RIGIDITY PROPERTIES OF LOCALLY SCALING FRACTALS

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Abstract

A set has local scaling if in a neighborhood of a point the structure of the set can be mapped onto a finer scale structure of the set. These scaling transformations are compact sets of locally affine contractions (that is, contractions with uniformly α-Hölder continuous derivatives). In this setting, without the open set condition or any other assumption on the spacing of these contractions, we show that the measure of the set is an upper semi-continuous function of the scaling transformations in the C^0-topology. With a restriction on the 'non-conformality' of the scaling transformations we show that the Hausdorff dimension is a lower semi-continuous function in the C^1-topology. We include some examples to show that continuity does not follow in either case. [1]

1 Introduction

We consider a certain class of contractions on the space $H(\bar{I})$ of compact subsets of the closed unit ball $\bar{I}$ in $\mathbb{R}^n$. For the purpose of this introduction, it is convenient to think of such a contraction as a finite collection of contracting diffeomorphisms as in the examples at the end of this section. More general and precise definitions will be given in section 2. For the moment the following definition of the contraction on the space of compact sets will suffice:

Definition 1.1 *Let $\{f_j\}_{j\in J}$ be a finite collection of contracting homeomorphisms. The associated iterated function system is the map $\mathcal{F} : H(\bar{I}) \to H(\bar{I})$, determined by*

- $\mathcal{F}(x) = \cup_{j\in J} f_j(x)$.
- $A \in H(\bar{I})$ *then* $\mathcal{F}(A) = \cup_{x\in A} \mathcal{F}(x)$.

It will be convenient, and it will not lead to confusion, to use the symbol $\mathcal{F}$ for the collection of functions as well as for the iterated function system to which it gives rise. We will do so frequently throughout the paper.

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The fact that such iterated function systems are contractions was observed by [5] and has a particularly important consequence: Let $A \in H(\bar{I})$, then $\{\mathcal{F}^n(A)\}_n$ converges uniformly to a uniquely defined compact set $\Lambda(\mathcal{F})$.

The object of our study is the fixed point $\Lambda(\mathcal{F})$ of these maps. By definition, then, this fixed point is a compact set for which we have local scaling as expressed by the fixed point equation:

$$\mathcal{F}(\Lambda) = \Lambda \quad .$$

Our interest centers on the question of how the measure and the Hausdorff dimension of this set vary as we vary the contraction $\mathcal{F}$.

To this end, we define spaces $\mathbb{G}^0$ and $\mathbb{G}^1$ of such contractions and put topologies on them. The space $\mathbb{G}^0$ is intended for a study of the Lebesgue measure of Λ, and is given a topology induced by the C^0-topology on the space of homeomorphisms. $\mathbb{G}^1$ is the subspace consisting of iterated function systems generated by contracting diffeomorphisms with uniform Hölder continuous derivatives, and is intended for considerations related to the Hausdorff dimension of Λ. We define two functions:

Definition 1.2 *Let $\mu : \mathbb{G}^0 \to \mathbb{R}^+$ be the function that assigns the Lebesgue measure of $\Lambda(\mathcal{F})$ to $\mathcal{F}$.*

Definition 1.3 *Let* Hdim $: \mathbb{G}^1 \to \mathbb{R}^+$ *be the function that assigns the Hausdorff dimension of $\Lambda(\mathcal{F})$ to $\mathcal{F}$.*

In section 2, we define our notions (in particular the spaces of contractions on $H(\bar{I})$) carefully. The space we define to deal with the measure theoretic question is in fact much more general than here indicated. In this setting we then prove in section 3 the following result.

Theorem 1.4 *The function μ is upper semi-continuous.*

The derivatives of the functions $\{f_j\}_{j\in J}$ that constitute the map $\mathcal{F}$ are of vital importance for dimension calculations. So we must assume that the functions generating $\mathbb{G}^1$ have Hölder continuous derivatives. This space is also defined in section 2.

As it happens, the theory involving derivatives is much subtler, and in addition requires the assumption that the system is semi-conformal.

Definition 1.5 *The subset $\mathbb{S} \subset \mathbb{G}^1$ is the collection of semi-conformal systems $\mathcal{F}$. A system $\mathcal{F} \in \mathbb{G}^1$ is called semi-conformal if*

$$\lim_{n\to\infty} \max_{f_i \in \mathcal{F},\, x_0 \in \bar{I}} \frac{1}{n} \ln \| (Df_n \cdots f_1|_{x_0}) \| \cdot \| (Df_n \cdots f_1|_{x_0})^{-1} \| = 0 \quad .$$

Note that if the derivatives of the functions are constant and equal, then the assumption of semi-conformality is equivalent to the assumption that the eigenvalues are equal in modulus. The assumption of semi-conformality is closely related to the main asumption in the definition of an asymptotic Moran symbolic construction in [14].

The proof of the main theorem passes through estimates of ratios of derivatives of long compositions taken at different points. These distortion calculations have been done before only in the case of one-dimensional systems. In section 4, we do these calculations for arbitrary dimension.

Definition 1.6 *The subset* $\mathbb{O} \subset \mathbb{G}^1$ *is the collection of systems* $\mathcal{F}$ *satisfying the open set condition. That is,* $\mathcal{F}$ *consists of finitely many functions* f_i *and there is an open set* V *containing* $\Lambda(\mathcal{F})$ *such that* $f_i(V) \cap f_j(V)$ *is empty whenever* $i \neq j$.

In section 5, we prove the following:

Theorem 1.7 *The function* Hdim *is continuous on* $\mathbb{S} \cap \mathbb{O} \cap \mathbb{G}^1$ *and lower semi-continuous on* $\mathbb{S} \cap \mathbb{G}^1$.

Corollary 1.8 *On* $\mathbb{S} \cap \mathbb{G}^1$ *the Hausdorff dimension and the limit capacity are equal.*

The definition of limit capacity is given at the end of this section. It is an important concept because it lends itself to numerical calculations and because it is the concept used in embedding theorems (see [17]).

The principal result here is the semi-continuity. The examples below will show that continuity does not hold. We note that in previous works ([8], [12], and [20]), continuity of the dimension has been proved if the measure function μ is restricted to conformal systems and if $\mathcal{F}$ satisfies a certain condition similar to the open set condition. In these cases smoothness of the dimension can also be proved (see for example [15]). Other workers have related the Hausdorff dimension to other quantities, such as the Lyapunov exponents [24]. In the absence of the open set condition, it is not clear how to define Lyapunov exponents.

Here are some examples to illustrate the subtlety of the problem we are studying. Note that they establish that the functions mentioned are not continuous.

For $t \in [0, 1/2]$, let $\mathcal{F}_t$ be given by $\{f_i\}_{i=0}^2$ where

$$\begin{cases} f_0(x) &= \dfrac{x}{3} \\ f_1(x) &= \dfrac{x+t}{3} \\ f_2(x) &= \dfrac{x+1}{3} \end{cases}$$

Note that each function maps the unit interval into itself, thus the system is a contraction of $H([0,1])$ into itself.

The dimension and measure of the invariant now depend only on the parameter t.

Theorem 1.9 *Let* $\mathcal{F}_t$ *be the system just described. Then*
i) If $t = p/q$ *is rational and* $pq \bmod 3 = 2$ *then* $\mu(t) = 1/q$.
ii) If $t = p/q$ *is rational and* $pq \bmod 3 \neq 2$ *then* $\text{Hdim}\,(t) < 1$.
iii) For all irrational t, $\mu(t) = 0$.
iv) For almost all t, $\text{Hdim}\,(t) = 1$.

