# A Compactification of Hénon Mappings in $\mathbf{C}^2$ as Dynamical Systems

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## I. Introduction

Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be a Hénon mapping:

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}, \; a \neq 0$$

where $p$ is a polynomial of degree $d \geq 2$, which without loss of generality we may take to be monic.

In [HO1], it was shown that there is a topology on $\mathbb{C}^2 \sqcup S^3$ homeomorphic to a 4-ball such that the Hénon mapping extends continuously. That paper used a delicate analysis of some asymptotic expansions, for instance, to understand the structure of forward images of lines near infinity. The computations were quite difficult, and it is not clear how to generalize them to other rational maps.

In this paper we will present an alternative approach, involving blow-ups rather than asymptotics. We apply it here only to Hénon mappings and their compositions, but the method should work quite generally, and help to understand the dynamics of rational maps $f : \mathbb{P}^2 \rightsquigarrow \mathbb{P}^2$ with points of indeterminacy. The application to compositions of Hénon maps proves a result suggested by Milnor, involving embeddings of solenoids in $S^3$ which are topologically different from those obtained from Hénon mappings. In the papers [Ves], [HHV], the method is applied to some other families of rational maps.

The approach consists of three steps, which we describe below.

–**Resolving points of indeterminacy** The general theory asserts that a "rational map" $f : \mathbb{P}^2 \rightsquigarrow \mathbb{P}^2$ is defined except at finitely many points, and that after a finite number of blow-ups at these points, the map becomes well-defined [S2, IV.3, Thm. 3]. Let us denote by $\tilde{X}_f$ the space obtained after these blow-ups, and $\tilde{f} : \tilde{X}_f \to \mathbb{P}^2$ the lifted morphism. For Hénon mappings, this is done in Section III. Section II is an introduction to blow-ups. Every book on algebraic geometry ([Har] for instance) has a treatment of this subject, but we are aiming the paper at people who work in Dynamical Systems, and who will often find this literature inpenetrable; in any case the theory is usually stated in terms of the functor **Proj** and we find that computations are quite difficult in that language. So we have included the definitions and basic properties, as well as some fundamental examples.

–**The complex sequence space** The mapping $\tilde{f}$ cannot be considered a dynamical system, since the domain and range are different. One way to obtain a dynamical system

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is to blow up the inverse images of the points you just blew up to construct $\tilde{X}_f$, then their inverse images, etc. On the the projective limit, we finally obtain a dynamical system $f_\infty : X_\infty \to X_\infty$.

The notation to keep track of successive blow-ups grows exponentially and soon becomes intractable. There is an alternative description of the projective limit in terms of sequence spaces, which is much easier to describe. We learned of this construction from [F1]; something analogous was constructed by Hirzebruch [Hirz] when resolving the cusps of Hilbert modular surfaces, and also considered by Inoue, Dloussky and Oeljeklaus [Dl], [DO].

The space $X_\infty$ will usually have a big subset $X_\infty^*$ which is an algebraic variety, but some points which are extremely singular, in the sense that every neighborhood has infinite dimensional homology.

We will construct $X_\infty$ for Hénon mappings in Section IV, and its topology (homology, etc.) is studied in Section V. There is a natural way to complete the homology $H_2(X_\infty^*, \mathbb{C})$ to a Hilbert space, so that $f_\infty$ induces an operator, and we hope in a future publication to show that the spectral invariants of this operator interacts with the dynamics.

–**The real oriented blow-up** In order to "resolve" the terrible singularities of $X_\infty$, we will use a further *real* blow-up, in which we consider complex surfaces as 4-dimensional real algebraic spaces, and divisors as real surfaces in them. One way of thinking of this blow-up of a surface $X$ along a divisor $D$ is to take an open tubular neighborhood $W$ of $D$ in $X$, together with a projection $\pi : \overline{W} \to D$. Excise $W$, to form $X' = X - W$. If $W$ is chosen properly, $X'$ will be a real 4-dimensional manifold with boundary $\partial X' = \partial \overline{W}$. The interior of $X'$ will be homeomorphic to $X - D$. Then $X'$ is some sort of blow-up of $X$ along $D$, with $\pi : \partial X' \to D$ the exceptional divisor. This construction is topologically correct, but non-canonical; the real oriented blow-up is a way of making it canonical.

We can pass to the projective limit with these real oriented blow-ups, constructing a space $\mathcal{B}_{\mathbb{R}}^+(X_\infty)$, which is topologically much simpler and better behaved than the original compactification, being a 4-dimensional manifold with boundary. The real interest is in the inner structures of the boundary 3-manifold, where we find solenoids (in this paper and in [Ves]), tori with irrational foliations (in [HHV]), etc.

The definition and first properties of these blow-ups are given in Section VI, and the methods needed to construct them are given in Section VII. Theorems VII.5 and VII.9 are the principal tools for understanding real oriented blow-ups, and we expect that they will be useful for many examples besides the ones explored in this paper.

In Section VIII, we show that the classical Hopf fibration is an example of a real oriented blow-up, in two different ways. In Section IX, we construct the real oriented blow-up for Hénon maps. This turns out to be quite an exciting space, and we further explore its structure in Section X, using toroidal decompositions. These results allow us to prove (Theorem X.8) that extensions of the Hénon maps to their sphere at infinity are not all conjugate, even when they have the same degree; the conjugacy class of the extension remembers the argument of the Jacobian of the Hénon mapping.

In Section XI, we construct the real-oriented blow-up for compositions of Hénon maps, obtaining a 3- sphere with two embedded solenoids $\Sigma^+$ and $\Sigma^-$, but such that the incompressible tori in $S^3 - (\Sigma^+ \cup \Sigma^-)$ are different from the incompressible tori we obtain from just one Hénon mapping. This difference was first conjectured by Milnor [Mi2].

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**Outline of the paper.**



## II. Blowing up a closed subspace of an analytic space

**Hypersurfaces and divisors.**

Much of this paper will be about making appropriate blow-ups. In Section III we will simply be blowing up surfaces at points, but in Section VI we will need to make much more elaborate blow-ups, of singular surfaces in 4-dimensional manifolds. Everything in this section is presumably standard, but we found it difficult to carry out our computations in the too elegant language of **Proj**, and we find the present treatment, in terms of affine equations, better adapted to our needs. The reader who feels comfortable with the simplest blow-ups, of surfaces at points, should go directly to the next section, and refer back when needed (which may well occur in Section VI). It seems that in the generality in which we will be describing the construction, the results of this section are due to Hironaka [Hir].

Let $X$ be an analytic space with structure sheaf $\mathcal{O}_X$, and $Y \subset X$ a closed subspace. The definition of the blow-up $\tilde{X}_Y$ of $X$ along $Y$ requires carefully distinguishing between hypersurfaces and divisors.

**Definition II.1.** A hypersurface in $X$ is a subspace defined by an ideal which locally requires only one generator; i.e., it is the locus locally defined by a single equation.

**Definition II.2.** A divisor is a hypersurface locally defined by a single equation which is not a zero divisor. More precisely, a subspace $Z \subset X$ is a divisor if every point $x \in X$ has a neighborhood $U$ such that $Z \cap U$ is defined by the principal ideal generated by some section $f \in \mathcal{O}_X(U)$ which is not a zero-divisor in the ring $\mathcal{O}_X(U)$.

Consider for instance $X = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C} \subset \mathbb{C}^2$, or alternately, the subspace of $\mathbb{C}^2$ defined by the equation $xy = 0$. The $x$-axis alone is a hypersurface in $X$, defined by the equation $y = 0$, but it is not a divisor, since $y$ is a zero divisor in $\mathcal{O}_X(X) = \mathbb{C}[x, y]/(xy)$. This example is typical: geometrically, saying that a function on a space is a zero divisor is saying that the space has several components, and that the function vanishes identically on some of them.

**Definition of the blow-up.**

Now we can define $\tilde{X}_Y$ as the "universal way" of replacing $Y$ by a divisor.

**Definition II.3.** The blow-up $\tilde{X}_Y$ of an analytic space $X$ along a closed subspace $Y$ is an analytic space $\tilde{X}_Y$, which comes with a morphism $\pi : \tilde{X}_Y \to X$ (called the canonical projection), such that $\pi^{-1}(Y)$ is a divisor (called the exceptional divisor) in $\tilde{X}_Y$, and if $g : Z \to X$ is a morphism such that $g^{-1}(Y)$ is a divisor in $Z$, there exists a unique morphism $\tilde{g} : Z \to \tilde{X}_Y$ such that $\pi \circ \tilde{g} = g$.

We will prove that the blow-up exists (and is then obviously unique up to unique isomorphism). If we can construct the blow-up locally, and prove that it has the right universal property, then the existence of a global blow-up will follow, since the local blow-ups will patch uniquely.

**Local construction of a blow-up.**

So we may assume that $Y \subset X$ is defined by the equation $Y = f^{-1}(0)$, where $f : X \to \mathbb{C}^{n+1}$ is an analytic mapping (i.e., a section of $\mathcal{O}_X^{n+1}$).

We define the blow-up $\tilde{X}_Y$ as follows. Recall that $\mathbb{P}^n$ is the space of lines $l \subset \mathbb{C}^{n+1}$ through the origin, and that it carries the tautological line bundle

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr] & & \mathbb{C}^{n+1} \times \mathbb{P}^n \arrow[dl] \\ & \mathbb{P}^n & \end{tikzcd}$$

where $E = \left\{(x, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \,|x \in l\right\}$.

First consider the locus $X'_Y \subset X \times \mathbb{P}^n$ defined as

$$X'_Y = \{(x, l) \in X \times \mathbb{P}^n \,|f(x) \in l\}\,.$$

This sounds set-theoretic rather than ideal theoretic, but we can easily define $X'_Y$ by equations. Let $[U_0 : \cdots : U_n]$ be homogeneous coordinates on $\mathbb{P}^n$. Then $X'_Y$ is defined by the ideal (a mixed ideal, homogeneous with respect to the $U_i$, affine with respect to the $f_j$), generated by all

$$U_i f_j - U_j f_i \tag{2.1}$$

for $i \neq j$.

Let $\pi : X'_Y \to X$ be induced by the projection onto the first factor. We now will see that locally $\pi^{-1}(Y)$ is locally defined by a single equation.

Let $E \to \mathbb{P}^n$ be the tautological line bundle, and $F \to X'_Y$ be the pull-back of $E$ by the composition

$$X'_Y \to X \times \mathbb{P}^n \to \mathbb{P}^n.$$

This leads to the following commutative diagram:

$$\begin{array}{ccccc}
 & & X \times \mathbb{C}^{n+1} \times \mathbb{P}^n & & \mathbb{C}^{n+1} \times \mathbb{P}^n \\
 & & \cup & & \cup \\
F & \hookrightarrow & X \times E & \to & E \\
\downarrow & & \downarrow & & \downarrow \\
X'_Y & \hookrightarrow & X \times \mathbb{P}^n & \to & \mathbb{P}^n
\end{array}$$

where both squares are fiber products.

Therefore the line bundle $F$ is a subbundle of the trivial bundle $X'_Y \times \mathbb{C}^{n+1} \to X'_Y$, and the map $X'_Y \to \mathbb{C}^{n+1}$ induced by $f$ is a section $f'$ of $F$, so that locally on $X'_Y$ the set $Y'$ defined by $f' = 0$ is a hypersurface.

More precisely, let us denote by $f'' : X'_Y \to \mathbb{C}^{n+1}$ the composition

$$X'_Y \to X \times \mathbb{P}^n \to X \to \mathbb{C}^{n+1}.$$

In the chart $U_i \neq 0$ on $\mathbb{P}^n$, with affine coordinates $u_j = U_j/U_i,\ j \neq i$, a non-vanishing section of the tautological bundle is given by

$$\sigma_i : \begin{bmatrix} u_0 \\ \vdots \\ u_{i-1} \\ u_{i+1} \\ \vdots \\ u_n \end{bmatrix} \mapsto \begin{bmatrix} u_0 \\ \vdots \\ u_{i-1} \\ 1 \\ u_{i+1} \\ \vdots \\ u_n \end{bmatrix}$$

with the 1 in the $i$th position. This section lifts and restricts to a section $\tau_i$ of $F$ above $X'_Y \cap \{U_i \neq 0\}$. Evidently, $f' = \tau_i f''_i$ in this chart, where $f''_i$ is the $i$th coordinate of $f''$, so $Y'$ is given by the single equation $f''_i = 0$ in this chart.

However, in even moderately complicated cases, the space $X'_Y$ has parasitic components on which $f'$ vanishes identically, so that $f'$ is a zero divisor, and we must get rid of these components.

Let $\mathcal{A}_{f'} \subset \mathcal{O}_{X'_Y}$ be the sheaf of ideals generated by functions which are annihilated by some power of $f'$, and let $\tilde{X}_Y \subset X'_Y$ be the subspace defined by $\mathcal{A}_{f'}$. This annihilator is the ideal of functions which vanish identically on the components of $X'_Y$ on which no power of $f'$ vanishes identically, so $\tilde{X}_Y$ is obtained from $X'_Y$ by removing the "parasitic components" on which some power of $f'$ vanishes identically. In the best cases, $\mathcal{A}_{f'}$ will be the zero sheaf of ideals, so $\tilde{X}_Y = X'_Y$, but in other cases, $\tilde{X}_Y$ will be exactly the union of the components of $X'_Y$ on which no power of $f'$ vanishes identically. Thus the restriction $\tilde{f}$ of $f'$ to $\tilde{X}_Y$ is not a zero divisor, and the subspace $\tilde{Y} \subset \tilde{X}_Y$ defined by $\tilde{f}$ is a divisor.

We will prove in II.7 that the space $\tilde{X}_Y$ is the blow-up of $X$ along $Y$, and that $\tilde{Y}$ is the exceptional divisor. First we will give some examples.

**Blowing up a point in a surface.**

The easiest example of a blow-up, and also the only one we will use until Section VI, is the blow-up of a surface at a point. Because of the universal property, it is enough to understand the blow-up in one chart, i.e., to understand the blow-up of $\mathbb{C}^2$ at the origin.

**Example II.4.** Let $x, y$ be the coordinates of $\mathbb{C}^2$; the origin is of course defined by the equations $x = y = 0$, so the blow-up is contained in $\mathbb{C}^2 \times \mathbb{P}^1$. If we use $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ as homogeneous coordinates in $\mathbb{P}^1$, then the equation expressing that the point $\begin{bmatrix} x \\ y \end{bmatrix}$ is on the line corresponding to $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ is $xU_1 = yU_0$. This equation is not a zero-divisor, so we have computed the blow-up.

This is covered by two affine coordinate charts:

–One in which $U_1 \neq 0$ and $u_0 = U_0/U_1$; in this chart the blow-up is given by the equation $x = yu_0$; clearly it is a non-singular surface parametrized by $y$ and $u_0$. The exceptional divisor is given in this chart by the single equation $y = 0$.

–One in which $U_0 \neq 0$ and $u_1 = U_1/U_0$; in this chart the blow-up is given by the equation $y = xu_1$; clearly it is a non-singular surface parametrized by $x$ and $u_1$. The exceptional divisor is given in this chart by the single equation $x = 0$.

So in this case the effect of blowing up is to replace the origin by the projective line $\mathbb{P}^1_{\mathbb{C}}$. More generally, a similar computation will show that if you blow up a smooth manifold $X$ along a smooth submanifold $Y \subset X$, then the fibers $\pi^{-1}(x)$ of the projection $\pi : \tilde{X}_Y \to X$ are points if $x \notin Y$, and the projective space $\mathbb{P}(T_xX/T_xY)$ associated to the "normal space" $T_xX/T_xY$ if $x \in Y$.

**Two more complicated examples.**

**Example II.5.** Let $X = \mathbb{C}^2$, and $Y$ be the subset defined by the equations $\{x^2 = 0, y = 0\}$. Clearly in this case $n = 1$, and the space $X'_Y$ is the subspace of $X \times \mathbb{P}^1$, given by the equation

$$x^2 U_1 = y U_0,$$

where $[U_0 : U_1]$ are homogeneous coordinates on $\mathbb{P}^1$.

In the chart where $U_0 \neq 0$, if you set $u_1 = U_1/U_0$, the space $X'_Y$ is defined by the equation $x^2 u_1 = y$, and $Y'$ is defined by the single equation $x^2 = 0$. In the chart where $U_1 \neq 0$ and $u_0 = U_0/U_1$ is an affine coordinate, $X'_Y$ is the space defined by the equation $y u_0 = x^2$, and $Y'$ is defined by the single equation $y = 0$. Since neither equation is a zero-divisor, we see that $\tilde{X}_Y = X'_Y$, and $\tilde{Y} = Y'$. In particular, the blow-up is a cone, with a singular point at the vertex $x = y = u_0 = 0$, and $\tilde{Y}$ is a "double line" on that cone.

Our next example is a special case of the following general fact. Let $X$ be an analytic space, $Y \subset X$ be a closed analytic subspace defined by an ideal $\mathcal{I}_Y$, and denote by $pY$ the subspace defined by $\mathcal{I}^p$. Then $\tilde{X}_{pY} = \tilde{X}_Y$, and the exceptional divisor in $\tilde{X}_{pY}$ is $p\tilde{Y}$, i.e., if $\tilde{Y}$ is locally defined by the single equation $f = 0$, then the exceptional divisor of $\tilde{X}_{pY}$ is locally defined by $f^p = 0$. This fact is easy to check from the universal property, but considerably harder to see from the explicit computation, even in the case when $X = \mathbb{C}^2$ and $Y$ is the origin.

**Example II.6.** Let $X = \mathbb{C}^2$, $\mathcal{J} \subset \mathcal{O}_X$ be the ideal generated by $\{x, y\}$, consider the subset $Y$ defined by the ideal $\mathcal{J}^2$, or alternately $Y = f^{-1}(0)$, where

$$f : \mathbb{C}^2 \to \mathbb{C}^3 \quad \text{is given by} \quad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}.$$

This time we required three equations, so that $X'_Y \subset X \times \mathbb{P}^2$ is given (in homogeneous coordinates $[U_0 : U_1 : U_2]$ on $\mathbb{P}^2$) by the equations $x^2 U_1 = xy U_0, x^2 U_2 = y^2 U_0$ and $xy U_2 = y^2 U_1$.

In the chart $U_0 \neq 0$, using the coordinates $u_1, u_2$ as above, $X'_Y$ is defined by the equations

$$x^2 u_1 = xy \quad , \quad x^2 u_2 = y^2 \quad , \quad xy u_2 = y^2 u_1,$$

and it is easy to see that the third equation is a consequence of the first two.

Further, $Y'$ is defined in this chart by the single equation $x^2 = 0$. In this case, $x^2$ is a zero-divisor in $\mathcal{O}_{X'_Y}$, since $x^2(u_1 y - u_2 x) = 0$.

Further computations, in which we were helped by J.-P. Soublin (by hand) and A. Sausse (using REDUCE), shows that $X'_Y$ has two primary components, one corresponding to the ideal $Q$ generated by the seven polynomials

$$x^2 U_1 - xy U_0 \ , \ x^2 U_2 - y^2 U_0 \ , \ xy U_2 - y^2 U_1 \ , \ x^3 \ , \ x^2 y \ , \ xy^2 \ , \ y^3$$

with radical $(x, y)$, and the other to the ideal $P$ generated by

$$U_1^2 - U_0U_2\ ,\ U_0y - U_1x\ ,\ U_2x - U_1y,$$

which is its own radical. The annihilator of $f'$ is exactly $P$. So $\tilde{X}_Y$ is the locus in $\mathbb{C}^2 \times \mathbb{P}^2$ defined by the mixed ideal $P$. This is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin defined by $\mathcal{J}$. Indeed, consider the mapping $X \times \mathbb{P}^1 \to X \times \mathbb{P}^2$ given in homogeneous coordinates by

$$(x, y, V_0, V_1) \mapsto (x, y, V_0^2, V_0V_1, V_1^2).$$

The equation $U_0U_2 = U_1^2$ defines the image of this embedding, and is one of the equations of $\tilde{X}_Y$, so the inverse image of $\tilde{X}_Y$ is defined by the mixed ideal generated by

$$V_0^2y - V_0V_1x \quad , \quad V_1^2x - V_0V_1y,$$

which is the same as the principal ideal generated by $V_0y - V_1x$. But this ideal also defines the blow-up of $\mathbb{C}^2$ at the origin defined by the ideal $\mathcal{J}$. The proper transform of $Y$, i.e., the locus defined by $\tilde{f}$, is the exceptional divisor $x = y = 0$ as a double line.

**Proof of the universal property.**

We must now prove that our blow-up $\tilde{X}_Y$ has the right universal property.

**Theorem II.7.** *If $g : Z \to X$ is a mapping such that $g^{-1}(Y)$ is a divisor in $Z$, then there exists a unique morphism $\tilde{g} : Z \to \tilde{X}_Y$ such that $\pi \circ \tilde{g} = g$.*

**Proof.** By definition, $g^{-1}(Y)$ is the locus defined by the equation $f \circ g = 0$. On the other hand, locally on $Z$, $g^{-1}(Y)$ is defined by a single equation $h = 0$. More precisely, any point $z_0 \in Z$ has a neighborhood $V \subset Z$ in which $g^{-1}(Y) \cap V$ is defined by the ideal $h\mathcal{O}_V$, where $h$ is a function, not a zero-divisor. Since each $g \circ f_i$ is in the ideal generated by $h$, we can find functions $\phi_i$ such that $g \circ f_i = \phi_i h$. Moreover, the $\phi_i$ do not all vanish at any point of $V$ since they generate the unit ideal. So we can find a morphism $\tilde{g}_V : V \to X \times \mathbb{P}^n$ by the formula

$$\tilde{g}_V(z) = (g(z), [\phi_0(z) : \cdots : \phi_n(z)]).$$

First, observe that this morphism does not depend on the choice of $V$ and $h$. If $(V', h')$ and $(V'', h'')$ are two open sets in which $g^{-1}(Y)$ are defined by a single function, then on $V' \cap V''$, we have that $h' = h''u$, where $u$ is invertible (i.e., a unit in $\mathcal{O}(V' \cap V'')$). With the obvious notation,

$$(\phi_0', \ldots, \phi_n') = u(\phi_0'', \ldots, \phi_n''),$$

so the corresponding homogeneous coordinates define the same mapping into $X \times \mathbb{P}^n$.

Thus we have a morphism $\tilde{g} : Z \to X \times \mathbb{P}^n$; we must show that the image is contained in $\tilde{X}_Y$.

First, it is clear that the image lies in $X'_Y$, or more precisely, that

$$\tilde{g}^*(a_i f_j - a_j f_i) = \phi_i(f_j \circ g) - \phi_j(f_i \circ g) = h((f_i \circ g)(f_j \circ g) - (f_j \circ g)(f_i \circ g)) = 0.$$

Next, if $\alpha \in \mathcal{A}_{f'}$, we need to know that $\tilde{g}^*\alpha = 0$. Since $\alpha(f')^m = 0$ for some $m$, we have

$$(\alpha \circ \tilde{g})(f' \circ \tilde{g})^m = 0,$$

which certainly implies that $f' \circ \tilde{g}$ is a zero-divisor if $\alpha \circ \tilde{g} \neq 0$. But $f' \circ \tilde{g}$ defines $f^{-1}(Y)$, which was assumed to be a divisor, so defined by a single equation, not a zero-divisor. $\square$

**Corollary II.8.** *If $f : X_1 \to X_2$ is a morphism of analytic spaces, $Y_2 \subset X_2$ is an analytic subspace, and $Y_1 = f^{-1}Y_2$, then there exists a unique morphism $\tilde{f} : (\tilde{X_2})_{Y_2} \to (\tilde{X_1})_{Y_1}$ which makes the diagram*

$$\begin{array}{ccc} (\tilde{X_2})_{Y_2} & \xrightarrow{\tilde{f}} & (\tilde{X_1})_{Y_1} \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_1 \end{array}$$

*commute.*

**Proof.** Clearly under the composition $(\tilde{X_2})_{Y_2} \to X_2 \to X_1$, the inverse image of $Y_1$ is the exceptional divisor, hence we can apply Theorem II.7. $\square$

We only defined $\tilde{X}_Y$ when $Y$ is defined by global equations. However, the universal mapping properties guarantees that if we perform blow-ups locally, they will glue together in a unique fashion, so that this actually constructs a global blow-up of an arbitrary analytic space along a closed subspace.

**An example in $\mathbb{C}^4$.**

Let us denote by $x_1, x_2, y_1, y_2$ the coordinates of $\mathbb{C}^4$. In Section 6, we will need to understand the blow-up $\mathbb{C}^4$ along the union

$$Y = \mathbb{C}^2 \times \{0\} \cup \{0\} \times \mathbb{C}^2$$

of the $(x_1, x_2)$ and the $(y_1, y_2)$ coordinate planes. This also provides an example where many of the complications of the previous sections occur and thus illustrates what all these zero-divisors and annihilators really mean.

Although $Y$ is of codimension 2, it requires four equations for its definition:

$$x_1y_1 = 0, x_2y_1 = 0, x_1y_2 = 0 \text{ and } x_2y_2 = 0,$$

i.e., $Y = f^{-1}(0)$ where $f : \mathbb{C}^4 \to \mathbb{C}^4$ is given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1y_1 \\ x_2y_1 \\ x_1y_2 \\ x_2y_2 \end{bmatrix}.$$

The fact that it is not defined by two equations will follow from the fact that the blow-up cannot be embedded in $X \times \mathbb{P}^1$, and the fact that four equations are really required will follow when we see that the blow-up cannot be embedded in $X \times \mathbb{P}^2$ either.

Since the ideal defining the union of the coordinate planes is generated by

$$x_1y_1, x_2y_1, x_1y_2, x_2y_2,$$

the variety $X'_Y \subset \mathbb{C}^4 \times \mathbb{P}^3$, is defined by the equation $f(x) \in \ell$, and if we use homogeneous coordinates $U_1, U_2, U_3, U_4$ on the second factor, this means that the vectors

$$\begin{bmatrix} x_1y_1 \\ x_2y_1 \\ x_1y_2 \\ x_2y_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

are linearly dependent. Although locally this locus is defined by three equations, globally it requires 6 equations:

$$\begin{aligned} x_1y_1U_4 &= x_2y_2U_1 \\ x_2y_1U_4 &= x_2y_2U_2 \\ x_1y_2U_4 &= x_2y_2U_3 \\ x_1y_1U_3 &= x_1y_2U_1 \\ x_2y_1U_3 &= x_1y_2U_2 \\ x_1y_1U_2 &= x_2y_1U_1. \end{aligned} \tag{2.2}$$

Let us denote by $\pi : X'_Y \to \mathbb{C}^4$ the projection induced by the projection $X \times \mathbb{P}^3 \to X$ onto the first factor. Locally, the locus $Y' = \pi^{-1}(Y)$ is defined by a single equation, as it should. For instance, in the chart $U_4 = 1$, the space $X'_Y$ is defined in $X \times \mathbb{P}^3$ by the 3 equations

$$\begin{aligned} x_1y_1 &= x_2y_2u_1 \\ x_2y_1 &= x_2y_2u_2\,, \\ x_1y_2 &= x_2y_2u_3 \end{aligned}$$

where $u_i = U_i/U_4, i = 1, 2, 3$, and $Y'$ is defined by the single equation $x_2y_2 = 0$. However, the functions

$$x_2y_2, x_2y_1, x_1y_1, x_1y_2$$

are zero-divisors in $\mathcal{O}(X'_Y)$: for instance,

$$x_2y_2(U_4y_1 - U_2y_2) = 0$$

in the ring of functions on $X'_Y$.

A careful analysis of equations (2.2) shows that $X'_Y$ is the union of five irreducible components, the four 4-dimensional linear spaces $Z_1, \cdots, Z_4$ with equations

$$\begin{aligned} x_1 = x_2 = y_1 = 0 \\ x_1 = y_1 = y_2 = 0 \\ x_1 = x_2 = y_2 = 0 \\ x_2 = y_1 = y_2 = 0 \end{aligned}$$

and the 4-dimensional space $Z_5$ with equation

$$\begin{aligned} U_1U_4 &= U_2U_3 \\ y_1U_4 &= y_2U_2 \\ x_1U_4 &= x_2U_3 \\ y_1U_3 &= y_2U_1 \\ x_1U_2 &= x_2U_1 \end{aligned} \tag{2.3}$$

We can now see that $\tilde{X}_Y$ is in fact $Z_5$. If we embed $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ by the Veronese mapping

$$\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right) \mapsto \begin{bmatrix} X_1Y_1 \\ X_2Y_1 \\ X_1Y_2 \\ X_2Y_2 \end{bmatrix},$$

where $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ are homogeneous coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$, we observe that the image has equation $U_1U_4 = U_2U_3$, which is satisfied identically on $\tilde{X}_Y$. Thus the blow-up is actually contained in $X \times (\mathbb{P}^1)^2$, and it is defined by the equations

$$\begin{aligned} y_1X_2Y_2 &= y_2X_2Y_1 \\ x_1X_2Y_2 &= x_2X_1Y_2 \\ y_1X_1Y_2 &= y_2X_1Y_1 \\ x_1X_2Y_1 &= x_2X_1Y_1. \end{aligned} \tag{2.4}$$

From the first equation we can see that either $X_2 = 0$ or $Y_2y_1 = Y_1y_2$. But if $X_2 = 0$, then $X_1 \neq 0$, so we can cancel $X_1$ in the third equation and get the same equation $Y_2y_1 = Y_1y_2$. Therefore the first and third equation are equivalent to the equation $Y_2y_1 = Y_1y_2$.

We can apply the same procedure to the second and fourth equations and see that they are equivalent to $X_2 x_1 = X_1 x_2$. Thus equations (2.4) are equivalent to equations

$$\begin{aligned} y_1 Y_2 &= y_2 Y_1 \\ x_1 X_2 &= x_2 X_1 \end{aligned} \tag{2.5}$$

The last two equations show that $\tilde{X}_Y$ is a manifold. For instance, in the chart $X_2 = Y_2 = 1$, $\tilde{X}_Y$ is given by the equations $x_1 = X_1 x_2$ and $y_1 = Y_1 y_2$, clearly parametrized by $x_2, y_2, X_1, Y_1$, and $\tilde{Y} = \pi^{-1}(Y)$ in this chart is given by the single equation $x_2 y_2 = 0$. We see in particular that the exceptional divisor $\tilde{Y}$ is not smooth, above $(0,0,0,0)$, it has the local structure of $W \times \mathbb{C}^2$, where $W \subset \mathbb{C}^2$ is the singular curve with equation $xy = 0$, i.e., the union of the axes.

## III. Making Hénon Mappings well-defined

Consider the Hénon mapping

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix} \tag{3.1}$$

with $a \neq 0$, which we will consider as a birational mapping $\mathbb{P}^2 \rightsquigarrow \mathbb{P}^2$ given in homogeneous coordinates as

$$H \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tilde{p}(x,z) - ayz^{d-1} \\ xz^{d-1} \\ z^d \end{bmatrix}, \tag{3.2}$$

where $\tilde{p}(x,z) = z^d p(x/z) = x^d + \dots$ is a homogeneous polynomial of degree $d$ in the two variables $x$ and $z$.

**Lemma III.1.** *a) The mapping $H$ has a unique point of indeterminacy at $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and collapses the line at infinity $l_\infty$ to the point $\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.*

*b) The mapping $H^{-1}$ has a unique point of indeterminacy at $\mathbf{q}$, and collapses $l_\infty$ to the point $\mathbf{p}$.*

**Proof.** A point of indeterminacy of a mapping written in homogeneous coordinates without common factors is a point where all coordinate functions vanish. In order for this to happen, we must have $z = 0$ of course, and the only remaining term is then $x^d$, so that at a point of indeterminacy, we also have $x = 0$. Thus $\mathbf{p}$ is the unique point of indeterminacy of $H$. Clearly any other point of $l_\infty$ is mapped to $\mathbf{q}$. Part (b) is similar. $\square$

**Self-intersections.** It is often necessary to know the self-intersection numbers of lines obtained when you make successive blow-ups. The rules for computing these numbers are simple, in this case where we are blowing up a surface at a smooth point $\mathbf{x}$ [S2, IV.3, Thm. 2 and its Corollaries]:

- The exceptional divisor has self-intersection $-1$;

- The proper transform $C'$ of any smooth curve $C$ passing through $\mathbf{x}$ has its self-intersection decreased by 1.

We will now go through a sequence of $2d-1$ blow-ups, required to make the Hénon mapping (3.2) well-defined. The results are summarized at the end in Theorem III.3, which we cannot state without the terminology which we create during the construction. We will denote $H_1, \dots, H_{2d-1}$ the extension of the Hénon mapping to the successive blow-ups.

To focus on the point of indeterminacy, we will begin work in the coordinates $u = x/y, v = 1/y$ so that the point of indeterminacy $\mathbf{p}$ is the point $u = v = 0$. The Hénon mapping, written in the affine coordinates $u, v$ in the domain and homogeneous coordinates in the range, is written

$$H : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{bmatrix} \tilde{p}(u,v) - av^{d-1} \\ uv^{d-1} \\ v^d \end{bmatrix}.$$

At a point of indeterminacy all three homogeneous coordinates vanish; we already knew that this happens only at $\mathbf{p}$, but it is clear again from this formula that it happens only at $u = v = 0$.

**The first blow-up.** Blow up $\mathbb{P}^2$ at the point $u = v = 0$, look in the chart $v = X_1 u$; i.e., use $u$ and $X_1$ as coordinates and discard $v$.

The extension of the Hénon mapping, written in the affine coordinates $u, X_1$ in the domain and homogeneous coordinates in the range, is written

$$H_1 : \begin{pmatrix} u \\ X_1 \end{pmatrix} \mapsto \begin{bmatrix} \tilde{p}(u, X_1 u) - a(X_1 u)^{d-1} \\ u^d X_1^{d-1} \\ u^d X_1^d \end{bmatrix} = \begin{bmatrix} uq(X_1) - a{X_1}^{d-1} \\ uX_1^{d-1} \\ u{X_1}^d \end{bmatrix}, \tag{3.3}$$

where we have set $\tilde{p}(1, X) = q(X)$, so that $q$ is a polynomial of degree $d$ whose constant term is 1.

Again, the only point where all three homogeneous coordinates vanish is where $u = X_1 = 0$. We invite the reader to check that the one point of the blow-up not covered by the chart $u, X_1$ is not a point of indeterminacy. The self-intersection numbers are as indicated in Figure 1: originally $l_\infty$ has self-intersection number 1, after one blow-up its proper transform acquires self-intersection number 0, and the exceptional divisor has self-intersection number -1.

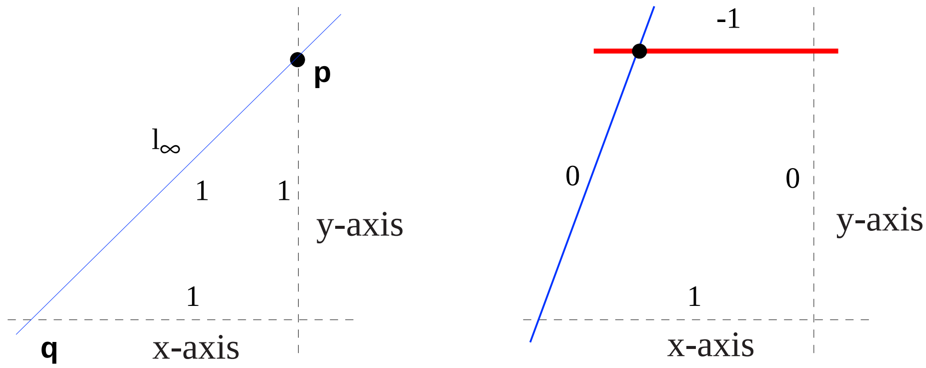


FIGURE 1. The original configuration of the axes (dotted) and the line at infinity in $\mathbb{P}^2$, and the configuration after the first blow-up. The last exceptional divisor is denoted by a thick line; the heavy dots are the points of indeterminacy of $H$ and $H_1$. The numbers labeling components are self-intersection numbers.

**Several more blow-ups.** We will now make a sequence of $d-1$ further blow-ups, setting successively

$$u = X_1 X_2,\ X_2 = X_1 X_3,\ \dots,\ X_{d-1} = X_1 X_d.$$

The Hénon mapping in these coordinates is given by the formula

$$H_k : \begin{pmatrix} X_1 \\ X_k \end{pmatrix} \mapsto \begin{bmatrix} X_1 X_k q(X_1) - a X_1^{d-k+1} \\ X_1^d X_k \\ X_1^{d+1} X_k \end{bmatrix} = \begin{bmatrix} X_k q(X_1) - a X_1^{d-k} \\ X_1^{d-1} X_k \\ X_1^d X_k \end{bmatrix} \tag{3.4}$$

and we see that when $k < d$, the unique point of indeterminacy is the point $X_1 = X_k = 0$, but when $k = d$, the unique point of indeterminacy is the point $X_1 = 0, X_d = a$. At each step, there is one point $X_k = \infty$ which is not in the domain of our chart; we leave it to the reader to check that this is never a point of indeterminacy.

Again the self-intersection numbers are as indicated in Figure 2. At each step we are blowing up the intersection of the last exceptional divisor with the proper transform of the first exceptional divisor. Therefore after this sequence of blow-ups, the last exceptional divisor (now next-to last) acquires self-intersection -2, and the first exceptional divisor has its self-intersection number decreased by 1, going from $-1$ to $-d$.

