $[f'_n]$ be $[f_n]$ normalized and let $k(n) = |S_Q \cap \{1, \ldots, n\}|$. Then

$$[f'_n] = \mathcal{R}^{k(n)}(f).$$

Hence renormalization acts on towers by shifting. Let $\mathcal{T}_n$ denote the tower $\mathcal{T}$ shifted by n so that f_n is normalized and has index 0. By compactness there exists a limiting tower $\mathcal{T}'$ and by Theorem 1.4 the germ $[f_0]$ is uniquely specified by the combinatorics of $\mathcal{T}'$: a bi-infinite sequence of $\sigma \in \Omega_p^{cpt}$. Hence if f has essentially period tripling combinatorics the germs $\mathcal{R}^k(f)$ converge to a unique germ F, which proves Theorem 1.1.

To prove Theorem 1.2 suppose $\bar{\sigma} \in \Sigma_p$ is a bi-infinite sequence of shuffles and ends in Ω_p^{cpt}. Let $\sigma_n = \pi_n(\bar{\sigma})$ denote the n-th element of $\bar{\sigma}$. For each σ_n let $\sigma_{m,n}$ be a sequence in Ω_p converging to σ_n. Define the sequence $\bar(\tau) \in \Pi_0^\infty \Omega_p$ by

$$\bar{\tau} = (\sigma_{0,0}, \sigma_{1,-1}, \sigma_{1,0}, \sigma_{1,1}, \sigma_{2,-2}, \sigma_{2,-1}, \sigma_{2,0}, \sigma_{2,1}, \sigma_{2,2}, \ldots, \sigma_{n,-n}, \ldots, \sigma_{n,n}, \ldots)$$

and let $\bar{\tau}_n = \omega^{j(n)}(t\bar{a}u)$ where ω is the left-shift operator and $j(n) = 1+3+5+\cdots+(2n-1)+n$. Then by construction $\bar{\tau}_n \to \bar{\sigma}$. Let f be a real quadratic polynomial with shuffle sequence $\bar{\tau}$ and let $\mathcal{T}$ be a tower in $Tow(\kappa)$ constructed from f. By compactness of towers let $\mathcal{T}' = \{f'_n, g'_n\}$ be a limiting tower of $\mathcal{T}_{j(n)}$. Define the function $h : \Sigma_p \to GQuad(m)$ by

$$h(\bar{\sigma}) = [f'_0].$$

From Theorem 1.4 h is well-defined and is continuous. The other properties of h are clear.

References

[deF] E. de Faria, *Proof of universality for critical circle mappings.* Thesis, CUNY 1992.

[DD] R.L. Devaney, A. Douady, "Homoclinic Bifurcations and Infinitely Many Mandelbrot Sets," preprint.

[D] A. Douady, "Does a Julia set depend continuously on the Polynomial?" In: *Complex Dynamical Systems: The Mathematics Behind the Mandelbrot and Julia Sets*, R.L. Devaney (editor), Proceedings of Symposia in Applied Mathematics, Vol. 49, Amer. Math. Soc., 1994, pp. 91-138.

[D2] A. Douady, "Chirurgie sur les applications holomorphes," in Proc. ICM Berkeley, 1986, pp. 724 - 738.

[DH1] A. Douady, J.H. Hubbard, "Etude Dynamique des Polynômes Complexes I-II," *Pub. Math. d'Orsay*, 1984.

[DH2] A. Douady, J.H. Hubbard, "On the dynamics of polynomial-like mappings," *Ann. Sci. École Norm. Sup.*, **18**, 1985, pp. 287-343.

[E] A. Epstein. *Towers of Finite Type Complex Analytic Maps.* Thesis, CUNY

[Lan] O.E. Lanford III, "A computer assisted proof of the Feigenbaum conjectures," Bull. Amer. Math. Soc., v.6, 1982, pp. 427-434.

[La] P. Lavaurs. *Systèmes dynamiques holomorphes: explosion de points pèriodiques paraboliques.* Thesis Universite de Paris-Sud, Centre d'Orsay, 1989.

[LV] O. Lehto, K. J. Vertanen. *Quasiconformal Maps in the Plane.* Springer-Verlag, 1973.

[LS] G. Levin, S. van Strien, "Local connectivity of the Julia set of real polynomials," preprint, 1995.

[LY] M. Lyubich, M. Yampolsky, "Dynamics of quadratic polynomials: Complex bounds for real maps," MSRI preprint #034-95.

[L1] M. Yu. Lyubich, "The dynamics of rational transforms: the topological picture," *Russ. Math. Surveys* **41**, 1986, pp. 35-95.

[L2] M. Yu. Lyubich, "Geometry of quadratic polynomials: moduli, rigidity, and local connectivity," preprint IMS at Stony Brook #1993/9.

[L3] M. Yu. Lyubich, "Dynamics of quadratic polynomials I-II," MSRI preprint #026-95, preprint IMS at Stony Brook #1995/14.To appear in Acta Math.

[L4] M. Yu. Lyubich, "Dynamics of Quadratic polynomials III: Parapuzzle and SBR measures," preprint IMS at Stony Brook #1996/5.

[L5] M. Yu. Lyubich, "Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjuecture," preprint IHES/M/96/61, September 1996.

[L6] M. Yu. Lyubich, "Combinatorics, geometry and attractors of quasi-quadratic maps," Annals Math., v. 140 (1994), pp. 347-404.

[McM1] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math. Studies 135, Princeton University Press, 1994.

[McM2] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, Annals of Math. Studies 142, Princeton University Press, 1994.

[McMS] C. McMullen, D. Sullivan, "Quasiconformal Homeomorphisms and Dynamics III: The Teichmüller space of a holomorphic dynamical system," preprint, 1995.

[MMS] R. Mañè, P. Sad, and D. Sullivan, "On the dynamics of rational maps," *Ann. Sci. Èc. Norm. Sup.* **16**, 1983, pp. 193-217.

[M1] J. Milnor, "Dynamics in One Complex Variable: Introductory Lectures," preprint IMS at Stony Brook #1990/5.

[MT] J. Milnor & W. Thurston. *On iterated maps of the interval.* Lecture Notes in Mathematics 1342. Springer Verlag (1988), 465-563.

[MvS] W. de Melo, S. van Strien. *One Dimensional Dynamics*, Springer-Verlag, 1993.

[Sh] M. Shishikura, "The Parabolic Bifurcation of Rational Maps," Colóquio **19** Brasileiro de Matemática.

[S] D. Sullivan, "Bounds, quadratic differentials, and renormalization conjectures," In: *Mathematics into Twenty-first Century: 1988 Centennial Symposium*, F. Browder editor, Amer. Math. Soc. 1992, pp. 417-466.

[Y] M. Yampolsky, "Complex bounds for critical circle maps," preprint IMS at Stony Brook #1995/12.