We remark that in the rational case ii), dimension calculations, although apparently feasible, are unknown to us.

The first result is a special case of a result proved in [4], with a more geometrical proof in [21]. The second statement is implied by Theorem 4.1 of [6] and Theorem 2.3 of [22]. The third part of the theorem is due to Odlyzko [11] for almost all t. In [7] this was generalized to include all irrational t. The last part is a special case of Theorem 9.12 of [1].

Another example is given by the following family of systems $\mathcal{G}_\lambda$ given by $\{f_i\}_{i=0}^2$ where

$$\begin{cases} f_0(x) &= \lambda x \\ f_1(x) &= \lambda(x+1) \\ f_2(x) &= \lambda(x+3) \end{cases}$$

Using techniques very different from ours, Pollicott and Simon [13], in answer to a question posed by Keane, have recently shown that for almost all $\lambda < 1/3$ the Hausdorff dimension $\operatorname{Hdim}(\mathcal{G}_\lambda)$ is equal to $-\ln 3/\ln\lambda$, while there is a dense subset of $[1/4, 1/3]$ such that if λ belongs to this set then $\operatorname{Hdim}(\mathcal{G}_\lambda)$ is strictly less than $-\ln 3/\ln\lambda$. (This is related to a problem posed by Erdös: for which $\lambda \in [1/3, 1]$ is the invariant density related to this system singular with respect to Lebesgue measure (see [18])).

For completeness, we list the definition of the Hausdorff dimension of a set here.

Definition 1.10 *Let $\mathcal{V}_\delta$ be the collection of covers of a set C whose members have diameter less than or equal to δ. Let*

$$\mathcal{H}^d_\delta(C) = \inf_{\mathcal{V}\in\mathcal{V}_\delta} \sum_{V_i\in\mathcal{V}} |V_i|^d \quad .$$

The Hausdorff dimension $\operatorname{Hdim}(C)$ *of C is given by:*

$$\operatorname{Hdim}(C) = \inf\{d \text{ such that } \lim_{\delta\to 0} \mathcal{H}^d_\delta(C) = 0\} \quad .$$

Definition 1.11 *Let C be a compact set in $\mathbb{R}^n$. The limit capacity (also known as the upper box counting dimension) $d_c(C)$ of C is given by:*

$$d_c(C) = \limsup_{\delta\to 0} \frac{\ln \nu_C(\delta)}{-\ln\delta} \quad ,$$

where $\nu_C(\delta)$ is the minimum number of balls of radius δ needed to cover C.

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2 Definitions

Let $\bar{I}$ be the compact unit ball in $\mathbb{R}^n$. The space $H(\bar{I})$ is the collection of compact subsets of $\bar{I}$. For any compact A, denote its ϵ-neighborhood in $\bar{I}$ by $N_\epsilon(A)$. We topologize $H(\bar{I})$ by endowing it with the so-called Hausdorff metric Hd:

$$\text{If } A_1, A_2 \in H(\bar{I}) \text{then } \mathrm{Hd}\,(A_1, A_2) = \max\{\epsilon_1, \epsilon_2\} \ \text{ where } \ \epsilon_i = \inf\{\epsilon | N_\epsilon(A_i) \supset A_j | j \neq i\} \quad .$$

Observe that this metric is induced by the usual Euclidean distance $|\cdot|$ in $\mathbb{R}^n$. With this topology $H(\bar{I})$ becomes a complete, compact and metric space (see [5]).

The Hausdorff distance satisfies the following (strong) property:

Lemma 2.1 *Let $A = \cup_i A_i$ and $B = \cup_i B_i$ be elements of $H(\bar{I})$. Then*
i) $\mathrm{Hd}\,(A, B) \leq \sup_i \ \mathrm{Hd}\,(A_i, B_i)$.
ii) $\mathrm{Hd}\,(A, B) \leq \sup_i \ \mathrm{Hd}\,(A, B_i)$.

Proof: Note that ii) follows immediately from i). To prove i), note that for all ϵ greater than the right hand side, we have

$$A_i \subseteq N_\epsilon(B_i) \Rightarrow \cup_i A_i \ \subseteq \ \cup_i N_\epsilon(B_i) = N_\epsilon(\cup_i B_i),$$

and vice versa. ■

Definition 2.2 *A generalized iterated function system $\mathcal{F}$, is a map $\mathcal{F} : H(\bar{I}) \to H(\bar{I})$ with the following properties:*

- *If $A \in H(\bar{I})$ then $\mathcal{F}(A) = \cup_{x \in A} \mathcal{F}(x)$.*
- $\exists L \in [0, 1)$ *such that* $\mathrm{Hd}\,(\mathcal{F}(x), \mathcal{F}(y)) \leq L \cdot Hd(x, y)$.

Remark: One can convince oneself that not every generalized iterated function system is an iterated function system by considering the function $z \to \lambda z^3$ on the unit disk, where $\lambda < 1/3$. It is still an open question whether every generalized iterated function system can be generated by continuous functions (as opposed to homeomorphisms) in the manner of Definition 1.1.

The last of the two properties in Definition 2.2 will be referred to as contractiveness, since it implies that $\mathcal{F}$ is a contraction on $H(\bar{I})$.

Lemma 2.3 *A generalized iterated function system $\mathcal{F}$ is a contraction on $H(\bar{I})$. In particular, $\mathcal{F}$ is continuous.*

Proof: Recall that ϵ-neighborhoods in $\bar{I}$ are denoted by N_ϵ. Suppose $B \subseteq N_\epsilon(A)$. By symmetry, it is sufficient to prove that

$$\mathrm{Hd}\,(\mathcal{F}(A), \mathcal{F}(\overline{N_\epsilon(A)})) \leq L \cdot \epsilon$$

Now, using Definition 2.2 and then applying Lemma 2.1 twice, one calculates:

$$\begin{aligned}\mathrm{Hd}\,(\mathcal{F}(A),\mathcal{F}(\overline{N_\epsilon(A)})) &\leq \mathrm{Hd}\,(\cup_{a\in A}\mathcal{F}(a),\cup_{a\in A}\mathcal{F}(\overline{N_\epsilon(a)})) \leq \sup_{a\in A}\ \mathrm{Hd}\,(\mathcal{F}(a),\mathcal{F}(\overline{N_\epsilon(a)}))\\ &\leq \sup_{a\in A}\ \sup_{b\in N_\epsilon(a)}\ \mathrm{Hd}\,(\mathcal{F}(a),\mathcal{F}(b)) \leq L\cdot\epsilon \quad .\end{aligned}$$

■

Definition 2.4 *The space $\mathbb{G}^0$ is the space of generalized iterated function systems together with the following metric:*

$$d_0(\mathcal{F},\mathcal{G}) = \max_{x\in\bar{I}}\ \mathrm{Hd}\,(\mathcal{F}(x),\mathcal{G}(x)) \quad .$$

$\mathbb{G}^0$ is complete and compact. This is not difficult to show. However, we will not use this fact, so we omit the proof.

One usually defines an iterated function system ([1]) as a system $\mathcal{F}$ consisting of a finite set of homeomorphisms. Even if one allows only finite sets of affine transformations, the theory of iterated function systems is very rich and has many applications (see [19] for an overview). Let $D^0_L(\bar{I})$ be space of contractions on $\bar{I}$ with Lipschitz constant L and equipped with the usual sup-metric $\|\cdots\|_0$.