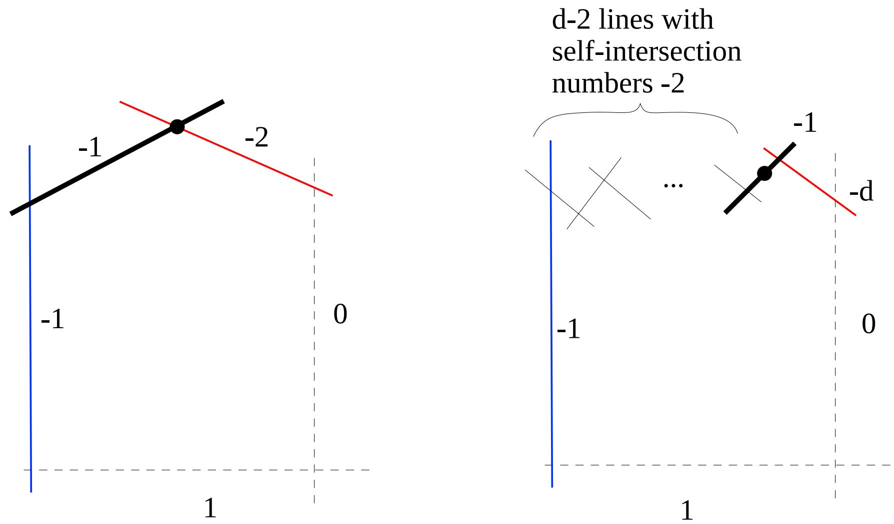


FIGURE 2. The configuration after the second and after the $d$-th of the blow-up. The heavy dots are the points of indeterminacy of $H_2$ and $H_d$; note that the dot on the right is an ordinary point; all the earlier ones except the very first were double points. Again, the numbers labeling components are the self-intersection numbers.

**The blow-ups which depend on the coefficients of $p$.**

The next $d-1$ blow-ups, although not really more difficult than the earlier ones, have rather more unpleasant formulas, because each occurs at a smooth point of the last exceptional divisor, and we need to specify this point. To lighten the notation, we will define by induction the polynomials

$$q_0(X) = q(X) \quad \text{and} \quad q_{k+1}(X) = \frac{q_k(X) - q_k(0)}{X}, \quad k = 1, \dots$$

and the numbers $Q_k$ (they are really coordinates of points on exceptional divisors)

$$Q_0 = 1 \quad \text{and} \quad Q_{k+1} = -\sum_{j=0}^{k} Q_j q_{k-j+1}(0).$$

Set $Y_0 = X_d$, and make the successive blow-ups

$$Y_k - aQ_k = X_1 Y_{k+1}, \; k = 0, \dots, d-2. \tag{3.5}$$

**Remark.** This means that $Y_{k+1}$ is the slope of a line through the point $X_1 = 0, Y_k = aQ_k$. In that sense the number $Q_k$ (or rather $aQ_k$), is really the coordinate on the line parametrized by $Y_k$ of the next point at which to blow up.

**Lemma III.2.** *a) In the coordinates $X_1, Y_k$, the Hénon mapping is given by the formula*

$$H_{d+k} : \begin{pmatrix} X_1 \\ Y_k \end{pmatrix} \mapsto \begin{bmatrix} X_1 Y_k q_1(X_1) + a\sum_{j=0}^{k-1} Q_j q_{k-j}(X_1) + Y_k \\ X_1^{d-k-1}(X_1^k Y_k + a\sum_{j=0}^{k-1} Q_j X_1^j) \\ X_1^{d-k}(X_1^k Y_k + a\sum_{j=0}^{k-1} Q_j X_1^j) \end{bmatrix}, \; k = 1, \dots, d-1. \quad (3.6)$$

*b) The mapping $H_{k+d}$ has the unique point of indeterminacy $X_1 = 0,\ Y_k = aQ_k$, for $k = 1, \dots, d-2$.*

*c) The mapping $H_{2d-1}$ has no point of indeterminacy, and maps the last exceptional divisor to $l_\infty \subset \mathbb{P}^2$ by an isomorphism.*

**Proof.** This is an easy induction: all the work was in finding the formula. To start the induction, compute the extension of the Hénon mapping in the chart $X_d - a = X_1 Y_1$ (using formula (3.4):

$$H_{d+1} : \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \mapsto \begin{bmatrix} (a + X_1 Y_1)(q(X_1) - 1) - X_1 Y_1 \\ X_1^{d-1}(a + X_1 Y_1) \\ X_1^d(a + X_1 Y_1) \end{bmatrix} = \begin{bmatrix} X_1 Y_1 q_1(X_1) + a q_1(X_1) + Y_1 \\ X_1^{d-2}(X_1 Y_1 + a) \\ X_1^{d-1}(X_1 Y_1 + a) \end{bmatrix}, \quad (3.7)$$

where we have used $q(X_1) - 1 = X_1 q_1(X_1)$, and factored out $X_1$. Observe that formula (3.7) is exactly the case $k = 1$ of formula (3.6).

Now suppose Lemma III.2 is true for $k$, and substitute $Y_k = X_1 Y_{k+1} + aQ_k$ from Equation (3.6). For the first coordinate of $H_{k+1}$ we find

$$\begin{aligned} X_1(X_1 Y_{k+1} + aQ_k)q_1(X_1) &+ a\sum_{j=0}^{k-1} Q_j q_{k-j}(X_1) + X_1 Y_{k+1} + aQ_k \\ &= X_1^2 Y_{k+1} q_1(X_1) + aX_1 Q_k q_1(X_1) + a\sum_{j=0}^{k-1} Q_j(q_{k-j}(X_1) - q_{k-j}(0)) + X_1 Y_{k+1} \\ &= X_1^2 Y_{k+1} q_1(X_1) + aX_1 Q_k q_1(X_1) + aX_1\sum_{j=0}^{k-1} Q_j q_{k-j+1}(X_1) + X_1 Y_{k+1} \\ &= X_1\left(X_1 Y_{k+1} q_1(X_1) + a\sum_{j=0}^{k} Q_j q_{k-j+1}(X_1) + Y_{k+1}\right). \end{aligned}$$

The second and third coordinates are similar. In particular, we see that we can factor out $X_1$, until $k = d - 1$. This proves part (a).

At each step, any points of indeterminacy must be on the last exceptional divisor, of equation $X_1 = 0$. But if we substitute $X_1 = 0$ in the first coordinate, we find $Y_k = aQ_k$, so that indeed there is only one point of indeterminacy, proving (b).

The restriction to the last exceptional divisor of the mapping $H_{2d-1}$ is given by

$$Y_{d-1} \mapsto \begin{bmatrix} Y_{d-1} - aQ_{d-1} \\ a \\ 0 \end{bmatrix}$$

and since $a \neq 0$, the map is well-defined. Moreover, since $Y_{d-1}$ appears in the first coordinate with degree 1, this last exceptional divisor maps by an isomorphism to the line at infinity. $\square$

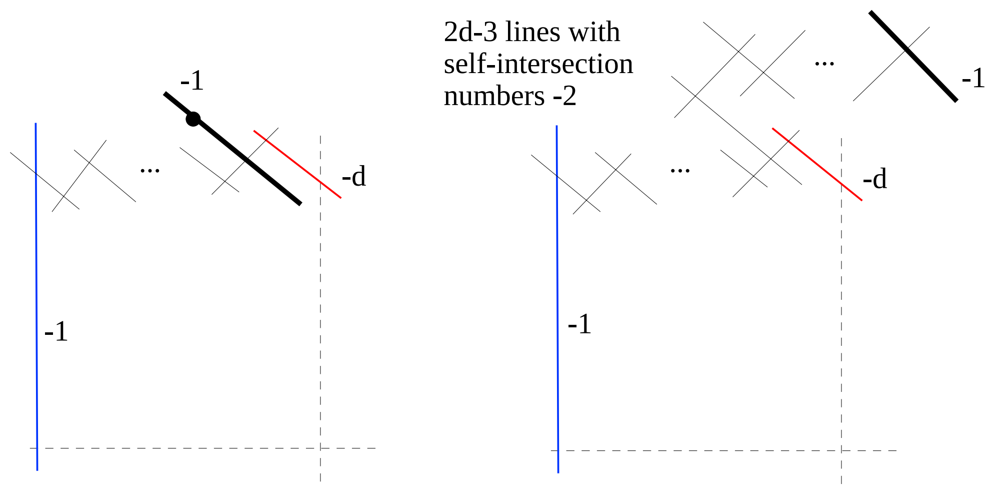


FIGURE 3. The configuration after the $(d+1)$-st blow-up, and after all $2d-1$ blow-ups.

The self-intersection numbers are marked in Figure 3. We are now always blowing up an ordinary point of the last exceptional divisor, so this last exceptional divisor (now next-to-last) acquires self-intersection -2. At the end, the very last exceptional divisor keeps self-intersection -1.

To summarize, we have proved the following result. Denote by $\tilde{X}_H$ the space obtained from $\mathbb{P}^2$ be the sequence of $2d-1$ blow-ups described above.

**Theorem III.3.** *The Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ extends to a morphism $H_{2d-1} = \tilde{H} : \tilde{X}_H \to \mathbb{P}^2$, and maps the divisor at infinity $\tilde{D} = \tilde{X}_H - \mathbb{C}^2$ to $l_\infty$, mapping all of $\tilde{D}$ to the point $\mathbf{q}$ except the last exceptional divisor which is mapped to $l_\infty$ by an isomorphism.*

**Terminology.**

To state the next result and later on, we need to give names to the irreducible components of $\tilde{D}$. Let us label $A'$ the proper transform of the line at infinity, then $B'$ the proper transform of the first exceptional divisor, then in order of creation $L_1, L_2, \ldots, L_{2d-3}$, and finally $\tilde{A}$ the components of the divisor $\tilde{D}$. The line $\tilde{A}$, i.e. the last exceptional divisor, will play a special role.

Since $A'$ is the proper transform of $l_\infty$, the projection $A' \to l_\infty$ is an isomorphism, and we can define $\mathbf{p}', \mathbf{q}'$ the points of $A'$ which correspond to $\mathbf{p}, \mathbf{q}$. Note that $\{\mathbf{p}'\} = L_1 \cap A'$.

The points $\tilde{\mathbf{p}} = \tilde{H}^{-1}(\mathbf{p})$, $\tilde{\mathbf{q}} = \tilde{H}^{-1}(\mathbf{q})$ will play a parallel role; note that $\{\tilde{\mathbf{q}}\} = L_{2d-3} \cap \tilde{A}$. This terminology is illustrated in Figure 4, which represents $\tilde{D}$, i.e., Figure 3(right), redrawn in a more symmetrical way.

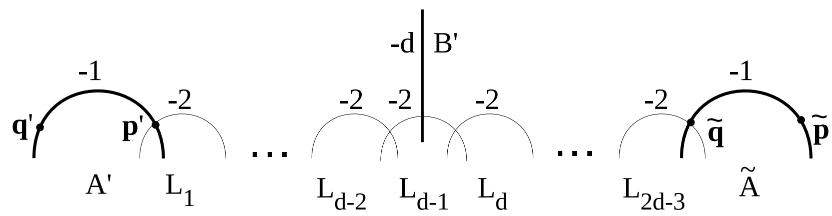


FIGURE 4. The divisor $\tilde{D}$. The numbers labeling the components are the self-intersection numbers.

**The rational mapping $H^\sharp$.**

In the next section, we will want to consider $\tilde{H}$ as a birational map $H^\sharp : \tilde{X}_H \rightsquigarrow \tilde{X}_H$.

**Theorem III.4.** *The rational map $H^\sharp : \tilde{X}_H \rightsquigarrow \tilde{X}_H$ is defined at all points except $\tilde{\mathbf{p}}$. It collapses $\tilde{D} - \tilde{A}$ to $\tilde{\mathbf{q}}$, maps $\tilde{A} - \tilde{\mathbf{p}}$ to $A' - \mathbf{p}'$ by an isomorphism.*

*The rational map $(H^\sharp)^{-1}$ is defined at all points except $\mathbf{q}'$. It collapses $\tilde{D} - A'$ to $\tilde{\mathbf{p}}$ and maps $A' - \mathbf{q}'$ to $\tilde{A} - \tilde{\mathbf{q}}$ by an isomorphism.*

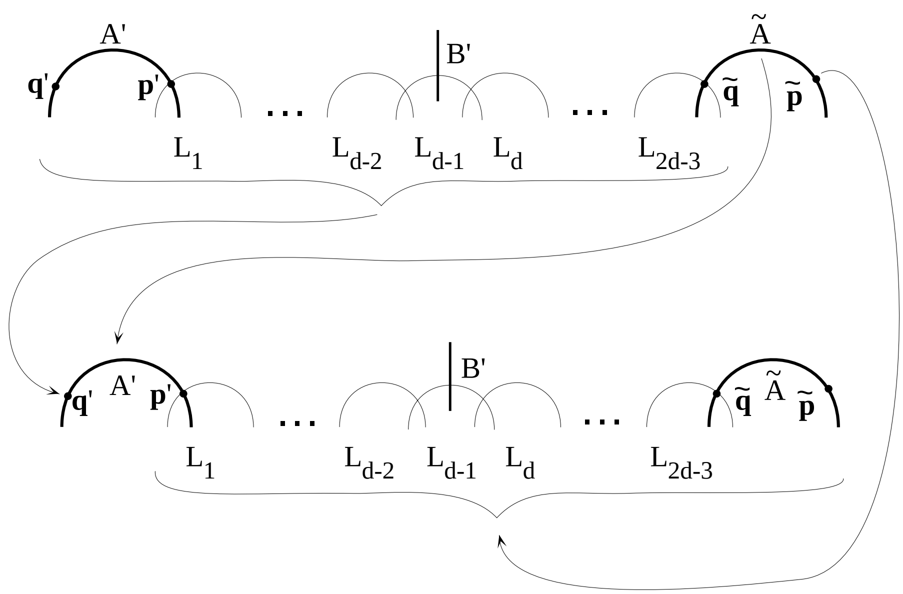


FIGURE 5. The "mapping" $H^\sharp$ acting on the divisor $\tilde{D}$.

**Proof.** This is really a corollary of Theorem III.3. Clearly, $H^\sharp$ is well defined on $\tilde{X}_H - \tilde{H}^{-1}(\mathbf{p})$, i.e., on $\tilde{X}_H - \{\tilde{\mathbf{p}}\}$, and coincides with $\tilde{H}$ there.

Further $\tilde{H}$ is an isomorphism from a neighborhood of $\tilde{\mathbf{p}}$ to a neighborhood of $\mathbf{p}$. So if you perform any sequence of blow-ups at $\mathbf{p}$, $\tilde{H}$ will become undetermined at $\tilde{\mathbf{p}}$. The statements about the inverse map are similar. □

## IV. Closures of graphs and sequence spaces

We now have a well-defined map $\tilde{H} : \tilde{X}_H \to \mathbb{P}^2$, but that does not solve our problem of compactifying $H : \mathbb{C}^2 \to \mathbb{C}^2$ as a dynamical system. We cannot consider $\tilde{H}$ as a dynamical system, since the domain and the range are different. Neither does $H^\sharp$ solve our problem, since it still has a point of indeterminacy. In this section we will show how to perform infinitely many blow-ups so that in the projective limit we do get a dynamical system. We will construct this infinite blow-up as a sequence space, as this simplifies the presentation and proof (this description was inspired by Friedland [Fr1], who considered the analog in $\left(\mathbb{P}^2\right)^{\mathbb{Z}}$). To make this construction, we need to analyze the graph of $H^\sharp$.

Let $X, Y$ be compact smooth algebraic surfaces, and $f : X \rightsquigarrow Y$ be a birational transformation. Let us suppose that it is undefined at $\mathbf{p}_1, \dots, \mathbf{p}_n$, and that $f^{-1}$ is undefined at $\mathbf{q}_1, \dots, \mathbf{q}_m$. Let

$$\Gamma_f \subset (X - \{\mathbf{p}_1, \dots, \mathbf{p}_n\}) \times Y$$

be the graph of $f$, and $\overline{\Gamma}_f \subset X \times Y$ its closure.

**Lemma IV.1.** *The space $\overline{\Gamma}_f$ is a smooth manifold, except perhaps at points $(\mathbf{x}, \mathbf{y}) \in \overline{\Gamma}_f$ such that*

$$\mathbf{x} \in \{\mathbf{p}_1, \dots, \mathbf{p}_n\} \quad \text{and} \quad \mathbf{y} \in \{\mathbf{q}_1, \dots, \mathbf{q}_m\}.$$

**Proof.** Clearly $pr_1 : \overline{\Gamma}_f \to X$ is locally an isomorphism near $(\mathbf{x}, \mathbf{y})$ if $\mathbf{x} \notin \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, and $pr_2 : \overline{\Gamma}_f \to Y$ is locally an isomorphism near $(\mathbf{x}, \mathbf{y})$ unless $\mathbf{y} \in \{\mathbf{q}_1, \dots, \mathbf{q}_m\}$. □

**Example IV.2.** If you have points $(\mathbf{p}_i, \mathbf{q}_j) \in \overline{\Gamma}_f$, they can genuinely be quite singular. For instance, if $X = Y = \mathbb{P}^2$ and $f = H$ is a Hénon mapping, then $f$ (resp. $f^{-1}$) has a unique point of indeterminacy $\mathbf{p}$ (resp. $\mathbf{q}$) (see III.1). The pair $(\mathbf{p}, \mathbf{q})$ is in $\overline{\Gamma}_H$, and near $(\mathbf{p}, \mathbf{q})$ we can find equations of $\overline{\Gamma}_H$ as follows.

In local coordinates

$$u = \frac{x}{y},\ v = \frac{1}{y} \quad \text{near } \mathbf{p}, \qquad s = \frac{y}{x},\ t = \frac{1}{x} \quad \text{near } \mathbf{q},$$

the space $\overline{\Gamma}_H$ is given by the two equations

$$\begin{aligned} v^d &= t\left(\tilde{p}(u, v) - a v^{d-1}\right) \\ ut &= sv \end{aligned}$$

which is quite singular at the origin indeed; one way to understand Section III is as a resolution of this singularity, as Proposition IV.3 shows.

Let $H$ be a Hénon mapping, $\tilde{X}_H$ be the blow-up on which $\tilde{H} : \tilde{X}_H \to \mathbb{P}^2$ is well-defined, and $H^\sharp : \tilde{X}_H \rightsquigarrow \tilde{X}_H$ be $\tilde{H}$ viewed as a rational mapping from $\tilde{X}_H$ to itself.

**Proposition IV.3.** *The closure $\overline{\Gamma}_{H^\sharp} \subset \tilde{X}_H \times \tilde{X}_H$ is a smooth submanifold.*

**Proof.** The mapping $H^\sharp : \tilde{X}_H \rightsquigarrow \tilde{X}_H$ is birational, and as we saw in Theorem III.4, it has a unique point of indeterminacy at $\tilde{\mathbf{p}} = \tilde{H}^{-1}(\mathbf{p})$, and the inverse birational mapping $(H^\sharp)^{-1}$ also has a unique point of indeterminacy $\mathbf{q}'$. But the point $(\tilde{\mathbf{p}}, \mathbf{q}')$ is *not* in $\overline{\Gamma}_{H^\sharp}$, so $\overline{\Gamma}_{H^\sharp}$ is a smooth (compact) manifold by Lemma IV.1. $\square$

There is another description of $\overline{\Gamma}_{H^\sharp}$, which we will need in a moment.

**Proposition IV.4.** *The space $\overline{\Gamma}_{H^\sharp}$, together with the projections $pr_1$ and $pr_2$ onto the first and second factor respectively, make the diagram*

$$\begin{array}{ccc} \overline{\Gamma}_{H^\sharp} & \xrightarrow{pr_2} & \tilde{X}_H \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ \tilde{X}_H & \xrightarrow{\tilde{H}} & \mathbb{P}^2 \end{array}$$

*a fibered product in the category of analytic spaces* [Dou].

**Proof.** The diagram

$$\begin{array}{ccc} \tilde{X}_H \times \tilde{X}_H & \xrightarrow{pr_2} & \tilde{X}_H \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ \tilde{X}_H & \xrightarrow{\tilde{H}} & \mathbb{P}^2 \end{array}$$

evidently commutes on the graph $\Gamma_H$, and also evidently commutes on a closed set, hence it commutes on $\overline{\Gamma}_{H^\sharp}$.

Since all the spaces involved are manifolds, it is enough to prove that the diagram is a fibered product in the category of analytic manifolds, i.e., set-theoretically. Since $\pi(\mathbf{y}) = \tilde{H}(\mathbf{x})$ on $\Gamma_H$ this is still true on the closure $\overline{\Gamma}_{H^\sharp}$. $\square$

It is time to construct one of our main actors. The space $X_\infty$, constructed below, is a compact space, which contains $\mathbb{C}^2$ as a dense open subset, and such that $H : \mathbb{C}^2 \to \mathbb{C}^2$ extends to $H_\infty : X_\infty \to X_\infty$.

The locus $D_\infty = X_\infty - \mathbb{C}^2$ is an infinite divisor at infinity, the geometry of which encodes the behavior of $H$ at infinity.

**Definition IV.5.** Let $X_\infty \subset (\tilde{X}_H)^{\mathbb{Z}}$ be the set of sequences $\underline{\mathbf{x}} = (\ldots, \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1, \ldots)$ such that successive pairs belong to $\overline{\Gamma}_{H^\sharp} \subset \tilde{X}_H \times \tilde{X}_H$ above.

Let $H_\infty : X_\infty \to X_\infty$ be the shift map

$$(H_\infty(\underline{\mathbf{x}}))_k = \mathbf{x}_{k+1},$$

where $\underline{\mathbf{x}} = (\dots, \mathbf{x}_1, \mathbf{x}_0, \mathbf{x}_1, \dots)$ is a point of $X_\infty$.

Clearly $X_\infty$ a compact space, since it is a closed subset of a product of compact sets, and $H_\infty$ is a homeomorphism $X_\infty \to X_\infty$. We will see below why $H_\infty$ can be understood as an extension of $H$.

**Proposition IV.6.** *a) The points of* $X_\infty$ *are of one of three types:*

(1) *Sequences with all entries in* $\mathbb{C}^2$*;*
(2) *Sequences of the form* $(\dots, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \mathbf{a}, \mathbf{b}, \mathbf{q}', \mathbf{q}', \dots)$ *with* $\mathbf{a} \in \tilde{D}$*,* $\mathbf{a} \neq \tilde{\mathbf{p}}$*;*
(3) *The two sequences* $\mathbf{p}^\infty = (\dots, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \dots)$ *and* $\mathbf{q}^\infty = (\dots, \mathbf{q}', \mathbf{q}', \dots)$*.*

*b) The sequences of type (1) are dense in* $X_\infty$*.*

**Proof.** If a sequence has any entry in $\mathbb{C}^2$, then it is the full orbit of that point, forwards and backwards. Otherwise, all entries are in the divisor $\tilde{D} = \tilde{X}_H - \mathbb{C}^2$. If these entries are not all $\tilde{\mathbf{p}}$, or all $\mathbf{q}'$, then there is a first entry $\mathbf{a}$ which is not $\tilde{\mathbf{p}}$; it must be preceded by all $\tilde{\mathbf{p}}$'s. What follows it is the orbit of $\mathbf{a}$, which is well-defined. Note that $\mathbf{b} = \tilde{H}(\mathbf{a})$ may be $\mathbf{q}'$ (this will happen unless $\mathbf{a} \in \tilde{A}$), and all the successive terms must be $\mathbf{q}'$. This proves (a).

For part (b), we must show that a sequence $\underline{x}$ of type 2 or 3 can be approximated by an orbit, i.e., that for any $\epsilon > 0$ and any integer $N$, there is a point $\mathbf{y} \in \mathbb{C}^2$ such that $d(H^n(x) - y_n) < \epsilon$ when $|n| < N$. If $\underline{x}$ is of type 2, we may assume that $x(0) \neq \tilde{\mathbf{p}}, \mathbf{q}'$. Then all iterates of $\tilde{H}$ and of $\tilde{H}^{-1}$ are defined and continuous in a neighborhood of $x(0)$, so any point in this neighborhood and close to $x(0)$ will have a long stretch of forward and backwards orbits close to $\underline{x}$; but every neighborhood of $x(0)$ contains points of $\mathbb{C}^2$, which is dense in $\tilde{X}$.

Similarly, the orbit of a point with $|x|$ very large and $y = 0$ will approximate $\mathbf{q}^\infty$, and a point with $|y|$ large and $x = 0$ will approximate $\mathbf{q}^\infty$. $\square$

**Example IV.7.** The fact that $\mathbb{C}^2$ is dense in $X_\infty$ is not quite so obvious as one might think, and there are examples of birational maps where it doesn't happen. For instance, consider the mapping

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ x \end{bmatrix},$$

a priori well defined on $(\mathbb{C}^*)^2$. Denote $\mathbf{p}$ and $\mathbf{q}$ the points at infinity on the $x$-axis and $y$-axis respectively. Then the pairs $(r, \mathbf{q})$ belong to $\overline{\Gamma}_F \subset \mathbb{P}^2 \times \mathbb{P}^2$ when $r$ is in the line at infinity, as do the pairs $(\mathbf{q}, r)$. Thus the sequence space contains points like

$$(\dots, \mathbf{q}, r, \mathbf{q}, \mathbf{q}, r, \mathbf{q}, r, \mathbf{q}, \mathbf{q}, \dots)$$

with symbols $\mathbf{q}$ and $r$ in any order. Such sequences cannot be approximated by orbits in $(\mathbb{C}^*)^2$; they will also form subsets which have dimension equal to the number of appearances of $r \neq \mathbf{q}$, which may be infinite. Such sequence spaces are a little scary, as well as pathological, and irrelevant to the original dynamical system.

In our case, if we had not blown up $\mathbb{P}^2$, $\mathbb{C}^2$ would still have been dense in the sequence space, but it would have had bad singularities.

**Proposition IV.8.** *The space $X^*_\infty = X_\infty - \{\mathbf{p}^\infty, \mathbf{q}^\infty\}$ is an algebraic manifold.*

*More precisely,*

(1) *The projection $\pi_0$ onto the 0-th coordinate induces an isomorphism of the space of orbits of the first type to $\mathbb{C}^2$;*

(2) *If $\underline{\mathbf{x}} = (\dots, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \mathbf{a}, \mathbf{b}, \mathbf{q}', \mathbf{q}', \dots)$ is a point of the second type, and $\mathbf{a}$ appears in the $k$-th position, then the projection $\pi_k$ onto the $k$-th position induces a homeomorphism of a neighborhood of $\underline{\mathbf{x}}$ onto $\tilde{X}_H - \{\tilde{\mathbf{p}}, \mathbf{q}'\}$.*

**Proof.** The first part is clear. For the second, if a point $\underline{\mathbf{y}}$ satisfies $\mathbf{y}_k \neq \tilde{\mathbf{p}}, \mathbf{q}'$, then the entire forward and backwards orbit of $\mathbf{y}_k$ is defined: forwards it will never land on $\tilde{\mathbf{p}}$, and backwards it will never land on $\mathbf{q}'$.

Let us call $\phi_k : \tilde{X}_H - \{\tilde{\mathbf{p}}, \mathbf{q}') \to X_\infty$ the map which maps $\mathbf{x}$ to the unique sequence $\underline{\mathbf{x}} \in X_\infty$ with $\mathbf{x}_k = \mathbf{x}$. The change of coordinate map $\phi_l^{-1} \circ \phi_k$ is then simply $H^{l-k}$ on $\mathbb{C}^2$.

This shows that the coordinate changes are algebraic on the intersections of coordinate neighborhoods, except for one detail. The set $\mathbb{C}^2 \subset X_\infty$ is exactly the intersection of the images of $\phi_k$ and $\phi_l$ when $|l-k| \geq 2$, but when $l = k+1$, the intersection then contains the sequences with entries $(\dots, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \mathbf{a}, \mathbf{b}, \mathbf{q}', \mathbf{q}', \dots)$ with $\mathbf{a} \in \tilde{A} - \{\tilde{\mathbf{p}}\}$ and $\mathbf{b} \in A' - \{\mathbf{q}'\}$, in this case also the change of coordinates is given by $H^\sharp$ and is still algebraic. $\square$

We can now see why $H_\infty$ is an extension of $H$. On the subset isomorphic to $\mathbb{C}^2$ formed of sequences in $\mathbb{C}^2$, with $\pi_0$ the isomorphism, we have

$$\pi_0\big(H_\infty(\underline{\mathbf{x}})\big) = H\big(\pi_0(\underline{\mathbf{x}})\big);$$

i.e., $\pi_0$ conjugates $H_\infty$ to $H$ on that subset.

**Notation.**

We will systematically identify $\mathbb{C}^2$ with $\phi_0(\mathbb{C}^2) \subset X_\infty$. With this identification, $H_\infty$ does extend $H$ continuously, and algebraically in $X^*_\infty$. Moreover, we will set $D_\infty = X_\infty - \mathbb{C}^2$; a picture of $D_\infty$ is given in Figure 6.

The lines denoted by $A_i$, $i \in \mathbb{Z}$ are formed of those sequences whose $i$-th entry is in $A'$; each such line connects the sequences whose $i$-th entry is in $L_1$ (denoted by $L_{i,1}$) with those whose $(i-1)$-st entry is in $L_{2d-3}$. In particular, the points $\mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_1, \mathbf{p}_1 \in X_\infty$ correspond to the sequences

$$\begin{array}{lllllllll} \mathbf{q}_0 = ( & \dots & \tilde{\mathbf{p}}, & \tilde{\mathbf{q}}, & \mathbf{q}', & \mathbf{q}', & \mathbf{q}', & \dots & ) \\ & \dots & -2, & -1, & 0, & 1, & 2, & \dots & \end{array} \qquad \begin{array}{lllllllll} \mathbf{p}_0 = ( & \dots & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \mathbf{p}', & \mathbf{q}', & \mathbf{q}', & \dots & ) \\ & \dots & -2, & -1, & 0, & 1, & 2, & \dots & \end{array}$$

$$\begin{array}{lllllllll} \mathbf{q}_1 = ( & \dots & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \tilde{\mathbf{q}}, & \mathbf{q}', & \mathbf{q}', & \dots & ) \\ & \dots & -2, & -1, & 0, & 1, & 2, & \dots & \end{array} \qquad \begin{array}{lllllllll} \mathbf{p}_1 = ( & \dots & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \mathbf{p}', & \mathbf{q}', & \dots & ) \\ & \dots & -2, & -1, & 0, & 1, & 2, & \dots & \end{array}$$

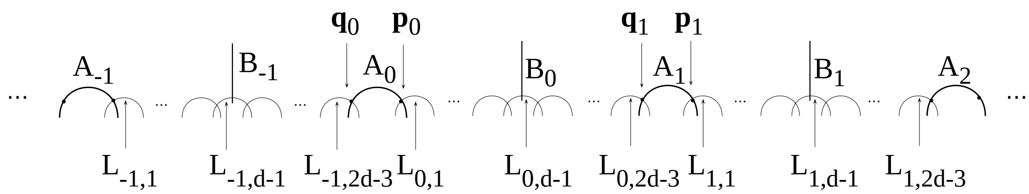


FIGURE 6. The divisor $D_\infty$.

## V. THE HOMOLOGY OF $X_\infty^*$

In this section we will study the homology groups $H_i(X_\infty^*)$, and more particularly $H_2(X_\infty^*)$ and the quadratic form on it coming from the intersection product.

Although nasty spaces (solenoids, etc.) are lurking around every corner, here we will compute only the homology groups of manifolds, being careful to exclude the nasty parts. All homology theories coincide for such spaces, and we may use singular homology, for instance. Unless stated otherwise, we use integer coefficients; at the end we will use complex coefficients. Using $d$-torsion coefficients would give quite different results, which can easily be derived using the universal coefficient theorem.

### Inductive limits.

It is fairly easy to represent $X_\infty^*$ as an increasing union of subsets whose homology can be fairly easily computed. Since inductive limits and homology commute, it is enough to understand these subsets.

First some terminology. If $G$ is an Abelian group, then $G^{\mathbb{N}}$ is the product of infinitely many copies of $G$, indexed by $\mathbb{N}$, i.e., the set of all sequences $(g_1, g_2, \dots)$ with $g_i \in G$. The group $G^{(\mathbb{N})} \subset G^{\mathbb{N}}$ is the set of sequences with only finitely many non-zero terms; in the category of Abelian groups, this is the *sum* of copies of $G$ indexed by $\mathbb{N}$; it is also easy to show that it is the inductive limit of the diagram

$$G \to G^2 \to G^3 \to \dots$$

where the map

$$G^k \to G^{k+1} \quad \text{is} \quad (g_1, \dots, g_k) \mapsto (g_1, \dots, g_k, 0).$$

The particular inductive limit we will encounter is not quite elementary, and we will start with an example, which has many features in common with our direct limit of homology groups.

**Example V.1.** Consider the inductive system

$$\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}^2 \xrightarrow{f_2} \mathbb{Z}^3 \xrightarrow{f_3} \dots,$$

where $f_n : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ is defined by

$$f_n(\mathbf{e}_{n,i}) = \begin{cases} \mathbf{e}_{n+1,i} & \text{if } i < n \\ \mathbf{e}_{n+1,n} + \mathbf{e}_{n+1,n+1} & \text{if } i = n, \end{cases}$$

using the standard basis $\mathbf{e}_{n,1}, \dots \mathbf{e}_{n,n}$ of $\mathbb{Z}^n$.

It certainly seems as if the inductive limit of this system should be $\mathbb{Z}^{(\mathbb{N})}$ of sequences of integers which are eventually 0. But this is not true, and the inductive limit is bigger.

**Proposition V.2.** *a) If $(v_m, v_{m+1}, \dots)$ represents an element of $\lim\limits_{\substack{\to \\ n}} (\mathbb{Z}^n, f_n)$, with $v_m \in \mathbb{Z}^m$, then for any $j$, the coordinate $(v_m)_j$ is constant as soon as $m > j$. This defines a map*

$$\lim_{\substack{\to \\ n}} (\mathbb{Z}^n, f_n) \to \mathbb{Z}^{\mathbb{N}}$$

*which is easily seen to be injective.*

*b) The image of $\lim\limits_{\substack{\to \\ n}} (\mathbb{Z}^n, f_n)$ in $\mathbb{Z}^{\mathbb{N}}$ consists of the sequences $(a_j)_{j\in\mathbb{N}}$ which are eventually constant.*

**Proof.** Any element of the inductive limit has a representative $v_m \in \mathbb{Z}^m$ for some $m$. The $m$-th entry of $v_m$ will be replicated as both the $m$-th and $(m+1)$-st entry of $v_{m+1}$, and then as the last three entries of $v_{m+2}$, etc. Clearly the image in $\mathbb{Z}^{\mathbb{N}}$ will be constant from the $m$-th term on. □

Thus there is an exact sequence

$$0 \to \mathbb{Z}^{(\mathbb{N})} \to \lim_{\substack{\to \\ n}} (\mathbb{Z}^n, f_n) \to \mathbb{Z} \to 0$$

where the third arrow associates to an eventually constant sequence the value of that constant.

We see that there is an extra generator to the inductive limit, which one may take to be the constant sequence of 1's in $\mathbb{Z}^{\mathbb{N}}$.

**Example V.3.** Now let us elaborate our example a little. Modify $f_n : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ so that $f_n(\mathbf{e}_{n,n}) = \mathbf{e}_{n+1,n} + d\mathbf{e}_{n+1,n+1}$ for some integer $d \geq 1$.

Most of the computation above still holds, except that a sequence

$$\underline{v} = (v_1, v_2, \dots) \in \mathbb{Z}^{\mathbb{N}}$$

belongs to the inductive limit if and only if it is eventually geometric with ratio $d$. We will denote $\mathbb{Z}[1/d]$ the rational numbers with only powers of $d$ in the denominator, i.e., the sub-ring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $1/d$. If we set $\underline{v}^+ = (1, d, d^2, \dots)$, then $\underline{v}$ belongs to the inductive limit if and only if there exists $a \in \mathbb{Z}[1/d]$ such that $\underline{v} - a\underline{v}^+$ has only finitely many non-zero entries. In other word, there is an exact sequence

$$0 \to \mathbb{Z}^{(\mathbb{N})} \to \lim_{\substack{\to \\ N}} (\mathbb{Z}^n, f_n) \to \mathbb{Z}[1/d] \to 0.$$

Note that our inductive limit is still a free Abelian group, for there is a theorem [Gri, Thm. 138] which asserts that a countable subgroup of $\mathbb{Z}^{\mathbb{N}}$ is free Abelian. In our case, the elements

$$(1, d, d^2, d^3, \dots), (0, 1, d, d^2, \dots), (0, 0, 1, d, \dots), \dots$$

form a basis. On the other hand $\mathbb{Z}[1/d]$ is not free (it is divisible by $d$).

**The homology of blow-ups.**

Before attacking the homology of $X_\infty^*$, we will remind the reader of some well-known facts about the homology of algebraic surfaces.