Lemma 2.5 *If $\mathcal{F}$ is a compact subset of $D^0_L(\bar{I})$, then $\mathcal{F}$ satisfies the hypotheses for a generalized iterated function system.*

Proof: We first prove that $\mathcal{F}: H(\bar{I}) \to H(\bar{I})$ is well-defined; that is, that the image of a compact set is compact.

To this end, define

$$e: \bar{I}\times D^0_L(\bar{I}) \to H(\bar{I})$$

by

$$e(x,f) = \{f(x)\} \quad .$$

To prove that e is continuous we let V be open and $(x,f)\in e^{-1}(V)$. Then $\{f(x)\}\in V$, and for some $\epsilon>0$, V contains an $(1+L)\epsilon$-neighbourhood of $\{f(x)\}$. Now suppose $(z,g)\in N_\epsilon(x)\times N_\epsilon(f)$. Then

$$|g(z)-f(x)| \leq |g(z)-f(z)|+|f(z)-f(x)| \leq (1+L)\epsilon.$$

That is, $(z,g)\in e^{-1}(V)$. This shows that e is continuous.

Thus for compact sets $A\subset\bar{I}$ and $\{f_j\}_{j\in J}\subset D^0_L(\bar{I})$, the set $\cup_{x\in A}\cup_{j\in J} f_j(x)$ is also compact and $\mathcal{F}$ is well-defined.

The remaining requirements are easy: For $A\in H(\bar{I})$ it is true by definition that $\mathcal{F}(A) = \cup_{x\in A}\mathcal{F}(x)$. Furthermore, $\mathrm{Hd}\,(\mathcal{F}(x),\mathcal{F}(y)) \leq \sup_{f\in\mathcal{F}}(f(x),f(y)) \leq L\cdot|x-y|$. It now follows immediately that $\mathrm{Hd}\,(\mathcal{F}(A),\mathcal{F}(B)) \leq L\cdot\ \mathrm{Hd}\,(A,B)$, and thus that $\mathcal{F}$ is continuous. ■

It is not clear to us how to define, in any natural way, the notion of differentiability of a generalized iterated function system. So, for our discussion of the Hausdorff dimension we consider only systems that consist of compact sets of diffeomorphisms (Definition 2.8). Note, however, that the space we obtain is not complete.

Definition 2.6 *The metric space $D_L^{1+\alpha}(\bar{I})$ is the set of diffeomorphisms f from $\bar{I}$ to $f(\bar{I}) \subseteq I$, with Lipschitz constant $L < 1$, and with α-Hölder continuous derivatives. We use the C^1-metric on $D_L^{1+\alpha}(\bar{I})$:*

$$\| f - g \|_1 = \sup_{x \in \bar{I}} (|f(x) - g(x)| + \| Df|_x - Dg|_x \|)$$

Definition 2.7 *A differentiable iterated function system is a map $\mathcal{F} : H(\bar{I}) \to H(\bar{I})$, defined by*

$$\mathcal{F}(x) = \cup_{j \in J} f_j(x) \quad ,$$

where $\{f_j\}_{j \in J}$ a compact set in $D_L^{1+\alpha}(\bar{I})$.

The C^1-metric on $D_L^{1+\alpha}(\bar{I})$ induces a corresponding Hausdorff metric Hd^* on the space of compact subsets of $D_L^{1+\alpha}(\bar{I})$. In the following definition we use the same symbols $\mathcal{F}$ and $\mathcal{G}$ for compact sets in $D_L^{1+\alpha}(\bar{I})$ and for the differentiable iterated function systems they generate:

Definition 2.8 *The space $\mathbb{G}^1$ is the space of differentiable iterated function systems together with the metric $\mathrm{Hd}^*(\mathcal{F}, \mathcal{G})$.*

3 The Measure Estimate

This section deals with the measure theoretic properties of the invariant sets. In particular, we prove that a generalized iterated function system $\mathcal{F}$ is always a point of upper semi-continuity of the function that associates with $\mathcal{F}$ the Lebesgue measure of its invariant set. The modulus of semi-continuity is estimated in terms of the limit capacity of the boundary of the invariant set. (Counter-examples to continuity of the measure function are discussed in the introduction).

As is well-known, the fixed point is a continuous function of $\mathcal{F}$. More precisely:

Proposition 3.1 $\mathrm{Hd}\,(\Lambda(\mathcal{F}),\Lambda(\mathcal{G})) \leq \dfrac{d_0(\mathcal{F},\mathcal{G})}{1-L}$.

Proof: Observe that by the triangle inequality

$$\begin{aligned}\mathrm{Hd}\,(\Lambda(\mathcal{F}),\Lambda(\mathcal{G})) &= \mathrm{Hd}\,(\mathcal{F}\Lambda(\mathcal{F}),\mathcal{G}\Lambda(\mathcal{G})) \leq \mathrm{Hd}\,(\mathcal{F}\Lambda(\mathcal{F}),\mathcal{F}\Lambda(\mathcal{G})) + \mathrm{Hd}\,(\mathcal{F}\Lambda(\mathcal{G}),\mathcal{G}\Lambda(\mathcal{G}))\\ &\leq L\cdot\ \mathrm{Hd}\,(\Lambda(\mathcal{F}),\Lambda(\mathcal{G})) + d_0(\mathcal{F},\mathcal{G}) \quad .\end{aligned}$$

■

Theorem 3.2 *The function μ is upper semi-continuous.*

Proof: The function μ is the composition of Λ (which is continuous), and the Lebesgue measure function on $H(\bar{I})$, which we will also denote by μ since no confusion is possible. It suffices to prove that the latter is semi-continuous.

Suppose $\Lambda_0 \in H(\bar{I})$ and suppose $\epsilon > 0$ is given.

Clearly, the neighborhoods $N_{1/n}(\Lambda_0)$ form a collection of monotone decreasing sets with

$$\lim_{n\to\infty} N_{1/n}(\Lambda_0) = \Lambda_0 \quad .$$

Since μ is a continuous measure this implies that

$$\lim_{n\to\infty} \mu(N_{\frac{1}{n}}) = \mu(\lim_{n\to\infty}(N_{\frac{1}{n}})) = \mu(\Lambda_0) \quad .$$

Therefore, if n is large enough, $\mathrm{Hd}\,(\Lambda,\Lambda_0) < \frac{1}{n}$, and so $\mu(\Lambda) < \mu(\Lambda_0) + \varepsilon$. ■

Notice, that the semi-continuity is not uniform (it can't be according to the examples in the introduction). But we can, in fact, estimate its modulus.

In the following, $\partial\Lambda$ denotes the boundary of the set Λ.

Proposition 3.3 *Suppose $\Lambda_0 \in H(\bar{I})$, and $d > d_0 = d_c(\partial\Lambda_0)$. Then for any $\epsilon > 0$ and sufficiently small $\Delta > 0$ the following is true: If $\Lambda \in N_\Delta(\Lambda_0)$, then*

$$\mu(\Lambda) \leq \mu(\Lambda_0) + \epsilon\Delta^{n-d} \quad .$$

Proof: Observe that $\partial\Lambda_0$ can be covered by $p(\Delta)\Delta^{-d_0}$ balls of radius Δ, where by definition

$$\limsup_{\Delta\to 0}\frac{\ln p(\Delta)}{-\ln\Delta}=0 \quad .$$

If we increase the radius of each of these balls to 3Δ, keeping their centers fixed, then the larger balls, together with Λ_0 will cover $N_\Delta(\Lambda_0)$. Thus, if K_n is the volume of the unit ball in $\mathbb{R}^n$,

$$\begin{aligned}\mu(\Lambda) &\le \mu(\Lambda_0)+\mu(N_\Delta(\partial\Lambda_0))\\ &\le \mu(\Lambda_0)+(3\Delta)^n K_n\cdot p(\Delta)\Delta^{-d_0}.\\ &= \mu(\Lambda_0)+(3\Delta)^n K_n\cdot(p(\Delta)\Delta^{\delta})\cdot\Delta^{-d_0-\delta} \quad .\end{aligned}$$

The term $(p(\Delta)\Delta^{\delta})$ tends to zero. Hence, for $\delta=d-d_0$ and Δ sufficiently small we have that the product $3^n\cdot K_n\cdot(p(\Delta)\Delta^{\delta})$ is less than ϵ. ∎

Corollary 3.4 *Suppose $\mathcal{F}_0\in\mathbb{G}^0$. Then for $d>d_c(\partial\Lambda(\mathcal{F}_0))$, and for any $\epsilon>0$ and sufficiently small $\Delta>0$ the following is true:*

$$d_0(\mathcal{F},\mathcal{F}_0)\le\Delta\Rightarrow\mu(\Lambda(\mathcal{F}))\le\mu(\Lambda(\mathcal{F}_0))+\epsilon\Delta^{n-d} \quad .$$

The corollary can be used to estimate the limit capacity in cases such as the example of Theorem 1.9, where we know the measure of the invariant set but not its dimension.

4 The Distortion Estimate

In one dimension, there is an elegant theory to obtain distortion estimates. This theory is described in various research papers and expository works (for example [3], [10] and [23]). The first step in this line of thought is the following. Consider the forward orbit of an interval I_0 under a function f and write $I_i = f(I_{i-1})$. Let $|\ln |Df||_\alpha$ denote the α-Hölder norm of the logarithm of the derivative Df of f (restricted to the forward orbit of I_0). The distortion is the ratio of derivatives of high iterates of f on a small interval I_0. It is given by the following expression (see the references listed above).

$$\sup_{x_0,y_0\in I_0} |\ln \left|\frac{Df^n(x_0)}{Df^n(y_0)}\right|| \leq |\ln|Df||_\alpha \sum_i |I_i|^\alpha \quad . \tag{1}$$

In higher dimension, there is no convenient theory for the calculus of distortions. In this section, the beginnings of such a calculus are developed. Our theory mimics the derivation of the above estimate for the one-dimensional calculus, but its elaboration is more awkward. Also we will iterate not necessarily by the same function, but each time by a function picked from an a priori fixed (compact) set of functions. The latter generalization complicates the notation, but not the mathematics. We will need this theory in the next section where the dimension estimates are done.

Let us start by outlining the general idea of the estimates. Suppose $\mathcal{F}$ is a compact set of diffeomorphisms in $D^1_{L,C}(\bar{I})$. Pick a ball B_0 of unit size and a sequence of contractions $f_i \in \mathcal{F}$. Define $f_i(B_{i-1}) = B_i$. We will express the distortion or nonlinearity as a sum of the diameters of the regions B_i, just as in the one-dimensional case, except that now there will be a penalty for non-conformality. More precisely, choose two points x_0 and y_0 in B_0 and denote the images of x_{i-1} and y_{i-1} under f_i by x_i and y_i. The usual operator norm is written as $\| \cdot \|$. Define

$$C_n = C_n(f_n, \cdots f_1; x_0, y_0) = (D(f_n \cdots f_1)|_{x_0})^{-1} \cdot (D(f_n \cdots f_1)|_{y_0}) \quad . \tag{2}$$

The idea is to obtain estimates for the logarithm of $\sup_{x_0,y_0\in\bar{I}, f_j\in\mathcal{F}} \| C_n \|$.

Here is the main estimate of this section. To simplify the notation of its proof, we set for $n \geq 1$

$$\delta_n = (Df_n|_{x_{n-1}})^{-1}(Df_n|_{y_{n-1}}) - \mathrm{Id} \quad . \tag{3}$$

We also agree that for a set X in $\mathbb{R}^n$, we will denote its diameter by $|X|$. Finally, we simplify the notation for the conformality estimates.

Definition 4.1 *Suppose $\mathcal{F} \in \mathbb{G}^1$ and define*

$$Q(n) \equiv \max_{i\leq n} \sup_{f_i\in\mathcal{F}, x_0\in\bar{I}} \ln \| (Df_i \cdots f_1|_{x_0}) \| \cdot \| (Df_i \cdots f_1|_{x_0})^{-1} \| \quad .$$

Note that by definition 1.5, $\mathcal{F}$ is called semi-conformal if $Q(n)/n$ tends to zero.

Theorem 4.2 *Let $\mathcal{F} \in \mathbb{G}^1$. Then there is a constant C such that for all vectors $v \in S^{n-1}$*

$$|\ln|C_n v|| \leq C \sum_{j=0}^{n-1} e^{Q(j)} \cdot |B_{j-1}|^\alpha \quad .$$

Proof: Observe that

$$\begin{aligned}
C_{n+1} &= (D(f_n \cdots f_1)|_{x_0})^{-1} \cdot (\text{Id } + \delta_{n+1}) \cdot (D(f_n \cdots f_1)|_{y_0}) \\
&= (D(f_n \cdots f_1)|_{x_0})^{-1} \cdot (\text{Id } + \delta_{n+1}) \cdot (D(f_n \cdots f_1)|_{x_0}) C_n \\
&= (\text{Id } + (D(f_n \cdots f_1)|_{x_0})^{-1} \cdot \delta_{n+1} \cdot D(f_n \cdots f_1)|_{x_0}) C_n \\
&= (\text{Id } + (D(f_n \cdots f_1)|_{x_0})^{-1} \cdot \delta_{n+1} \cdot D(f_n \cdots f_1)|_{x_0}) \\
&\quad \cdot (\text{Id } + (D(f_{n-1} \cdots f_1)|_{x_0})^{-1} \cdot \delta_n \cdot D(f_{n-1} \cdots f_1)|_{x_0}) \cdots (\text{Id } + \delta_1) \quad .
\end{aligned}$$

Further, because $\mathcal{F}$ and $\bar{I}$ are compact, there is a uniform upper estimate for $\|(Df|_x)^{-1}\|$ on $\mathcal{F} \times \bar{I}$. By the assumption of uniform Hölder continuity it then follows that

$$\| \delta_i \| \leq C |B_{i-1}|^\alpha \quad .$$

Now, using Schwarz' inequality (twice) and the triangle inequality, we obtain that

$$\| C_n \| \leq \prod_{j=1}^{n} \left(1 + \| (Df_j \cdots f_1|_{x_0})^{-1} \| \cdot \| \delta_{j+1} \| \cdot \| Df_j \cdots f_1|_{x_0} \|\right) \quad . \tag{4}$$

Note that the matrices C_n are invertible and that

$$(C_n(f_n, \cdots f_1; x_0, y_0))^{-1} = C_n(f_n, \cdots f_1; y_0, x_0) \quad .$$

Thus $\| C_n \|^{-1}$ also satisfies equation (4). Since

$$\| C_n^{-1} \|^{-1} \leq |C_n v| \leq \| C_n \| \quad ,$$

we now obtain the estimate for $|\ln |C_n v||$ upon taking logarithms. ∎

Remark: One will notice that the estimate also holds for expanding diffeomorphisms. The problem is that in this case the estimate is bad, since the factor multiplying the largest of the diameters is uncontrollable. It is, however, possible to derive a theorem along the same lines for expanding diffeomorphisms. This is done by redefining C_n and δ_n as follows:

$$\begin{aligned}
C_n &= (D(f_n \cdots f_1)|_{x_0}) \cdot (D(f_n \cdots f_1)|_{y_0})^{-1} \quad . \\
\text{and} \quad \delta_n &= (Df_n|_{x_{n-1}}) \cdot (Df_n|_{y_{n-1}})^{-1} - \text{Id} \quad .
\end{aligned}$$

(Notice that in both cases we write the expanding term first.) Now one derives from the recursion:

$$C_{n+1} = (\text{Id } + \delta_{n+1}) \cdot (Df_{n+1}|_{y_n})^{-1} \cdot C_n \cdot (Df_{n+1}|_{y_n}) \quad ,$$

and continues the reasoning as in the above proof.