**Proposition V.4.** *If $X$ is a smooth algebraic surface (or more generally a four-dimensional topological manifold), and $Z \subset X$ is a finite subset, then the inclusion $X - Z \hookrightarrow X$ induces an isomorphism on 1- and 2-dimensional homology.*

**Proof.** Consider the long exact sequence of the pair $(X, X - Z)$, which gives in part

$$\dots \to H_3(X, X - Z) \to H_2(X - Z) \to H_2(X) \to H_2(X, X - Z) \to \dots;$$

it is enough to show that the first and last term vanish. Let $(U_{\mathbf{z}})_{\mathbf{z} \in Z}$ be a set of neighborhoods of the points of $Z$ homeomorphic to 4-balls; by excision, we have

$$H_k\big(X, X - \{\mathbf{z}\}\big) = \bigoplus_{\mathbf{z} \in P} H_k\big(U_{\mathbf{z}}, U_{\mathbf{z}} - \{\mathbf{z}\}\big).$$

The long exact sequence of such a pair $(U_{\mathbf{z}}, U_{\mathbf{z}} - \{\mathbf{z}\})$ gives in part

$$\begin{aligned} \dots \to & H_3(U_{\mathbf{z}}) \to H_3\big(U_{\mathbf{z}}, U_{\mathbf{z}} - \{\mathbf{z}\}\big) \to H_2(U_{\mathbf{z}} - \{\mathbf{z}\}) \to \\ H_2(U_{\mathbf{z}}) \to & H_2\big(U_{\mathbf{z}}, U_{\mathbf{z}} - \{\mathbf{z}\}\big) \to H_1\big(U_{\mathbf{z}} - \{\mathbf{z}\}\big) \to \dots \end{aligned}$$

The first, third, fourth and sixth terms vanish, since $U_{\mathbf{z}}$ is contractible and $U_{\mathbf{z}} - \{\mathbf{z}\}$ has the homotopy type of a 3-sphere. The result for 2-dimensional homology follows; the proof for one-dimensional homology is similar. $\square$

**Proposition V.5.** *If $X$ is a smooth algebraic surface (or more generally an orientable 4-dimensional topological manifold), and $Z \subset X$ is a finite subset, then the inclusion $X - Z \hookrightarrow X$ induces an exact sequence*

$$0 \to H_4(X) \to \mathbb{Z}^Z \to H_3(X - Z) \to H_3(X) \to 0.$$

*In particular, if $X$ is compact and $Z$ is a single point, then the inclusion induces an isomorphism $H_3(X - Z) \to H_3(X)$.*

The proof comes from considering the same exact sequences as above; we omit it.

We will now see that if you blow up a point of a surface, you increase the 2-dimensional homology by the class of the exceptional divisor.

Let $X$ be a surface, and $\mathbf{z}$ a smooth point. Let $\pi : \tilde{X}_{\mathbf{z}} \to X$ be the canonical projection, and $E = \pi^{-1}(\mathbf{z})$ be the exceptional divisor.

Consider the homomorphism

$$i : H_2(X) \to H_2(\tilde{X}_{\mathbf{z}}) \tag{5.1}$$

given by the composition

$$H_2(X) \to H_2\big(X - \{\mathbf{z}\}\big) \to H_2(\tilde{X}_{\mathbf{z}});$$

first the inverse of the isomorphism $H_2\big(X - \{\mathbf{z}\}\big) \to H_2(X)$ in Proposition V.4, followed by the map induced by inclusion.

**Proposition V.6.** *The map*

$$H_2(X) \oplus \mathbb{Z} \to H_2(\tilde{X}_{\mathbf{z}})$$

*given by $(\alpha, m) \mapsto i(\alpha) + m[E]$ is an isomorphism.*

**Proof.** This is an application of the Mayer-Vietoris exact sequence. Let $U$ be an open neighborhood of $\mathbf{z}$ in $X$ homeomorphic to a 4-ball. Clearly, $\pi$ is a homeomorphism $\tilde{X}_{\mathbf{z}} - E \to X - \{\mathbf{z}\}$. So applying Mayer-Vietoris to the open cover $\tilde{X}_{\mathbf{z}} - E$ and $\tilde{U}_{\mathbf{z}}$ of $\tilde{X}_{\mathbf{z}}$ gives in part

$$\dots \to H_2\big(U - \{\mathbf{z}\}\big) \to H_2\big(X - \{\mathbf{z}\}\big) \oplus H_2(\tilde{U}_{\mathbf{z}}) \to H_2(\tilde{X}_{\mathbf{z}}) \to H_1\big(U - \{\mathbf{z}\}\big) \to \dots.$$

The first and last terms are zero, because $U - \{\mathbf{z}\}$ has the topology of a 3-sphere, so the middle mapping is an isomorphism

$$H_2\big(X - \{\mathbf{z}\}\big) \oplus H_2(\tilde{U}_{\mathbf{z}}) \to H_2(\tilde{X}_{\mathbf{z}}).$$

The result follows by applying Proposition V.4, since $\tilde{U}_{\mathbf{z}}$ deforms onto the exceptional divisor, which is a projective line. $\square$

The same Mayer-Vietoris exact sequence, together with Proposition V.5, will also show the following result.

**Proposition V.7.** *If $X$ is compact, the canonical projection induces isomorphisms $H_1(\tilde{X}_{\mathbf{z}}) \to H_1(X)$ and $H_3(\tilde{X}_{\mathbf{z}}) \to H_3(X)$.*

The next proposition will be the key to most of our computations.

**Proposition V.8.** *Consider the composition*

$$H_2(X) \to H_2\big(X - \{\mathbf{z}\}\big) \to H_2(\tilde{X}_{\mathbf{z}}).$$

*Let $C$ be a curve in $X$ which has $m$ smooth branches through $\mathbf{z}$. Then the image of $[C]$ in $H_2(\tilde{X}_{\mathbf{z}})$ is $[C'] + m[E]$, where $C'$ is the proper transform of $C$ in $\tilde{X}_{\mathbf{z}}$.*

**Proof.** Let $\hat{C}$ be the normalization of $C$, which is in particular a smooth 2-dimensional differentiable manifold, and $f : \hat{C} \to C$ the normalizing map. The mapping $f$ can be deformed (differentiably, but perhaps not analytically) to a map $f' : \hat{C} \to X$ which avoids $\mathbf{z}$; so $f'$ lifts to a map $\tilde{f}' : \hat{C} \to \tilde{X}_{\mathbf{z}}$.

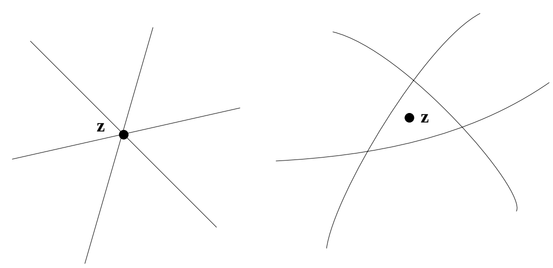


FIGURE 7. A curve with 3 smooth branches through $\mathbf{z}$, and a deformation which avoids $\mathbf{z}$.

Then $[\tilde{f}'(\hat{C})]$ is the image of $[C]$ in $H_2(\tilde{X}_{\mathbf{z}})$. The homology class $[\tilde{f}'(\hat{C})]$ is of the form $[\tilde{C}'] + n[E]$ for some $n$: indeed, $f'$ can be chosen so that $[\tilde{f}'(\hat{C})]$ is contained in a small neighborhood of $C' \cup E$, which will retract onto $C' \cup E$, and hence whose 2-dimensional homology is generated by $[C']$ and $[E]$. We discover what $n$ is by observing that the intersection number $[\tilde{f}'(\hat{C})] \cdot [E]$ vanishes, since the corresponding cycles are disjoint. So

$$0 = [\tilde{f}'(\hat{C})] \cdot [E] = \big([C'] + n[E]\big) \cdot [E] = m - n,$$

since each branch of $C$ through $\mathbf{z}$ contributes 1 to $[C'] \cdot [E]$. $\square$

**The finite approximations to $X_\infty$.**

Now let us consider the set

$$X_{[N,M]} \subset \prod_{i=N}^{M} \tilde{X}_H, \quad N \leq M,$$

of finite sequences $(\mathbf{x}_N, \mathbf{x}_{N+1}, \dots, \mathbf{x}_M)$ with pairs of successive points in $\overline{\Gamma}_{H^\sharp}$, and $D_{[N,M]} \subset X_{[N,M]}$ the subset with all coordinates in $\tilde{D}$.

The set $D_{[N,M]}$ contains the point $\mathbf{p}^{[N,M]}$ all of whose coordinates are $\tilde{\mathbf{p}}$, and the point $\mathbf{q}^{[N,M]}$ all of whose coordinates are $\mathbf{q}'$. Let us set

$$X^*_{[N,M]} = X_{[N,M]} - \left\{\mathbf{p}^{[N,M]}, \mathbf{q}^{[N,M]}\right\} \quad \text{and} \quad D^*_{[N,M]} = X^*_{[N,M]} \cap D_{[N,M]}.$$

**Proposition V.9.** *If $-\infty \leq N' \leq N \leq M \leq M' \leq \infty$, the natural projection*

$$X_{[N',M']} \to X_{[N,M]}$$

*has an inverse*

$$X^*_{[N,M]} \to X_{[N',M']}$$

*defined on $X^*_{[N,M]}$.*

**Proof.** For any point of $X^*_{[N,M]}$, the $N$-th coordinate is not $\mathbf{q}'$, so has a well-defined backwards orbit, and the $M$-th coordinate is not $\tilde{\mathbf{p}}$, so it has a well-defined forwards orbit. These orbits define an inclusion of $X^*_{[N,M]}$ into $X_\infty$. □

The point of this proposition is that we can compute the homology of $X^*_{[N,M]}$. If $V$ is an algebraic variety, let $\mathrm{Irr}(V)$ denote the set of irreducible components of $V$.

**Proposition V.10.** *a) The space $X_{[N,M]}$ is a smooth algebraic surface, and $D_{[N,M]}$ is a divisor in $X_{[N,M]}$.*

*b) The divisor $D_{[N,M]}$ consists of $M+1-N$ ordered blocks, each consisting of $2d$ projective lines, with the last line of one block coinciding with the first of the next, as in Figure 8.*

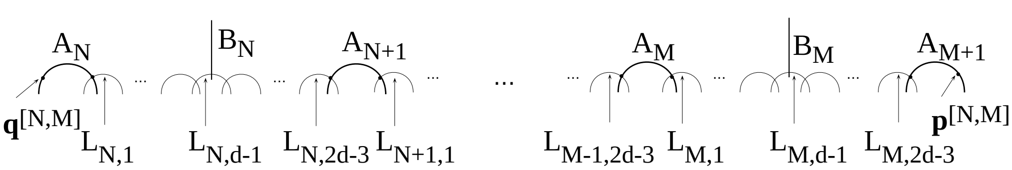


FIGURE 8. The divisor $D_{[N,M]}$.

**Proof.** Part (a) is more or less obvious, except perhaps for the points $\mathbf{p}^{[N,M]}, \mathbf{q}^{[N,M]}$. The projection onto the $M$-th coordinate gives an isomorphism of a neighborhood of $\mathbf{p}^{[N,M]}$ onto a neighborhood of $\tilde{\mathbf{p}}$, and the projection onto the $N$-th coordinate works for $\mathbf{q}^{[N,M]}$.

A point of $D_{[N,M]}$ will be a sequence of points at infinity in $\tilde{X}_H$. Such a sequence will consist of either

- all $\tilde{\mathbf{p}}$ or all $\mathbf{q}'$, or

- a certain number of $\tilde{\mathbf{p}}$'s (perhaps none), then a first element different from $\tilde{\mathbf{p}}$, then something (perhaps $\mathbf{q}'$), then all $\mathbf{q}'$s.

Let us denote by $D_k$ the $k$-th block

$$D_k = \left\{ \underline{\mathbf{x}} \in D_{[N,M]} \,\middle|\, \mathbf{x}_k \in (\tilde{D} - \tilde{A}) \cup \{\tilde{\mathbf{q}}\} \text{ and } \mathbf{x}_{k-1} \neq \mathbf{q}' \right\},$$

for $N \leq k \leq M$ (if $k = N$, then condition $x_{k-1} \neq \mathbf{q}'$ is void). This set is parametrized by $\mathbf{x}_k\mathbf{x}_k \in (\tilde{D} - \tilde{A}) \cup \{\tilde{\mathbf{q}}\}$, and every point of $D_{[N,M]}$ belongs to precisely one $D_k$, except as follows.

- The points whose $M$-th coordinate belong to $\tilde{A} - \{\tilde{\mathbf{q}}\}$; these form a projective line denoted $A_{M+1}$.

- The points $\mathbf{q}_k,\ k = N+1, \dots, M$ whose $k$-th coordinate is $\mathbf{q}'$ and whose $k-1$-st coordinate is $\tilde{\mathbf{q}}$. The point $\mathbf{q}_k$ is simultaneously the left-most point of $D_k$ and the right-most point of $D_{k-1}$;

- The point $\mathbf{q}_{M+1} = A_{M+1} \cap D_M$. $\square$

All lines have the same self-intersection numbers as the corresponding lines have in $\tilde{D}$, except for the connecting lines, i.e., the lines $A_k,\ k = N+1, \dots, M$ where $\mathbf{x}_k \in A'$ and $x_{k-1} \neq \mathbf{q}'$, which have self-intersection $-3$, as indicated in the Figure 9. This is proved as part of the proof of V.11.

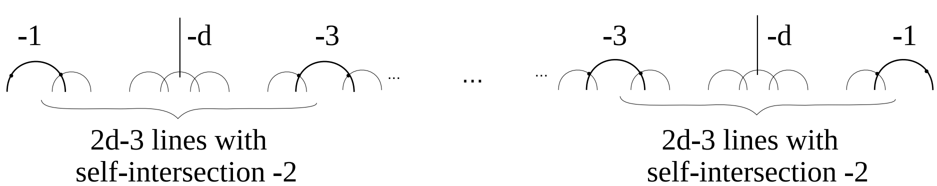


FIGURE 9. The self-intersections of the components of $D_{[N,M]}$.

**Proposition V.11.** a) *The map which associates to each irreducible component of $D_{[N,M]}$ the 2-dimensional homology class which it carries induces an isomorphism*

$$\mathbb{Z}^{\mathrm{Irr}(D_{[N,M]})} \to H_2\big(X_{[N,M]}\big),$$

*when $-\infty < N \leq M < \infty$.*

*b) The inclusion $X^*_{[N,M]} \to X_{[N,M]}$ induces an isomorphism on 2-dimensional homology.*

Before we prove this proposition, note that it represents the second homology group of $X^*_\infty$ as

$$\lim_{\substack{\rightarrow \\ N}} \mathbb{Z}^{\mathrm{Irr}(D_{[-N,N]})}, \tag{5.2}$$

since an increasing union of open sets is a inductive limit in the category of topological spaces, and homology commutes with inductive limits ([Spa], Chap. IV, 1.7). This is very similar to example V.3 above, and we will need to look carefully at the inclusions.

**Proof.** (a) With a slightly different definition of $X_{[N,M]}$, this follows from Proposition V.6.

We need to know that $X_{[N,M]}$ is obtained from the projective plane $\mathbb{P}^2$ by a sequence of blow-ups, corresponding naturally to the irreducible components of $D_{[N,M]}$. First notice that we may assume that $N = 0$: clearly shifting the indices gives an isomorphism $X_{[N,M]} \to X_{[0,M-N]}$.

Next observe that $X_{[0,0]} = \tilde{X}_H$, which as we saw is obtained from $\mathbb{P}^2$ by a sequence of blow-ups, each of which creates one component of $\tilde{D} = D_{[0,0]}$ other than $A' = A_0$. The component $A'$ is the proper transform of $l_\infty$ which was there to begin with and generated the homology $H_2(\mathbb{P}^2)$. So the theorem is true when $M = 0$.

If $M = 1$, notice that $X_{[0,1]} = \Gamma_{H^\sharp}$, so the diagram

$$\begin{tikzcd} X_{[0,1]} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X_{[0,0]} \arrow[d, "\pi"'] \\ X_{[0,0]} \arrow[r, "\tilde{H}"] & \mathbb{P}^2 \end{tikzcd}$$

is a fibered product by Proposition IV.4. But the bottom mapping $\tilde{H}$ is an isomorphism on a neighborhood of $\tilde{\mathbf{p}}$, mapping $\tilde{\mathbf{p}}$ to $\mathbf{p} = [0 : 1 : 0]$. Thus the inverse image by $pr_1$ of this neighborhood maps under $pr_1$ to its image just as the inverse image of the neighborhood of $\mathbf{p}$ maps under $\pi$.

This same argument shows that the component $A_1$ of $D_{[0,1]}$ has self-intersection 3. Indeed, the line $A_1 \subset D_{[0,0]}$ has self-intersection $-1$, and the first two blow-ups required to build $X_{[0,1]}$ are blow-ups of points of $A_1$.

Apply the same argument, using the diagram

$$\begin{tikzcd} X_{[0,2]} \arrow[r] \arrow[d] & X_{[1,2]} \arrow[d] \\ X_{[0,1]} \arrow[r] & \tilde{X}_H \end{tikzcd}$$

to show that $X_{[0,2]}$ is constructed from $X_{[0,1]}$ by a sequence of blow-ups, etc.

As above, we see that $A_2$, which has self-intersection $-1$ in $X_{[0,1]}$, has twice a point blown up, and has self-intersection $-3$ in $X_{[0,2]}$; by induction $A_k$ will have self-intersection $-1$ in $X_{[0,k+1]}$ and self-intersection $-3$ in $X_{[0,k+2]}$.

(b) This follows immediately from Propositions V.4 and V.10. □

Next, we need to compute the homomorphism $H_2\left(X_{[-N,N]}\right) \to H_2\left(X_{[-(N+1),N+1]}\right)$ induced by the composition of the isomorphism

$$H_2\left(X_{[-N,N]}\right) \to H_2\left(X^*_{[-N,N]}\right)$$

and the mapping

$$H_2\left(X^*_{[-N,N]}\right) \to H_2\left(X_{[-(N+1),(N+1)]}\right)$$

induced by the inclusion.

**Proposition V.12.** *The homomorphism*

$$i_N : H_2\left(X_{[N,M]}\right) \to H_2\left(X_{[(N-1),M+1]}\right)$$

*described above is given by the following formula:*

$$\begin{aligned}
i_N[C] &= [C] \quad \text{if } C \neq A_{M+1}, A_N \\
i_N[A_{M+1}] &= [A_{M+1}] + [B_{M+1}] + 2[L_{2d-3,M+1}] + 3[L_{2d-4,M+1}] + \cdots + \\
&\qquad d\Big([L_{d-1,M+1}] + [L_{d-2,M+1}] + \cdots + [L_{1,M+1}] + [A_{M+2}]\Big), \\
i_N[A_N] &= [A_N] + [B_{N-1}] + 2[L_{1,N-1}] + 3[L_{2,N-1}] \cdots + \\
&\qquad d\Big([L_{d-1,N-1}] + [L_{d,N-1}] + \ldots\cdots + [L_{2d-3,N-1}] + [A_{N-1}]\Big).
\end{aligned}$$

**Proof.** This is a straightforward verification, using Proposition V.8. The following sequence of figures should explain exactly the sequence of blow-ups.

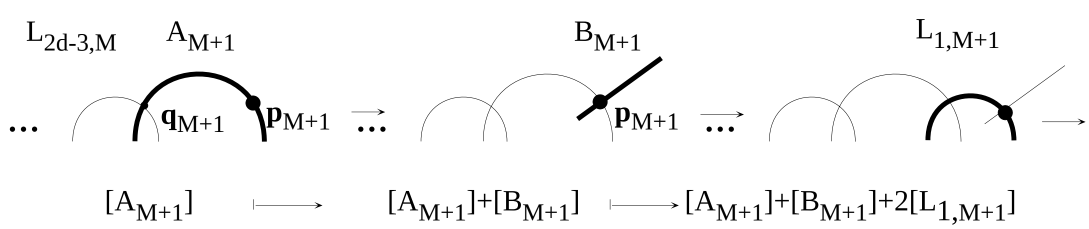


FIGURE 10. The first two blow-ups performed on $A_{M+1} \subset X_{[N,M]}$. Note that each of $A_{M+1}$ and $B_{M+1}$ contribute 1 to the coefficient of the exceptional divisor $L_{1,M+1}$.

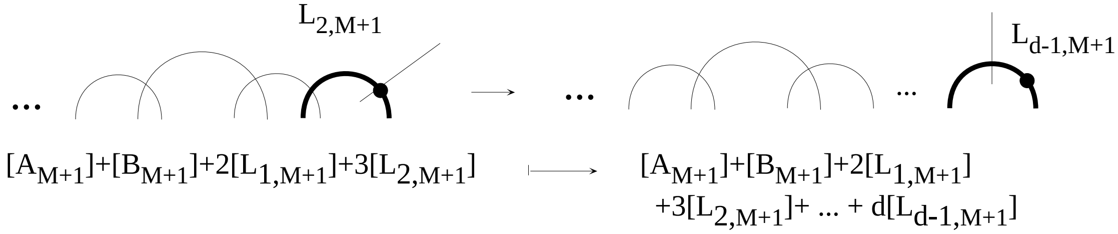


FIGURE 11. The next blow-up and the configuration after $d$ blow-ups. For the figure on the right, $B_{M+1}$ contributes 1 and $2L_{1,M+1}$ contributes 2 to the coefficient of the exceptional divisor $L_{2,M+1}$.

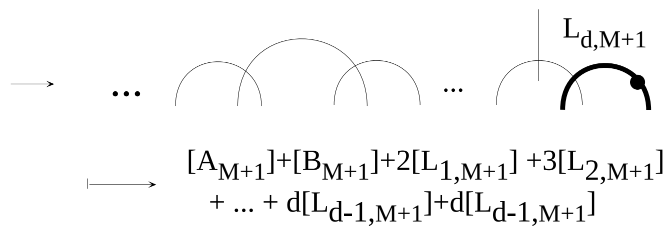


FIGURE 12. The configuration after $d+1$ blow-ups. $dL_{d-1,M+1}$ contributes $d$ to the coefficient of the exceptional divisor $L_{d,M+1}$, and it is the only contribution since this time we are blowing up an ordinary point.

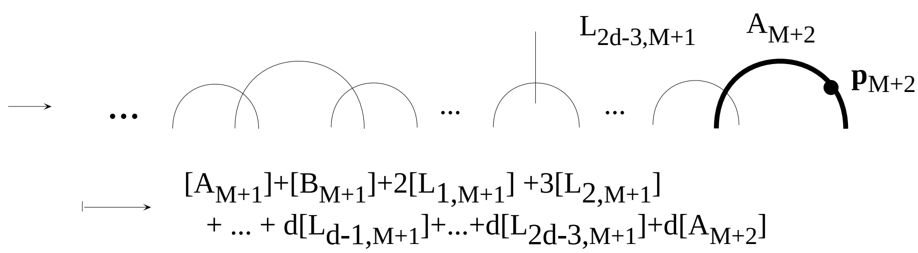


FIGURE 13. The configuration after all the blow-ups required to pass from $X_{[N,M]}$ to $X_{[N,M+1]}$ have been made. We have blown up ordinary points on lines with weight $d$, so the new exceptional divisor always has weight $d$.

$\square$

**Theorem V.13.** *a) The inductive limit*

$$\lim_{\substack{\rightarrow \\ N}} H_2(X_{[-N,N]}) = H_2(X^*_\infty)$$

*embeds naturally in* $\mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$.

*b) If* $\underline{v} \in \mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$ *is an element of* $H_2(X^*_\infty)$, *then the limits*

$$\nu^+(\underline{v}) = lim_{n\to\infty} \frac{\underline{a}(A_n)}{d^n} \quad \text{and} \quad \nu^-(\underline{v}) = lim_{n\to\infty} \frac{\underline{a}(A_{-n})}{d^n}$$

*both exist, since the sequences are eventually constant.*

*c) The sequence*

$$0 \to \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))} \to H_2(X^*_\infty) \overset{(\nu^+,\nu^-)}{\to} Z[1/d] \oplus Z[1/d] \to 0 \tag{5.3}$$

*is exact.*

**Proof.** a) Any element $\underline{v}$ of the inductive limit is the image of some $\underline{v}_N \in H_2(X_{[-N,N]}) = \mathbb{Z}^{\mathrm{Irr}(D_{[-N,N]})}$ for all sufficiently large $N$, and the coefficient $\underline{v}_N(L)$ of any irreducible component $L \in \mathrm{Irr}(D_{[-N,N]})$ is then the same as the coefficient $\underline{v}_{N'}(L)$ for all $N' \geq N$ by Proposition V.12. This proves (a).

The element $\underline{v}$ of the limit is entirely determined by the corresponding element of $v_N \in H_2(X_{[-N,N]})$. In particular, $v_N$ assigns some integer weights $\alpha$ to $[A_{-N}]$ and $\beta$ to $[A_{N+1}]$. Then, again by Proposition V.12, we see that

$$\begin{aligned}
&\underline{v}(A_{-N-1}) = d\alpha,\ \underline{v}(A_{-N-2}) = d^2\alpha, \dots \\
&\underline{v}(A_{N+2}) = d\beta,\ \underline{v}(A_{N+3}) = d^2\beta, \dots .
\end{aligned}$$

In particular, the sequences defining $\nu^-$ and $\nu^+$ are constant after $N$, so the limits exist. This proves (b).

For any element $\underline{v} \in \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$, there exists $N$ such that $\underline{v}$ has coefficient 0 for all irreducible components $L \in \mathrm{Irr}(D^*_\infty$ which do not belong to $D_{[-N,N]}$. Then $\underline{v}$ is in the image of $H_2(X_{[-(N+1),N+1]}$, and we see that $\mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$ is included in $H_2(X^*_\infty)$. Clearly it is the kernel of the mapping $(\nu^-,\nu^+)$, which is surjective. $\square$

The exact sequence (5.3) is naturally split: call $v^-$, $v^+ \in H_2(X^*_\infty)$ the images of $[A']$ and $[\tilde{A}]$ in $H_2(X_{[0,0]}) = H_2(\tilde{X}_H)$ under the inclusions

$$H_2(X_{[0,0]}) \to H_2(X_{[-1,1]}) \to H_2(X_{[-2,2]}) \to \dots .$$

Then $\nu^+(v^+) = \nu^-(v^-) = 1$, and another way of stating part (c) is that for any element $\underline{v}$ of the inductive limit $\varinjlim_N H_2(X_{[-N,N]})$ there exists a unique pair $(a,b) = (\nu^-(\underline{v}), \nu^+(\underline{v})) \in \mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d]$ such that $v - av^- - bv^+$ belongs to $\mathbb{Z}^{(\mathrm{Irr}(D_\infty))}$.

**Theorem V.14.** *The Hénon mapping $H_\infty : X^*_\infty \to X^*_\infty$ induces a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))} & \longrightarrow & H_2(X^*_\infty) & \xrightarrow{(\nu^-,\nu^+)} & \mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d] & \longrightarrow & 0 \\
\downarrow & & \alpha\downarrow & & H_2(H_\infty)\downarrow & & \downarrow\beta & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))} & \longrightarrow & H_2(X^*_\infty) & \longrightarrow & \mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d] & \longrightarrow & 0,
\end{array}$$

*where $\alpha$ is the shift*

$$\alpha([A_k]) = [A_{k-1}] \quad , \quad \alpha([B_k]) = [B_{k-1}] \quad , \quad \alpha([L_{i,k}]) = [L_{i,k-1}]$$

*and $\beta$ is the mapping $\beta(a, b) = (a/d, bd)$.*

**Proof.** The action of $H$ on the homology is induced by shifting (to the left) by one block in $\mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$. Clearly, this induces the same shift on $\mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$, and the statement about $\alpha$ is true. The see that $\beta$ is correct, consider a homology class $x \in \mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$ in the image of $H_2\left(X_{[-N,N]}\right)$. It will satisfy

$$x_{A_{N+1}} = b,\ x_{A_{N+2}} = db,\ x_{A_{N+3}} = d^2 b, \dots,$$

for some $b \in \mathbb{Z}$, and $\nu^+(x) = b/d^N$. The sequence $(H_2(H_\infty))(x)$ is the same sequence shifted, so that

$$\big(H_2(H_\infty)\big)(x)_{A_{N+1}} = db,\ \big(H_2(H_\infty)\big)(x)_{A_{N+2}} = d^2 b,\ \big(H_2(H_\infty)\big)(x)_{A_{N+3}} = d^3 b, \dots,$$

and

$$\nu^-\big(H_2(H_\infty)\big)(x) = \frac{db}{d^N} = d\nu^-(x).$$

The computation for $\nu^+$ is similar. $\square$

One way of understanding the exact sequence (5.3) is as part of the homology exact sequence of the pair $D^*_\infty \subset X^*_\infty$.

**Proposition V.15.** *a) There exists a unique isomorphism*

$$\mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d] \to H_2(X^*_\infty, D^*_\infty)$$

*which makes the diagram*

$$\begin{tikzcd} H_2(X^*_\infty) \arrow[r] \arrow[d, "Id"'] & \mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d] \arrow[d] \\ H_2(X^*_\infty) \arrow[r] & H_2(X^*_\infty, D^*_\infty) \end{tikzcd}$$

*commute.*

*b) Both $H_3(X^*_\infty)$ and $H_3(X^*_\infty, D^*_\infty)$ are isomorphic to $\mathbb{Z}$, and the canonical map*

$$H_3(X^*_\infty) \to H_3(X^*_\infty, D^*_\infty)$$

*is an isomorphism.*

*c) Both $H_1(X^*_\infty)$ and $H_1(X^*_\infty, D^*_\infty)$ are zero.*

**Remark.** We will see in Section IX that the homology group $H_2(X^*_\infty, D^*_\infty)$ can also be understood as $H_1(S^3 - (\Sigma^+ \cup \Sigma^-))$, where $\Sigma^+$ and $\Sigma^-$ are solenoids embedded in a 3-sphere obtained by an appropriate real oriented blow-up. A classical result of algebraic topology asserts that for the standard $d$-adic solenoid $\Sigma_d$ embedded in the 3-sphere in the standard way, $H_1(S^3 - \Sigma_d) = \mathbb{Z}[\frac{1}{d}]$. This explains why these bizarre groups are appearing in this complex-analytic setting, by making precise sense of the sentence "at infinity, $D^*$ has two solenoids".

**Proof.** The exact sequence of the pair $D^*_\infty \subset X^*_\infty$ reads in part

$$H_2(D^*_\infty) \to H_2(X^*_\infty) \to H_2(X^*_\infty, \mathcal{D}^*_\infty) \to H_1(D^*_\infty),$$

Clearly the first term is precisely $\mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$, and the last terms vanishes, since $D^*_\infty$ is a union of 2-spheres identified at points, with the quotient topology from the disjoint union. This proves (a).

Another part of the long exact sequence reads

$$H_3(D^*_\infty) \to H_3(X^*_\infty) \to H_3(X^*_\infty, D^*_\infty) \to H_2(D^*_\infty) \to H_2(X^*_\infty)$$

and the fact that the last map is injective and that $H_3(\mathcal{D}^*_\infty) = 0$ says that the canonical map

$$H_3(X^*_\infty) \to H_3(X^*_\infty, D^*_\infty)$$

is an isomorphism.

To see what it is an isomorphism between, notice first that $H_3(\mathbb{P}^2) = 0$, and it then follows from V.7 that $H_3(X_{[N,M]}) = 0$ for all $N, M$. Next, the inclusion

$$X^*_{[N,M]} \in X_{[N,M]}$$

induces (still by V.7 for the final 0) an exact sequence

$$0 \to \underset{\mathbb{Z}}{\underset{\|}{H_4(X_{[N,M]})}} \to \underset{\mathbb{Z}^2}{\underset{\|}{H_4(X_{[N,M]}, X^*_{[N,M]})}} \to H_3(X^*_{[N,M]}) \to \underset{0}{\underset{\|}{H_3(X_{[N,M]})}}.$$

Thus $H_3(X^*_{[N,M]})$ is canonically the quotient of $\mathbb{Z}^2$ by the image of $\mathbb{Z}$ under the diagonal map, i.e., it is isomorphic to $\mathbb{Z}$.

The argument for (c) is similar but easier. □

**The intersection form on the homology.**

The homology space $H_2\left(X^*_\infty\right)$ carries a quadratic form coming from intersection. We can make it explicit as follows.

**Proposition V.16.** *On $\mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$, the quadratic form is determined by the self-intersections and mutual intersections of the irreducible components of $D^*_\infty$.*

*The classes $v^+$ and $v^-$ satisfy the following rules:*

$$
\begin{gathered}
v^+ \cdot v^+ = v^- \cdot v^- = -1 \quad , \; v^+ \cdot v^- = 0 \\
v^+ \cdot [L_{2d-3,0}] = v^- \cdot [L_{1,0}] = 1, \; v^+ \cdot [A_1] = v^- \cdot [A_0] = -1
\end{gathered}
\tag{5.4}
$$

*with all other intersections* 0.

**Proof.** The statement about $\mathbb{Z}^{\mathrm{Irr}(D^*_\infty)}$ should be clear.

For the other classes, one way to do it is to construct a differentiable surface $C^+ \subset \tilde{X}_H$ (not an algebraic curve), which represents $\tilde{A}$, and which avoids $\tilde{\mathbf{q}}$ and $\mathbf{q}'$. Note that $C^+$ cannot be algebraic (or analytic): the self-intersection of $\tilde{A}$ is $-1$ so it is *rigid* as an algebraic curve. The curve $C^+$ is then contained in $X^*_\infty$ and represents $v^+$. But a neighborhood of the curve $C^+$ is also contained in $X^*_\infty$, so $v^+$ only intersects curves of $D_\infty$ which $C^+$ intersects. Thus $v^+ \cdot v^+ = C^+ \cdot [A_0] = C^+ \cdot C^+ = -1$, and $v^+ \cdot [L_{2d-3,0}] = C^+ \cdot [L_{2d-3,0}] = 1$.

Similarly, construct a differentiable surface $C^- \subset \tilde{X}_H$ which is a deformation of $A'$; clearly we can take $C^+ \cap C^- = \phi$. $\square$

Of course the quadratic form is invariant under the action of $H_\infty$, since this is a homeomorphism of $X^*_\infty$. It certainly isn't obvious from the formulas. Let us check one case, just for consistency's sake. Take $d = 2$. Since $H_\infty$ induces the shift, we see that

$$
(H_\infty)_*(v^+) = 2v^+ + [A_0] + [B_0] + 2[L_{1,0}];
$$

The intersection product gives

$$
\begin{aligned}
\big((H_\infty)_*(v^+)\big)^2 &= 4(v^+)^2 + (A_0)^2 + (B_0)^2 + 4(L_{1,0})^2 + \\
&\qquad 4B_0 \cdot L_{1,0} + 4A_0 \cdot L_{1,0} + 8v^+ \cdot L_{1,0} \\
&= -4 - 3 - 2 - 8 + 4 + 4 + 8 = -1
\end{aligned}
$$

as it should.

This quadratic form on $H_2(X^*_\infty)$ is of course neither positive nor negative definite. For instance $\Delta$, the closure of the diagonal of $\mathbb{C}^2$ in $X^*_\infty$, has self-intersection $+1$, whereas all the irreducible components of $\mathcal{D}^*_\infty$ have negative self-intersection. The following proposition says that the form is mainly negative.

**Theorem V.17.** *The intersection form is negative definite on* $\mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$.

**Proof.** We will give two proofs, one conceptual and one computational. Each proves a stronger (but not the same stronger) result.

**First proof.** An element $v \in \mathbb{Z}^{(\mathrm{Irr}(\mathcal{D}^*_\infty))}$ comes from the homology of some $X_{[N,M]}$ which assigns coefficient 0 to the first and the last exceptional divisor. Its self-intersection in

$X_{[N,M]}$ and in $X_\infty^*$ coincide. We will in fact prove that if $v$ has coefficient 0 with respect to one of these, then $(v \cdot v) < 0$ unless $v = 0$.

Indeed, the complement of the last exceptional divisor in $D_{[N,M]}$, can be blown down to a point, so by a theorem of Grauert [G], the intersection matrix of this divisor is negative definite, hence $v \cdot v$ is negative if $v \neq 0$.

**Second proof.** Let us call $a_n, b_n, x_{i,j}$ the coefficients of $A_n, B_n, L_{i,j}$ respectively. Thus we are considering the quadratic form

$$\begin{aligned} \dots - 3a_n^2 + 2a_n x_{n,1} - 2x_{n,1}^2 + 2x_{n,1}x_{n,2} + \dots - 2x_{n,d-2}^2 + 2x_{n,d-2}x_{n,d-1} \\ - db_n^2 + 2b_n x_{n,d-1} - 2x_{n,d-1}^2 + 2x_{n,d-1}x_{n,d} + \dots + 2x_{n,2d-3}a_{n+1} - 3a_{n+1}^2 + \dots . \end{aligned}$$

It is clearly enough to show that the quadratic form obtained by allocating half the coefficient $a_n$ to the next term and half to the previous term is negative definite, i.e., that the quadratic term in $2d$ variables

$$\begin{aligned} &-\frac{3}{2}a_0^2 + 2a_0x_1 - 2x_1^2 + 2x_1x_2 + \dots - 2x_{d-2}^2 + 2x_{d-2}x_{d-1} \\ &- db^2 + 2bx_{d-1} - 2x_{d-1}^2 + 2x_{d-1}x_d + \dots + 2x_{2d-3}a_1 - \frac{3}{2}a_1^2 \end{aligned}$$

is negative definite. This is something like working in one block at a time.

If we isolate the terms containing $b$, and complete squares, we find that this quadratic form can be written

$$\begin{aligned} &-\left(db^2 - 2bx_{d-1} + \frac{1}{d}x_{d-1}^2\right) \\ &-\frac{3}{2}a_0^2 + 2a_0x_1 - 2x_1^2 + 2x_1x_2 + \dots - 2x_{d-2}^2 + 2x_{d-2}x_{d-1} \\ &-\left(2 - \frac{1}{d}\right)x_{d-1}^2 + 2x_{d-1}x_d - 2x_d^2 \cdots + 2x_{2d-3}a_1 - \frac{3}{2}a_1^2. \end{aligned}$$

If we complete squares from both ends, we can write this as

$$\begin{aligned} &-\left(db^2 - 2bx_{d-1} + \frac{1}{d}x_{d-1}^2\right) \\ &-\left(\frac{3}{2}a_0^2 - 2a_0x_1 + \frac{2}{3}x_1^2\right) - \left(\frac{3}{2}a_1^2 - 2a_1x_{2d-3} + \frac{2}{3}x_{2d-3}^2\right) \\ &-\left(\frac{4}{3}x_1^2 - 2x_1x_2 + \frac{3}{4}x_2^2\right) - \left(\frac{4}{3}x_{2d-3}^2 - 2x_{2d-3}x_{2d-2} + \frac{3}{4}x_{2d-2}^2\right) \\ &- \dots - \dots \\ &-\left(\frac{d+1}{d}x_{d-2}^2 - 2x_{d-2}x_{d-1} + \frac{d}{d+1}x_{d-1}^2\right) - \left(\frac{d+1}{d}x_d^2 - 2x_dx_{d-1} + \frac{d}{d+1}x_{d-1}^2\right) \\ &-\frac{d-1}{d(d+1)}x_{d-1}^2. \end{aligned}$$

□

It works, with a tiny bit to spare, so we actually get a slightly stronger result:

**Proposition V.18.** *There exists $K$ depending only on $d$ such that for any $v \in \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))}$, we have*

$$\frac{1}{K}\sum v_i^2 \le -v\cdot v \le K\sum v_i^2.$$

Thus we can complete $\mathbb{C}^{(\mathrm{Irr}(D^*_\infty))}$ with respect to the intersection inner product, to get a Hilbert space, which we denote $\hat{H}_2^-(X_\infty - \{\mathbf{p}^\infty, \mathbf{q}^\infty\}; \mathbb{C})$. By Proposition V.18, this intersection product norm is equivalent to the $l_2$ norm on the space of sequences.

The exact sequence

$$0 \to \mathbb{Z}^{(\mathrm{Irr}(D^*_\infty))} \to H_2(X^*_\infty) \to \mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d] \to 0$$

gives, tensoring with $\mathbb{C}$,

$$0 \to \mathbb{C}^{(\mathrm{Irr}(D^*_\infty))} \to H_2(X^*_\infty;\mathbb{C}) \to \mathbb{C}\oplus\mathbb{C} \to 0$$

so it is natural to complete the entire homology, i.e., to set

$$\hat{H}_2(X^*_\infty;\mathbb{C}) = \hat{H}_2^-(X^*_\infty;\mathbb{C}) \oplus \mathbb{C}v^+ \oplus \mathbb{C}v^-.$$

On this completed homology space (unlike homology with infinite chains, dual of cohomology with compact supports), the inner product is still defined (for instance by the formulas (5.4)).

Clearly the subspace $\mathbb{C}^{(\mathrm{Irr}(D^*_\infty))} \subset H_2(X^*_\infty;\mathbb{C})$ is invariant under the Hénon mapping $H$, which is simply a shift in $D_\infty$, so it induces a unitary operator on the Hilbert space $\hat{H} : \hat{H}_2^-(X_\infty - \{p^\infty, q^\infty\}; \mathbb{C})$. This unitary operator has only continuous spectrum, on the unit circle, and with spectral density $2d-1$. There are in addition two eigenvectors of

$$(H_\infty)_* : \hat{H}_2(X^*_\infty;\mathbb{C}) \to \hat{H}_2(X^*_\infty;\mathbb{C}),$$

one with eigenvalue $d$ and one with eigenvalue $1/d$. One way of defining them is as

$$w^+ = \lim_{n\to\infty}\frac{1}{d^n}(H_\infty)^n_*(v^+) \text{ and } w^- = \lim_{n\to\infty}\frac{1}{d^n}(H_\infty)^{-n}_*(v^-).$$

These do belong to the completed homology (but not to the homology), since $w^+$ is $v^+$ on the positive part of $\mathrm{L}^*_\infty$, and decreases like a geometric series on the negative part.

These homology classes are already well known in the theory: they are the homology classes of the currents $\mu^-$ and $\mu^+$, as defined by [BS1].

## VI. Real oriented blow-ups of complex divisors in surfaces

In this section we will describe a technique which will allow us, among other things, to understand the structure of $X_\infty$ at the bad points $\mathbf{p}^\infty, \mathbf{q}^\infty$. Given a divisor $D \subset X$ in a surface, the idea is to take an open tubular neighborhood $W$ of $D$ in $X$, together with a projection $\pi : \bar{W} \to D$. Excise $W$, to form $X' = X - W$. If $W$ was chosen properly, $X'$ will be a real 4-dimensional manifold with boundary $\partial X' = \partial \bar{W}$. The interior of $X'$ will be homeomorphic to $X - D$. We will think of $X'$ as some sort of blow-up of $X$ along $D$, with $\pi : \partial X' \to D$ the exceptional divisor.

This picture is the right one to keep in mind, but it is non-canonical. Further, the spaces $\partial X', X'$ have no natural algebraic structure (or even analytic). In this section we will formalize the construction above when $W$ is the "first-order infinitesimal tubular neighborhood" of $D$. This will involve considering $X$ and $D$ as *real algebraic objects*, a 4-dimensional smooth variety $X_\mathbb{R}$ and a singular surface $D_\mathbb{R} \subset X_\mathbb{R}$.

In our definition II.3 of blow-ups, and in the associated construction, we never used that we were working over an algebraically closed field; section II applies just as well if the underlying field is $\mathbb{R}$ (or any field). Let $Y \subset X$ be a complex algebraic manifold and a subspace. We will call $X_\mathbb{R}$ the manifold $X$ seen as a real algebraic manifold, $Y_\mathbb{R} \subset X_\mathbb{R}$ the subspace defined by the ideal $I_\mathbb{R}(Y)$ of real-algebraic functions on $X_\mathbb{R}$ which vanish on the real-algebraic subset $Y$.

**Remark.** This construction of $Y_\mathbb{R}$ does not coincide with the more obvious idea of taking a chart $U \subset X$ with $U \subset \mathbb{C}^n$, and complex equations $f_j : U \to \mathbb{C}$ such that $Y \cap U$ corresponds to the ideal generated by $f_1, \dots, f_k$. Then consider the subset of $U \subset \mathbb{R}^{2n}$ with coordinates $x_1 + iy_1 = z_1, \dots, x_n + iy_n = z_n$ and the locus defined by the $2k$ equations $\mathrm{Re} f_1 = 0, \mathrm{Im}(f_1) = 0, \dots, \mathrm{Re}(f_k) = 0, \mathrm{Im}(f_k) = 0$, or equivalently by the ideal generated by the $\{\mathrm{Re}(f_j),\ \mathrm{Im}(f_j)\}$. Even in simple cases, these two ideals are different.

**Example VI.1.** Consider $X = \mathbb{C}^2$, with $Y$ given by the equation $xy = 0$. Then the construction above yields the locus in $\mathbb{R}^4$ defined by the equations

$$x_1 x_2 - y_1 y_2 = 0 \quad \text{and} \quad x_1 y_2 + x_2 y_1 = 0. \tag{6.1}$$

But the algebraic set $Y \subset \mathbb{R}^4$ is the union of the $x_1, y_1$-plane and the $x_2, y_2$-plane; the ideal $I_\mathbb{R}(Y)$ is generated by the four elements

$$x_1 x_2,\ x_1 y_2,\ y_1 x_2,\ y_1 x_2. \tag{6.2}$$

In Section II we saw that this ideal cannot be generated by fewer elements, since the blow-up of $\mathbb{C}^4$ along the subspace it defines is not a subset of $\mathbb{P}^1$ or $\mathbb{P}^2$. Thus the ideals are different, and in fact the subset defined by the elements (6.1) consists in $\mathbb{C}^4$ of four planes, two of which are our real planes, and the others intersect the real locus only at the origin. Our space $Y_\mathbb{R}$ is the one given by the ideal generated by the elements (6.2).

The ideal $Y_{\mathbb{R}}$ appears to be quite difficult to compute in general. For instance, if $X = \mathbb{C}^2$ and $Y$ is the curve of equation $x^2 + y^3 = 0$, we do not know how to write generators for the corresponding ideal $I_{\mathbb{R}}(Y)$. $\triangle$

We will study the blow-up of $X_{\mathbb{R}}$ along $Y_{\mathbb{R}}$. The objects we obtain this way are very different from the blow-up of $X$ along $Y$. In particular, when $Y$ is a divisor in $X$, the complex blow-up doesn't change it at all, whereas as a real variety it is of codimension 2, and the real blow-up is quite different from the original space, as we will see. We will denote this blow-up by $\mathcal{B}_{\mathbb{R}}(X,Y)$, rather than $(\widetilde{X_{\mathbb{R}}})_{Y_{\mathbb{R}}}$, both to lighten notation, and to emphasize the different, more topological way we think of real blow-ups.

**Remark.** In section II, we were working with analytic spaces, with structure sheaves which might contain nilpotents, etc. This is if anything even more important here: the origin in $\mathbb{R}^2$ defined by the two equations $x = 0,\ y = 0$ and the origin defined by the single equation $x^2 + y^2 = 0$ are completely different objects, and their blow-ups have no obvious relation. $\triangle$

The space $\mathcal{B}_{\mathbb{R}}(X,Y)$ is a real-algebraic space, but it is not the boundary of a tubular neighborhood we are after; we will need to modify it a bit. To avoid interrupting the formal definition, we isolate a purely topological construction we will need: *prime ends.*

**Prime Ends.**

**Definition VI.2.** Let $Z \subset U$ be a topological space and a closed subset. Define the set of *prime ends* $P_z(U,Z)$ of $U - Z$ at $z \in Z$ to be the projective limit

$$P_z(U,Z) = \lim_{\leftarrow} \pi_0(V - Z)$$

where the projective limit is over all neighborhoods $V$ of $z$ in $U$ with respect to inclusions, and $\pi_0$ is the functor which associates to a topological space its set of connected components.

Now consider the *prime end modification* of $U$ along $Z$, which is the disjoint union

$$\mathcal{P}(U,Z) = (U - Z) \sqcup \coprod_{z \in Z} P_z(U,Z),$$

where the topology on $U - Z$ in unchanged, and a basis of neighborhoods of a point $\tilde{z} \in P_z(U,Z)$ is obtained by choosing a basis of neighborhoods of $z$ in $U$, and taking the closure in $U$ of the component of $U - Z$ corresponding to $\tilde{z}$. This space comes with an obvious projection $\mathcal{P}(U,Z) \to U$, which is continuous. With slight abuse of notation, we will denote $\mathcal{P}(Z) \subset \mathcal{P}(U,Z)$ the inverse image of $Z$ in $\mathcal{P}(U,Z)$.

The following properties of the prime end modification are left to the reader:

- The prime end modification of a Hausdorff space is Hausdorff.

• When $U$ is a locally finite simplicial complex, and $Z \subset U$ is a simplicial subset, $P_z(U,Z)$ is finite, and the projective limit is achieved by taking $V$ the star of $z$, i.e., the union of the open simplices with $z$ in their closure.

• The prime end modification of a simplicial complex along a subcomplex is a simplicial complex.

• If $F : U_1 \to U_2$ is continuous, $Z_2$ is closed in $U_2$ and $Z_1 = F^{-1}(Z_2)$, then there exists a unique mapping

$$\mathcal{P}(F) : \mathcal{P}(U_1, Z_1) \to \mathcal{P}(U_2, Z_2) \tag{6.3}$$

making the diagram

$$\begin{array}{ccc} \mathcal{P}(U_1, Z_1) & \xrightarrow{\mathcal{P}(F)} & \mathcal{P}(U_2, Z_2) \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 \end{array}$$

commute.

**Definition of the real oriented blow-up.**

For real algebraic varieties, it makes sense to speak of the oriented blow-up. This takes two steps:

**Step 1.** Let $X$ be an analytic space, and $Y$ a closed subspace. If $Y = f^{-1}(0)$, where $f : X \to \mathbb{C}^{n+1}$ is analytic, recall from Section II that the real blow-up is a subspace $\mathcal{B}_{\mathbb{R}}(X,Y) \subset X_{\mathbb{R}} \times \mathbb{P}^{2n+1}_{\mathbb{R}}$, locally defined by a section $\tilde{f}$ of the tautological line bundle over $X_{\mathbb{R}} \times \mathbb{P}^{2n+1}_{\mathbb{R}}$ which is not a zero-divisor. We will be mainly concerned with the case $n = 0$, so $Y$ is a divisor in $X$, but $Y_{\mathbb{R}}$ is of codimension 2 in $X_{\mathbb{R}}$.

The sphere $S^{2n+1}$ is naturally a double cover of $\mathbb{P}^{2n+1}_{\mathbb{R}}$: to every $s \in S^{2n+1}$ we can consider the line $l_s = s\mathbb{R} \in \mathbb{P}^{2n+1}$, and the double cover is simply $s \mapsto l_s$. Consider the inverse image $\mathcal{B}^*_{\mathbb{R}}(X,Y)$ of $\mathcal{B}_{\mathbb{R}}(X,Y)$ in $X_{\mathbb{R}} \times S^{2n+1}$, now defined by a section $f^*$ of the tautological bundle on $X \times S^{2n+1}$ (note that the tautological line bundle over $S^{2n+1}$ is simply the normal bundle, so that $f^*$ is a normal vector field on $S^{2n+1}$). Since this bundle is oriented (in fact trivial), it makes sense to consider the subset

$$\mathcal{B}^{**}_{\mathbb{R}}(X,Y) \subset \mathcal{B}^*_{\mathbb{R}}(X,Y)$$

defined by $f^* \geq 0$, with $\mathcal{B}^{**}_{\mathbb{R}}(Y) \subset \mathcal{B}^{**}_{\mathbb{R}}(X,Y)$ defined by $f^* = 0$ as the exceptional divisor.

**Remark.** The object of passing to the double cover is to make the tautological bundle oriented, so as to be able to speak of $f^* \geq 0$. $\triangle$

**Step 2.** The pair

$$\mathcal{B}^{**}_{\mathbb{R}}(Y) \subset \mathcal{B}^{**}_{\mathbb{R}}(X,Y)$$

is almost the oriented real blow-up, but sometimes it has singularities which we want to eliminate, using prime ends. Define the *oriented real blow-up* $\mathcal{B}^+_{\mathbb{R}}(X,Y)$ and the exceptional divisor $\mathcal{B}^+_{\mathbb{R}}(Y) \subset \mathcal{B}^+_{\mathbb{R}}(X,Y)$ to be the prime end modification

$$\mathcal{B}^+_{\mathbb{R}}(Y) = \mathcal{P}(\mathcal{B}^{**}_{\mathbb{R}}(Y)) \subset \mathcal{P}\big(\mathcal{B}^{**}_{\mathbb{R}}(X,Y), \mathcal{B}^{**}_{\mathbb{R}}(Y)\big) = \mathcal{B}^+_{\mathbb{R}}(X,Y).$$

of $\mathcal{B}^{**}_{\mathbb{R}}(X,Y)$ along $\mathcal{B}^{**}_{\mathbb{R}}(Y)$.

We will denote by $p_{(X,Y)} : \mathcal{B}^+_{\mathbb{R}}(X,Y) \to X$ the canonical projection, and we will omit the subscript when there is no ambiguity.

**Example VI.3.** Let us construct the real oriented blow-up of the origin $Y$ in $\mathbb{C}$. Set $z = x + iy$, so that as a real algebraic subspace the origin $Y_{\mathbb{R}}$ is defined by the equations $x = y = 0$, and $\mathcal{B}_{\mathbb{R}}(\mathbb{C}, Y) \subset \mathbb{C} \times \mathbb{P}^1_{\mathbb{R}}$ is the space

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2 \times \mathbb{P}^1_{\mathbb{R}} \mid xv = uy \right\}.$$

The set $\mathcal{B}^*_{\mathbb{R}}(\mathbb{C}, Y)$ comes from replacing $\mathbb{P}^1_{\mathbb{R}}$ by $S^1$, i.e., it is the set

$$\mathcal{B}^*_{\mathbb{R}}(\mathbb{C}, Y) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2 \;\Big|\; u^2 + v^2 = 1 \text{ and } xv = uy \right\}.$$

The set $\mathcal{B}^{**}_{\mathbb{R}}(\mathbb{C}, Y) \subset \mathcal{B}^*_{\mathbb{R}}(\mathbb{C}, Y)$ is the subset where $\begin{pmatrix} x \\ y \end{pmatrix}$ is a non-negative multiple of $\begin{pmatrix} u \\ v \end{pmatrix}$, in other words, it is the set

$$\mathcal{B}^{**}_{\mathbb{R}}(\mathbb{C}, Y) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2 \;\Big|\; u^2 + v^2 = 1,\; xv = uy \text{ and } ux \geq 0 \right\}.$$

The mapping $(\mathbb{R}/2\pi\mathbb{Z}) \times [0, \infty) \to \mathcal{B}^{**}_{\mathbb{R}}(\mathbb{C}, Y)$ given by

$$(\theta, r) \mapsto \left( r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right)$$

is a homeomorphism; so every point of the exceptional divisor $(\mathbb{R}/2\pi\mathbb{Z}) \times \{0\}$ corresponds to a unique prime end, and $\mathcal{B}^+_{\mathbb{R}}(\mathbb{C}, Y) = \mathcal{B}^{**}_{\mathbb{R}}(\mathbb{C}, Y)$. You might prefer to think of the real oriented blow-up as passing to polar coordinates, without identifying the different angles when the radius is 0.

We will refer to points of the exceptional divisor $\mathbb{R}/2\pi Z$ as points "infinitesimally near the origin", whose modulus is zero, but which still have a polar angle.

In this case, the exceptional divisor is the circle $\mathbb{R}/2\pi\mathbb{Z}$, but the identification with the circle depends on the chart. In general, if you take the real oriented blow-up of a Riemann surface at a point, the circle $\mathbb{R}/2\pi\mathbb{Z}$ will act freely and transitively on the exceptional divisor, but there will be no distinguished point on this divisor. More generally yet, the real oriented blow-up of a surface along a smooth curve will be a principle circle-bundle, but not naturally trivial (and usually topologically non-trivial: see Section VIII).

**Naturality of the real oriented blow-up.**

This definition seems (and perhaps it is) a little ad-hoc. Still: it has some nice properties. For instance, the real oriented blow-up is always homeomorphic to a simplicial complex. Indeed, the space $\mathcal{B}^{**}_{\mathbb{R}}(X,Y)$ is obviously subalgebraic, hence triangulable by a theorem of Hironaka [Hi2]. Since the prime-end modification of a simplicial complex along a subcomplex is still a simplicial complex, the result follows.

Next, notice that it does have a certain weak naturality.

**Proposition VI.4.** *Let $F : X_1 \to X_2$ be an algebraic isomorphism of complex algebraic spaces, $Y_2 \subset X_2$ an algebraic subspace, and $Y_1 = F^{-1}(Y_2)$. Then there exists a unique homeomorphism $\mathcal{B}^+_{\mathbb{R}}(F) : \mathcal{B}^+_{\mathbb{R}}(X_1,Y_1) \to \mathcal{B}^+_{\mathbb{R}}(X_2,Y_2)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{B}^+_{\mathbb{R}}(X_1,Y_1) & \xrightarrow{\mathcal{B}^+_{\mathbb{R}}(F)} & \mathcal{B}^+_{\mathbb{R}}(X_2,Y_2) \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ X_1 & \xrightarrow{F} & X_2 \end{array}$$

*commutes, where the $p_i$ are the canonical projections.*

**Remark.** The mapping $\mathcal{B}^+_{\mathbb{R}}(F)$ is actually an algebraic isomorphism, after an appropriate definition has been given.

**Proof.** It is enough to prove the statement locally on $X_1$, so we may suppose that $(Y_2)_{\mathbb{R}}$ is given by equations. But clearly if $p \in I_{\mathbb{R}}(Y_2)$, then $p \circ F \in I_{\mathbb{R}}(Y_1)$, and since $F$ is invertible, this means that $F^*(I_{\mathbb{R}}(Y_2)) = I_{\mathbb{R}}(Y_1)$. Then Corollary II.8 gives a mapping

$$\mathcal{B}_{\mathbb{R}}(F) : \mathcal{B}_{\mathbb{R}}(X_1,Y_1) \to \mathcal{B}_{\mathbb{R}}(X_2,Y_2)$$

which lifts uniquely to the double cover

$$\mathcal{B}^*_{\mathbb{R}}(F) : \mathcal{B}^*_{\mathbb{R}}(X_1,Y_1) \to \mathcal{B}^*_{\mathbb{R}}(X_2,Y_2)$$

so that oriented lines are mapped to oriented lines. Thus the region where $f^* \geq 0$ is preserved and the map lifts to

$$\mathcal{B}^{**}_{\mathbb{R}}(F) : \mathcal{B}^{**}_{\mathbb{R}}(X_1,Y_1) \to \mathcal{B}^{**}_{\mathbb{R}}(X_2,Y_2).$$

Finally, we find our lift $\mathcal{B}^+_{\mathbb{R}}(F)$ from the naturality of the prime end modification (see 6.3). $\square$

The authors do not know in what generality it is true that the real oriented blow-up of a complex manifold along a closed subspace yields a PL manifold with boundary: is this perhaps always true? In any case, we will require the result only when $X$ is a complex surface and $Y = D = \cup D_i$ is a divisor consisting of a locally finite union of curves intersecting transversally, i.e., a divisor with normal crossings. In that case it is true: the real blow-ups admit the following detailed description.

**Theorem VI.5.** *a) The real blow-up $\mathcal{B}_{\mathbb{R}}(X, D)$ is a 4-dimensional manifold, and $\mathcal{B}_{\mathbb{R}}^{**}(D)$ is a singular 3-dimensional subvariety.*

*b) The real oriented blow-up $\mathcal{B}_{\mathbb{R}}^{+}(X, D)$ is a 4-dimensional manifold with boundary, and the boundary is a manifold with corners.*

Before we give the proof, note the following corollary.

**Corollary VI.6.** *There is a tubular neighborhood $W$ of $D$ in $X$, together with a projection $\bar{p} : \overline{W} \to D$ extending the identity on $D$, and a homeomorphism $h : \mathcal{B}_{\mathbb{R}}^{+}(D) \to \partial W$ making the diagram*

$$\begin{array}{ccc} \mathcal{B}_{\mathbb{R}}^{+}(D) & \xrightarrow{h} & \partial W \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle \bar{p}}\big\downarrow \\ D & \longrightarrow & D \end{array}$$

*commute.*

**Proof of VI.6.** Since $\mathcal{B}_{\mathbb{R}}^{+}(X, D)$ is a manifold with boundary, there exists a continuous mapping $\phi : \mathcal{B}_{\mathbb{R}}^{+}(D) \times [0, 1] \to \mathcal{B}_{\mathbb{R}}^{+}(X, D)$ which is a homeomorphism onto its image, which forms a tubular neighborhood of the boundary. Let $\overline{W}$ be the image of the projection of $\phi(\mathcal{B}_{\mathbb{R}}^{+}(D) \times [0, 1])$ into $X$; it is a neighborhood of $X$ which retracts onto $D$, i.e., a tubular neighborhood of $D$, and clearly its boundary is $\mathcal{B}_{\mathbb{R}}^{+}(D) \times \{1\}$, hence naturally homeomorphic to $\mathcal{B}_{\mathbb{R}}^{+}(D)$. $\square$

**Remark VI.7.** It follows from VI.5 that $\mathcal{B}_{\mathbb{R}}^{+}(D)$ is an orientable manifold, being the boundary of an oriented 4-manifold. However, because we tend to think of $\mathcal{B}_{\mathbb{R}}^{+}(D)$ rather as the boundary of a tubular neighborhood of $D$ than as the boundary of its complement, we will give $\mathcal{B}_{\mathbb{R}}^{+}(D)$ the opposite of the boundary orientation of $\partial\mathcal{B}_{\mathbb{R}}^{+}(X, D)$.

**Proof of VI.5.** The theorem only needs to be proved locally, so it is enough to prove it when $X = \mathbb{C}^2$ and $D$ is the union of the coordinate axes. However, remember that we are considering these as real algebraic varieties, so $X = \mathbb{R}^4$, and $D = \mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2$. We have already computed equations for this blow-up, in equations 2.3:

$$\begin{aligned} U_1U_4 &= U_2U_3 \\ U_4y_1 &= U_2y_2 \\ U_4x_1 &= U_3x_2 \\ U_3y_1 &= U_1y_2 \\ U_2x_1 &= U_1x_2. \end{aligned} \tag{6.4}$$

When we computed these equations, we were working in the complex, but they are just as true in the real.

In particular, since $\widetilde{\mathbb{C}^4}_{\mathbb{C}^2\times 0\cup 0\times\mathbb{C}^2}$ is smooth, so is $\mathcal{B}_{\mathbb{R}}(X,D)$, and $\mathcal{B}_{\mathbb{R}}(D)$ is singular, having locally the structure of two 3-dimensional submanifolds intersecting transversally along a (real) surface in $\mathbb{R}^4$ (see 2.5).

Now, to understand $\mathcal{B}^*_{\mathbb{R}}(X,D)$, we simply have to consider the locus with the same equation, but think of $(U_1,U_2,U_3,U_4)$ as the coordinates of a point in $S^3$ rather than homogeneous coordinates of a point in $\mathbb{P}^3_{\mathbb{R}}$. Note that $\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix}$ is a positive section of the tautological (i.e., normal) bundle to $S^3$, and $\mathcal{B}^*_{\mathbb{R}}(X,D)$ is the locus where

$$\begin{pmatrix}x_1y_1\\x_2y_1\\x_1y_2\\x_2y_2\end{pmatrix} \quad \text{is a multiple of} \quad \begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix}$$

whereas $\mathcal{B}^{**}_{\mathbb{R}}(X,D)$ is the locus where this multiple is $\geq 0$.

**Lemma VI.8.** *The mapping* $\Phi : [0,\infty)^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2 \to \mathbb{R}^4 \times S^3$ *which sends*

$$\big(r_1, r_2 \in [0,\infty), \theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}\big) \quad \textit{to} \quad \left\{\begin{array}{rr} x_1 = r_1\cos\theta_1 & x_2 = r_1\sin\theta_1 \\ y_1 = r_2\cos\theta_2 & y_2 = r_2\sin\theta_2 \\ U_1 = \cos\theta_1\cos\theta_2 & U_2 = \sin\theta_1\cos\theta_2 \\ U_3 = \cos\theta_1\sin\theta_2 & U_4 = \sin\theta_1\sin\theta_2 \end{array}\right\}$$

*maps surjectively to* $\mathcal{B}^{**}_{\mathbb{R}}(X,D)$, *and only the points*

$$(0,0,\theta_1,\theta_2) \quad \textit{and} \quad (0,0,\theta_1+\pi,\theta_2+\pi)$$

*are identified.*

**Proof of VI.8.** Clearly the equations 6.4 are satisfied on the image of $\Phi$. Moreover, on the image of $\Phi$, we have

$$\begin{pmatrix}x_1y_1\\x_2y_1\\x_1y_2\\x_2y_2\end{pmatrix} = r_1r_2\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix}$$

and since $r_1r_2 \geq 0$, the image of $\Phi$ is contained in $\mathcal{B}^{**}_{\mathbb{R}}(X,D)$.

We will try to define an inverse $\Phi^{-1} : \mathcal{B}^{**}_{\mathbb{R}}(X,D) \to [0,\infty)^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2$; our attempt will show the remainder of the lemma.

Clearly if $(x_1, x_2) \neq (0,0)$ and $(y_1, y_2) \neq (0,0)$, it is easy to reconstruct $r_1, r_2, \theta_1, \theta_2$ by representing these points in polar coordinates.

You can still compute the polar coordinates if one of $(x_1, x_2)$ or $(y_1, y_2)$ is the origin of $\mathbb{R}^2$. Suppose for instance it is the first. The difficulty is then to compute $\theta_1$, but this can be found from the second lot of equations, for instance $\cos\theta_1 = U_1/\cos\theta_2$. But when both points are the origin, then the second set of equations only determine $(\theta_1, \theta_2)$ up to the given ambiguity, since the real "Veronese" mapping

$$V : \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \to S^3 \quad , \quad (\theta_1, \theta_2) \mapsto \begin{pmatrix} \cos\theta_1 \cos\theta_2 \\ \sin\theta_1 \cos\theta_2 \\ \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \end{pmatrix}$$

is a double cover of its image. □ VI.8

This shows that the local structure of $\mathcal{B}^{**}_{\mathbb{R}}(X, D)$ is the union of two opposite quadrants, multiplied by a smooth surface, since the domain of $\Phi$ is the first quadrant times a torus, and (corner $\times$ torus) is a double cover of its image. The open quadrants correspond to the locus $r_1 r_2 > 0$, and the prime end closure consists exactly of doubling the singular locus, to make the oriented blow-up a manifold with boundary, the boundary of which is a manifold with corners. In particular, the map $\Phi$ lifts to a homeomorphism of the oriented blow-up. □

We saw that we could understand the real oriented blow-up of $\mathbb{C}$ at 0 in terms of polar coordinates. We will now show that this is still true for the real oriented blow-up of $\mathbb{C}^2$ along $D = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$. We will denote $\mathbb{C}^* = \mathbb{C} - \{0\}$.

**Corollary VI.9.** *The real oriented blow-up $\mathcal{B}^+_{\mathbb{R}}(\mathbb{C}^2, D)$ is the closure of the set*

$$\left\{ (w_1, w_2, \theta_1, \theta_2) \in (\mathbb{C}^*)^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2 \,\middle|\, \frac{w_1}{|w_1|} = e^{i\theta_1}, \ \frac{w_2}{|w_2|} = e^{i\theta_2} \right\}$$

*in* $\mathbb{C}^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2$.

**Proof.** We saw in Lemma VI.8 that the real oriented blow-up is parametrized by $[0,\infty)^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2$. □

From this description, we see that in the real oriented blow-up, when one of the variables is 0, its polar angle remains. More precisely, let $p : \mathcal{B}^+_{\mathbb{R}}(D) \to D$ be the projection of the exceptional divisor.

**Corollary VI.10.** *Let $D \subset$ of $X$ be a divisor with normal crossings in a surface, and $p : \mathcal{B}^+_{\mathbb{R}}(X, D) \to X$ be the canonical projection. The inverse images of points by $p$ admit the following canonical description:*

*a) If $z \in D$ is an ordinary point, $p^{-1}(z)$ is canonically homeomorphic to $\mathbb{S}(T_zX/T_zD)$, where $\mathbb{S}(E)$, the sphere of $E$, is the space of oriented lines in $E$ (i.e., the double cover of $\mathbb{P}_{\mathbb{R}}(E)$). In particular, it is homeomorphic to a circle, and is principal under the circle $\mathbb{R}/2\pi\mathbb{Z}$ acting by rotations.*

*b) Let $z \in D$ be a double point, say locally $z = D_1 \cap D_2$. Then $p^{-1}(z)$ is canonically homeomorphic to*

$$\mathbb{S}\big(T_zX/T_zD_1\big) \times \mathbb{S}\big(T_zX/T_zD_2\big).$$

*In particular, it is homeomorphic to a torus, and principal under $(\mathbb{R}/2\pi\mathbb{Z})\times(\mathbb{R}/2\pi\mathbb{Z})$ acting coordinatewise on the two factors.*

**Corollary VI.11.** *If $X$ is a complex surface, and $D \subset X$ is a smooth curve, then the canonical mapping $p : \mathcal{B}^+_{\mathbb{R}}(D) \to D$ is a principal circle-bundle.*

**Remarks VI.12.** 1) We are using the fact that $T_zX/T_zD$ is a complex vector space of dimension 1 and not just a real vector space of dimension 2 to make the rotations above well-defined.

2) In part (b), we could also say that $p^{-1}(z)$ is canonically homeomorphic to $\mathbb{S}(T_zD_2) \times \mathbb{S}(T_zD_1)$, since there is a canonical homeomorphism $T_zD_2 \to T_zX/T_zD_1$. This seems simpler, but it doesn't help when we want to think of a double point as a limit of ordinary points.

3) Let $D'$ denote $D$ with the double points removed. It follows from this theorem that $p^{-1}(D')$ is a principal circle-bundle over $D'$.

## VII. The effect of complex blow-ups on real oriented blow-ups

Let $z \in D \subset X$ be a point of a divisor in a surface. We now want to show exactly how a complex blow-up of $X$ at $z$ affects the real oriented blow-up of the divisor. This will relate $\mathcal{B}^+_{\mathbb{R}}(X, D)$ and $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$, where $\tilde{D} = \pi^{-1}(D)$ and $\pi : \tilde{X}_z \to X$ is the canonical projection. Clearly $\mathcal{B}^+_{\mathbb{R}}(X, D)$ and $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$ have the same overall topology: the boundary of a tubular neighborhood of the divisor $D$ is still the boundary of a tubular neighborhood of $\tilde{D}$. What is changed is the way the circle acts.

The description will be given in Theorems VII.5 and VII.9, which deal with blowing up a smooth point and a double point of the divisor respectively. Before we state these results, we will isolate a result in differential topology, which is somehow almost obvious, but which we found surprisingly hard to prove. It will be essential to the proof of Theorem VII.11.

**Opening Manifolds.** In this section, we will show that you can "open a manifold along a submanifold", squeezing the outside arbitrarily little. The difficulty of the results comes from the fact that the opening is constrained to respect certain coordinates, whereas the metric with respect to which we are measuring the squeezing is unrelated to those coordinates.

Let $M$ be a compact smooth $k$-dimensional manifold, $E$ a vector space with a Euclidean norm, and let $\mu$ be a continuous Riemannian metric on $M \times E$; we will denote $|\xi|_\mu$ the length of tangent vectors with respect to this metric, and $d_\mu(u,v)$ the distance between $u \in M \times E$ and $v \in M \times E$. Note that we are not assuming any particular relation between the norm $|\ |$ on $E$ and the Riemannian metric $\mu$.

Let us denote by $N_r$ the closed subset

$$N_r = \{(m,x) \in M \times E \mid |x| \leq r\}$$

of $M \times E$, so that $N_0 = M \times \{0\}$, and $N_r$ is a closed tubular neighborhood of $N_0$ when $r > 0$.

**Lemma VII.1.** *For any $\alpha < 1$ and any $R > 0$, there exist $r > 0$ and a $C^\infty$ diffeomorphism*

$$F : (M \times E) - N_0 \to (M \times E) - N_r$$

*such that*

(1) *$F$ maps the fiber $\{m\} \times E$ to itself for every $m \in M$;*
(2) *$F$ is the identity outside $N_R$;*
(3) *$d_\mu(F(u),F(v)) \geq \alpha d_\mu(u,v)$ for all pairs of points $u,v$ in $M \times E - N_0$.*

We will refer to such a mapping $F$ as "opening" $M \times E$ along $M \times \{0\}$.

**Proof.** Fix $\alpha$, $0 < r_0 < R$. Choose a family of increasing diffeomorphisms $\tau_\rho : (0,\infty) \to (\rho,\infty)$ for $0 < \rho$ sufficiently small, such that

(i) All $\tau_\rho(t) = t$ when $t > R$;

(ii) $\tau_\rho$ converges to the identity in the $C^1$ topology on compact subsets of $(0,\infty)$ as $\rho \to 0$.

(iii) $\tau'_\rho(t) > 1$ for $t \leq r_0$.

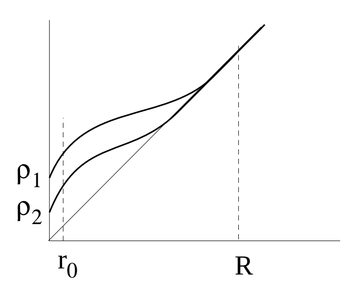


FIGURE 14. The graphs of $\tau_{\rho_1}$ and $\tau_{\rho_2}$.

Consider

$$F_\rho : M \times E - N_0 \to M \times E - N_\rho, \quad (m,x) \mapsto \left(m, \tau_\rho(|x|)\frac{x}{|x|}\right).$$

We claim that $F_\rho$ will do the trick for $\rho$ sufficiently small. Only the third condition poses any problem.

**Sublemma VII.2.** *Recall that $\alpha < 1$ and $R > 0$ are fixed. There exists $\rho_0 > 0$ such that*

$$d_\mu\big(F_\rho(u), F_\rho(v)\big) \geq \sqrt{\alpha}\, d_\mu(u,v)$$

*when $u, v \in N_{\rho_0} - N_0$ and for all $\rho < \rho_0$.*

**Proof of VII.2.** Choose a number $\beta < 1$ such that $\beta^2 > \sqrt{\alpha}$. For each point $(m,0) \in M \times \{0\}$, find $U \subset \mathbb{R}^k, V \subset E$ and a local coordinate $\phi : U \times V \to M \times E$ at $(m,0)$ such that the diagram

$$\begin{array}{ccc} U \times V & \longrightarrow & M \times E \\ {\scriptstyle pr_1}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ U & \longrightarrow & M \end{array}$$

commutes.

By making our coordinate neighborhood small enough, we can assume that the ratio of the constant (Euclidian) Riemannian metric $\mu_0$ on $U \times V$, given by the Riemannian metric on $T_{(m,0)}M \times E$, and the given Riemannian metric $\mu$ (viewed as a metric on $U \times V$ via $\phi$) satisfy

$$\beta < \frac{|\xi|_{\mu_0}}{|\xi|_\mu} < \frac{1}{\beta},$$

for all vectors $\xi$ tangent to $M \times E$ at a point in $\phi(U \times V)$.

By taking $\rho$ sufficiently small, we can assume that there exists $V' \subset V$, such that the mapping

$$\phi^{-1} \circ F_\rho \circ \phi : U \times V' \to U \times V$$

is defined and non-decreasing on tangent vectors, for the metric $\mu_0$, so it will decrease the length of tangent vectors at most by a factor of $\beta^2$ for the metric $\mu$. Since $M$ is compact, such coordinate neighborhoods will cover $N_{\rho_0}$ for $\rho_0 > 0$ sufficiently small. □ VII.2

Now take $u, v \in M \times E$, and let $\delta$ be a length-minimizing geodesic joining $F(u)$ to $F(v)$ (which we will temporarily assume exists). Let $\gamma = F^{-1}(\delta)$; of course, there is no reason to think that $\gamma$ is length-minimizing.