Remark: Notice that we assume that derivatives and their inverses exist, precluding singularities. The one-dimensional theory also includes a calculus of distortions in the presence of singularities in the derivative by using for example Koebe estimates or cross-ratios (reviewed in [10]). At present there are no tools to study the analogous problem in higher dimension.

Remark: The term $e^{Q(j)}$ in the theorem has value one if the f's are conformal. In reality this factor is a penalty for deviation from conformality. It is easy to see that if the moduli of the eigenvalues of $D(f_n \cdots f_1)$ bunch together sufficiently, then the lack of conformality will be compensated for by the exponential decrease of the $|B_i|$. In particular, it is clear that a semi-conformal system has bounded distortion.

Definition 4.3 *For $\mathcal{F} \in \mathbb{G}^1$, define the distortion as*

$$D(n) = \max_{i \leq n} \sup_{x,y \in \bar{I}, f_j \in \mathcal{F}} |\ln \|C_i(f_i, \cdots f_1; x, y)\|| \quad .$$

The system is said to have bounded distortion if $D(n)/n$ tends to zero.

Note that this is slightly more general than the usual requirement which is that $D(n)$ is uniformly bounded.

Corollary 4.4 *Let $\mathcal{F} \in \mathbb{G}^1$. Then for all x_0 and y_0 in $\bar{I}$ and each unit-vector v there is a unit-vector $w(v)$ such that*

$$\left| \ln \frac{|D(f_n \cdots f_1)|_{x_0}(w(v))|}{|D(f_n \cdots f_1)|_{y_0}(v)|} \right| \leq D(n) \quad .$$

Proof: Note that

$$D(f_n \cdots f_1)|_{x_0}(C_n v) = D(f_n \cdots f_1)|_{y_0}(v)$$

We choose $w(v) = \frac{C_n v}{|C_n v|}$. Then

$$\frac{|D(f_n \cdots f_1)|_{x_0}(w(v))|}{|D(f_n \cdots f_1)|_{y_0}(v)|} = \frac{1}{|C_n v|} \quad .$$

Now take logarithms and apply the theorem. ■

To obtain good bounds in the dimension calculations, we need to know that the asymptotic rate of contraction is independent of the direction in the tangent space. This is implied by semi-conformality (see definition 1.5) and the following result:

Corollary 4.5 *Let $\mathcal{F} \in \mathbb{G}^1$. Then for all unit-vectors v and u:*

$$\left| \ln \frac{|D(f_n \cdots f_1)|_{x_0}(u)|}{|D(f_n \cdots f_1)|_{y_0}(v)|} \right| \leq D(n) + Q(n) \quad .$$

Proof: We write this as

$$\left| \ln \frac{|D(f_n \cdots f_1)|_{x_0}(w(v))|}{|D(f_n \cdots f_1)|_{y_0}(v)|} \right| + \ln \left| \frac{|D(f_n \cdots f_1)|_{y_0}(u)|}{|D(f_n \cdots f_1)|_{y_0}(w(v))|} \right| \quad .$$

To the first term we apply the previous result. The second is calculated with the help of

$$\| (Df_n \cdots f_1|_{x_0})^{-1} \|^{-1} |v| \leq |(Df_n \cdots f_1|_{x_0})v| \leq \| Df_n \cdots f_1|_{x_0} \| |v| \quad ,$$

and the definition of $Q(n)$. ■

In the remainder of this section, we discuss the relation between derivatives and sizes of domains. In particular, we derive a version of the mean value theorem to to obtain better estimates of the diameter of the iterate of a region.

Lemma 4.6 *Let A and B be connected compact sets in $\mathbb{R}^n$ and suppose in addition that A is convex. Suppose that $g : A \to B$ is a diffeomorphism. Then there is a point $a_+ \in A$ and a $v_{a_+} \in T_{a_+}A$ (the tangent space of A at a_+) such that*

$$\frac{|Dg|_{a_+} v_{a_+}|}{|v_{a_+}|} \geq \frac{|B|}{|A|} \quad .$$

Proof: Let w and z in B such that $|w-z| = |B|$ and let $x = g^{-1}(w)$ and $y = g^{-1}(z)$. Connect x and y by a straight segment $\gamma \in A$ (by the convexity of A) and parametrize this curve by arclength ($|D\gamma| = 1$). Then

$$|B| = \left| \int_0^{|x-y|} Dg(\gamma(t)) \cdot D\gamma(t)\, dt \right| \leq |A| \cdot \max_{x \in A} \| Dg_x \| \quad .$$

Now choose a_+ to be the point where the maximum is assumed. ■

Lemma 4.7 *Let A be a closed ball and B be a set in $\mathbb{R}^n$. Suppose that $g : A \to B$ is a diffeomorphism. Then there is a point $a_- \in A$ and a $v_{a_-} \in T_{a_-}A$ such that*

$$\frac{|Dg_{a_-} v_{a_-}|}{|v_{a_-}|} \leq \frac{|B|}{|A|} \quad .$$

Proof: From elementary calculus, we know that

$$\int_A \frac{|\det Dg_x|}{\text{vol}(A)}\, d^n x = \frac{\text{vol}(B)}{\text{vol}(A)} \quad .$$

The right hand side of this equation is the average of the positive function $|\det Dg_x|$. Thus there is a $a_- \in A$ such that

$$|\det Dg|_{a_-}| \leq \frac{\text{vol}(B)}{\text{vol}(A)} \quad .$$

Denote the eigenvalues of $Dg|_{a_-}$ by $\{\lambda_i\}_{i=1}^n$ (counting multiplicity). Observe that vol(B) is no greater than the volume of a ball with diameter $|B|$. Thus

$$\prod_{i=1}^n |\lambda_i| \leq \frac{|B|^n}{|A|^n} \quad .$$

By taking logarithms and dividing by n, it becomes obvious that the average of $\{\ln|\lambda_i|\}_{i=1}^n$ is no greater than $\frac{|B|}{|A|}$. Thus there must be an eigenspace V_{a_-} of Dg_{a_-} satisfying the lemma. ■

These two lemmas imply the higher dimensional version of the mean value theorem that we will use in the next section.