Consider $\gamma_1$ the part of $\gamma$ in $N_{\rho_0}$ and $\gamma_2$ the remainder. Then the $\mu$-length of $F_\rho(\gamma_1)$ is contracted at most by $\sqrt{\alpha}$ when $\rho < \rho_0$, whereas the $\mu$-length of $F_\tau(\gamma_2)$ is contracted arbitrarily little when $\rho$ is sufficiently small, since on $M \times E - M_{\rho_0}$ the mappings $F_\rho$

converge to the identity in the $C^1$ topology. Thus for any $\epsilon$, we may assume that $\gamma$ is contracted by $\sqrt{\alpha}-\epsilon$, which we may take to be $>\alpha$ by taking $\epsilon$ sufficiently small. Then we have

$$d_\mu\big(F(u),F(v)\big)=l_\mu(\delta)\geq\alpha l_\mu(\gamma)\geq\alpha d_\mu(u,v).$$

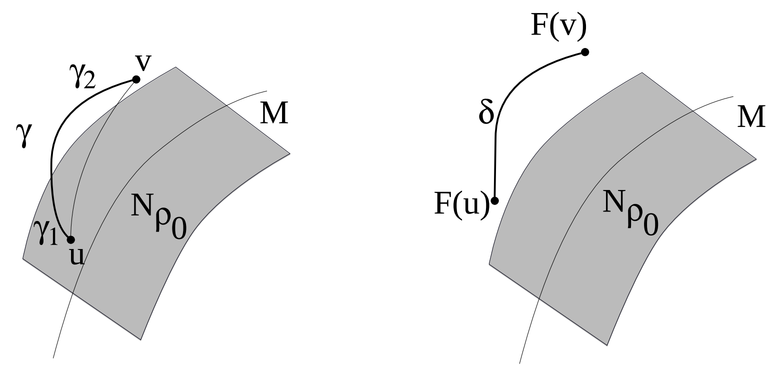


FIGURE 15. A picture of $\gamma=\gamma_1\cup\gamma_2$, together with the true shorter geodesic joining $u$ to $v$, and the image $\delta=F(\gamma)$.

If geodesics do not exist, it is easy to modify the argument above, using curves which almost minimize distances. $\square$

**Blowing up smooth points of divisors.** Let $D\subset X$ be a divisor with normal crossings in a surface, and $\mathcal{B}^+_{\mathbb{R}}(D)\subset\mathcal{B}^+_{\mathbb{R}}(X,D)$ be the real oriented blow-up, so that the diagram

$$\begin{array}{ccc}\mathcal{B}^+_{\mathbb{R}}(D) & \longrightarrow & \mathcal{B}^+_{\mathbb{R}}(X,D)\\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow\\ D & \longrightarrow & X\end{array}$$

commutes. Let $z\in D$ be a point, $\tilde{X}_z$ be the complex blow-up of $X$ at $z$, $\pi:\tilde{X}_z\to X$ the canonical projection, and set $\tilde{D}=\pi^{-1}(D)$. The divisor $\tilde{D}$ is the union of the exceptional divisor $E$ and the proper transform $D'$ of $D$; we will call their point of intersection $\tilde{z}=E\cap D'$.

Let $w_1,\ w_2$ be local coordinates on $X$ at $z$, mapping a neighborhood $W$ of $z$ isomorphically onto the region $V_R\times V_R$ of $\mathbb{C}^2$, where $V_R$ is the open disc of radius $R$ in $\mathbb{C}$. To lighten notation, we set $V_R^*=V_R-\{0\}$. Within $W$, we will assume that $D$ is given by the equation $w_2=0$, so that $D$ has no double points in $W$. Note that $\tilde{W}_z=\pi^{-1}(W)$ can be understood either as the inverse image of $W$ in $\tilde{X}_z$, or as $W$ blown-up at $z$.

Recall (see Example II.4) that $\tilde{W}_z\subset W\times\mathbb{P}^1_{\mathbb{C}}$ is defined by the equation $w_1U_2=w_2U_1$, where $(w_1,w_2)\in W$ and $\begin{bmatrix}U_1\\U_2\end{bmatrix}$ are homogeneous coordinates in $\mathbb{P}^1_{\mathbb{C}}$. We will also use the affine coordinates $u_1=U_1/U_2$ and $u_2=U_2/U_1$ on appropriate parts of $\tilde{W}_z$; note that $w_1$ and $u_2$ are local coordinates near $\tilde{z}$.

Denote by $\tilde{p} : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to \tilde{X}_z$ the canonical projection from the real oriented blow-up, and use the same name for the restriction to $\mathcal{B}^+_{\mathbb{R}}(\tilde{D})$ to the boundary.

We will need to parametrize $\mathcal{B}^+_{\mathbb{R}}(W, D)$ and $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D})$. By Corollary VI.9 the choice of $w_1,\ w_2$ induces a homeomorphism of $p^{-1}(W)$ with the closure of the set

$$\left\{ (w_1, w_2, \theta_2) \in V_R \times V_R^* \times \mathbb{R}/2\pi\mathbb{Z} \;\middle|\; \frac{w_2}{|w_2|} = e^{i\theta_2} \right\} \tag{7.1}$$

in $V_R \times V_R \times \mathbb{R}/2\pi\mathbb{Z}$. In this closure, when $w_2 = 0$, its polar angle remains, so $(w_1, |w_2|, \theta_2)$ parametrize $\mathcal{B}^+_{\mathbb{R}}(W, D)$.

By VI.9 again, a neighborhood of $\tilde{p}^{-1}(\tilde{z})$ is homeomorphic to the closure of the set of

$$\left\{ (w_1, u_2, \theta_1, \phi_2) \in V_R^* \times \mathbb{C}^* \times (R/2\pi\mathbb{Z})^2 \;\middle|\; \theta_1 = \frac{w_1}{|w_1|},\ \phi_2 = \frac{u_2}{|u_2|} \right\}$$

in $V_R \times \mathbb{C} \times (R/2\pi\mathbb{Z})^2$. So $(|w_1|, |u_2|, \theta_1, \phi_2)$ parametrize a neighborhood of $\tilde{z}$ in $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D})$; in particular, by Corollary VI.10, the torus $\tilde{p}^{-1}(\tilde{z})$ is parametrized by $\theta_1$ and $\phi_2$.

**Proposition VII.3.** *The mapping $\pi : \tilde{X}_z \to X$ lifts to a unique mapping $\tilde{\pi} : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to \mathcal{B}^+_{\mathbb{R}}(X, D)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) & \xrightarrow{\tilde{\pi}} & \mathcal{B}^+_{\mathbb{R}}(X, D) \\ {\scriptstyle \tilde{p}}\downarrow & & {\scriptstyle p}\downarrow \\ \tilde{X}_z & \xrightarrow{\pi} & X \end{array}$$

*commutes.*

**Proof.** It is enough to extend $\pi$ above $W$. Let us compute $\tilde{\pi}$ near $\tilde{z}$. The space $\tilde{W}_z$ is parametrized by $w_1$ and $u_2$, and $\pi(w_1, u_2) = (w_1, w_1 u_2)$.

Thus if we set $w_1 = r_1 e^{i\theta_1},\ u_2 = \rho_2 e^{i\phi_2}$, then the map $\pi$ is

$$\pi : \begin{pmatrix} r_1 e^{i\theta_1} \\ \rho_2 e^{i\phi_2} \end{pmatrix} \mapsto \begin{pmatrix} r_1 e^{i\theta_1} \\ r_1 \rho_2 e^{i\phi_2 + \theta_1} \end{pmatrix}.$$

As a map of real oriented blow-ups, the domain is parametrized by $(r_1, \theta_1, \rho_2, \phi_2)$ and the range by $(w_1, |w_2| \in [0, \infty), \arg w_2 \in \mathbb{R}/2\pi\mathbb{Z})$, and the map is written

$$(r_1, \theta_1, \rho_2, \phi_2) \mapsto (r_1 e^{i\theta_1}, r_1\rho_2, \theta_1 + \phi_2) \tag{7.2}$$

and is obviously continuous. The proof near the one omitted point $u_2 = \infty$ is easier. □

**Proposition VII.4.** *The mapping $\tilde{\pi}$ maps the torus $\tilde{p}^{-1}(\tilde{z})$( parametrized by $(\theta_1, \phi_2)$) to the circle $p^{-1}(z)$ (parametrized by $\theta_2$), by the mapping*

$$(\theta_1, \phi_2) \mapsto \theta_1 + \phi_2.$$

**Proof.** Since the projection $\pi$ is simply $(w_1, u_2) \mapsto (w_1, w_1 u_2)$, and our variable $\theta_2$ corresponds to the argument of the second entry, the result follows. □

We will now show how the real oriented blow-up $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$ can be constructed from $\mathcal{B}^+_{\mathbb{R}}(X, D)$. Recall that $\mathcal{B}^+_{\mathbb{R}}(X, D)$ is a 4-dimensional manifold with boundary, and that the boundary is a 3-manifold containing tori, corresponding to the double points of $D$, and with a circle action on the complement of these tori. We need to understand the corresponding structure on $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$.

Roughly speaking, it can be understood as follows. If $z \in D$ is a smooth point of $D$, then the fiber $p^{-1}(z)$ is a circle. Thicken this circle to make a solid torus, invariant under the existing circle action. Keep the old circle action on the outside of the solid torus, and modify it inside, so that the oriented circle orbits on the boundary of the solid torus are the "sums" of the old ones and of boundaries of discs $\Delta$ in the solid torus, which are oriented so that $p : \Delta \to D$ is orientation preserving.

This description is too imprecise to be proved in this form: Theorem VII.5 makes it precise.

Call $Z$ the space $\mathcal{B}^+_{\mathbb{R}}(X, D)$, but with a modified circle action on part of the boundary $\mathcal{B}^+_{\mathbb{R}}(D)$. Choose $r < R$, set $\Delta_r \subset D$ the region $|w_1| < r$, and in $p^{-1}(\Delta_r)$ use the circle action

$$\Theta * (w_1, 0, \theta_2) = (w_1 e^{i\Theta}, 0, \theta_2 + \Theta), \tag{7.3}$$

where we are using the coordinates (7.1) in $\mathcal{B}^+_{\mathbb{R}}(X, D)$.

**Theorem VII.5.** *There exists a homeomorphism $h : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to Z$, which coincides with $\tilde{\pi}$ outside of $\tilde{p}^{-1}(\tilde{W}_z)$, and which respects the circle actions.*

**Proof.** We only need to construct $h$ in $\tilde{p}^{-1}(\tilde{W}_z) \subset \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$, so long as $h$ coincides with $\tilde{\pi}$ outside a compact subset of $\tilde{p}^{-1}(\tilde{W}_z)$. We will begin by defining our homeomorphism on $\partial\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) = \mathcal{B}^+_{\mathbb{R}}(\tilde{D})$, and then extend it to the interior.

A neighborhood of $\tilde{p}^{-1}(E - \{\tilde{z}\})$ can be described as the closure of the set of

$$\left\{ (u_1, w_2, \theta_2) \in \mathbb{C} \times V_R^* \times \mathbb{R}/2\pi\mathbb{Z} \;\middle|\; e^{i\theta_2} = \frac{w_2}{|w_2|} \right\}$$

in $\mathbb{C} \times V_R \times \mathbb{R}/2\pi\mathbb{Z}$, and $\tilde{p}^{-1}(E)$ is the subset of equation $w_2 = 0$. Again the polar angle of $w_2$ remains when $w_2 = 0$; we will write $u_1 = |u_1|e^{i\phi_1}$, but when $u_1 = 0$ it does not have an argument.

Choose a homeomorphism $\eta_1 : [0,\infty) \to [r,\infty)$, such that $\eta_1(t) = t$ for $t > R$; it will need to satisfy further properties in order for Theorem VII.6 to be true, which we will spell out when we come to it. Let us set (see Figure 16)

$$\eta_2(t) = \frac{tr}{t+r}.$$

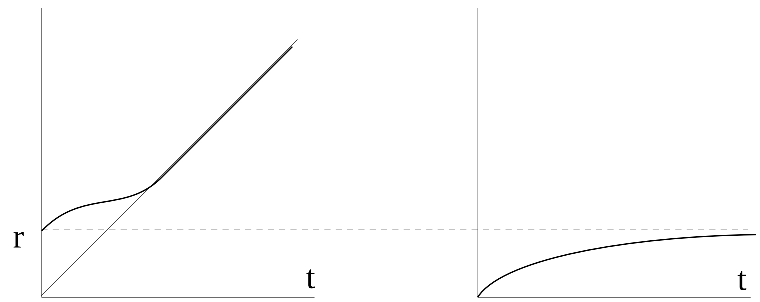


FIGURE 16. The graph of $\eta_1$ (left) and the graph of $\eta_2$ (right).

Define $h : \tilde{p}^{-1}(D' - \{\tilde{z}\}) \to \mathcal{B}^+_{\mathbb{R}}(D)$ by the formula

$$h : (w_1, 0, \theta_1, \theta_2) \mapsto \big(\eta_1(|w_1|)e^{i\theta_1}, 0, \theta_2\big). \tag{7.4}$$

As a set, $\partial Z = \mathcal{B}^+_{\mathbb{R}}(D)$. Using parametrization (7.1) in the range, we define $h : \tilde{p}^{-1}(E - \{\tilde{z}\}) \to \partial Z$ by the formula

$$h : (u_1, 0, \phi_1, \theta_2) \mapsto \big(\eta_2(|u_1|)e^{i(\phi_1+\theta_2)}, 0, \theta_2\big). \tag{7.5}$$

To check the continuity of these two definitions on $\tilde{p}^{-1}(\tilde{z})$, we choose local coordinates $(u, v)$ on $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$ near $\tilde{z}$, related to the previous coordinates by

$$\begin{cases} u = w_1 \\ v = \dfrac{w_2}{w_1} \end{cases} \qquad \text{on a neighborhood of } \tilde{p}^{-1}(D' - \{\tilde{z}\})$$

$$\begin{cases} u = w_2 u_1 \\ v = \dfrac{1}{u_1} \end{cases} \qquad \text{on a neighborhood of } \tilde{p}^{-1}(E - \{\tilde{z}\})$$

By Corollary VI.9, in the coordinates $u, v$, the space $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$ is canonically homeomorphic to the closure of the set

$$\left\{(u, v, \beta, \gamma) \in V_R^* \times \mathbb{C}^* \times (\mathbb{R}/2\pi\mathbb{Z})^2 \;\middle|\; e^{i\beta} = \frac{u}{|u|},\ e^{i\gamma} = \frac{v}{|v|}\right\}$$

in $V_R \times \mathbb{C} \times (\mathbb{R}/2\pi\mathbb{Z})^2$; when the variables $u$ or $v$ or both vanish, their polar angles both remain. The locus $\tilde{p}^{-1}(\tilde{D}) \cap \tilde{W}_z$ is given by the equation $uv = 0$.

The expression (7.4) of $h$ on $p^{-1}(D' - \{\tilde{z}\})$ becomes, in these coordinates,

$$h : (u, 0, \beta, \gamma) \mapsto (\eta_1(|u|)e^{i\beta}, 0, \beta + \gamma).$$

Similarly, the expression (7.5) of $h$ on $\tilde{p}^{-1}(E - \{\tilde{z}\})$ becomes

$$h : (0, v, \beta, \gamma) \mapsto \left(\eta_2\left(\frac{1}{|v|}\right)e^{-i\gamma}e^{i(\beta+\gamma)}, 0, \beta + \gamma\right) = \left(\eta_2\left(\frac{1}{|v|}\right)e^{i\beta}, 0, \beta + \gamma\right).$$

We have used the computation

$$\theta_2 = \arg w_2 = \arg uv = \beta + \gamma$$

of $\theta_2$ in the coordinates $u, v$.

Now to check the continuity:

$$\lim_{\rho \to 0} h(\rho e^{i\beta}, 0, \beta, \gamma) = (re^{i\beta}, 0, \beta + \gamma)$$
$$\lim_{\rho \to 0} h(0, \rho e^{i\gamma}, \beta, \gamma) = (re^{i\beta}, 0, \beta + \gamma).$$

It works; our homeomorphism is well-defined on $\mathcal{B}^+_{\mathbb{R}}(\tilde{D}) = \partial \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$.

We need to check that $h$ transforms the circle action of $\mathcal{B}^+_{\mathbb{R}}(\tilde{D})$ into the circle action of $Z$. The only place that needs checking is in $\tilde{p}^{-1}(E)$. Using the notation of Equation (7.5), we see that in the domain the circle action is

$$\Theta * (u_1, 0, \phi_1, \theta_2) = (u_1, 0, \phi_1, \theta_2 + \Theta);$$

by (7.3), the circle action in the range is

$$\Theta * \left(\eta_2(|u_1|)e^{i(\phi_1+\theta_2)}, 0, \theta_2\right) = \left(\eta_2(|u_1|)e^{i(\phi_1+\theta_2+\Theta)}, 0, \theta_2 + \Theta\right).$$

Clearly $h$ transforms one into the other.

Now we extend $h$ to the interior. Let us consider the subset $W' \subset \tilde{W}_z$ given by the inequality

$$W' = \left\{(w_1, w_2) \,\middle|\, |w_1|^2 + (|w_2| - r)^2 < r^2\right\}, \quad 2r < R,$$

as shown in Figure 17.

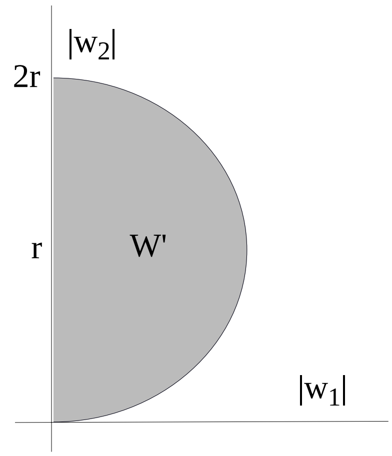


FIGURE 17. The region in the $|w_1|$, $|w_2|$-plane defining $W'$.

The set $W'$ can be considered as a subset of $W$, or of $\tilde{W}_z$, or of $\mathcal{B}^+_{\mathbb{R}}(W, D)$, or of $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D})$, since the canonical projections are all isomorphisms on the inverse image of $W'$. As a subset of $\tilde{W}_z$, it is a neighborhood of $E - \{\tilde{z}\}$, and we can use $w_1/w_2, w_2$ as parameters in $W'$, which extend the parameters $u_1, w_2$ above. As a subset of $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D})$, it is a neighborhood of $\tilde{p}^{-1}(E - \{\tilde{z}\})$.

We can now extend $h$ by the formulas

$$h : \begin{cases} (w_1, w_2) \mapsto (\eta_{1,|w_2|}(|w_1|)\frac{w_1}{|w_1|}, w_2) & \text{if } (w_1, w_2) \notin W' \\ (u_1, w_2) \mapsto (\eta_2(|u_1|)\frac{u_1}{|u_1|}\frac{w_2}{|w_2|}, w_2) & \text{if } (u_1, w_2) \in W', \end{cases} \tag{7.6}$$

where we are using the coordinates $(w_1, w_2)$ in the range in both cases. The second line above could also be written

$$(w_1, w_2) \mapsto (\eta_2\left(\frac{|w_1|}{|w_2|}\right)\frac{w_1}{|w_1|}, w_2).$$

We opted to write it in terms of $u_1$ instead because that variable extends to the boundary. Note that since $W'$ does not intersect the boundary, we do not need the polar angles as variables in either domain or range; of course, they will return when we check continuity.

The functions $\eta_{1,\rho}$ and $\eta_2$ are illustrated in Figure 18.

We need to check the continuity of the mapping, both on the part of $\partial W'$ where $w_2 \neq 0$, and on $\partial Z$.

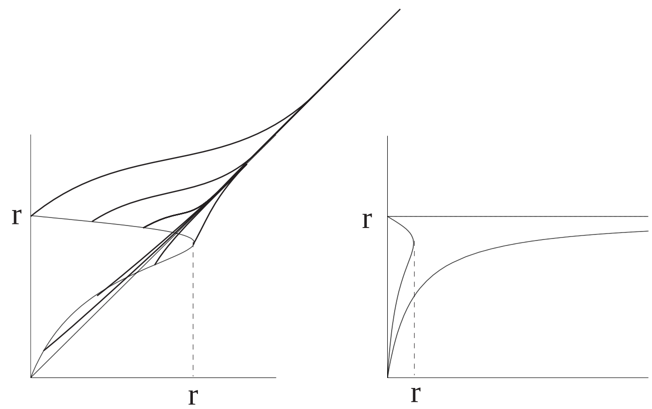


FIGURE 18. On the left, the graphs of functions $\eta_{1,\rho}$ for various values of $\rho \in [0, 2r]$. They are defined for $t \geq \sqrt{r^2 - (\rho - r)^2}$, and the domain is mapped homeomorphically to

$$\left[\eta_2(\sqrt{r^2 - (\rho - r)^2}/\rho), \infty\right);$$

thus all the graphs start on the curve given parametrically by

$$\rho \mapsto \left(\sqrt{r^2 - (\rho - r)^2},\ \eta_2(\sqrt{r^2 - (\rho - r)^2}/\rho)\right)$$

which is also represented on the right, where the horizontal scale and the vertical scale are different.

The second will be true if $\eta_{1,0}(t) = \eta_1(t)$, as is clear from Equations (7.6) and (7.5), since the argument of $w_2$ remains as a variable when $w_2 = 0$. For the first, we have $|w_1|^2 = r^2 - (|w_2| - r)^2$, and we see that we must have

$$\left(\eta_{1,|w_2|}(|w_1|)\frac{w_1}{|w_1|}, w_2\right) = \left(\eta_2\left(\left|\frac{w_1}{w_2}\right|\right)\frac{w_1}{|w_1|}, w_2\right)$$

when $|w_1|^2 = r^2 - (|w_2| - r)^2$. Setting $\rho = |w_2|$ to clarify the notation, we need

$$\eta_{1,\rho}(\sqrt{r^2 - (\rho - r)^2}) = \eta_2\left(\frac{\sqrt{r^2 - (\rho - r)^2}}{\rho}\right). \tag{7.7}$$

It is now easy to choose $\eta_{1,\rho}(t)$, defined for $t \geq \sqrt{r^2 - (\rho - r)^2}$, so that

(1) Equation (7.7) is satisfied;
(2) for each $\rho$, the function $\eta_{1,\rho}(t)$ is monotone increasing;
(3) $\eta_{1,0}(t) = \eta_1$;
(4) $\eta_{1,2r}(t) = t$.

These conditions will guarantee that the mapping $h$ defined by (7.6) is a homeomorphism as required. $\square$

To prove Theorem VII.11, we will need some control on how the homeomorphism $h : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to Z$, constructed above, distorts distances. Note that $h$ is not canonical: it depends on the local coordinates $w_1, w_2$, and on the choice of $r$, $\eta_1$ and $\eta_2$.

**Theorem VII.6.** *If $\mu$ is any Riemannian metric on $\mathcal{B}^+_{\mathbb{R}}(D)$ which is continuous in the $(w_1, \theta_2)$ coordinates, then for any $\epsilon > 0$, $\alpha < 1$ and any neighborhood $W$ of $z$ in $X$, the homeomorphism $h$ can be chosen to coincide with $\tilde{\pi}$ except on $\mathcal{B}^+_{\mathbb{R}}\left(\tilde{W}_z, \tilde{D} \cap \tilde{W}_z\right)$, and such that for any $x, y \in \mathcal{B}^+_{\mathbb{R}}(\tilde{D})$,*

$$d_\mu(h(x), \tilde{\pi}(x)) < \epsilon$$

*and*

$$d_\mu\big(h(x), h(y)\big) \geq \alpha d_\mu\big(\tilde{\pi}(x), \tilde{\pi}(y)\big).$$

Note that all the distances are being measured in $\mathcal{B}^+_{\mathbb{R}}(D) = \partial Z$ with respect to the Riemannian metric $\mu$.

**Proof.** This is a matter of carefully choosing the number $r$ and the function $\eta_1$. Clearly $d_\mu(h(x), \tilde{\pi}(x))$ will be arbitrarily small if $r$ is chosen sufficiently small. The second inequality follows from Lemma VII.1, on the outside of $\tilde{\pi}^{-1}(E)$, we have precisely a map of the form specified there, and we have proved the needed result.

With our choice of coordinates, the region $p^{-1}(\Delta_R)$ is canonically $V_R \times S^1$, parametrized by $w_1$ and $\theta_2$, with $|w_1| \leq R$. We can modify the Riemannian metric $\mu$ on this region by setting $\mu_1(w_1, \theta_2) = \mu(0, \theta_2)$. With respect to $\mu_1$, the projection $\bar{\pi} : (w_1, \theta_2) \mapsto (0, \theta_2)$ is obviously length-decreasing. But since $\mu$ and $\mu_1$ coincide on $p^{-1}(z)$, we have

$$\alpha \leq \left|\frac{\mu}{\mu_1}\right| \leq \frac{1}{\alpha}$$

on $p^{-1}(\Delta_r)$ for $r$ sufficiently small (we only need the inequality on the right). So the projection $\bar{\pi}$ satisfies $\alpha d_\mu(\bar{\pi}(u), \bar{\pi}(v)) \leq d_\mu(u, v)$ for $u, v \in p^{-1}(\Delta_r)$.

Now take the value of $r$ in the definition of $h$ to be the $r$ above. Since $h$ conjugates $\tilde{\pi}$ with $\bar{\pi}$, we see that the second requirement is satisfied when $x, y \in \tilde{p}^{-1}(E)$, so that $h(x), h(y) \in p^{-1}(\Delta_r)$.

If $x \in p^{-1}(E)$ and $y \notin p^{-1}(E)$, it suffices to put the two arguments above together. Join $h(x)$ to $h(y)$ by a length-minimizing geodesic $\gamma$, and consider the parts $\gamma_1 = \gamma \cap p^{-1}(\Delta_r)$ and $\gamma_2 = \gamma - \gamma_1$. Then we have the string of inequalities

$$\begin{aligned} d_\mu\big(h(x), h(y)\big) &= l_\mu(\gamma_1) + l_\mu(\gamma_1) \geq \alpha\Big(l_\mu\big(\tilde{\pi} \circ h^{-1}(\gamma_1)\big)\Big) + \alpha\Big(l_\mu\big(\tilde{\pi} \circ h^{-1}(\gamma_2)\big)\Big) \\ &\geq \alpha_\mu\big(\tilde{\pi}(x), \tilde{\pi}(y)\big). \end{aligned}$$

□

**Blowing up double points of divisors.** We now will give the corresponding description for double points of divisors. Let

$$D \subset X,\ \mathcal{B}^+_{\mathbb{R}}(D) \subset \mathcal{B}^+_{\mathbb{R}}(X, D),\ p : \mathcal{B}^+_{\mathbb{R}}(X, D) \to X$$

be as above. Let $z \in D$ be a double point, and denote by

$$\pi : \tilde{X}_z \to X,\ \tilde{p} : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to \tilde{X}_z$$

the canonical projections (complex and real respectively).

Let $w_1$ and $w_2$ be local coordinates centered at $z$, giving an isomorphism of a neighborhood $W$ of $z$ with the bidisc $V_R \times V_R$ so that $D \cap W$ is given by the equation $w_1 w_2 = 0$; in particular, in $W$ there is no other double point of $D$. Let $D_1 = W \cap \{w_2 = 0\}$, $D_2 = W \cap \{w_1 = 0\}$ be the two irreducible components of $D$ in $W$.

The space $\tilde{W}_z$ is the locus in $W \times \mathbb{P}^1$ given by the equation $w_1 U_2 = w_2 U_1$, where $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ are homogeneous coordinates; again we will use the affine coordinates $u_1 = U_1/U_2$ and $u_2 = U_2/U_1$ where they are defined. The divisor $\tilde{D} = \pi^{-1}(D)$ is the union of the proper transforms $D_1'$ of $D_1$ and $D_2'$ of $D_2$, and the exceptional divisor $E$, as shown in Figure 19.

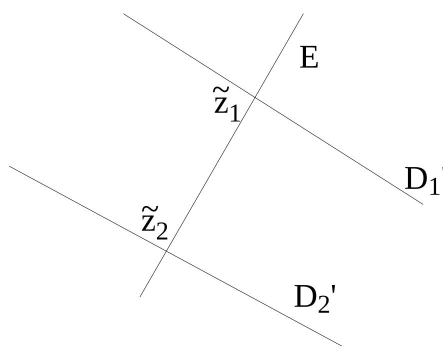


FIGURE 19. The configuration in $\tilde{W}_z$.

By Corollary VI.9, we can identify $\mathcal{B}^+_{\mathbb{R}}(W, D \cap W) = p^{-1}(W)$ with the closure of the set

$$\left\{ (w_1, w_2, \theta_1, \theta_2) \in (V_R^*)^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2 \,\middle|\, e^{i\theta_1} = \frac{w_1}{|w_1|},\ e^{i\theta_2} = \frac{w_2}{|w_2|} \right\}$$

in

$$V_R^2 \times (\mathbb{R}/2\pi\mathbb{Z})^2$$

so that when each coordinate function vanishes, its polar angle remains. This is a manifold with boundary, where the boundary is defined by $w_1 w_2 = 0$.

The fiber $p^{-1}(z)$ is a torus, which is canonically principal under the group $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, and which is parametrized by $\theta_1, \theta_2$.

Similarly, a neighborhood of $\tilde{z}_1 \in \mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D} \cap \tilde{W}_z)$ is parametrized by

$$|u_2| \in [0, \infty),\ \phi_2 = \arg u_2, |w_1| \text{ and } \theta_1 = \arg w_1,$$

with the torus $\tilde{p}^{-1}(\tilde{z}_1)$ parametrized by $\phi_2$ and $\theta_1$. The torus $\tilde{p}^{-1}(\tilde{z}_2)$ can be parametrized in a similar way.

Proposition VII.7 is very similar to Proposition VII.3.

**Proposition VII.7.** *There exists a unique lift $\tilde{\pi} : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to \mathcal{B}^+_{\mathbb{R}}(X, D)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) & \xrightarrow{\tilde{\pi}} & \mathcal{B}^+_{\mathbb{R}}(X, D) \\ {\scriptstyle \tilde{p}}\Big\downarrow & & {\scriptstyle p}\Big\downarrow \\ \tilde{X}_z & \xrightarrow{\pi} & X \end{array}$$

*commutes.*

**Proof.** Begin by working near $\tilde{z}_1$. The computation starts as in Proposition VII.3, but in the range of equation (7.2), we must use $(r_1, \theta_1, r_2, \theta_2)$ as coordinates, since the arguments of $w_1$ and $w_2$ both remain when $r_1$ or $r_2$ tend to 0. So the mapping $\tilde{\pi}$ in these coordinates becomes (see Equation (7.2))

$$(r_1, \theta_1, \rho_2, \phi_2) \mapsto (r_1, \theta_1, r_1\rho_2, \theta_1 + \phi_2);$$

again this is obviously continuous. The computation near $\tilde{z}_2$ is similar. $\square$

**Proposition VII.8.** *The mapping $\tilde{p}$, mapping the torus $\tilde{p}^{-1}(z_1)$, parametrized by $\theta_1, \phi_2$, to the torus $p^{-1}(z)$, parametrized by $\theta_1$ and $\theta_2$, is given by the formula*

$$\begin{bmatrix} \theta_1 \\ \phi_2 \end{bmatrix} \mapsto \begin{bmatrix} \theta_1 \\ \theta_1 + \phi_2 \end{bmatrix}.$$

**Proof.** Near $\tilde{z}_1$, we can use the local coordinates $w_1$ and $u_2$; again the mapping $\pi$ is given by the formula

$$(w_1, u_2) \mapsto (w_1, u_2 w_1).$$

Thus the argument of the first coordinate is $\theta_1$, and the argument of the second is $\theta_1 + \phi_2$. $\square$

As in the case of simple points, we will need a much more precise description of the real oriented blow-up $\mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D})$: it can be constructed from $\mathcal{B}^+_{\mathbb{R}}(X, D)$ as follows. Thicken the torus $p^{-1}(z)$, so that the thickened torus is invariant under the circle actions. Keep the old circle actions on the outside of the thickened torus, and modify it inside, so that the oriented circle orbits on the boundary torus are the "sums" of an orbit of $\mathbb{R}/2\pi\mathbb{Z} \times \{0\}$ and an orbit of $\{0\} \times \mathbb{R}/2\pi\mathbb{Z}$.

Again, this description is too imprecise to be proved; Theorem VII.9 should make everything precise.

Let $Z$ be $\mathcal{B}^+_{\mathbb{R}}(X, D)$, with a new circle action the boundary $\partial Z = \mathcal{B}^+_{\mathbb{R}}(D)$ defined as follows. Let $P$ be the region

$$P = \{|w_1| \le r, w_2 = 0\} \cup \{w_1 = 0, |w_2| \le r\}$$

and on this region, define the circle action

$$\begin{aligned} \Theta * (w_1, 0, \theta_1, \theta_2) &= (w_1, 0, \theta_1 + \Theta, \theta_2 + \Theta) \\ \Theta * (0, w_2, \theta_1, \theta_2) &= (0, w_2, \theta_1 + \Theta, \theta_2 + \Theta). \end{aligned}$$

Keep the previous circle action outside $P$; this gives as it should two circle actions on the two tori $|w_1| = r$ and $|w_2| = r$.

**Theorem VII.9.** *There exists a homeomorphism of $h : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_z, \tilde{D}) \to Z$ which respects the circle actions on the boundaries.*

**Proof.** The proof is quite similar to that of VII.5, and we will be a bit sketchier. As before, we will begin by defining the map $h$ on the boundary, and then extend it to the interior. The divisor $\tilde{D} \cap \tilde{W}_z$ consists of three components: the proper transforms $D'_1, D'_2$ of $D_1$ and $D_2$, and the exceptional divisor $E$. Correspondingly, the boundary of $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D} \cap \tilde{W}_z)$ consists of $\tilde{p}^{-1}(D'_1)$, $\tilde{p}^{-1}(D'_2)$ and $\tilde{p}^{-1}(E)$. Let $\tilde{z}_1 = E \cap D'_1$ and $\tilde{z}_2 = E \cap D'_2$, as shown in Figure 19.

Define $h$ on $\tilde{p}^{-1}(D'_1) \cup \tilde{p}^{-1}(D'_2)$ by the formulas

$$h(w_1, 0, \theta_1, \theta_2) = \left(\eta_1(|w_1|)\tfrac{w_1}{|w_1|},\ 0,\ \theta_1,\ \theta_2\right) \quad \text{in} \quad \tilde{p}^{-1}(D'_1) \tag{7.8}$$

$$h(0, w_2, \theta_1, \theta_2) = \left(0,\ \eta_1(|w_2|)\tfrac{w_2}{|w_2|},\ \theta_1,\ \theta_2\right) \quad \text{in} \quad \tilde{p}^{-1}(D'_2), \tag{7.9}$$

where $\eta_1$ is the same function which we used in the proof of Theorem VII.5 (see Figure 20). This leaves $\tilde{p}^{-1}(E)$. Recall that $\tilde{W}_z$ is the locus in $W \times \mathbb{P}^1$ given by the equation $w_1 U_2 = w_2 U_1$, where $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ are homogeneous coordinates. We will break it into two pieces:

- $E_1$ where $|U_2| \le |U_1|$, which we will parametrize by $(u_2,\ w_1)$, setting $u_2 = U_2/U_1$, and
- $E_2$ where $|U_1| \le |U_2|$, which we will parametrize by $(u_1,\ w_2)$, setting $u_1 = U_1/U_2$.

The space $\tilde{p}^{-1}(E_1)$ thus has coordinates $(u_2, \phi_2, \theta_1)$ and the space $\tilde{p}^{-1}(E_2)$ thus has coordinates $(u_1, \phi_1, \theta_2)$. Using these coordinates, we will define $h$ on $\tilde{p}^{-1}(E)$ as follows :

$$h(u_2, \phi_2, \theta_1) = \left(\eta_2(|u_2|)e^{i\theta_1},\ 0,\ \theta_1,\ \theta_1 + \phi_2\right) \quad \text{in} \quad \tilde{p}^{-1}(E_1) \tag{7.10}$$

$$h(u_1, \phi_1, \theta_2) = \left(0,\ \eta_2(|u_1|)e^{i\theta_2},\ \phi_1 + \theta_2,\ \theta_2\right) \quad \text{in} \quad \tilde{p}^{-1}(E_2), \tag{7.11}$$

where $\eta_2(t) = r - tr$ (see Figure 20).

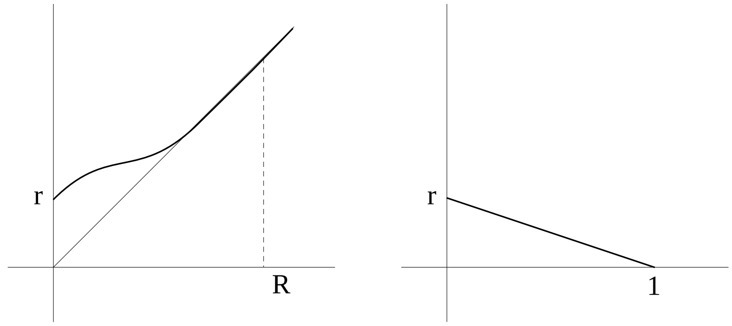


FIGURE 20. The graphs of $\eta_1$ (left) and of $\eta_2$ (right).

There are three places where we must check the compatibility of these formulas: above $\tilde{z}_1$, above $\tilde{z}_2$ and above the circle $|u_1| = |u_2| = 1$.