Corollary 4.8 *Let A be a closed ball and B a set in $\mathbb{R}^n$. Suppose that $g : A \to B$ is a diffeomorphism. Then there is a point $a \in A$ and a $v \in T_a A$ such that*

$$\frac{|Dg|_a v|}{|v|} = \frac{|B|}{|A|} \quad .$$

Proof: Note that the transformation

$$T : A \times TA \to \mathbb{R}^+$$

defined by

$$T(x, v) = \frac{|Dg|_x v|}{|v|}$$

is continuous and $A \times TA$ is path-connected. The result is thus a consequence of the previous lemmas. ■

We now use these results to derive a general statement about scalings in contracting maps.

Theorem 4.9 *Let $\mathcal{F} \in \mathbb{G}^1$ and suppose that f_a is a composition of at most n functions of $\mathcal{F}$ and f_b is a composition of arbitrary length. Then for any ball B we have*

$$\left| \ln \left(\frac{|f_a f_b(B)|}{|f_a(B)|} \cdot \frac{|B|}{|f_b(B)|} \right) \right| \leq 2Q(n) + 2D(n) \quad .$$

Proof: The expression in the theorem can be written as:

$$\left| \ln \left(\frac{|f_a f_b(B)|}{|B|} \cdot \frac{|B|}{|f_a(B)|} \cdot \frac{|B|}{|f_b(B)|} \right) \right|$$

With the help of Corollary 4.8, we get

$$\frac{|f_a f_b(B)|}{|B|} = \frac{|(Df_a \cdot Df_b)|_x \; v_x|}{|v_x|} = \frac{|Df_a|_y v_y|}{|v_y|} \; \frac{|Df_b|_x v_x|}{|v_x|} \quad ,$$

where v_y is a unit vector in the direction of $Df_b|_x v_x$. The other derivatives can also be calculated with the help of the same corollary, to give

$$\left(\frac{|f_a f_b(B)|}{|f_a(B)|} \cdot \frac{|B|}{|f_b(B)|} \right) = \frac{|Df_a|_{x_1}(v_1)|}{|Df_a|_{x_2}(v_2)|} \cdot \frac{|Df_b|_{y_1}(w_1)|}{|Df_b|_{y_2}(w_2)|} \quad .$$

The result now follows from Corollary 4.5. ■

5 The Dimension Estimate

We prove that if $\mathcal{F} \in \mathbb{G}^1$ is a semi-conformal differentiable iterated function system, then it is a point of lower semi-continuity of the function that evaluates the Hausdorff dimension. We note here that if $\mathcal{F}$ is a one-dimensional system with finitely many branches and satisfying a strong condition on the distance of the individual branches, then the Hausdorff dimension varies continuously (see [20] or [16] for more information). Without that condition, it is clear that the dimension is not continuous as observed in the introduction. Nonetheless, the proof of the semi-continuity given here bears resemblance to Takens' proof.

For a given system $\mathcal{F} \in \mathbb{G}^1$, we choose positive constants $K = K(\mathcal{F})$ and $k = k(\mathcal{F})$ such that for all $f \in \mathcal{F}$ and $x \in \overline{I}$,

$$\begin{array}{ll} \left(\| (Df|_x)^{-1} \|\right)^{-1} & > e^{-K} \quad ; \\ \| Df|_x \| & < e^{-k} \quad . \end{array} \tag{1}$$

Note that by continuity (1) automatically holds for all $\mathcal{F}'$ in a $\mathbb{G}^1$-neighbourhood of $\mathcal{F}$ (see Definition 2.8).

A dynamic cover $\mathcal{U}$ of $\Lambda(\mathcal{F})$ is a finite cover by open sets each of which can be written as $f_n \cdots f_1(I)$.

Lemma 5.1 *Suppose $\mathcal{F} \in \mathbb{G}^1$. Then for each $n > 0$, there is a dynamic cover $\mathcal{U}_n$ of $\Lambda(\mathcal{F})$ such that for all $U \in \mathcal{U}_n$:*

$$2e^{-2Q(n)-2D(n)-K-nk} \leq |U| < 2e^{-nk} \quad .$$

Furthermore, all elements of $\mathcal{U}_n$ are of the form $f_m \cdots f_1(I)$ with $m \leq n$.

Proof: Since $\Lambda(\mathcal{F}) \subset I$, $\{f(I)\}_{f\in\mathcal{F}}$ covers $\Lambda(\mathcal{F})$. Let $\mathcal{U}_1$ be a finite subcover. It clearly satisfies the lemma. Denote the finite set of contractions selected in this process by $\mathcal{F}_1$. Thus

$$\Lambda(\mathcal{F}) \subset \{f(I)\}_{f\in\mathcal{F}_1} = \mathcal{F}_1(I) \quad .$$

We now continue by induction. Suppose that for $i \in \{1, \cdots, n\}$, we have finite covers $\mathcal{U}_i = \mathcal{F}_i(I)$ of $\Lambda(\mathcal{F})$ satisfying the lemma. To construct $\mathcal{U}_{n+1}$, we will for each $W = f_w(I) = f_\ell \cdots f_1(I) \in \mathcal{U}_n$ find a finite cover of $W \cap \Lambda(\mathcal{F})$ by sets of the form $f_w f_\alpha(I)$ where each f_α is also a composition of elements of $\mathcal{F}$.

If $W = \Lambda(\mathcal{F})$ already satisfies the inequalities of the lemma for level $n+1$, we accept W itself as a member of $\mathcal{U}_{n+1}$. If this is not the case, we have

$$2e^{-(n+1)k} \leq |W| < 2e^{-nk}. \tag{2}$$

Note that for any set $W = f_w(I)$ satisfying (2) we necessarily have $|w| \leq n$, where $|w|$ denotes the length of the composition f_w.

We now replace $f_w(I)$ by $\{f_w f(I)\}_{f\in\mathcal{F}_1}$. The latter is clearly a covering of $W\cap\Lambda(\mathcal{F})$. For some $m\leq n$ we have, by Theorem 4.9, the following situation:

$$\begin{aligned} |f_w f(I)| &= C_w \frac{|f(I)|}{|I|}|f_w(I)| \quad , \\ |\ln C_w| &\leq 2Q(m)+2D(m) \quad , \\ e^{-K} \leq & \frac{|f(I)|}{|I|} < e^{-k} \quad . \end{aligned} \tag{3}$$

Combining this with (2), we obtain

$$2e^{-(n+1)k-K-2Q(m)-2D(m)} \leq |f_w f(I)| < 2e^{-(n+1)k+2Q(m)+2D(m)} \quad . \tag{4}$$

In particular, we have the necessary lower bound for $|f_w f(I)|$. If the upper bound is also satisfied we accept $f_w f(I)$ as a member of $\mathcal{U}_{n+1}$. If not, then $f_w f(I)$ also satisfies (2). This means that we can proceed, as for $f_w(I)$, by replacing $f_w f(I)$ by the covering $\{f_w f g(I)\}_{g\in\mathcal{F}_1}$ of $f_w f(I)\cap\Lambda(\mathcal{F})$, and treating its members as before.