Above $\tilde{z}_1$, the functions $u_2$ and $w_1$ are local coordinates, and since $w_2 = u_2 w_1$, we have $\theta_2 = \phi_2 + \theta_1$. This means that under the mapping (7.8), the point $(0, 0, \theta_1, \theta_2)$ in $\tilde{p}^{-1}(D'_1)$ maps to $(re^{i\theta_1}, 0, \theta_1, \theta_2)$, whereas the same point viewed in $\tilde{p}^{-1}(E_1)$ has coordinates $(0, \phi_2, \theta_1)$ and maps under (7.10) to $(re^{i\theta_1}, 0, \theta_1, \theta_1 + \phi_2)$, which is the same point.

The analogous check above $\tilde{z}_2$ is left to the reader.

Above the circle $u_1 = u_2 = 1$, we must pass from the coordinates $(u_2, w_1)$ to the coordinates $(u_1, w_2)$. Under (7.10), the point $(e^{i\phi_2}, \phi_2, \theta_1)$ in $\tilde{p}^{-1}(E_1)$ maps to $(0, 0, \theta_1, \theta_1 + \phi_2)$, whereas under (7.11), the same point, viewed as an element of $\tilde{p}^{-1}(E_2)$, has coordinates $(e^{i\phi_1}, \phi_1, \theta_2) = (e^{-i\phi_2}, -\phi_2, \theta_1 + \phi_2)$, and also maps to $(0, 0, \theta_1, \theta_1 + \phi_2)$.

Finally, observe that the map $h$ as constructed does transform the circle action on $\mathcal{B}^+_{\mathbb{R}}(\tilde{D})$ into the circle action on $Z$. In $\tilde{p}^{-1}(E_1)$, the circle action is

$$\Theta * (u_2, \phi_2, \theta_2) = (u_2, \phi_2, \theta_2 + \Theta),$$

which under (7.10) becomes

$$\Theta * (w_1, 0, \theta_1, \theta_2) = (w_1, 0, \theta_1 + \Theta, \theta_2 + \Theta).$$

The computation for $\tilde{p}^{-1}(E_2)$ is identical.

This ends the construction of $h$ on $\partial Z$; now to extend it to the interior.

We will call $B$ the subset of $W$ in which

$$\left(|w_1|^2 + |w_2|^2\right)^2 < 4r^2 |w_1 w_2|.$$

Again, we can think of $B$ as a subset of $W$, or of $\tilde{W}_z$, or of $\mathcal{B}^+_{\mathbb{R}}(W, D \cap W)$, or of $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D} \cap \tilde{W}_z)$, since all the canonical projections are isomorphisms on the inverse images of $B$.

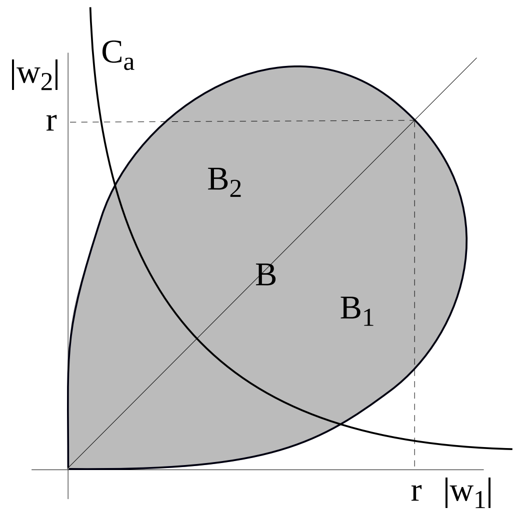


FIGURE 21. The region in $|w_1|, |w_2|$ defining $B$.

This region contains $E$ in its closure, since every line through the origin has a neighborhood of the origin in $B$, except for the axes themselves. This is illustrated in Figure 21.

Let $W_1, W_2 \subset W$ be the regions where $|w_1| \geq |w_2|$ and $|w_1| \leq |w_2|$ respecively. Then the space $B$ is the union of the two subsets $B_1 = W_1 \cap B$ and $B_2 = W_2 \cap B$; these have $E_1$ and $E_2$ in their closures in $\tilde{W}_z$ (resp. $\tilde{p}^{-1}(E_1)$ and $\tilde{p}^{-1}(E_1)$ in their closures in $\mathcal{B}^+_{\mathbb{R}}(\tilde{W}_z, \tilde{D} \cap \tilde{W}_z)$).

The curve $C_a$ of equation $w_1 w_2 = a^2$ intersects $B$ in an annulus $B_a$ for $0 < |a| < r$, which shrinks to a circle when $|a| = r$ and is empty for $|a| > r$. We will extend $h$ so that it maps each $C_a$ into itself. Moreover, we will use the local coordinates $u_2, w_1$ in $B_1$ and $u_1, w_2$ in $B_2$, and keep the coordinates $w_1, w_2$ outside of $B$. Throughout, we will use the coordinates $w_1, w_2$ in the range.

In these coordinates, our mapping $h$ is written

$$h(w_1, w_2) = \left( \eta_{1,|a|}(|w_1|) \frac{w_1}{|w_1|}, \ \frac{|a|^2}{\eta_{1,|a|}(|w_1|)} \frac{w_2}{|w_2|} \right) \quad \text{in} \quad W_1 - B_1 \tag{7.12}$$

$$h(u_2, w_1) = \left( \eta_{2,|a|}(|u_2|) e^{i\theta_1}, \ \frac{a^2}{\eta_{2,|a|}(|u_2|)} e^{-i\theta_1} \right) \quad \text{in} \quad B_1 \tag{7.13}$$

$$h(u_1, w_2) = \left( \frac{a^2}{\eta_{2,|a|}(|u_1|)} e^{-i\theta_2}, \ \eta_{2,|a|}(|u_1|) e^{i\theta_2} \right) \quad \text{in} \quad B_2 \tag{7.14}$$

$$h(w_1, w_2) = \left( \frac{|a|^2}{\eta_{1,|a|}(|w_2|)} \frac{w_1}{|w_1|}, \ \eta_{1,|a|}(|w_2|) \frac{w_2}{|w_2|} \right) \quad \text{in} \quad W_2 - B_2, \tag{7.15}$$

where $\eta_{2,|a|}(t) = r - (r - |a|)t$ and

$$\eta_{1,\rho} : \left[ \sqrt{\rho(r + \sqrt{r^2 - \rho^2})}, \ \infty \right) \to \left[ \eta_{2,\rho}\left( \frac{(r - \sqrt{r^2 - \rho^2})}{\rho} \right), \ \infty \right)$$

is a diffeomorphism, to be chosen below.

There are now three compatibilities to check: on the boundary $\partial Z$, on the boundary of $B$ and when $|u_1| = |u_2| = 1$. The first will be satisfied if we require that

$$\lim_{|a|\to 0} \eta_{1,|a|} = \eta_1 \quad \text{and} \quad \lim_{|a|\to 0} \eta_{2,|a|} = \eta_2.$$

For the third, our extension of $h$ is precisely the identity on this locus. Indeed, using formula (7.13), we see

$$\eta_{2,|a|}(|u_2|)e^{i\theta_1} = \eta_{2,|a|}(1)e^{i\theta_1} = |a|e^{i\theta_1} = w_1$$

and

$$\frac{a^2}{\eta_{2,|a|}(|u_2|)}e^{-i\theta_1} = \frac{a^2}{w_1} = w_2$$

there; the check for (7.14) is identical.

For the second, we need to guarantee that

$$\eta_{1,|a|}\left(\sqrt{|a|\left(r+\sqrt{r^2-|a|^2}\right)}\right) = \eta_{2,|a|}\left(\frac{r-\sqrt{r^2-|a|^2}}{|a|}\right).$$

A look at the curve on which the graph of $\eta_{1,\rho}$ must start (Figure 22) shows that this is easy to accomplish.

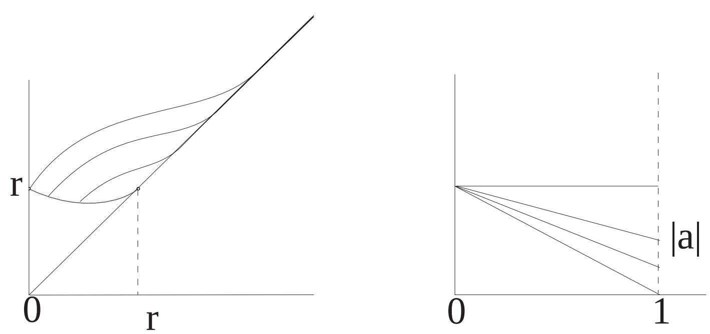


FIGURE 22. The graphs of the functions $\eta_{1,\rho}$ (left) and $\eta_{2,\rho}$ (right) for various values of $\rho$.

□

Again, in order to prove Theorem VII.11 we will need to control how much $h$ distorts distances.

**Theorem VII.10.** *For any Riemann metric $\mu$ on $\mathcal{B}^+_{\mathbb{R}}(D)$ which is continuous in the coordinates $\rho, \theta, \Theta$, and for any $\epsilon > 0$ and $\alpha < 1$, we can choose $h$ so that*

$$d_\mu(h(x), \tilde{\pi}(x)) < \epsilon$$

*and*

$$d_\mu\big(h(x), h(y)\big) \geq \alpha d_\mu\big(\tilde{\pi}(x), \tilde{\pi}(y)\big)$$

*for any* $x, y$ *in* $\mathcal{B}^+_{\mathbb{R}}(\tilde{D})$.

**Proof.** This is a matter of choosing $r$ and $\eta_1$ correctly in the constructions above.

In our coordinates, the mapping $\tilde{\pi}$ is written

$$\begin{array}{ll}
\tilde{\pi} \circ h^{-1}(w_1, 0, \theta_1, \theta_2) = (0, 0, \theta_1, \theta_2) & \text{if } |w_1| \leq r \\
\tilde{\pi} \circ h^{-1}(w_1, 0, \theta_1, \theta_2) = (\eta_1(|w_1|)\frac{w_1}{|w_1|}, 0, \theta_1, \theta_2) & \text{if } |w_1| \geq r \\
\tilde{\pi} \circ h^{-1}(0, w_2, \theta_1, \theta_2) = (0, 0, \theta_1, \theta_2) & \text{if } |w_2| \leq r \\
\tilde{\pi} \circ h^{-1}(0, w_2, \theta_1, \theta_2) = (\eta_1(|w_2|)\frac{w_2}{|w_2|}, 0, \theta_1, \theta_2) & \text{if } |w_1| \geq r.
\end{array}$$

This map collapses the thickened torus $|w_1|, |w_2| \leq r$ back onto the torus above $z$. Clearly

$$d_\mu(h(x), \tilde{\pi}(x)) < \epsilon$$

will be small if $r$ is small and $\eta_1$ is uniformly close to the identity. Again the second part is an application of Lemma VII.1. □

**Infinitely many blow-ups.** Suppose that we repeat infinitely many times the following procedure, as in Section IV.

Take a surface $X_0$ containing a divisor with normal crossings $D_0 \subset X_0$; although it is not essential, we will assume that $X_0$ is compact. Choose a point $z_0 \in D_0$, blow it up to create a surface $X_1 = (\tilde{X_0})_{z_0}$, with a projection $\pi_1 : X_1 \to X_0$; set $D_1 = \pi_1^{-1}(D_0)$. Now choose a new point $z_1 \in D_1$, etc. Denote by

$$\tilde{\pi}_i : \mathcal{B}^+_{\mathbb{R}}(X_i, D_i) \to \mathcal{B}^+_{\mathbb{R}}(X_{i-1}, D_{i-1})$$

the map induced from $\pi_i$.

Theorems VII.5 and VII.9 assert that at each stage the pair $\mathcal{B}^+_{\mathbb{R}}(D_i) \subset \mathcal{B}^+_{\mathbb{R}}(X_i, D_i)$ is homeomorphic to the pair $\mathcal{B}^+_{\mathbb{R}}(D_0) \subset \mathcal{B}^+_{\mathbb{R}}(X_0, D_0)$. It seems reasonable to think that the same will remain true in the limit, and it is.

**Theorem VII.11.** *The projective limit of pairs*

$$\varprojlim(\mathcal{B}^+_{\mathbb{R}}(D_i), \tilde{\pi}_i) \subset \varprojlim((\mathcal{B}^+_{\mathbb{R}}(X_i, D_i), \tilde{\pi}_i)$$

*is homeomorphic to the pair* $\mathcal{B}^+_{\mathbb{R}}(D_0) \subset \mathcal{B}^+_{\mathbb{R}}(X_0, D_0)$, *and the canonical map*

$$\varprojlim(\mathcal{B}^+_{\mathbb{R}}(X_i, D_i), \tilde{\pi}_i) \to \mathcal{B}^+_{\mathbb{R}}(X_0, D_0)$$

*can be approximated by homeomorphisms.*

**Proof.** Theorems VII.5 and VII.9 guarantee that we can choose homeomorphisms

$$h_i : \mathcal{B}^+_{\mathbb{R}}(X_i, D_i) \to \mathcal{B}^+_{\mathbb{R}}(X_{i-1}, D_{i-1}).$$

By induction, we will give $\mathcal{B}^+_{\mathbb{R}}(X_i, D_i)$ the metric which makes $h_i$ an isometry; and Theorems VII.6 and VII.10 then show that each $h_i$ can further be chosen so that for any sequences $\epsilon_i > 0,\ 0 < \alpha_i < 1$,

(1) the composition $\tilde{\pi}_i \circ h_i^{-1}$ is $\epsilon_i$ close to the identity map (of $\mathcal{B}^+_{\mathbb{R}}(X_{i-1}, D_{i-1})$);

(2) if $x_i, y_i \in \mathcal{B}^+_{\mathbb{R}}(D_i)$ and $x_{i-1} = \tilde{\pi}_i(x_i),\ y_{i-1} = \tilde{\pi}_i(y_i)$, then

$$d(h_i(x_i), h_i(y_i)) \geq \alpha_i d(x_{i-1}, y_{i-1}).$$

Moreover, again by choosing the $r$ sufficiently small, we can assume that for any compact subset $K \subset X - D$, there exists $m$ such that the mapping $h_i$ coincides with $\tilde{\pi}$ on $K$ for $i > m$.

**Remark.** The open set $X - D$ is naturally embedded as a subset in all $\mathcal{B}^+_{\mathbb{R}}(X_i, D_i)$; in terms of this natural embedding, the choice of $r$ above guarantees that $h_i$ is the identity on the compact subset $K$ for $i > m$.

Denote by $g_n$ the composition

$$\varprojlim_i \mathcal{B}^+_{\mathbb{R}}(X_i, D_i) \to \mathcal{B}^+_{\mathbb{R}}(X_n, D_n) \stackrel{h_1 \circ \cdots \circ h_n}{\to} \mathcal{B}^+_{\mathbb{R}}(X_0, D_0),$$

where the first map is the canonical projection. The first condition above guarantees that the $g_n$ converge uniformly if the $\epsilon_n$ form a convergent series. Call $g = \lim_{n\to\infty} g_n$.

The mapping $g$ is obviously surjective. The second condition guarantees that $g$ is injective on $\varprojlim \mathcal{B}^+_{\mathbb{R}}(D_i)$ if $\prod \alpha_n > 0$. Moreover, it is injective on $X - D$ since the sequence is eventually constant and injective on every compact subset. Therefore it is injective.

Finally, since $g$ is bijective, continuous and the domain is compact (a projective limit of compact sets is compact), it is a homeomorphism. $\square$

## VIII. Real oriented blow-ups and the Hopf fibration

The Hopf fibration $p : S^3 \to S^2$ is a famous example from topology, where the circle acts on the 3-sphere, and the quotient is the 2-sphere. If we think of $S^3$ as the unit sphere in $\mathbb{C}^2$, and $S^2$ as $\mathbb{P}^1_{\mathbb{C}}$, then it can be written as

$$p : (z_1, z_2) \mapsto [z_1 : z_2].$$

It can also be thought of as the quotient of the 3-sphere by the circle action

$$\mathbb{R}/2\pi\mathbb{Z} \times S^3 \to S^3, \quad \Theta * (z_1, z_2) = (e^{i\Theta} z_1, e^{i\Theta} z_2)$$

The Hopf fibration seems a natural candidate to be the real oriented blow-up of a projective line in a surface, and it is. But it comes up in two different settings, which differ in a subtle but essential way.

More specifically, we will consider a line in $\mathbb{P}^2_{\mathbb{C}}$, for instance the line at infinity $l_\infty$, and the exceptional divisor $E$ that you get by blowing up the origin. The plane $\mathbb{P}^2_{\mathbb{C}}$ has a real oriented blow-up along both of these lines; we can imagine these real oriented blow-ups as the boundaries of tubular neighborhoods of the two lines respectively. If we take the tubular neighborhood of $E$ to be the open unit ball, and the tubular neighborhood of $l_\infty$ to be the complement of the closed ball, then both have the same boundary $S^3$. We will see in Propositions VIII.1 and VIII.2 that there really is a homeomorphism of the 3-sphere with $\mathcal{B}^+_{\mathbb{R}}(E)$ and $\mathcal{B}^+_{\mathbb{R}}(l_\infty)$, and that the first homeomorphism transforms the canonical circle action on $\mathcal{B}^+_{\mathbb{R}}(E)$ into the Hopf action above and that the other transforms the standard circle action into

$$\Theta * (z_1, z_2) = (e^{-i\Theta} z_1, e^{-i\Theta} z_2).$$

If you think of rotating a line in $\mathbb{C}^2$ in the direction of the boundary of a disc, or the boundary of the outside of a disc, you will see where the minus sign comes from.

You might think that this minus sign is the difference between the two cases, but it isn't: the antipodal map $S^3 \to S^3$ is an orientation-preserving diffeomorphism which transforms one circle action into the other, so these two circle actions are essentially the same. The actual difference between the two cases is the orientation of $S^3$, as the boundary of the unit ball in the case of $\mathcal{B}^+_{\mathbb{R}}(E)$, and as the boundary of the outside of the unit ball in the case of $\mathcal{B}^+_{\mathbb{R}}(l_\infty)$. This may seem a minor problem, but it turns out to have essential consequences.

Indeed, if you further blow up a point $z \in l_\infty$, making

$$\pi : \widetilde{(\mathbb{P}^2_{\mathbb{C}})}_z \to \mathbb{P}^2_{\mathbb{C}},$$

set $D = \pi^{-1}(l_\infty)$, and still denote by $p : \mathcal{B}^+_{\mathbb{R}}(D) \to D$ the projection, then there are fibers of $p$ which link with linking number 0, and there are none which link with linking number $\pm 2$.

On the other hand, if you blow up a point $z \in E$, call $E_1$ the new exceptional divisor and $E'$ the proper transform of $E$, set $D = E' \cup E_1$, and again denote by $p : \mathcal{B}^+_{\mathbb{R}}(D) \to D$ the projection, then there are fibers which link with linking number 2, and there are none which link with linking number 0.

These two situations differ geometrically, not just by a conventional sign. Let us now justify these claims.

**The real blow-up of an exceptional divisor.** Let $\pi : \widetilde{\mathbb{C}^2}_0 \to \mathbb{C}^2$ be the plane $\mathbb{C}^2$ blown up at the origin, and $E = \pi^{-1}(0,0)$ be the exceptional divisor. Recall the two charts

$$u_1 = x,\ v_1 = \frac{y}{x} \quad \text{and} \quad u_2 = y,\ v_2 = \frac{x}{y}$$

which provide coordinates on overlapping open sets which cover $\widetilde{\mathbb{C}^2}_0$. In each of these charts, $\mathcal{B}^+_{\mathbb{R}}(\widetilde{\mathbb{C}^2}_0, E)$ is the closure of the set

$$\left\{(u_j, v_j, \theta_j) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z} \,\middle|\, \frac{u_j}{|u_j|} = e^{i\theta_j}\right\}, \; j = 1, 2$$

in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z}$, and that $p^{-1}(E)$ is given by the equation $u_j = 0$.

Thus $\mathcal{B}^+_{\mathbb{R}}(E)$ is parametrized by the two charts $(v_1, \theta_1)$ and $(v_2, \theta_2)$, and on the overlap where $v_1, v_2 \neq 0$, we have

$$v_2 = \frac{1}{v_1}, \; \theta_2 = \theta_1 + \arg v_1.$$

**Proposition VIII.1.** *The mapping $S^3 \to \mathcal{B}^+_{\mathbb{R}}(E)$ given by*

$$(z_1, z_2) \mapsto \begin{cases} v_1 = \frac{z_2}{z_1}, \; \theta_1 = \arg z_1 & \text{if } z_1 \neq 0 \\ v_2 = \frac{z_1}{z_2}, \; \theta_2 = \arg z_2 & \text{if } z_2 \neq 0 \end{cases}$$

*is a diffeomorphism which takes the orientation of $S^3$ as the boundary of the unit ball in $\mathbb{R}^4$ to the chosen orientation of $\mathcal{B}^+_{\mathbb{R}}(E)$, and which transforms the Hopf circle action*

$$\Theta * (z_1, z_2) = (e^{i\Theta} z_1, e^{i\Theta} z_2)$$

*into the canonical circle action*

$$\Theta * (v_j, \theta_j) = (v_j, \theta_j + \Theta)$$

*on $\mathcal{B}^+_{\mathbb{R}}(E)$.*

**Proof.** The main thing to check is that the mapping is compatible with the identification $\theta_2 = \theta_1 + \arg v_1$, which becomes $\arg z_2 = \arg z_1 + \arg(z_2/z_1)$.

The map is injective: if we know $v_1, \theta_1$, then from $z_2 = v_1 z_1$ and the equation $|z_1|^2 + |z_2|^2 = 1$ we see

$$|z_1|^2 = \frac{1}{1 + |v_1|^2},$$

and since we also know the argument of $z_1$, we know $z_1$, hence $z_2$.

The surjectivity is also clear from the argument above.

The compatibility with the circle action is

$$\frac{e^{i\Theta} z_1}{e^{i\Theta} z_2} = \frac{z_1}{z_2} \quad , \quad \arg e^{i\Theta} z_1 = \arg z_1 + \Theta.$$

The vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a direct basis of $T_{(1,0)}S^3$ if you orient $S^3$ as the boundary of the ball. The first of these vectors is tangent to the oriented orbit through $(1, 0)$, whereas the last two project under $p$ to a direct basis of $T_{p(1,0)}E$. $\square$

**The real oriented blow-up of a line in $\mathbb{P}^2_{\mathbb{C}}$.**

We now examine the blow-up of a line in $\mathbb{P}^2_{\mathbb{C}}$, which we will take to be the line at infinity $l_\infty$. This time, local coordinates on a neighborhood of $l_\infty \in \mathbb{P}^2_{\mathbb{C}}$ are

$$u_1 = \frac{1}{x},\ v_1 = \frac{y}{x} \quad \text{and} \quad u_2 = \frac{1}{y},\ v_2 = \frac{x}{y}.$$

This leads to the charts $\mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z} \to \mathcal{B}^+_{\mathbb{R}}(l_\infty)$ given by

$$(v_j, \theta_j),\ j = 1, 2.$$

On the overlap $v_1, v_2 \neq 0$ these coordinates are identified by

$$v_2 = \frac{1}{v_1}, \theta_2 = \theta_1 - \arg v_1.$$

This is also a variant of the Hopf fibration.

**Proposition VIII.2.** *a) The mapping $S^3 \to \mathcal{B}^+_{\mathbb{R}}(l_\infty)$ given by*

$$(z_1, z_2) \mapsto \begin{cases} v_1 = \frac{z_2}{z_1},\ \theta_1 = -\arg z_1 & \text{if } z_1 \neq 0 \\ v_2 = \frac{z_1}{z_2},\ \theta_2 = -\arg z_2 & \text{if } z_2 \neq 0 \end{cases}$$

*is a diffeomorphism which carries the orientation of $S^3$ as the boundary of the complement of the 4-ball to the standard orientation $\mathcal{B}^+_{\mathbb{R}}(l_\infty)$.*

*b) This diffeomorphism transforms the circle action*

$$\Theta * (z_1, z_2) = (e^{-i\Theta} z_1, e^{-i\Theta} z_2)$$

*into the canonical circle action*

$$\Theta * (v_j, \theta_j) = (v_j, \theta_j + \Theta).$$

The proof is identical to the case above.

**Stereographic projection.** What really makes these two cases different? It is the orientation which $S^3$ acquires. We invite the reader to check that with the definition of VI.7, $\mathcal{B}^+_{\mathbb{R}}(D)$ is oriented so that a tangent vector to an oriented orbit, followed by an oriented basis tangent to a section, defines the orientation of the space $\mathcal{B}^+_{\mathbb{R}}(D)$. Thus if we reverse the direction of the circle orbits (and keep the orientation of the base), we change the orientation of the space.

To better picture the difference of the two cases above, we will identify $S^3 - \{(0,0,0,1)\}$ to $\mathbb{R}^3$ via stereographic projection.

**Proposition VIII.3.** *Set $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$, $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$.*

*a) The stereographic projection $S^3 - \{north\ pole\} \to \mathbb{R}^3$ given by*

$$\begin{aligned} X &= \frac{x_1}{1-y_2} & \qquad & X = \frac{x_1}{1-y_2} \\ Y &= \frac{y_1}{1-y_2} & \textit{resp.} \qquad & Y = \frac{y_1}{1-y_2} \\ Z &= \frac{x_2}{1-y_2} & \qquad & Z = -\frac{x_2}{1-y_2} \end{aligned} \tag{8.1}$$

*maps the orientation of the 3-sphere as the boundary of the inside (resp. outside) of the unit ball in $\mathbb{C}^2$ to the standard orientation of $\mathbb{R}^3$.*

*b) The tori $T_r$, $0 \le r \le 1$, with parametric equations*

$$\begin{aligned} X &= \left(\frac{\sqrt{1-r^2}}{1-r\sin\alpha}\right)\cos\beta & \qquad & X = \left(\frac{\sqrt{1-r^2}}{1-r\sin\alpha}\right)\cos\beta \\ Y &= \left(\frac{\sqrt{1-r^2}}{1-r\sin\alpha}\right)\sin\beta & \textit{resp.} \qquad & Y = \left(\frac{\sqrt{1-r^2}}{1-r\sin\alpha}\right)\sin\beta \\ Z &= \frac{r\cos\alpha}{1-r\sin\alpha} & \qquad & Z = -\frac{r\cos\alpha}{1-r\sin\alpha} \end{aligned} \tag{8.2}$$

*are the images of the tori $|z_1| = \sqrt{1-r^2}$, $|z_2| = r$. They are tori of revolution around the $z$-axis, with two singular tori: the unit circle in the $(x,y)$-plane (for $r = 0$) and the $z$-axis (for $r = 1$). The other tori are obtained by rotating around the $z$-axis the circle in the $(X,Z)$-plane with center at $X = 1/\sqrt{1-r^2}, Z = 0$ and radius $r/\sqrt{1-r^2}$.*

*c) The circle actions*

$$\Theta * (z_1, z_2) = (e^{i\Theta} z_1, e^{i\Theta} z_2) \quad \textit{resp.} \quad \Theta * (z_1, z_2) = (e^{-i\Theta} z_1, e^{-i\Theta} z_2)$$

*become, in these coordinates,*

$$\Theta * (r,\alpha,\beta) = (r, \alpha+\Theta, \beta+\Theta) \quad \textit{resp.} \quad \Theta * (r,\alpha,\beta) = (r, \alpha-\Theta, \beta-\Theta). \tag{8.3}$$

**Proof.** We will prove only the parts concerning the blow-up of the line at infinity, which appear in the right-hand column, as those are the ones we are concerned with.

a) Recall that if $M$ is a smooth k-manifold with boundary and $m \in \partial M$, then

$$v_1, v_2, \ldots, v_{k-1} \in T_m \partial M$$

define the standard orientation of $\partial M$, if $v_0, v_1, \ldots, v_{k-1}$ define the standard orientation of $M$, where $v_0$ is a outward-pointing vector of $T_m M$. The vectors

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

are a direct basis of $\mathbb{R}^4$ with $v_0$ pointing out of the outside of the unit sphere at the south pole, so $v_1, v_2, v_3$ define the orientation of $S^3$ as the boundary of the outside. Under the stereographic projection (8.1) these vectors get mapped to the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which give the standard orientation of $\mathbb{R}^3$.

Rewriting the mapping (8.1) in polar coordinates

$$x_1 = \sqrt{1-r^2}\cos\beta, \;\; y_1 = \sqrt{1-r^2}\sin\beta, \;\; x_2 = r\cos\alpha, \;\; y_2 = r\sin\alpha, \;\; 0 \le r \le 1,$$

we will immediately get formulas (8.2) and statement c) of the proposition.

To prove part b) it is enough to check that the curves in the $(X, Z)$ plane given by the parametric equations

$$X = \frac{\sqrt{1-r^2}}{1 - r\sin\alpha} \quad \text{and} \quad Z = \frac{r\cos\alpha}{1 - r\sin\alpha}$$

are actually the circles with centers at $(1/\sqrt{1-r^2}, 0)$ and radii $r/\sqrt{1-r^2}$. Indeed,

$$\left(\frac{\sqrt{1-r^2}}{1-r\sin\alpha} - \frac{1}{\sqrt{1-r^2}}\right)^2 + \left(\frac{r\cos\alpha}{1-r\sin\alpha}\right)^2 = \frac{r^2(r-\sin\alpha)^2 + r^2\cos^2\alpha(1-r^2)}{(1-r\sin\alpha)^2(1-r^2)} = \frac{r^2}{1-r^2}$$

□

**Blowing up one point of a line.** Suppose we now want to blow up one point of $E$ or of $l_\infty$. We can symbolically represent the two variants of the Hopf fibration in the following picture.

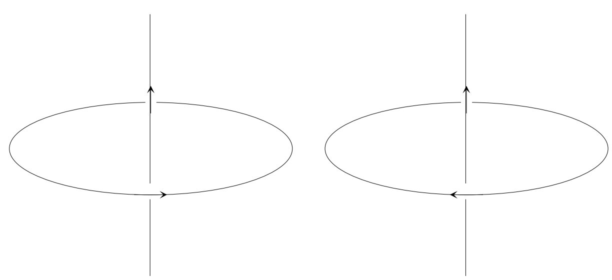

Figure 23. The real oriented blow-up of an exceptional divisor (left) and of the line at infinity (right).

In both $\mathcal{B}^+_{\mathbb{R}}(\tilde{\mathbb{P}}^2_0, E)$ and $\mathcal{B}^+_{\mathbb{R}}(\tilde{\mathbb{P}}^2_2, l_\infty)$, $S^3 - \{\text{north pole}\}$ is identified by stereographic projection with $\mathbb{R}^3$ with its standard orientation. In both, the $z$-axis oriented up is one oriented fiber, and the unit circle in the $x, y$-plane is another, oriented counterclockwise in the first, clockwise in the second. Now we will blow up one point $\mathbf{p}$, on $E$ or $l_\infty$ respectively, and see what happens to the real oriented blow-ups. Choose the unit circle in the $x, y$-plane to correspond to $\mathbf{p}$, and thicken it as in Theorem VII.5, corresponding to the region $r \leq \epsilon$ for some small $\epsilon > 0$. This solid torus $T$ intersects the $x, z$-plane in a disc which has a canonical orientation, since it projects homeomorphically to $E$ or $l_\infty$, and the boundary orientation of the boundary of the disc in both cases corresponds to $\alpha$ increasing from 0 to $2\pi$. This gives one generator $a$ of $H_1(\partial T)$. Letting $\beta$ increase from 0 to $2\pi$ defines a second generator $b$ of $H_1(\partial T)$; this curve also corresponds to the boundary of a disc on the outside of $T$, which also has a boundary orientation, and our conventions are consistent.

In these coordinates, an oriented orbit of the circle action on the torus has homology class $a + b$ in the case of $\mathcal{B}^+_{\mathbb{R}}(E)$, and homology class $-a - b$ in the case of $\mathcal{B}^+_{\mathbb{R}}(l_\infty)$.

If we change the circle action inside the torus, as in Theorem VII.5, the homology class of an orbit is $a + 2b$ in the case of $\mathcal{B}^+_{\mathbb{R}}(E)$, and $-b$ in the case of $\mathcal{B}^+_{\mathbb{R}}(l_\infty)$. This is not a difference in sign convention: fibers inside the torus link with linking number $-2$ in the first case, and in the second case have linking number 0 (we are following Milnor's convention [Mil4] for the sign of the linking number).

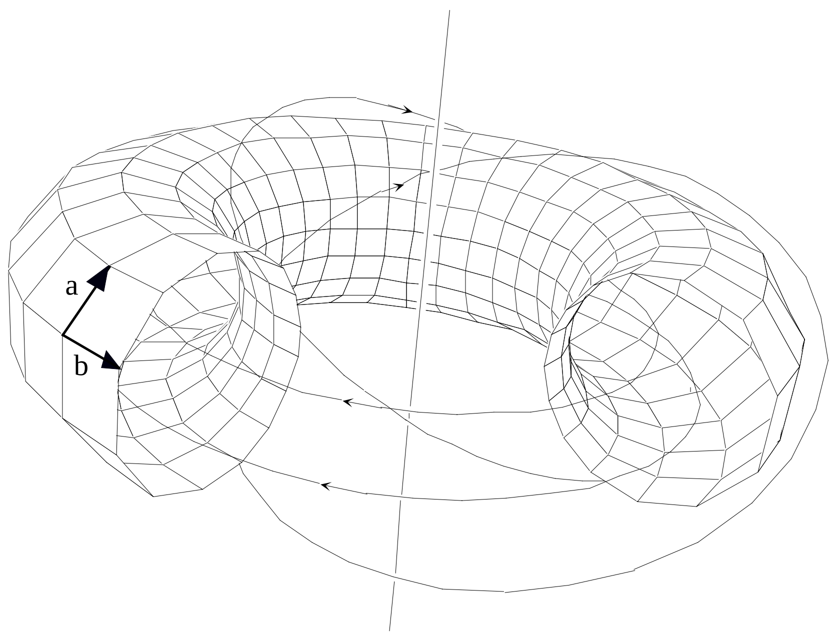


FIGURE 24. Blow up $\mathbb{P}^2$ at a point $z \in l_\infty$, and then take the real oriented blow-up of the divisor consisting $l_\infty$ and the exceptional divisor. You obtain a 3-sphere containing a torus, and a circle action on both components of the complement of the torus. This picture represents the stereographic projection of this space, with the inside of the torus corresponding to the exceptional divisor, where the circle orbits do not link, and the outside corresponding to the line at infinity, where the circles link with linking number 1. Curves describing the homology classes $a$ and $b$ are drawn on the torus; note that the circle orbits on the outside are in the class $-a - b$ and those inside are in the class $-b$.

So these two alternatives both exist, and even a trusting reader might wonder whether we have gotten it right: there were about six places where a wrong sign would lead to the two possibilities getting interchanged. It would be most reassuring to find some independent way of choosing the correct picture, and there is.

Indeed, the divisors $D$ under consideration consist of two lines: in one case they have respectively self-intersection $-2$ and $-1$, and in the other, 0 and $-1$ (in both cases, -1 corresponds to the new exceptional divisor). Figure 25 represents these two configurations.

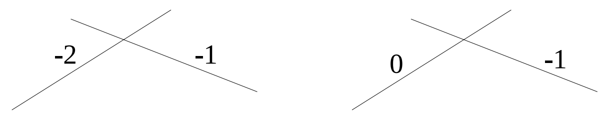


FIGURE 25. The divisor being blown up.

The self intersection number 0 really means that the normal bundle to the proper transform $l'_\infty$ is trivial. A limit on the torus $p^{-1}(\tilde{z})$ of circle orbits coming from the exceptional divisor is a vector tangent at $z$ to the last exceptional divisor, plus a normal vector, i.e., a vector tangent to $l_\infty$ which makes a full turn around.

This can be moved slightly to a circle in $l'_\infty$ turning around $z$, plus a constant normal vector. Since the constant normal vector in the small disc around $z$ can extended to a vector field on $l'_\infty$ normal to $l'_\infty$, we see the disc on the outside bounded by such an orbit. So our case, with self intersection of $l'_\infty$ zero, definitely corresponds to the proposed picture.

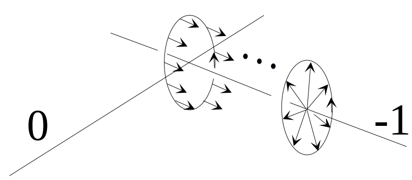


FIGURE 26. The circle at right represents a normal vector to the exceptional divisor, turning around the divisor, i.e., an orbit of the circle action. This circle can be deformed into the circle at left, on the original line (at infinity), turning around $z$, and with a constant normal vector.

## IX. Real oriented blow-ups for complex Hénon mappings

Each space $X_{[-N-1,N+1]}$ is obtained from $X_{[-N,N]}$ by a sequence of blow-ups, first at $\mathbf{q}_N$ and $\mathbf{p}_{N+1}$ and then at points of the most recent exceptional divisor; let

$$\pi_{N+1,N} : X_{[-N-1,N+1]} \to X_{[-N,N]}$$

denote the blow-down mapping. By Propositions VII.3 and VII.7, the mapping $\pi_{N+1,N}$ induces a projection

$$\tilde{\pi}_{N+1,N} : \mathcal{B}^+_{\mathbb{R}}\left(X_{[-(N+1),(N+1)]}, D_{[-(N+1),N+1]}\right) \to \mathcal{B}^+_{\mathbb{R}}\left(X_{[-N,N]}, D_{[-N,N]}\right),$$

which allows us to consider the projective limit

$$\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty) = \varprojlim \Big(\mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]}); \tilde{\pi}_{N+1,N}\Big).$$

There is a canonical inclusion $j_N : \mathbb{C}^2 \to X_{[-N,N]}$ (as in Proposition IV.6, b), which lifts to $\tilde{j}_N : \mathbb{C}^2 \to \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]})$ since the real oriented blow-up is taken along $D_{[-N,N]} = X_{[-N,N]} - \mathbb{C}^2$. These inclusions are compatible with $\tilde{\pi}_{N+1,N}$, leading to an inclusion $\tilde{j}_\infty : \mathbb{C}^2 \to \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$.