Since

$$|f_w f_\alpha(I)| < 2e^{-(n+1)k} \text{ if } |w|+|\alpha| = n+1 \quad ,$$

this iterative procedure must terminate in success after no more than $n+1-|w|$ steps, resulting in a dynamic cover $\mathcal{U}_{n+1}$ satisfying the lemma. ■

With each dynamic cover $\mathcal{U}_n = \mathcal{F}_n(I)$ of $\Lambda(\mathcal{F})$, associate a compact subset Λ_n of Λ in the following way: Let $\mathcal{V}_n$ be a maximal collection of disjoint members of $\mathcal{U}_n$. Thus by construction each $V\in\mathcal{V}_n$ is of the form $f(I)$ where f belongs to a subset $\mathcal{G}_n$ of $\mathcal{F}_n$. Clearly, $\mathcal{G}_n$ is a differentiable iterated function system consisting of a finite number, say N_n, of diffeomorphisms, and its invariant set $\Lambda_n = \Lambda(\mathcal{G}_n)$ is a subset of Λ. The main reason for introducing the systems $\mathcal{G}_n$, is, of course, that they satisfy the open set condition (definition 1.6). Therefore, the Hausdorff dimensions of their invariant sets are easy to calculate. These dimensions serve as approximations to the dimension of Λ (see Proposition 5.5).

Lemma 5.2 *For each V in $\mathcal{V}_n$, choose $x_V\in V$ and let $\tilde{V}$ be the ball with center x_V but with radius $4e^{-nk}$. Then the collection $\tilde{\mathcal{V}}_n$ of these sets covers Λ, and for any $V\in\mathcal{V}_n$ we have*

$$|\tilde{V}| \leq 4e^{2Q(n)+2D(n)+K}|V| \quad .$$

The proof is easy and is therefore left to the reader.

Lemma 5.3 *Suppose $\mathcal{F}\in\mathbb{S}\cap\mathbb{G}^1$ and let $\mathcal{G}_n$ be the differentiable iterated function system constructed as above, and suppose it consists of N_n diffeomorphisms. Then*

$$\text{Hdim}\,(\Lambda_n) \leq \text{Hdim}\,(\Lambda) \leq \limsup_{n\to\infty} \frac{\ln N_n}{nk} \quad .$$

Proof: The first of the inequalities is obvious because of the inclusion $\Lambda_n \subseteq \Lambda$. For the second inequality let $d_n = \frac{\ln N_n}{nk}$. Then for any $\epsilon > 0$,

$$\begin{aligned}\mathcal{H}^{\epsilon+d_n}_{8e^{-nk}}(\Lambda) &\leq \textstyle\sum_{\tilde{V}\in\tilde{\mathcal{V}_n}} |\tilde{V}|^{\epsilon+d_n} \leq \sum_{V\in\mathcal{V}_n} 4^{\epsilon+d_n} e^{(K+2Q(n)+2D(n))(\epsilon+d_n)} |V|^{\epsilon+d_n} \\ &\leq 8^{\epsilon+d_n} N_n e^{(K+2Q(n)+2D(n)-nk)(\epsilon+d_n)} \qquad (5) \\ &\leq 8^{\epsilon+d_n} e^{K+2Q(n)+2D(n))(\epsilon+d_n)-nk\epsilon} \quad .\end{aligned}$$

This tends to zero as n goes to infinity, and so establishes the upper bound for the Hausdorff dimension. ∎

The actual calculation of the Hausdorff dimension uses the following result (see [1]).

Proposition 5.4 *Let the system $\mathcal{H} \in \mathbb{G}^1$ be a set of N contractions satisfying the open set condition. Suppose further that*

$$0 < e^{\lambda_-} \leq \frac{|Dh \cdot v|}{|v|} \leq e^{\lambda_+} < 1 \quad ,$$

for all $h \in \mathcal{H}$. Then we have

$$\frac{-\ln N}{\lambda_-} \leq \text{ Hdim}\,(\Lambda(\mathcal{H})) \leq \frac{-\ln N}{\lambda_+} \quad .$$

Proposition 5.5 *Consider the invariant sets Λ_n of the systems $\mathcal{G}_n$ derived from the system $\mathcal{F} \in \mathbb{S} \cap \mathbb{G}^1$ and consisting of N_n contractions satisfying the open set condition. Let k be the constant defined in equation (1). Then we have:*

$$\lim_{n\to\infty} |\text{ Hdim}\,(\Lambda_n) - \frac{\ln N_n}{nk}| = 0 \quad .$$

Proof: The proof consists of estimating the numbers λ_- and λ_+ of the previous proposition for the sets $\Lambda(\mathcal{G}_n)$. Let $g : I \to V \in \mathcal{V}_n$ be a member of the finite family $\mathcal{G}_n$. By construction, the map g is a composition of $m \leq n$ diffeomorphisms $f \in \mathcal{F}$. For any $x \in \bar{I}$ and v in the tangent space $T_x\bar{I}$, we split the basic estimate of Proposition 5.4 into two parts:

$$\frac{|Dg_x \cdot v|}{|v|} = \frac{|Dg_a \cdot v_a|}{|v_a|} \cdot \frac{|Dg_x \cdot v|}{|Dg_a \cdot v_a|} \quad ,$$

Where a and the unit tangent vector v_a are selected so that the first ratio on the right hand side is estimated by applying the mean value theorem (Corollary 4.8). The second ratio is estimated by using Corollary 4.5. Using Lemma 5.1 these estimates give us

$$-nk - K - 3D(n) - 3Q(n) \leq \ln \frac{|Dg_x \cdot v|}{|v|} \leq -nk + D(n) + Q(n) \quad .$$

thus, by Proposition 5.4,

$$\frac{\ln N_n}{nk + K + 3D(n) + 3Q(n)} \leq \text{ Hdim}\,(\Lambda(\mathcal{G}_n)) \leq \frac{\ln N_n}{nk - D(n) - Q(n)} \quad . \qquad (6)$$

Using semi-conformality, as $n \to \infty$, this establishes the result. ■

Recall the definition of the limit capacity, given in Section 1. The limit capacity is always at least as big as the Hausdorff dimension, because for the former we insist that the covering sets all have the same diameter. The following is an immediate consequence of Proposition 5.5 and Lemma 5.3:

Corollary 5.6 *Suppose that $\mathcal{F} \in \mathbb{S} \cap \mathbb{G}^1$. Then the limit capacity of $\Lambda(\mathcal{F})$ is equal to its Hausdorff dimension.*

In the mean time we have everything in place to prove the extension of the continuity result (one part of Theorem 1.7). The methods are exactly the same as those used in the previous proposition.

Theorem 5.7 *Every point of $\mathbb{G}^1 \cap \mathbb{O} \cap \mathbb{S}$ is a point of continuity of the function* Hdim *on $\mathbb{G}^1 \cap \mathbb{O}$.*

Proof: For $\mathcal{F} \in \mathbb{G}^1 \cap \mathbb{O} \cap \mathbb{S}$ and $\mathcal{F}' \in \mathbb{G}^1 \cap \mathbb{O}$ we let $\mathcal{F}_n$ and $\mathcal{F}'_n$ be the collection of iterates associated with the dynamic covers of $\Lambda(\mathcal{F})$ and $\Lambda(\mathcal{F}')$ respectively. If $\mathcal{F}$ and $\mathcal{F}'$ are sufficiently close in the $\mathbb{G}^1$-topology, then they will satisfy (1) for the same constants k and K. Because $\mathcal{F}, \mathcal{F}' \in \mathbb{O}$ we have $\Lambda(\mathcal{F}_n) = \Lambda(\mathcal{F})$ and $\Lambda(\mathcal{F}'_n) = \Lambda(\mathcal{F}')$. We let $N_n = \#(\mathcal{F}_n)$, and we define $D(n)$ and $Q(n)$ for $\mathcal{F}$ as in Definitions 4.3 and 4.1. For $\mathcal{F}'$ we denote the corresponding quantities by $D'(n)$ and $Q'(n)$. Then by (6) in the proof of Proposition 5.5,

$$\frac{\ln N_n}{nk + K + 3D(n) + 3Q(n)} \leq \text{Hdim}\,(\Lambda(\mathcal{F})) \leq \frac{\ln N_n}{nk - D(n) - Q(n)}$$

for each n. A similar inequality holds for $\text{Hdim}\,(\Lambda(\mathcal{F}'))$.