**Theorem IX.1.** *a) The mapping $\tilde{j}_\infty$ is injective, with dense image, allowing us to think of $\mathbb{C}^2$ as a subset of $\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$.*

*b) The Hénon mapping $H : \mathbb{C}^2 \to \mathbb{C}^2$ extends continuously to an automorphism*

$$\mathcal{B}^+_{\mathbb{R}}(H_\infty) : \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty) \to \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty).$$

**Proof.** (a) The injectivity of $\tilde{j}_\infty$ is clear since all the $\tilde{j}_N$ are injective. Moreover, all the $j_N$ have dense image by Proposition IV.6, and so do the $\tilde{j}_N$ since the interior of a manifold with boundary is dense in the manifold. The density of the image of $\tilde{j}_\infty$ follows immediately, since the topology on the projective limit is inherited from the topology of the product. (This also follows from the much more general Theorem VII.11).

(b) Clearly the Hénon mapping, i.e., the shift, induces an isomorphism

$$X_{[-N,N+1]} \to X_{[-(N+1),N]},$$

hence a homeomorphism

$$\mathcal{B}^+_{\mathbb{R}}\big(X_{[-N,N+1]}, D_{[-N,N+1]}\big) \to \mathcal{B}^+_{\mathbb{R}}\big(X_{[-(N+1),N]}, D_{[-(N+1),N]}\big)$$

by Proposition VI.4. The result will follow since

$$\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty) = \varprojlim_{M,N\to\infty} \mathcal{B}^+_{\mathbb{R}}\big(X_{[-N,M]}, D_{[-N,M]}\big)$$

when $N$ and $M$ can go to infinity in any way one wants: the pairs $[-N,N]$ are cofinal in the projective system of pairs $[-N,M]$. $\square$

**Remark.** The existence of $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ is a bit less trivial than one might expect. For instance, there is *no well-defined* mapping $\mathcal{B}^+_{\mathbb{R}}(\tilde{H}) : \mathcal{B}^+_{\mathbb{R}}(\tilde{X}, \tilde{D}) \to \mathcal{B}^+_{\mathbb{R}}(\mathbb{P}^2, l_\infty)$. Indeed, although $\tilde{D} = \tilde{H}^{-1}(l_\infty)$ set theoretically, it is *not true* in the sense of ideals. In the sense of ideals, $\tilde{H}^{-1}(l_\infty)$ contains $A'$, $B$ and $L_1, \dots, L_d$ with multiplicity $d-1$, and $L_{d+k}$ with multiplicity $d-k-1$ for $k = 0, \dots, d-3$, and finally $\tilde{A}$ with multiplicity 1 (for the notation, see Figure 4). At the double points where two irreducible components of $\tilde{D}$ meet and their multiplicities are the same, the map $\tilde{H}$ extends anyway because of Example II.6. But where the two multiplicities are different, Theorem VI.4 does not guarantee that $\tilde{H}$ extends, and in fact it doesn't, as can be verified by explicit computation. $\triangle$

The remainder of this section is devoted to understanding the structure of $\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ in detail, as this is equivalent to understanding the dynamics of Hénon mappings at infinity.

**Theorem IX.2.** *(a) The pair $(\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty), \mathcal{B}^+_{\mathbb{R}}(D_\infty))$ is homeomorphic to the pair $(B_4, S^3)$, the closed 4-ball bounded by the 3-sphere.*

*b) The mapping $p : \mathcal{B}^+_{\mathbb{R}}(D_\infty) \to D_\infty$ has as its fibers:*

- *a circle above ordinary points.*
- *a torus above double points,*
- *a $d$-adic solenoid $\Sigma^-$ above $\mathbf{p}^\infty$ and a $d$-adic solenoid $\Sigma^+$ above $\mathbf{q}^\infty$.*

**Proof.** Part (a) has already been proved in Theorem VII.11. More precisely, we have seen that the real oriented blow-up of $\mathbb{P}^2$ along the line at infinity is a 4-ball bounded by a 3-sphere, and so $\varprojlim \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]})$ is also. Indeed, $\varprojlim \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]})$ is obtained by infinitely many times making a blow-up of a surface $X$ at a point $z$ of a divisor $D$, then taking the real oriented blow-up of the resulting surface $\tilde{X}_z$ along the inverse image $\tilde{D}$ of the divisor $D$. That is precisely the situation of Theorem VII.11. Moreover, the first two cases of part (b) follow immediately from Corollary VI.10.

The third statement is a bit more delicate. The point $\mathbf{p}^\infty$ is represented by the sequence

$$\mathbf{p}_1 \in X_{[0,0]}, \quad \mathbf{p}_2 \in X_{[-1,1]}, \quad \mathbf{p}_3 \in X_{[-2,2]}, \quad \ldots$$

and above this point we see the projective limit of the system of circles

$$p_0^{-1}(\mathbf{p}_1) \leftarrow p_1^{-1}(\mathbf{p}_2) \leftarrow p_2^{-1}(\mathbf{p}_3) \ldots .$$

**Remark.** We are adding an index to avoid ambiguity, calling

$$p_N : \mathcal{B}^+_{\mathbb{R}}\left(X_{[-N,N]}, D_{[-N,N]}\right) \to X_{[-N,N]}$$

the canonical projection; the added notation is necessary as the $\mathbf{p}_{N+1}$ can be viewed as points in all $X_{[-M,M]}$ with $M > N$ (or in $X_\infty$), but only in $X_{[-N,N]}$ are $\mathbf{p}_{N+1}$ and $\mathbf{q}_N$ simple points of $D_{[-N,N]}$. $\triangle$

There is a canonical parametrization of $p_N^{-1}(\mathbf{p}_{N+1}) \subset \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]})$, obtained from the composition

$$\begin{array}{ccccccc}
p_N^{-1}(\mathbf{p}_{N+1}) & \xrightarrow[N\text{th coordinate}]{\text{projection to}} & p^{-1}(\tilde{\mathbf{p}}) & \xrightarrow{\text{applying } \tilde{H}} & p^{-1}(\mathbf{p}) & \xrightarrow{\arg 1/y} & \mathbb{R}/2\pi\mathbb{Z}. \\
\bigcap & & \bigcap & & \bigcap & & \\
\mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]}) & & \mathcal{B}^+_{\mathbb{R}}(\tilde{X}, \tilde{D}) & & \mathcal{B}^+_{\mathbb{R}}(\mathbb{P}^2, l_\infty) & &
\end{array}$$

We will denote this coordinate by $\theta_N$.

There is also a natural parametrization of $p_N^{-1}(\mathbf{q}_{-N}) \subset \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, X_{[-N,N]})$, which is a bit simpler, obtained from the similar composition

$$\begin{array}{ccccc} p_N^{-1}(\mathbf{q}_{-N}) & \xrightarrow[-N\text{th coordinate}]{\text{projection to}} & p^{-1}(\mathbf{q}') \simeq & p^{-1}(\mathbf{q}) & \xrightarrow{\arg 1/x} \mathbb{R}/2\pi\mathbb{Z}. \\ \bigcap & & \bigcap & \bigcap & \\ \mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, D_{[-N,N]}) & & \mathcal{B}^+_{\mathbb{R}}(\tilde{X}, \tilde{D}) & \mathcal{B}^+_{\mathbb{R}}(\mathbb{P}^2, l_\infty) & \end{array}$$

We will denote this coordinate by $\phi_N$.

Now part (c) follows from Proposition IX.3.

**Proposition IX.3.** *We have*

$$\tilde{\pi}_{N+1,N}(\theta_{N+1}) = d\theta_{N+1} + \arg a \quad \textit{and} \quad \tilde{\pi}_{N+1,N}(\phi_{-N-1}) = d\phi_{-N-1}.$$

**End of proof of IX.2 using IX.3.** One description (see [HO1], Section 3) of the $d$-adic solenoid $\Sigma_d$ is as

$$\Sigma_d = \varprojlim(\mathbb{R}/2\pi\mathbb{Z}, \theta \mapsto d\theta).$$

Clearly the second part of Proposition IX.3 shows that precisely the space $\Sigma^+$ above $\mathbf{q}^\infty$ is canonically the $d$-adic solenoid.

For the point $\mathbf{p}^\infty$, observe that if we set $\psi = \phi + \arg a/(d-1)$, then the mapping $\phi \mapsto d\phi + \arg a$ becomes

$$\psi \mapsto d\left(\psi - \frac{\arg a}{d-1}\right) + \arg a + \frac{\arg a}{d-1} = d\psi.$$

Thus the subset $\Sigma^- \subset \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ above $\mathbf{q}^\infty$ is also a $d$-adic solenoid.

**Proof of IX.3.** Let us first compute the map

$$\begin{array}{ccc} p^{-1}(\tilde{\mathbf{p}}) & \to & p^{-1}(\mathbf{p}) \\ \bigcap & & \bigcap \\ \mathcal{B}^+_{\mathbb{R}}(\tilde{X}, \tilde{D}) & & \mathcal{B}^+_{\mathbb{R}}(\mathbb{P}^2, l_\infty) \end{array}$$

induced by the blow-down mapping $\tilde{X}_H \to \mathbb{P}^2$, where both domain and range are identified to $\mathbb{R}/2\pi\mathbb{Z}$:

- $p^{-1}(\mathbf{p})$ using $\arg 1/y$;
- $p^{-1}(\tilde{\mathbf{p}})$ using $(\arg 1/y) \circ \tilde{H}$.

In Section III, we began by using the coordinates $u = x/y, v = 1/y$ near $\mathbf{p}$, so we see that $\arg v$ is our parameter for $p^{-1}(\mathbf{p})$. Still in the notation of Section III, $\arg X_1$ gives the parametrization of $p^{-1}(\tilde{\mathbf{p}})$. Formula (3.6), for the case $k = d-1$, tells us that the point $\arg X_1 = \theta$ of $p^{-1}(\tilde{\mathbf{p}})$ is mapped by $\mathcal{B}^+_{\mathbb{R}}(\tilde{H})$ to the point where $v$ has argument

$$\arg \frac{X_1\big(X_1^{d-1}Y_{d-1} + a\sum_{j=0}^{d-2} Q_j X_1^j\big)}{X_1^{d-1}Y_{d-1} + a\sum_{j=0}^{d-2} Q_j X_1^j} = \arg X_1.$$

Thus we must see how the blow-down maps $p^{-1}(\tilde{\mathbf{p}})$ to $p^{-1}(\mathbf{p})$, working in the coordinate $\theta = \arg X_1$ in the domain, and $\theta = \arg v$ in the range, since these correspond under the Hénon mapping.

More precisely, consider the mapping from the circle $p^{-1}(\tilde{\mathbf{p}}) \subset \mathcal{B}^+_{\mathbb{R}}(\tilde{X}_H, \tilde{D})$ to the circle $p^{-1}(\mathbf{p}) \subset \mathcal{B}^+_{\mathbb{R}}(\mathbb{P}^2, l_\infty)$. Let $\mathbf{a}_0 = \mathbf{p}$, $\mathbf{a}_1, \dots, \mathbf{a}_{2d-2}$ be the successive points at which we performed blow-ups to get from $\mathbb{P}^2$ to $\tilde{X}_H$, and finally set $\mathbf{a}_{2d-1} = \tilde{\mathbf{p}}$. The point $\mathbf{a}_0$, and the points $\mathbf{a}_d, \dots, \mathbf{a}_{2d-1}$ are simple points of the divisor constructed so far; the points $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}$ are double points. The inverse images of these points are parametrized by

$$\begin{array}{rll} \arg v & \text{at } \mathbf{a}_0, & \\ \begin{pmatrix} \arg X_1 \\ \arg u \end{pmatrix} & \text{at } \mathbf{a}_1, & \\ \begin{pmatrix} \arg X_1 \\ \arg X_k \end{pmatrix} & \text{at } \mathbf{a}_k, & k = 2, \dots, d-1, \\ \arg X_1 & \text{at } \mathbf{a}_k, & k = d, \dots, 2d-1. \end{array} \tag{9.1}$$

In these coordinates, the blow-down mapping $\tilde{X}_H \to \mathbb{P}^2$ induces the composition

$$\begin{array}{l} \underset{\mathbf{a}_{2d-1}}{\theta} \mapsto \cdots \mapsto \underset{\mathbf{a}_d}{\theta} \mapsto \underset{\mathbf{a}_{d-1}}{\begin{bmatrix} \theta \\ \theta + \arg a \end{bmatrix}} \mapsto \underset{\mathbf{a}_{d-2}}{\begin{pmatrix} \theta \\ 2\theta + \arg a \end{pmatrix}} \mapsto \dots \\ \qquad \mapsto \underset{\mathbf{a}_1}{\begin{pmatrix} \theta \\ (d-1)\theta + \arg a \end{pmatrix}} \mapsto \underset{\mathbf{a}_0}{d\theta + \arg a}. \end{array} \tag{9.2}$$

All of these are straightforward applications of Propositions VII.4 and VII.8. Let us spell out the mapping which takes $\mathbf{a}_d$ to $\mathbf{a}_{d-1}$. In that case we are taking the circle above the point $X_1 = 0, X_d = a$, parametrized by $\arg X_1$, to the torus above $X_1 = 0, X_{d-1} = 0$. The blow-down mapping is

$$\begin{pmatrix} X_1 \\ X_d \end{pmatrix} \mapsto \begin{pmatrix} X_1 \\ X_{d-1} \end{pmatrix} = \begin{pmatrix} X_1 \\ X_1 X_d \end{pmatrix},$$

and in particular the circle $X_1 = \rho e^{i\theta}$, $X_d = a$ is mapped to the circle $X_1 = \rho e^{i\theta}, X_{d-1} = \rho a e^{i\theta}$. If we let $\rho \to 0$ and remember only the arguments, we get the desired formula.

This almost proves the first part of Proposition IX.3; by definition, the mapping $p_N^{-1}(\mathbf{p}_{N+1}) \to p_{N-1}^{-1}(\mathbf{p}_N)$ is precisely the mapping above, the domain and range being identified to $\mathbb{R}/2\pi\mathbb{Z}$ by $H^{\circ N}$ and $H^{\circ(N-1)}$ respectively.

There are two ways of approaching the first part of Proposition IX.3: to make the sequence of blow-ups at $\mathbf{q}$, repeating the material of section III to make $H^{-1}$ well-defined, or to make a change of variables to make $H^{-1}$ conjugate to a Hénon mapping, for a new polynomial $p_1$ and and a new Jacobian $a_1$, of course. Remember that we used the fact that the polynomial $p$ is monic in Section III, so $p_1$ must also be monic. We invite the reader to show that in the variables $x_1, y_1$, where $\zeta x_1 = y$ and $\zeta y_1 = x$ with $\zeta^{d-1} = a$, we have

$$H^{-1} : \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mapsto \begin{pmatrix} p_1(x_1) - y_1/a \\ x_1 \end{pmatrix}$$

with $p_1$ monic. Thus, in these coordinates the blow-down takes

$$\arg(1/y_1) \to d \arg(1/y_1) + \arg(1/a).$$

We invite the reader to check that this means exactly that in the original variables, the formula of Proposition IX.3 is satisfied. □

## X. The topology of $\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$

We now want to describe the mapping $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ more precisely. A first statement says that appropriate restrictions of $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ and $\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1}$ are solenoidal ([HO1], Section 3).

**Remark.** Actually, the definition of solenoidal which appears in [HO1] isn't quite the right one in our setting, because there the mappings are differentiable, and the notion of expanding and contracting is with respect to this structure; $\mathcal{B}^+_{\mathbb{R}}(D_\infty)$ doesn't have a natural differentiable structure, especially on the solenoids $\Sigma^\pm$ which are naturally accumulations of corners. However, the main result we will be using is Theorem 3.1 of [HO1] which uses none of this structure (and in the main case of interest, we will construct the mapping explicitly anyway, in Proposition X.1). The construction of Theorem VII.11 provides the necessary contraction in the fiber directions. △

Let us denote by $T_{\mathbf{p}_i}$ and $T_{\mathbf{q}_i}$ the tori $\tilde{p}^{-1}(\mathbf{p}_i)$ and $\tilde{p}^{-1}(\mathbf{q}_i)$. Each of these separates $\mathcal{B}^+_{\mathbb{R}}(D_\infty)$ into two pieces. We will denote by $T^+_{\mathbf{p}_i}$ the one which contains the attractive solenoid $\Sigma^+$, and by $T^-_{\mathbf{p}_i}$ the one which contains the repelling solenoid $\Sigma^-$, and similarly for $\mathbf{q}_i$.

**Proposition X.1.** *a) The mapping $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ maps $T^+_{\mathbf{p}_i}$ (resp. $T^+_{\mathbf{q}_i}$) into itself, and is solenoidal of degree $d$.*

*b) The mapping $\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1}$ maps $T^-_{\mathbf{p}_i}$ (resp. $T^-_{\mathbf{q}_i}$) into itself, and is solenoidal of degree $d$.*

**Proof.** We need to examine carefully the sequence of blow-ups which makes $H$ well-defined, to understand how the tori corresponding to the double points of $D_\infty$ are embedded in the 3-sphere $\mathcal{B}^+_{\mathbb{R}}(D_\infty)$. Recall that we called $\mathbf{a}_0 = \mathbf{p},\ \mathbf{a}_1, \dots, \mathbf{a}_{2d-2}$ the successive points at which we performed blow-ups to get from $\mathbb{P}^2$ to $\tilde{X}_H$, and finally set $\mathbf{a}_{2d-1} = \tilde{\mathbf{p}}$.

In Section VIII, we showed how to start, creating a torus $T_{\mathbf{a}_0}$ in the 3-sphere, separating the solid torus corresponding to $l_\infty$ from the solid torus corresponding to the first exceptional divisor (which will eventually be the irreducible component $B$ of $\tilde{D}$). See Figure 24 to understand how these solid tori are placed after stereographic projection; the words "inner" and "outer" will refer to this picture.

The next $d-1$ blow-ups are fairly easy to understand, now that we have started right. We thicken the torus $T_{\mathbf{a}_0}$, creating an inner torus $T_{\mathbf{a}_1}$ and an outer torus $T_{\mathbf{p}'} = T_{\mathbf{p}_0}$ (which we can call by its final name, since it will not be affected by further blow-ups). Then thicken the inner torus $T_{\mathbf{a}_1}$, creating an inner torus $T_{\mathbf{a}_2}$, and one which corresponds to $L_1 \cap L_2$ (between $T_{\mathbf{a}_2}$ and $T_{\mathbf{p}_0}$). Then thicken the inner torus $T_{\mathbf{a}_2}$ again, $d-1$ times in all. The inside of the torus $T_{\mathbf{p}_0}$ is $T^-_{\mathbf{p}_0}$. See Figure 27 for the case $d = 3$.

The circle orbits fibering the regions between the successive tori are contained in $T^-_{\mathbf{p}_0}$ with the innermost torus (corresponding to the component $B$) removed, which is a space with homology $\mathbb{Z}^2$, generated by $a$ and $b$ (see Figure 24). At the first thickening, an oriented circle orbit between the two tori is in the homology class $-a-2b$, at the next the new thickened torus is fibered by curves with homology class $-a-3b$, etc., ending up with a thickened torus fibered by circles in the homology class $-a-db$, and an inner solid torus (corresponding to $B$) with fibers in the homology class $-b$.

In summary, after the first $d$ blow-ups, we have an inner solid torus with fibers in the homology class $-b$, then a succession of thickened tori with fibers in the homology classes

$$-a-db, \quad -a-(d-1)b, \quad \dots, \quad -a-2b, \tag{10.1}$$

and finally the region $T^+_{\mathbf{p}_0}$ corresponding to $l_\infty$, fibered (by the old Hopf fibers) in the homology class $-a-b$. See Figures 27 and 28.

We must now make $d-1$ more blow-ups of ordinary points. The first of these can be realized by thickening a circle orbit in the region corresponding to $L_{d-1}$, which is fibered by circles in the homology class $-a-db$. We will then thicken a circle inside this torus, which we may take to be the "core circle", and repeat this $d-2$ times. The final torus we create this way is $T_{\mathbf{q}_1}$. All the solid tori are thickenings of the original circle, and hence in the homology class $-a-db$.

We need to start making the second series of blow-ups, as we don't yet have the torus $T_{\mathbf{p}_1}$. So thicken a fiber inside $T^-_{\mathbf{q}_1}$, creating a solid torus still in the homology class $-a-db$, and thicken it again $d-1$ times; the outermost torus of the series just created is $T_{\mathbf{p}_1}$. We will not need to describe the further blow-ups.

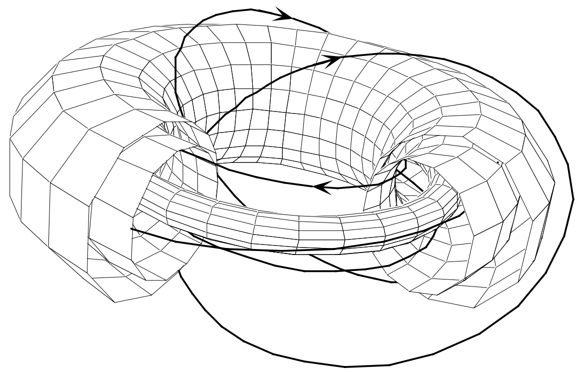

FIGURE 27. The picture above corresponds to the situation after $d$ blow-ups when $d = 3$, and after stereographic projection. The outside corresponds to the real oriented blow-up of the line at infinity (now $A'$), with the Hopf circle action, as shown. The inner torus corresponds to $B$, with the circle action where the orbits are not linked. The region between the inner torus and the next corresponds to $L_{d-1}$; that is where all the further action will take place, thickening a circle orbit (represented on the drawing, and going around 3 times in one direction as it turns once in the other direction). The region between the outer torus and the next corresponds to $L_1$; the circle action there has orbits which turn twice in one direction as they turn once in the other; we haven't drawn them to keep the drawing simpler.

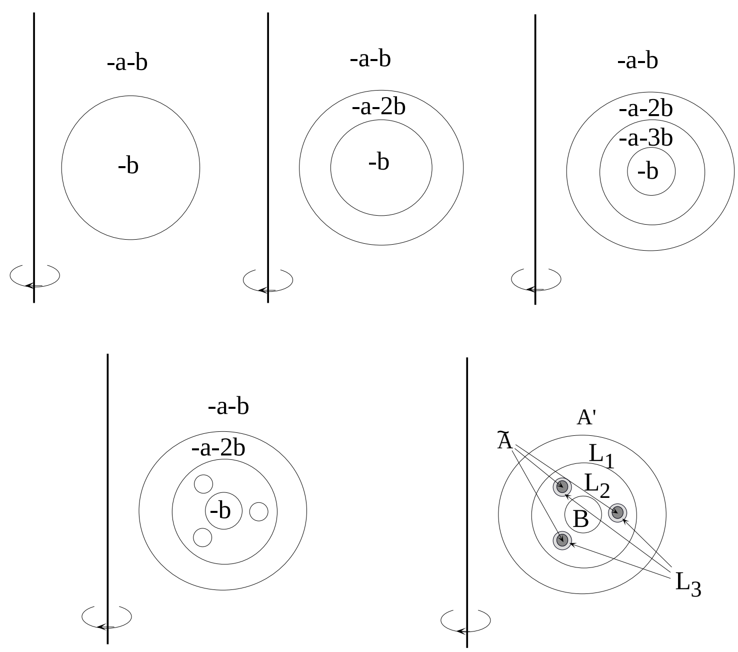


FIGURE 28. You can almost imagine constructing the pattern of tori in $S^3$ by rotating the figure above around the $z$-axis (shown as a heavy line). The case represented corresponds to $d = 3$. The "almost" is because the small circles are not actually rotated: as they turn around the $z$-axis they also turn in their annulus, so as to connect up and together form a single torus, as shown in Figure 27.

Moreover, $\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1}$ is a homeomorphism which maps the solid torus $T^-_{\mathbf{p}_0}$ to the solid torus $T^-_{\mathbf{p}_1}$. We claim that as a map $T^-_{\mathbf{p}_0} \to T^-_{\mathbf{p}_1}$ it is conjugate to the mapping $\tau_{d,0}$ as defined in ([HO1], Section 3). Recall that the mapping $\tau_{d,k} : S^1 \times D \to S^1 \times D$, where $S^1, D \subset \mathbb{C}$ are respectively the unit circle and the disc of radius 2, is given by the formula

$$\tau_{d,k}(\zeta, z) = \big(\zeta^d, \zeta + \epsilon z \zeta^{k+1-d}\big).$$

A "solenoidal" map $S^1 \times D \to S^1 \times D$ is one which is expanding in the circle direction and contracting in the disc direction. Certainly $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ expands in the circle direction; in fact it is $\beta \mapsto d\beta$ (this is the coordinate $\beta$ of the stereographic projection). By choosing our thickenings sufficiently small, the mapping will be contracting on the discs. $\square$

**Corollary X.2.** *The mappings $\mathcal{B}^+_{\mathbb{R}}(H_\infty) : T^+_{\mathbf{p}_i} \to T^+_{\mathbf{p}_i}$ and $\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1} : T^-_{\mathbf{p}_i} \to T^-_{\mathbf{p}_i}$ are conjugate to $\tau_{d,0}$.*

**Proof.** Theorem 3.11 of [HO1] asserts that every unbraided solenoidal mapping from a solid torus to itself is conjugate to precisely one of the $\tau_{d,k}$, and Propositions 4.1 and 4.6 assert that only $\tau_{d,0}$ extends to the 3-sphere. Indeed, Proposition 4.6 asserts that when $k \neq 0$, the forward images of $\tau_{d,k}$ are knotted; but no homeomorphism of $S^3$ can map an unknotted solid torus to a knotted one. Since our map $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$ extends to the 3-sphere $\mathcal{B}^+_{\mathbb{R}}(D_\infty)$, it is enough to prove that

$$\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1} : T^-_{\mathbf{p}_1} \to T^-_{\mathbf{p}_1}$$

is solenoidal and unbraided. The unbraided part follows from the homotopy class $-a - db$. $\square$

By Theorem 3.1 of [HO1], there are maps $\pi_i^+ : T^+_{\mathbf{p}_i} \to \mathbb{R}/2\pi\mathbb{Z}$ and $\pi_i^- : T^-_{\mathbf{p}_i} \to \mathbb{R}/2\pi\mathbb{Z}$ such that the diagrams

$$\begin{array}{ccc} T^+_{\mathbf{p}_i} & \xrightarrow{\mathcal{B}^+_{\mathbb{R}}(H_\infty)} & T^+_{\mathbf{p}_i} \\ {\scriptstyle \pi_i^+}\downarrow & & {\scriptstyle \pi_i^+}\downarrow \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow{\theta \mapsto d\theta} & \mathbb{R}/2\pi\mathbb{Z} \end{array} \qquad \begin{array}{ccc} T^-_{\mathbf{p}_i} & \xrightarrow{\mathcal{B}^+_{\mathbb{R}}(H_\infty)^{-1}} & T^-_{\mathbf{p}_i} \\ {\scriptstyle \pi_i^-}\downarrow & & {\scriptstyle \pi_i^-}\downarrow \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow{\theta \mapsto d\theta} & \mathbb{R}/2\pi\mathbb{Z} \end{array}$$

commute.

In our case, these functions $\pi_i^+$ and $\pi_i^-$ can be computed explicitly; they are given by proposition X.3. Before stating this proposition, notice that

- $T^+_{\mathbf{q}_i}$ is the set of $x \in \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ such that $\big(p_\infty(x)\big)_i = \mathbf{q}'$;
- $T^-_{\mathbf{q}_i}$ is the set of $x \in \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ such that $\big(p_\infty(x)\big)_{i-1} \in \tilde{A}$.
- $T^-_{\mathbf{p}_i}$ is the set of $x \in \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ such that $\big(p_\infty(x)\big)_{i-1} = \tilde{\mathbf{p}}$;

- $T^+_{\mathbf{p}_i}$ is the set of $x \in \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ such that $\big(p_\infty(x)\big)_i \in A'$.

There is a natural blow-down mapping $Q_i : X_\infty \to X_{[i,\infty)}$, which blows $D_\infty$ down onto $D_{[i,\infty)}$.

Since $Q_i$ is a blow-down, it induces a mapping

$$\mathcal{B}^+_{\mathbb{R}}(Q_i) : \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty) \to \mathcal{B}^+_{\mathbb{R}}\big(X_{[i,\infty)}, D_{[i,\infty)}\big).$$

More precisely, the blow-downs $X_{[-N,N]} \to X_{[i,N]},\ i < N$ induce mappings on the real oriented blow-ups $\mathcal{B}^+_{\mathbb{R}}(X_{[-N,N]}, X_{[-N,N]}) \to \mathcal{B}^+_{\mathbb{R}}(X_{[i,N]}, D_{[i,N]})$; to construct $Q_i$, we must pass to the projective limit.

The mapping $Q_i$ maps $T^+_{\mathbf{q}_i}$ to the circle above $\mathbf{q}_i$ in $\mathcal{B}^+_{\mathbb{R}}(D_{[i,\infty]}$. This circle is canonically parametrized, by $\phi_i$. Let us denote

$$\Phi_i = \phi_i \circ \mathcal{B}^+_{\mathbb{R}}(Q_i)|_{T^+_{\mathbf{q}_i}} : T^+_{\mathbf{q}_i} \to \mathbb{R}/2\pi\mathbb{Z}$$

the composition.

Exactly analogously, there is a natural blow-down $P_i : X_\infty \to X_{(-\infty,i]}$ , which blows $D_\infty$ down onto $D_{(-\infty,i]}$.

Again, since $P_i$ is a projective limit of blow-downs, it induces a mapping

$$\mathcal{B}^+_{\mathbb{R}}(P_i) : \mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty) \to \mathcal{B}^+_{\mathbb{R}}\big(X_{(-\infty,i]}, D_{(-\infty,i]}\big)$$

which maps $T^-_{\mathbf{p}_i}$ to the circle above $\mathbf{p}_i$. This circle is canonically parametrized, by $\psi_i$ (remember that $\psi_i = \theta_i + \arg(a/(d-1))$. Let us denote

$$\Psi_i = \psi_i \circ \mathcal{B}^+_{\mathbb{R}}(P_i)|_{T^-_{\mathbf{p}_i}} : T^-_{\mathbf{p}_i} \to \mathbb{R}/2\pi\mathbb{Z}$$

the composition.

**Proposition X.3.** *(a) We may choose $\pi^+_i = \Phi_i$.*

*(b) We may choose $\pi^-_i = \Psi_i$.*

**Proof.** (a) Clearly $\Phi_{i-1}(\mathcal{B}^+_{\mathbb{R}}(H_\infty)(x)) = \Phi_i(x)$ when $x \in T^+_{\mathbf{q}_i}$, as the left-hand side is just the right-hand side shifted one to the left. Moreover, Proposition IX.3 says that $\Phi_i(y) = d\Phi_{i-1}(y)$ for $y \in T^+_{\mathbf{q}_{i-1}}$. So

$$\Phi_i\big(\mathcal{B}^+_{\mathbb{R}}(H_\infty)(x)\big) = d\Phi_{i-1}\big(\mathcal{B}^+_{\mathbb{R}}(H_\infty)(x)\big) = d\Phi_i(x).$$

The argument for part (b) is similar. $\square$

**Remark.** We should discuss what happened to the $d-1$ choices of $\pi^+$ and $\pi^-$. For $\pi^+$, our particular choice was given by the coordinate system in $\mathbb{C}^2$, because ultimately,

$\psi = \arg 1/x$. If we conjugate a Hénon mapping by setting $x_1 = \zeta x, y_1 = \zeta y$ where $\zeta^{d-1} = 1$, it is easy to show that the Hénon mapping remains of the same form (the polynomial remains monic and the number $a$ is not changed). For $\pi^-$, we don't actually have a canonical choice. The coordinate $\theta$, which ultimately comes from $\arg 1/y$, is canonical, but $\psi = \theta + \arg(a/(d-1))$ is exactly ambiguous by a $d-1$ root of 1, as one would expect. $\triangle$

The point of these computations is that since $\pi^+$ and $\pi^-$ are conjugacy invariants (up to the ambiguity above) of the mapping $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$, we can use them to find a condition for when the restrictions of $\mathcal{B}^+_{\mathbb{R}}((H_1)_\infty)$ and $\mathcal{B}^+_{\mathbb{R}}((H_2)_\infty)$ to the spheres at infinity are conjugate, where $H_1$ and $H_2$ are Hénon mappings with corresponding polynomials $p_1$ and $p_2$, and Jacobians $a_1$ and $a_2$. In order to pin down our result, we need to know something about toroidal decompositions of 3-manifolds.

**Toroidal decompositions.** The 3-manifold $\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)$ has an interesting toroidal decomposition. We will give the definitions and basic properties, largely due to Jaco, Johannson, Shalen and Waldhausen. Our sources for this material are [Hemp] and especially [Hat2].

Let $M$ be an orientable irreducible 3-manifold with boundary. A properly embedded surface $S \subset M$ is *incompressible* if for any closed embedded disc in $D \subset M$ with $\partial D \subset S$, there is a disc $D' \subset S$ with $\partial D = \partial D'$. The manifold $M$ is atoroidal if each incompressible torus is isotopic to a boundary component.

Let $D \subset \mathbb{C}$ be the open unit disc. A *Seifert manifold* is a 3-dimensional manifold, foliated by circles, such that each leaf has a neighborhood homeomorphic to the quotient of $D \times [0,1]$ by the equivalence relation which identifies $(0,z)$ to $(1, e^{2\pi i p/q} z)$ for some rational number $p/q$, with the foliation induced by the lines $\{z\} \times [0,1]$. The set of leaves is then a surface with boundary $\Omega$, and the canonical mapping $M \to \Omega$ is referred to as a *Seifert fibration*. This is a locally trivial fibration over the subset $\Omega' \subset \Omega$ corresponding to the regular leaves; the singular leaves (like the one corresponding to $z = 0$ in the model) correspond to the discrete set $\Omega - \Omega'$.

The key results for us are the following:

**Theorem X.4.** *Let $M$ be a 3-dimensional compact orientable manifold with boundary. Then there exists a collection of disjoint incompressible tori $T_i \subset M$ such that each component of $M - \cup_i T_i$ is either atoroidal or a Seifert manifold, and a minimal such collection is unique up to isotopy.*

This is exactly Theorem 3.3 of [Hat2].

**Theorem X.5.** *A Seifert manifold with at least two boundary components has a unique Seifert fibration up to isomorphism.*

This follows immediately from Theorem 4.3 of [Hat2]. Indeed, Hatcher shows that the Seifert fibration is unique except for a list of exceptions, and all these exceptions have 1 or 0 boundary components.

**Theorem X.6.** *Let $f : M \to \Omega$ be a Seifert fibration, and let $\Omega' \subset \Omega$ be the complement of the points corresponding to singular fibers. Suppose that $M$ is connected and that $\partial M \neq \phi$. Then every incompressible surfaces in $M$ without boundary is isotopic to a surface of the form $f^{-1}(\gamma)$ for some curve $\gamma \subset \Omega'$, and the isotopy classes of such surfaces correspond exactly to the isotopy classes of such curves.*

This follows from Proposition 3.5 of [Hat2]. Hatcher proves that every incompressible and boundary-incompressible surface is isotopic to either a vertical or a horizontal surface. Horizontal surfaces have non-empty boundary, and surfaces without boundary are vacuously boundary-incompressible. So our surfaces are isotopic to vertical surfaces, i.e., surfaces of the form $f^{-1}(\gamma)$.

We will be interested in applying these notions to the manifold $\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)$, which comes with a family of tori $T_{\mathbf{p}_i}$, which we will see are incompressible.

**Theorem X.7.** *a) The tori $T_{\mathbf{p}_i} \subset \mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)$ are incompressible.*

*b) Every incompressible torus in $\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)$ is isotopic to exactly one of the $T_{\mathbf{p}_i}$.*

**Proof.** a) The homology $H_1(\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-))$ is isomorphic to $\mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d]$. This is proved for instance using the Alexander duality theorem ([Spa], 6.2, Thm. 16), which asserts that

$$H_1\left(\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)\right) = H^1(\Sigma^+) \oplus H^1(\Sigma^-).$$

The cohomology above is Čech cohomology (isomorphic to Alexander-Spanier cohomology), given by the inductive limit of the singular cohomology of a basis of neighborhoods of $\Sigma^\pm$. Using the system of neighborhoods $T^+_{\mathbf{p}_{-i}}$ of $\Sigma^+$, we see that

$$H^1(\Sigma^+) = \varinjlim(\mathbb{Z}, n \mapsto dn) = \mathbb{Z}[1/d].$$

(This is similar to but simpler than Example V.3.)

An isomorphism is specified by sending the generators $a$, $b$(see Figure 24) of $H_1 T_{\mathbf{p}_0}$ to $(0,1)$ and $(1,0)$ in $\mathbb{Z}[1/d] \oplus \mathbb{Z}[1/d]$. Under this isomorphism, the corresponding generators of $H_1(T_{\mathbf{p}_i})$ are sent to $(0, d^{-i})$ and $(d^i, 0)$. In particular, the inclusion is injective on the homology of such a torus. If a disc in $\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-)$ bounds a disc in $T_{\mathbf{p}_i}$, then the homology class of this boundary is zero in $H_1(\mathcal{B}^+_{\mathbb{R}}(D_\infty) - (\Sigma^+ \cup \Sigma^-))$, hence also in $H_1(T_{\mathbf{p}_i})$, so the curve bounds a disc in the torus, since any simple closed curve in a torus which is trivial in the homology bounds a disc. (This is not true on surfaces of higher genus, which is why incompressibility is not defined using injectivity of the inclusion on homology.) This is the definition of an incompressible torus.

b) Let $T'$ be such a torus. It is contained in the compact manifold $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_j}$ for $i$ sufficiently small and $j$ sufficiently large, which allows us to apply Theorem X.4, where $M$ must be compact. So it is enough to prove that $T_{\mathbf{p}_{i+1}}, \dots, T_{\mathbf{p}_{j-1}}$ is a minimal family of incompressible tori in $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_j}$ such that the components of the boundary are atoroidal or Seifert manifolds.