Fix n. Then for $\mathcal{F}'$ in a sufficiently small neighborhood $\mathcal{N}$ of $\mathcal{F}$ in $\mathbb{G}^0 \cap \mathbb{O} \cap \mathbb{S}$ it follows that $\#(\mathcal{F}'_n) = N_n$, and that $D'(n)$ and $Q'(n)$ are arbitrarily close to $D(n)$ and $Q(n)$ respectively. Thus, by choosing n large initially, and $\mathcal{N}$ small, we can make $|\,\text{Hdim}\,(\Lambda(\mathcal{F}) - \text{Hdim}\,(\Lambda(\mathcal{F}'))|$ as small as desired. ■

Here is the remaining half of the main result.

Theorem 5.8 *Every point of $\mathbb{G}^1 \cap \mathbb{S}$ is a point of lower semi-continuity of the function* Hdim *on $\mathbb{G}^1$.*

Proof: Recall that $\Lambda_n = \Lambda_n(\mathcal{F})$ is the invariant set of the system $\mathcal{G}_n = \mathcal{G}_n(\mathcal{F})$ satisfying the open set condition. For a given $\mathcal{F} \in \mathbb{G}^1 \cap \mathbb{S}$ choose n so that

$$|\,\text{Hdim}\,(\Lambda(\mathcal{F})) - \text{Hdim}\,(\Lambda_n(\mathcal{F}))| < \epsilon/2 \quad .$$

This is possible by Proposition 5.5. If we choose $\mathcal{F}' \in \mathbb{G}^1$ sufficiently close to $\mathcal{F}$, then $\mathcal{G}_n(\mathcal{F}')$ is close to $\mathcal{G}_n(\mathcal{F})$ and $\mathcal{G}_n(\mathcal{F}')$ also satisfies the open set condition. We assume, using Theorem 5.7, that $|\,\text{Hdim}\,(\Lambda_n(\mathcal{F}')) - \text{Hdim}\,(\Lambda_n(\mathcal{F}))| < \epsilon/2$. Then, using Lemma 5.3,

$$\begin{aligned} \text{Hdim}\,(\Lambda(\mathcal{F}')) &\geq \text{Hdim}\,(\Lambda_n(\mathcal{F}')) \\ &\geq \text{Hdim}\,(\Lambda_n(\mathcal{F})) - \epsilon/2 \\ &\geq \text{Hdim}\,(\Lambda(\mathcal{F})) - \epsilon \quad . \end{aligned}$$

■

Remark: We note that in the proof of this theorem the imposition of the most restrictive condition, namely semi-conformality, arises not from our limited knowledge of the calculus of distortions in higher dimension, but from the absence of methods for calculating the Hausdorff dimension for sets in $\mathbb{R}^n$ with $n > 1$. Admittedly there are some methods that apply to certain affinely generated sets (see [9]) but these cannot be used here. It appears likely that the semi-continuity of the theorem holds in a wider context; that is, that the requirement of semi-conformality can be relaxed or dropped altogether. To address this problem, a deeper knowledge of the function Hdim is required. Conceivably, this type of result is more readily proved using a different definition of the dimension.

References

[1] K. J. Falconer, *Fractal Geometry and its Applications*, John Wiley, 1990.

[2] A. Friedman *Introduction to Modern Analysis*, Dover, 1973.

[3] J. Graczyk, L. Jonker, G. Swiatek, F. M. Tangerman, J. J. P. Veerman, *Scalings in Circle Maps (III)*, Comm. Math. Phys. 173, 599-622, 1995.

[4] K. Gröchenig, A. Haas, *Self-similar Lattice Tilings*, J. Fourier An. Appl. 1, 131-170, 1994.

[5] J. E. Hutchinson, *Fractals and Self-Similarity*, Indiana Univ. Math. J., 30, No 5, 713-747, 1981.

[6] J. C. Lagarias, Y. Wang, *Integral Self-Affine Tilings in* $\mathbb{R}^n$ *I*, J. Lond. Math. Soc. (2), 54, 161-179, 1996.

[7] J. C. Lagarias, Y. Wang, *Tiling the Line with Translates of One Tile*, Invent. Math. 124, 341-365, 1996.

[8] H. McCluskey, A. Manning, *Hausdorff Dimension of Horseshoes*, Erg. Th. and Dyn. Sys. 3, 251-260, 1983.

[9] C. McMullen, *The Hausdorff Dimension of General Sierpinski Carpets*, Nagoya Math. J. 96, 1984, 1-9.

[10] W. De Melo, S. Van Strien, *One Dimensional Dynamics*, Springer Verlag, 1993.

[11] A. M. Odlyzko, *Non-Negative Digit Sets in Positional Number Systems*, Proc. London Math. Soc., 3rd series, 37, 213-229, 1978.

[12] J. Palis, M. Viana, *On the Continuity of the Hausdorff Dimension*, Lecture Notes in Math. 1331, 150-161, 1988.

[13] M. Pollicott, K. Simon, *The Hausdorff Dimension of* λ*-expansions with Deleted Digits*, Trans. A.M.S., 347, 967-983, 1995.

[14] Y. Pesin and H. Weiss, *On the Dimension of Deterministic and Random Cantor-like sets, Symbolic Dyunamics, and the Eckmann-Ruelle conjecture*, Preprint, Department of Mathematics, the Pennsylvania State University.

[15] D. Ruelle, *Bowen's Formula for the Hausdorff Dimension of Self-Similar Sets*, Erg. Th. Dyn. Sys., vol2, part 1, 99-109.

[16] D. Ruelle, *Bowen's Formula for the Hausdorff Dimension of Self-Similar Sets*, Progress in Physics 7 ("Scaling and Self-Similarity"), Birkhauser, 351-359, 1983.

[17] T. Sauer, J. A. Yorke, M. Casdagli, *Embedology*, J. Stat. Phys 65, 579-616, 1991.

[18] B. Solomyak, *On the Random Series* $\sum \pm\lambda^n$ *(an Erdös Problem)*, Ann. of Math., 142, 611-625, 1995.

[19] J. Stark, P. Bressloff, *Iterated Function Systems and their Applications*, in *Wavelets, Fractals and Fourier Transforms*, 65-90, Inst. Math. Appl. Conf. Ser., New Series 43, Oxford Univ. Press, New York, 1993

[20] F. Takens, *Limit Capacity and Hausdorff Dimension of Dynamically Defined Sets*, Lecture Notes in Math. 1331, 197-211, 1988.

[21] J. J. P. Veerman, *Two-Dimensional Generalizations of Haar Bases*, to appear, 1996.

[22] J. J. P. Veerman, *Intersecting Self-Similar Cantor Sets*, Bol. Soc. Bras. Mat. 26, 167-181, 1995.

[23] J. J. P. Veerman, *The Dynamics of Well-Ordered Orbits*, Lecture Notes, University of Barcelona, 1995.

[24] L. S. Young, *Dimension, Entropy, and Lyapunov Exponents*, Erg. Th. Dyn. Sys., vol 2, part 2, 109-124, 1982.