First observe that the components of

$$\left(T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_j}\right) - \left(\bigcup_{n=i+1}^{j-1} T_{\mathbf{p}_n}\right)$$

are in fact both atoroidal and Seifert manifolds. Indeed, the region $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+1}}$ is homeomorphic to the region $M_i$ bounded by the tori corresponding to $L_{i,d-2} \cap L_{i,d-1}$ and $L_{i,d} \cap L_{i,d-1}$. This region contains the solid torus corresponding to $B_i$, but that torus can be collapsed onto a circle without changing the homeomorphism type; call $M'_i$ the resulting manifold.

The manifold $M'_i$ is fibered by the natural circle action, and the circle corresponding to $B_i$ becomes a singular circle of type $(1, d)$. Thus $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+1}}$ is a Seifert manifold; let $f_i : T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+1}} \to \Omega_i$ be the corresponding projection to the set of leaves (the base). It is also atoroidal, by Theorem X.6, since $\Omega_i$ is an annulus with one distinguished point corresponding to the unique singular fiber. This is seen as follows. In Figure 28, the manifold $M_i$ corresponds to the annulus between the center circle and the next, with the $d$ small circles removed.

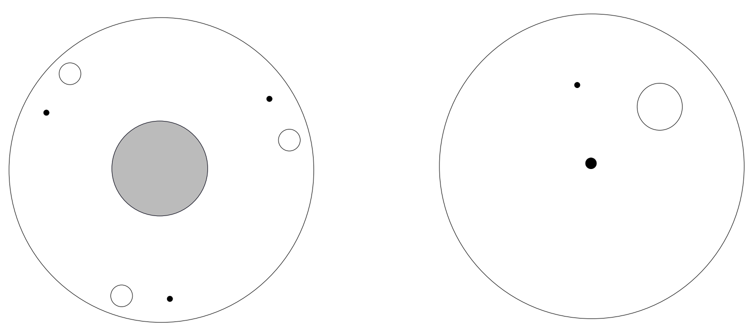

FIGURE 29. On the right, we have repeated the relevant part of Figure 28, showing the annulus corresponding to $M_i$ when $d = 3$; the three dots represent the intersection of the plane of the figure with a circle orbit in $M_i$. The right-hand side represents the first, after applying $z \mapsto z^d$ and collapsing the central disc to a point. Every point on the right corresponds to a unique circle orbit, i.e., the base is an annulus with a single singular fiber (corresponding to the central point).

If you parametrize the disc by $z$, and compose $z \mapsto z^d$ with a collapse of the central disc to a point (corresponding to collapsing the solid torus corresponding to $B$ to a circle), you manufacture a space which corresponds exactly to the set of leaves. See Figure 29.

Any simple closed curve on an annulus with a puncture is homotopic to a point or to a boundary component, so there are no incompressible tori in $M'_i$ by Theorem X.6.

Now we need to show that our family $T_{\mathbf{p}_{i+1}}, \dots, T_{\mathbf{p}_{j-1}}$ is minimal. It is clearly enough to show that $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}}$ is neither atoroidal nor Seifert. It clearly isn't atoroidal, since it contains $T_{\mathbf{p}_{i+1}}$, so we must show it isn't Seifert. Suppose $f : T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}} \to \Omega$ is a Seifert fibration, where $\Omega$ is some surface. The surface $\Omega$ must have a boundary consisting of two components, but otherwise we don't know much about it. By Theorem X.6, there is a curve $\gamma \subset \Omega'$, where $\Omega'$ is the complement of the projections of the singular fibers, such that $T_{\mathbf{p}_i}$ is isotopic to $T'_i = f^{-1}(\gamma)$; in particular, the restriction of $f$ to the components of $\left(T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}}\right) - T'_i$ is a Seifert fibration.

But each of these is already a Seifert fibration, and in fact in a unique way by Theorem X.5. It is then enough to show that the fibers of $f_i$ and $f_{i+1}$ on the torus $T_{\mathbf{p}_i}$ which is the intersection of their domains are not homotopic curves; since they should both be homotopic to the fibers of $f$, this contradicts the existence of such an $f$.

In the basis $a, b$ for $H_1(T_{\mathbf{p}_i})$, we have seen that a fiber of $f_0$ has the homology class $-a - db$. We claim that a fiber of $f_1$ has the homology class $-da - b$; knowing this will end the proof.

We need to repeat the construction of Proposition X.1, to understand the sequence of tori corresponding to the block $B_1$ of $D_\infty$. Take the 3-sphere, with the sequence of real oriented blow-ups corresponding to $\mathcal{B}^+_{\mathbb{R}}(\tilde{D})$, as represented in Figure 27.

The fiber above $\mathbf{q}$ is a circle orbit outside the torus corresponding to $\mathbf{q}'$, which we may take to be the $z$-axis. Thicken this torus; the fibers inside the thickened torus will now have the homology class $-a - b + b = -a$ by Theorem VII.5. Indeed, if you choose a small disc transverse to the $z$-axis, it projects to $\tilde{D}$ (in fact, to a neighborhood of $\mathbf{q}'$ in the line at infinity $A'$), and the induced orientation gives its boundary the orientation $+b$. Now, when we make $d-1$ more blow-ups, always of double points, the regions between the tori created have circle orbits in the classes $(-2a - b, \ \dots, -da - b)$. The region foliated by curves in the class $-da - b$ corresponds to the 3-manifold $M_1$. $\square$

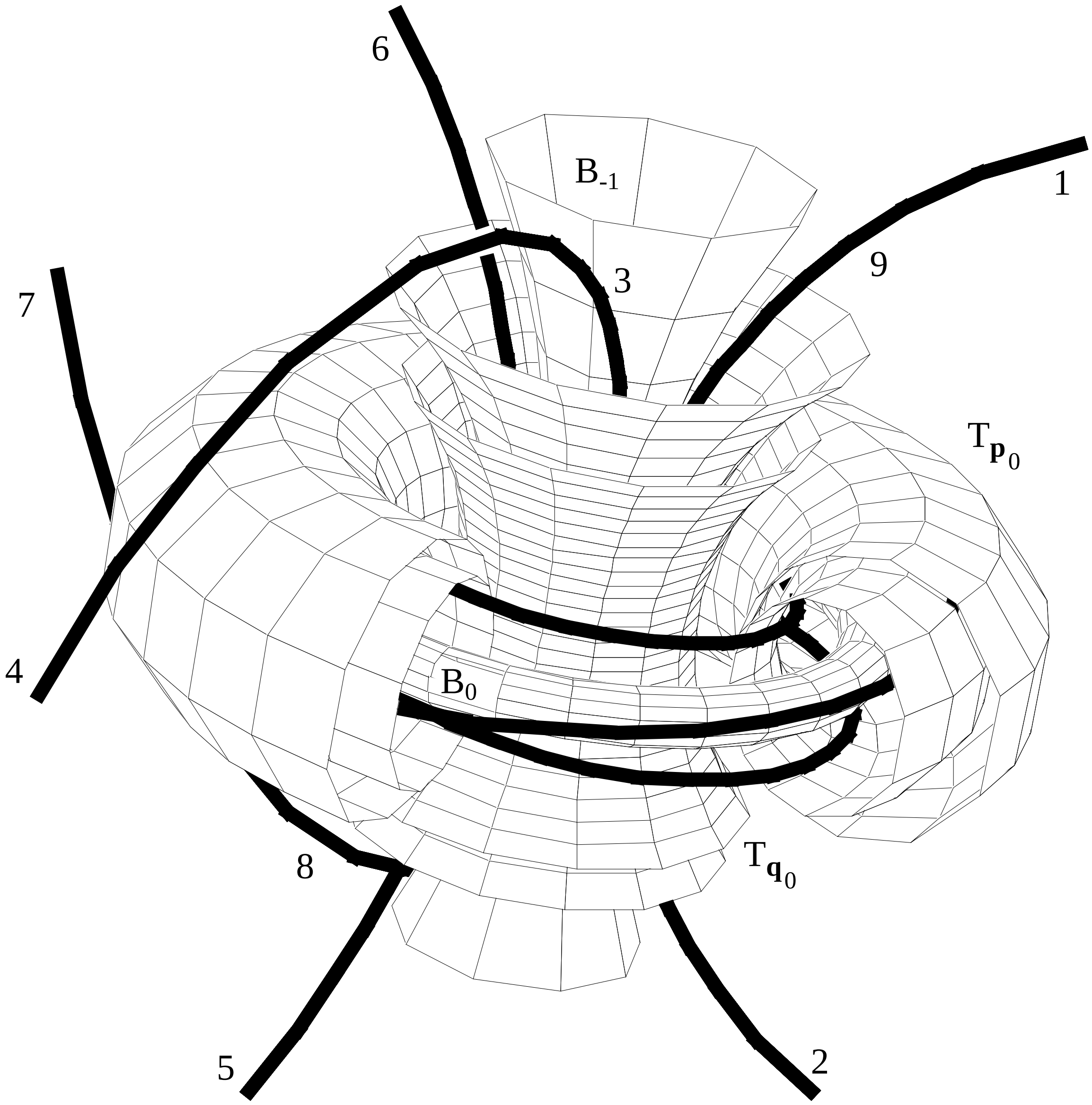


FIGURE 30. The configuration of tori in the 3-sphere at infinity between the torus corresponding to $B_0$ and $B_{-1}$, in the case $d = 3$. The torus corresponding to $L_{-1,1} \cap L_{-1,2}$, shown as a heavy curve, winds three times around the torus corresponding to $B_0$, in the figure a small thickening of the unit circle in the $x, y$-plane. The torus corresponding to $L_{0,2} \cap L_{1,3}$, also shown as a heavy curve, winds three times around the torus $B_{-1}$, represented in the figure as a thickening of the $z$-axis.

**Conjugacy invariants of $\mathcal{B}^+_{\mathbb{R}}(H_\infty)$.**

Let $H_1$ and $H_2$ be Hénon mappings. We will give a necessary condition for when the restrictions of $\mathcal{B}^+_{\mathbb{R}}((H_1)_\infty)$ and $\mathcal{B}^+_{\mathbb{R}}((H_2)_\infty)$ to the spheres at infinity $\mathcal{B}^+_{\mathbb{R}}(D_{1,\infty})$ and $\mathcal{B}^+_{\mathbb{R}}(D_{2,\infty})$ are conjugate. To (sort of) lighten notation, we will call these restrictions $\mathcal{B}^+_{\mathbb{R}}((H_i)'_\infty)$.

**Theorem X.8.** *In order for $\mathcal{B}^+_{\mathbb{R}}((H_1)'_\infty)$ and $\mathcal{B}^+_{\mathbb{R}}((H_2)'_\infty)$ to be topologically conjugate, it is necessary that they have the same degree, and that* $\arg a_1 \equiv \arg a_2 \mod 2\pi/(d-1)$.

**Proof.** That they must have the same degree is clear by counting fixed points in the solenoids.

We will first investigate the "critical locus" of the map

$$(\pi_1^+, \pi_0^-) : T^-_{\mathbf{p}_0} \cap T^+_{\mathbf{p}_1} \to (\mathbb{R}/2\pi\mathbb{Z})^2.$$

**Remark.** The notion of "critical locus" isn't quite right: $\mathcal{B}^+_{\mathbb{R}}(D_\infty)$ isn't naturally a differentiable manifold, it is naturally a *manifold with corners*, almost an object of the piecewise-linear category. What we will find is more PL than $C^1$: it will turn out that the set of points which have neighborhoods on which $\pi_1^+$ and $\pi_0^-$ differ by a constant is non-empty. In our setting, the critical locus will be the closure of this open set by definition. This is a much stronger notion of "critical" than one would expect: generically for a differentiable mapping from a 3-dimensional manifold to a surface, the critical locus should be a curve. △

**Lemma X.9.** *On $p_\infty^{-1}(B_0)$, we have the identity*

$$\pi_1^+ - \pi_0^- = -\arg a + \pi - \frac{\arg a}{d-1} = \pi - \frac{d}{d-1}\arg a.$$

**Proof.** The parametrized path $t \mapsto \begin{bmatrix} c \\ te^{-i\alpha} \end{bmatrix}$, $t > 0$, thought of as a path in $X_\infty$, approaches a specific point of $B_0$ with coordinate $c$. Thought of as a path in $\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$, it approaches the point $x$ above $c$ where $\Psi_0 = \alpha + (\arg a)/(d-1)$. Thus $\pi_0^-(x) = \alpha + (\arg a)/(d-1)$.

To compute $\pi_1^+(x)$, apply $H$ to the path: the path $t \mapsto \begin{bmatrix} p(c) - ate^{-i\alpha} \\ c \end{bmatrix}$, $t > 0$ approaches $\mathcal{B}^+_{\mathbb{R}}(H_\infty)(x) \in p_\infty^{-1}(B_{-1})$. Just as we needed the argument of $1/y$ to compute $\Psi_0$ (adjusted by $(\arg a)/(d-1)$), we need the argument of $1/x$ (unadjusted) to compute $\Phi_0(H(x) = \Phi_1(x)$; clearly this argument is $\alpha - \arg a + \pi$. Thus $\pi_1^+(x) = \alpha - \arg a + \pi$. □

This means that on this solid torus, the two functions $\pi_1^+$ and $\pi_0^-$ differ by the constant $\frac{d}{d-1}\arg a$.

**Lemma X.10.** *The solid torus $p_\infty^{-1}(B_0)$ is the critical locus of $\pi_1^+, \pi_0^-$.*

**Proof.** We will only outline how to do this, for points above $L_1$. Choose as above a curve in $X_\infty$ tending in $\mathcal{B}^+_{\mathbb{R}}(X_\infty, D_\infty)$ to a point above a point $c \in L_i$. For instance,

$t \mapsto \begin{bmatrix} t^{-i\alpha} \\ cte^{-2i\alpha} \end{bmatrix}$, $t > 0$ is a curve approaching a point $x$ above $L_1$. On this point, we have $\pi_0^-(x) = 2\alpha - \arg c + (\arg a)/(d-1)$. If we apply $H$ to this curve and compute $\arg(1/x)$, we find

$$\pi_1^+(x) = \begin{cases} d\alpha & \text{if } d > 2; \\ 2\alpha - \arg(1-ac) & \text{if } d = 2. \end{cases}$$

Since $\arg c$ shows up explicitly in the formula for $\pi_1^+ - \pi_0^-$, we see that such a point is not critical. $\square$

**Remark.** The point $ac = 1$ in the case $d = 2$ above corresponds to $B \cap L_1$; a similar point will show up above $L_{d-1}$ for every $d$.

We now need to see that the number $\frac{d}{d-1} \arg a$ from Lemma X.9 is (almost) a conjugacy invariant of $\mathcal{B}_{\mathbb{R}}^+(H_\infty)$.

Suppose that $F : \mathcal{B}_{\mathbb{R}}^+(D_{1,\infty}) \to \mathcal{B}_{\mathbb{R}}^+(D_{2,\infty})$ conjugates $\mathcal{B}_{\mathbb{R}}^+(H_{1,\infty}) \to \mathcal{B}_{\mathbb{R}}^+(H_{2,\infty})$, where we have used indices 1 and 2 to distinguish the objects created from $H_1$ and $H_2$. Then $F$ must send $\Sigma_1^\pm$ to $\Sigma_2^\pm$ since these are the closures of the set of periodic points of $\mathcal{B}_{\mathbb{R}}^+(H_{1,\infty})$ and $\mathcal{B}_{\mathbb{R}}^+(H_{2,\infty})$ respectively, so it must also send incompressible tori in the complement of the solenoids $\Sigma_1^\pm$ to incompressible tori in the complement of the solenoids $\Sigma_2^\pm$. Since $T_{\mathbf{p}_i}$ separates the $T_{\mathbf{p}_j}$ with $j > i$ from the $T_{\mathbf{p}_j}$ with $j < i$, we see that the order of the tori must be preserved, and there exists $k$ such that

$$F(T_{1,\mathbf{p}_i}) \quad \text{is isotopic to} \quad T_{2,\mathbf{p}_{i+k}}$$

By composing $F$ with $\mathcal{B}_{\mathbb{R}}^+((H_2)'_\infty)^{\circ k}$, we may assume that $k = 0$, and that

$$F(T_{1,\mathbf{p}_i}) \quad \text{is isotopic to} \quad T_{2,\mathbf{p}_i}$$

**Lemma X.11.** *The functions $\pi_{1,i}^+$ and $\pi_{1,i}^+ \circ F$ must differ by a multiple of $2\pi/(d-1)$ on their common domain of definition.*

**Proof.** By composing $\pi_{i,1}^+$ with an appropriate multiple of $2\pi/(d-1)$, we may assume that if $x_1 \in \Sigma_1^+$ is the fixed point of $\mathcal{B}_{\mathbb{R}}^+((H_1)'_\infty)$ with $\pi_{i,1}^+(x_1) = 0$, the $F(x_1) = x_2$ is the fixed point of $\mathcal{B}_{\mathbb{R}}^+((H_2)'_\infty)$ with $\pi_{i,1}^+(x_2) = 0$. After this change, we must show that $\pi_{1,i}^+$ and $\pi_{1,i}^+ \circ F$ coincide on their common domain.

Choose $j$ sufficiently small so that

$$T_{\mathbf{p}_{2,j}}^+ \subset T_{\mathbf{p}_2,i}^+ \cap F(T_{\mathbf{p}_1,i}^+).$$

Now both diagrams

$$\begin{array}{ccc} T_{\mathbf{p}_{2,j}}^+ & \xrightarrow{\mathcal{B}_{\mathbb{R}}^+(H_{2,\infty})} & T_{\mathbf{p}_{2,j}}^+ \\ \downarrow{\scriptstyle \pi_{2,i}^+} & & \downarrow{\scriptstyle \pi_{2,i}^+} \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow{\theta \mapsto d\theta} & \mathbb{R}/2\pi\mathbb{Z} \end{array} \qquad \begin{array}{ccc} T_{\mathbf{p}_{2,j}}^+ & \xrightarrow{\mathcal{B}_{\mathbb{R}}^+(H_{2,\infty})} & T_{\mathbf{p}_{2,j}}^+ \\ \downarrow{\scriptstyle \pi_{i,1}^+ \circ F^{-1}} & & \downarrow{\scriptstyle \pi_{i,1}^+ \circ F^{-1}} \\ \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow{\theta \mapsto d\theta} & \mathbb{R}/2\pi\mathbb{Z} \end{array}$$

commute. The uniqueness statement [HO1], Theorem 3.1 isn't quite enough to guarantee that the verical arrows coincide, since they aren't of degree 1. In that case, the proof guarantees that such maps differ by a multiple of $1/(d^{i-j}-1)$. This is enough to guarantee that if $\pi^+_{2,i}$ and $\pi^+_{i,1}\circ F^{-1}$ coincide at a single point, then they coincide everywhere; indeed, they do coincide at $x_2$. Now

$$\pi^+_{2,i} = \frac{1}{d^m}\circ \pi^+_{2,i}\circ \mathcal{B}^+_{\mathbb{R}}(H_{2,\infty}) \quad \text{and} \quad \pi^+_{i,1}\circ F^{-1} = \frac{1}{d^m}\circ \pi^+_{2,i}\circ \mathcal{B}^+_{\mathbb{R}}(H_{2,\infty});$$

any difference comes from different branches of $\frac{1}{d^m}$. Since $F(T_{\mathbf{p}_{1,i}})$ is isotopic to $T_{\mathbf{p}_{2,i}}$, and we can choose a branch continuously during the isotopy, we see that indeed $\pi^+_{2,i}$ and $\pi^+_{1,i}\circ F^{-1}$ (the latter adjusted at the beginning of the proof) must agree on their common domain of definition. $\square$

Now we need to know something about this common domain of definition.

**Lemma X.12.** *The interiors of the torus $F(T^+_{\mathbf{p}_{1,i}})$ and the interior of the torus $\mathcal{B}_{i-1}$ corresponding to $B_{2,i-1}$ must have non-empty intersection.*

**Proof.** An alternative way of saying this is to say that $\mathcal{B}_{i-1}$ cannot be isotoped, in $\mathcal{B}^+_{\mathbb{R}}(D_{2,\infty}) - (\Sigma^+_2 \cup \Sigma^-_2)$, to a torus outside of $F(T^+_{\mathbf{p}_{1,i}})$. This can be seen from linking numbers. The presumed isotopy will take place in the complement of $T^+_{\mathbf{p}_{2,j}}$ for $j$ sufficiently small. All curves (or unknotted solid tori) outside $T^+_{\mathbf{p}_{2,i}}$ have linking number some integer multiple of $d^{i-j}$ with $T^+_{\mathbf{p}_{2,j}}$. But $\mathcal{B}_{i-1}$ has linking number $d^{i-j-1}$ with $T^+_{\mathbf{p}_{2,j}}$ (See Figure 31).

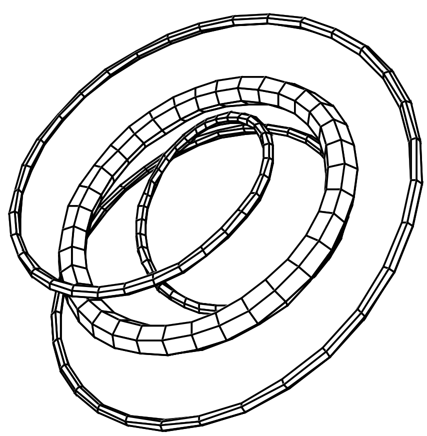

FIGURE 31. Represented here is a $3,1$-curve on the boundary of a solid torus, corresponding to 3 times the generator of the homology of the solid torus, and the core curve of the solid torus. Clearly they link with linking number 1, whereas any curve outside the torus links with the $3,1$-curve with linking number some multiple of 3. Thus this figure represents the case $i-j=1$ and $d=3$.

Since the linking number must be constant during the isotopy, this is a contradiction. $\square$

Thus $F$ must map some open subset of the torus corresponding to $B_{1,i}$ to some open subset of the torus corresponding to $B_{2,i}$, in such a way that

$$\pi^{\pm}_{2,i} \circ F = \pi_{1,i}$$

up to a multiple of $1/(d-1)$. This proves Theorem X.8. $\square$

**Remark.** It seems likely that the condition $\arg a_1 = \arg a_2$ is also sufficient for conjugacy. We will explore this in a future paper. It turns out that the conjugacy properties of mappings like $\mathcal{B}^+_{\mathbb{R}}((H)_\infty)$ (or the mapping $h_d$ of [HO1]) is quite subtle; and that there is an infinite-dimensional moduli space, even when the maps are hyperbolic on a neighborhood of the solenoids.

## XI. The compactification of compositions of Hénon mappings

A theorem of Friedland and Milnor asserts that any polynomial automorphism of $\mathbb{C}^2$ is either elementary, in the sense that we can find one variable which depends only on itself, or conjugate to a composition of Hénon maps. Therefore understanding the appropriate compactification of $\mathbb{C}^2$ to which such a composition extends is evidently important.

Milnor, in a personal communication, suggested what the 3-sphere at infinity should look like; we will now state and prove this conjecture.

Let

$$H_i : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p_i(x) - a_i y \\ x \end{bmatrix}, \quad i = 1 \cdots k$$

be $k$ Hénon mappings, with $a_i \neq 0$ and $p_i$ of degree $d_i \geq 2$. We will consider $G = H_k \circ \cdots \circ H_1$, which is a polynomial mapping of algebraic degree $d = d_1 \cdots d_k$.

Recall that $\Sigma_d = \varprojlim(\mathbb{R}/\mathbb{Z}, t \mapsto dt)$ is the $d$-adic solenoid, and $\sigma_d : \Sigma_d \to \Sigma_d$ the map induced by $t \mapsto dt$.

We will call the simplest link of two circles with linking number $d$ the one formed by the circles

$$\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \; 0 \leq t \leq 2\pi \quad \text{and} \quad \begin{pmatrix} (1 + \frac{1}{2}\cos dt)\cos t \\ (1 + \frac{1}{2}\cos dt)\sin t \\ \frac{1}{2}\sin dt \end{pmatrix}$$

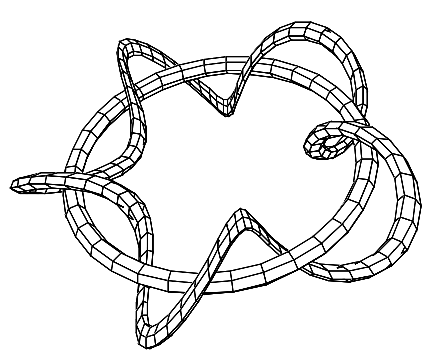

FIGURE 32. The simplest link of two circles linking with linking number 5.

**Theorem XI.1.** *a) There exists a topology on $\mathbb{C}^2 \sqcup S^3$ homeomorphic to the 4-ball, with $S^3$ corresponding to the boundary, such that $G$ extends continuously to a homeomorphism of $g : S^3 \to S^3$.*

*b) The homeomorphism $g$ has two invariant solenoids $\Sigma^+, \Sigma^-$, one attracting and one repelling, and both homeomorphic to $\Sigma_d$, and the homeomorphisms can be chosen to be conjugacies between the restriction $g|_{\Sigma^+}$ and $\sigma_d$, and between $g|_{\Sigma^-}$ and $\sigma_d^{-1}$.*

*c) The complement $M = S^3 - (\Sigma^+ \cup \Sigma^-)$ has a decomposition by incompressible tori $(T_i)_{i \in \mathbb{Z}}$, unique up to isotopy, into pieces $M_i$ bounded by $T_i$ and $T_{i+1}$, homeomorphic to the complement of the simplest link of two circles with linking number $d_{i \bmod k}$. Moreover, $M_i \cap M_j = \emptyset$ unless $|i - j| \leq 1$. The tori can be chosen so that $g(T_i) = T_{i+k}$.*

In particular, the topology of the sphere at infinity is different for a composition of Hénon maps with total degree $d$ and for a single such mapping: the solenoids are the same but they are embedded differently in the 3-sphere.

**Proof.** Let $\tilde{X}_G$ be the minimal blow-up of $\mathbb{P}^2$ on which $\tilde{G} : \tilde{X}_G \to \mathbb{P}^2$ is well-defined.

It can be constructed as follows: set $G_m = H_m \circ \cdots \circ H_1$, so that $G_1 = H_1$ and $G_k = G$, and define $\tilde{X}_{G_m}$ to be the minimal blow-up of $\mathbb{P}^2$ on which $\tilde{G}_m : \tilde{X}_{G_m} \to \mathbb{P}^2$ is well-defined. Further denote by $\pi_{G_m} : \tilde{X}_{G_m} \to \mathbb{P}^2$ the canonical projection, and $\tilde{D}_{G_m} = \pi_{G_M}^{-1}(l_\infty)$ the divisor at infinity of $\tilde{X}_{G_m}$.

We will construct $\tilde{X}_{G_m}$ by induction. Clearly $\tilde{X}_{G_1} = \tilde{X}_{H_1}$ is the space constructed in Section III. Suppose we have constructed $\tilde{X}_{G_{m-1}}$, together with $\tilde{G}_{m-1}$ and $\pi_{G_{m-1}}$.

Set $\tilde{X}_{G_m}$ to be such that the upper left-hand square of the diagram

$$\begin{tikzcd}
\tilde{X}_{G_m} \arrow[r] \arrow[d] & \tilde{X}_{H_m} \arrow[r, "\tilde{H}_m"] \arrow[d, "\pi_{H_m}"'] & \mathbb{P}^2 \\
\tilde{X}_{G_{m-1}} \arrow[r, "\tilde{G}_{m-1}"] \arrow[d, "\pi_{G_{m-1}}"'] & \mathbb{P}^2 & \\
\mathbb{P}^2 & &
\end{tikzcd}$$

is a fiber product in the category of analytic spaces. Then the top line is a mapping $\tilde{G}_m : \tilde{X}_{G_m} \to \mathbb{P}^2$, whereas the left-hand column represents $\tilde{X}_{G_m}$ as a modification of $\mathbb{P}^2$ at $\mathbf{p}$. Thus $\mathbb{C}^2$ is dense in $\tilde{X}_{G_m}$, and it is clear by induction that $\tilde{G}_m$ extends $G_m : \mathbb{C}^2 \to \mathbb{C}^2$. That it is the minimal modification of $\mathbb{P}^2$ to which $G_m$ extends follows from the fact that the construction of Section III is the minimal modification to which the individual Hénon maps extend. Thus $\tilde{X}_{G_m}$ is the required minimal blow-up.

The divisor above infinity

$$\tilde{D}_G = \tilde{X}_G - \mathbb{C}^2 = \pi_G^{-1}(l_\infty)$$

looks as follows.

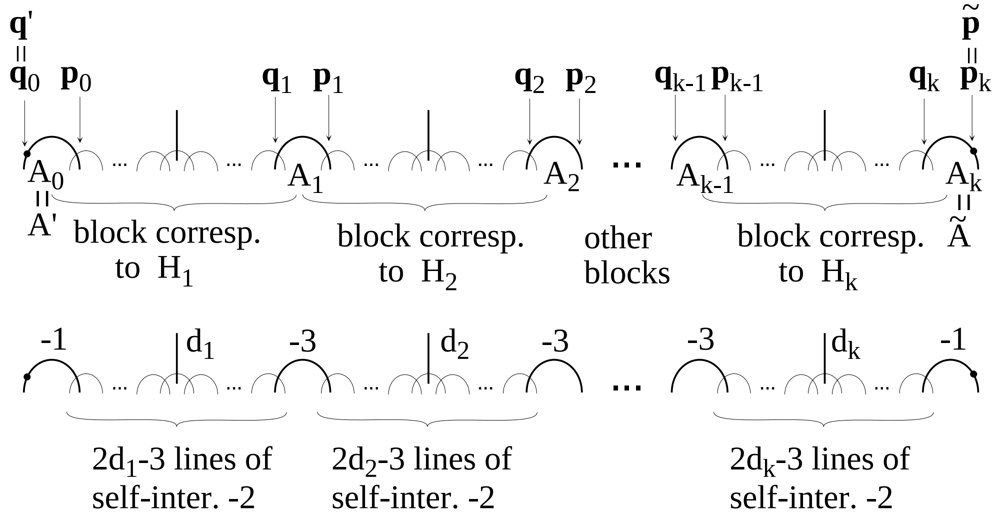


FIGURE 33. The divisor $\tilde{D}_G$. The top figure gives the labels of all the components and points to which we will need to refer, the second gives the self-intersections of all these components.

As before, we will avoid an infinite sequence of blow-ups by a considering a sequence space. Consider the rational mapping $G^\sharp : \tilde{X}_G \rightsquigarrow \tilde{X}_G$, which is $\tilde{G}$ wherever it is defined, and define

$$\overline{\Gamma}_G \subset \tilde{X}_G \times \tilde{X}_G$$

to be the closure of the graph $\Gamma_{G^\sharp} \subset \tilde{X}_G \times \tilde{X}_G$ of $G$.

**Lemma XI.2.** *A pair $(x, y)$ belongs to $\overline{\Gamma}_G$ if and only if either*

- *it is in $\Gamma_{G^\sharp}$, or*
- $x = \tilde{\mathbf{p}}, y \in (\tilde{D} - A') \cup \{\mathbf{p}'\}$.

The proof is analogous to that of Theorem III.4.

Now define the natural compactification of the composition of Hénon mappings $G$ as

$$X_\infty(G) = \Big\{(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \in \Big(\tilde{X}_G\Big)^{\mathbb{Z}} |(x_n, x_{n+1}) \in \overline{\Gamma}_G \quad \text{for all } n \in \mathbb{Z}\Big\}.$$

Using Lemma XI.2, this space is not so difficult to understand.

**Proposition XI.3.** *The space $X_\infty(G)$ is compact. The complement of two points $\mathbf{q}^\infty = (\dots, \mathbf{q}', \mathbf{q}', \mathbf{q}', \dots)$ and $\mathbf{p}^\infty = (\dots, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \dots)$ is an algebraic manifold.*

**Proof.** The proof is the same as that of Proposition IV.8.

Again, to understand the structure of the bad points, we will pass to the real oriented blow-ups.

We can define spaces $X_{[-N,M]}(G)$ analogously to the construction in Section V, with the divisors $D_{[-N,M]}(G)$; next we construct the real oriented blow-ups

$$\mathcal{B}_{\mathbb{R}}^{+}(X_{[-N,M]}(G), D_{[-N,M]}(G))$$

and take their projective limit

$$\mathcal{B}_{\mathbb{R}}^{+}(X_{\infty}(G), D_{\infty}(G)) = \varprojlim_{N\to\infty} (\mathcal{B}_{\mathbb{R}}^{+}(X_{[-N,N]}(G), D_{[-N,N]}(G)).$$

Note that again we are using Propositions VII.3 and VII.7 to construct the mappings implicit in the projective limit.

The pair $\mathcal{B}_{\mathbb{R}}^{+}(D_{\infty}(G)) \subset \mathcal{B}_{\mathbb{R}}^{+}(X_{\infty}(G), D_{\infty}(G))$ is the compactification of $\mathbb{C}^2$ promised in Theorem XI.1. By Theorem VII.11, the pair is homeomorphic to $(B^4, S^3)$, exactly as in the proof of Theorem IX.2. Moreover, the inclusion of $\mathbb{C}^2 \subset \mathcal{B}_{\mathbb{R}}^{+}(X_{\infty}(G), D_{\infty}(G))$ and the extension of $G$ to the real oriented blow-up is constructed exactly as in IX.1.

The proof of (b) is closely analogous to Proposition IX.3, but requires a bit of terminology. First, label components and points of $\tilde{D}_{\infty}$ as follows: let $A_0 = l_{\infty} \subset \mathbb{P}^2$, and define recursively $A_i \subset \tilde{X}_{G_i}$ to be the component of $H_i^{-1}(A_{i-1})$ which is sent by $H_i$ isomorphically onto $A_{i-1}$. Finally, Let $\mathbf{p} = \mathbf{p}_0$ and $\mathbf{q} = \mathbf{q}_0$; by induction each $A_i$ contains the two points $\mathbf{p}_i, \mathbf{q}_i$ which under the isomorphism $G_i|_{A_i}$ map to $\mathbf{p}_{i-1}$ and $\mathbf{q}_{i-1}$, as in Figure 33.

Next, we will label $\mathbf{p}_{m,i}$ and $\mathbf{q}_{m,i}$ the points of $D_{\infty}(G)$ whose $m$th entry is $\mathbf{p}_i$, this requires a bit of care when $i = 0$ and $i = k$, which we will leave to the reader.

Proposition IX.3 tells us that there are natural angles $\theta_j$ parametrizing $p^{-1}(\mathbf{p}_j) \subset \tilde{X}_{G_j}$, and that these angles correspond under the Hénon mappings (see Equation (9.1), where this angle appears as the argument of $v$ and the argument of $X_1$). Note that we are considering these fibers at the moment when they are created by the blow-up, so that each lies above a simple point of the divisor defined so far. Moreover, the same proposition (specifically, see Equation (9.2)) says that the composition of the Hénon mappings takes angles $\theta_j$ to angles $\theta_{j-1}$ as indicated in the following diagram:

$$\theta_k \underset{\tilde{X}_{G_k}\to\tilde{X}_{G_{k-1}}}{\overset{\text{blow-down}}{\mapsto}} d_k\theta_k + \arg a_k \underset{\tilde{X}_{G_{k-1}}\to\tilde{X}_{G_{k-2}}}{\overset{\text{blow-down}}{\mapsto}} d_k d_{k-1}\theta_k + d_{k-1}\arg a_k + \arg a_{k-1}$$

$$\underset{\tilde{X}_{G_{k-2}}\to\tilde{X}_{G_{k-3}}}{\overset{\text{blow-down}}{\mapsto}} \quad \dots \quad \underset{\tilde{X}_{G_1}\to\tilde{X}_{G_0}}{\overset{\text{blow-down}}{\mapsto}} d\theta_k + \beta,$$

where $d = d_k \dots d_1$ and

$$\beta = d_{k-1}d_{k-2}\dots d_1 \arg a_k + d_{k-2}d_{k-3}\dots d_1 \arg a_{k-1} + \dots + d_1 \arg a_2 + \arg a_1.$$

An analogous argument, using appropriate conjugates of the inverses of the $H_j$, will show that the similar parameter $\phi_j$ of $p^{-1}(\mathbf{q}_j)$ is simply multiplied by $d$. Now the proof ends in the same way as the proof of IX.3, showing that the fibers above $\mathbf{p}^\infty$ and $\mathbf{q}^\infty$ are both $d$-adic solenoids, in one case using the angles $\phi_n$, and in the other $\psi_n = \theta_n + \beta/(d-1)$.

Part (c) has substantially already been proved: The tori corresponding to the $\mathbf{p}_{n,j}$ do form a sequence of incompressible tori in $\mathcal{B}^+_{\mathbb{R}}(D_\infty(G)) - (\Sigma^+ \cup \Sigma^-)$, and the components of the complements are homeomorphic to the simplest link of two circles which link with linking number $d_j$. Moreover, the proof we have given in Theorem X.7 that this is the unique toroidal decomposition of $\mathcal{B}^+_{\mathbb{R}}(D_\infty(G)) - (\Sigma^+ \cup \Sigma^-)$ goes through without change. $\square$

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