A new advance in the Bernstein Problem

in mathematical genetics

Yuri I. Lyubich
Department of Mathematics, Technion
32000 Haifa, Israel

1 Introduction

Here we present a new result on a problem posed by S.N.Bernstein [1] in mathematical foundations of the population genetics. The problem is related to a statement called *The Stationarity Principle (S.P.).* Being valid under the Mendel Law this principle is consistent with some more general mechanisms of heredity. Bernstein suggested to describe all situations satisfying the S.P.. In mathematical terms this sounds as follows.

Let $\Delta^{n-1} \subset \mathbb{R}^n$ be the basis simplex,

$$\Delta^{n-1} = \{ x \in \mathbb{R}^n : s(x) \equiv \sum_{i=1}^{n} x_i = 1, \quad x = (x_i)_1^n \geq 0\}. \tag{1.1}$$

Consider a quadratic mapping $V : \Delta^{n-1} \to \Delta^{n-1}$,

$$x_j' \equiv (Vx)_j = \sum_{i,k=1}^{n} p_{ik,j} x_i x_k \quad (1 \leq j \leq n), \tag{1.2}$$

which is *stochastic* in the sense that

$$p_{ik,j} \geq 0, \quad \sum_{j=1}^{n} p_{ik,j} = 1. \tag{1.3}$$

Certainly, the symmetry $p_{ki,j} = p_{ik,j}$ is also supposed.

Stony Brook IMS Preprint #1996/9
August 1996

A mapping V is called *Bernstein* (or *stationary*) if $V^2 = V$, where $V^2 \equiv V \circ V$. This property is just the S.P. *The Bernstein problem is to explicitly describe all such mappings.* For $n = 3$ the problem was solved in [2], [3], [4]. The cases $n = 1, 2$ are trivial (see below).

Biologically, V is the evolutionary operator of an infinite population under certain conditions (see [17], Sections 1.1, 1.2 for a detailed explanation). Each individual from the population belongs to a biological *type (character).* The set of types is supposed to be finite, say $\{1, ..., n\}$, and the partition of the population into the types has to be *hereditary.* This means that for every triple (i, k, j) of types there exists a probability $p_{ik,j}$ for parents of types i and k to have an offspring of type j. In this sense $p_{ik,j}$ are the *inheritance coefficients.* Thus, we have (1.3) automatically in this context. The symmetry $p_{ki,j} = p_{ik,j}$ means that the sexual differentiation does not affect on the heredity.

The points $x \in \Delta^{n-1}$ are just the probability distributions on the set of types. Every such a point is a *state* of the population. If x is a state in a *parental generation* then $x' = Vx$ is the state in the *offspring generation.* With an *initial state* x the sequence $\{V^t x\}_{t=0}^{\infty}$ is the corresponding *trajectory* of the population considered as a dynamical system ([17], Section 1.2). In a simplest case x is a fixed point, $x = Vx$, so the trajectory is reduced to the point x. Such points are *equilibria* from the dynamical point of view. The only case all states are equilibria is $V = I$, the identity mapping. Note that this mapping $x_j' = x_j$ $(1 \le j \le n)$ can be also represented as a quadratic one, namely, $x_j' = x_j s(x)$ $(1 \le j \le n)$. This is also can be done for any linear stochastic mapping $x' = Tx$, namely, $x' = s(x)Tx$. The corresponding dynamical system is the Markov chain generated by T. If $(t_{ij})_{i,j=1}^n$ is the matrix of T then $p_{ik,j} = \frac{1}{2}(t_{is} + t_{kj})$ in the above mentioned quadratic representation.

It is also useful to note that any constant mapping $x' = c \in \Delta^{n-1}$ can be written as a quadratic one: $x' = cs^2(x)$, $x \in \Delta^{n-1}$.

It is easy to prove that *for $n \le 2$ every Bernstein mapping is constant or identity.*

The S.P. $V^2 = V$ means that every offspring state Vx is an equilibrium, so that the trajectory consists of x and Vx. Such a simplest dynamics should correspond to an "elementary" law of heredity. Just this philosophy compeled S.N.Bernstein to pose his problem. On the other hand, S.P. is a fortiori valid under the Mendel Law. Let us explain this in a more detail.

A simplest mechanism of heredity (discovered by Mendel) is determined by two *genes*, say **A** and **a**. Every individual has one of three possible genotypes: **AA**, **aa**, **Aa**. Each parent provides each offspring with one of these two genes. The genes of **Aa** are reproduced in offspring with probabilities $\frac{1}{2}$. Any offspring genotype appears as an independent random combination of two parental genes. This mechanism transforms a parental state $x = (x_1, x_2, x_3)$ of a population into the offspring state x' with

$$x_1' = p^2, \quad x_2' = q^2, \quad x_3' = 2pq \tag{1.4}$$

where

$$p = x_1 + \frac{1}{2}x_3, \quad q = x_2 + \frac{1}{2}x_3 \tag{1.5}$$

These p and q are the probabilities of the genes **A** and **a** at the state x. For the first time these formulas were independently obtained in [5] and [18] therefore the corresponding quadratic mapping $V : \Delta^2 \to \Delta^2$ is called the *Hardy-Weinberg mapping.* Obviously, $p+q=1$, so in the next generation

$$p' = x_1' + \frac{1}{2}x_3' = p^2 + pq = p \tag{1.6}$$

and $q' = q$ similarly. For this reason $x_1'' = p'^2 = p^2 = x_1'$ and $x_2'' = x_2'$, $x_3'' = x_3'$. This means that $V^2 = V$.

The relations $p' = p$ and $q' = q$ catch the phenomenon that the genes pass from parents to offsprings with no appearing or disappearing (usually occured under mutation and selection).

Following this classical pattern we introduced ([7]; [17], Section 4.1) a general concept of *stationary gene structure.* Let us reproduce it below.

Given an evolutionary operator $V : \Delta^{n-1} \to \Delta^{n-1}$, a linear form

$$f(x) = \sum_{i=1}^{n} a_i x_i \tag{1.7}$$

is called *invariant* if $f(Vx) = f(x)$ $(x \in \Delta^{n-1})$ or $f' = f$ for short, like (1.6). A trivial example is $s(x)$ or any multiple of s. Obviously, the set J of all invariant linear forms is a linear space, $1 \leq \dim J \leq n$. If V is constant then $\dim J = 1$. If $V = I$ (and only in this case) then $\dim J = n$.

We say that V *has a stationary gene structure (s.g.s)* ($\equiv V$ is *regular*) if this mapping can be written as

$$x_j' = \sum_{i,k=1}^{r} c_{ik,j} f_i(x) f_k(x) \quad (1 \leq j \leq n) \tag{1.8}$$

where $f_1, ..., f_r$ are some invariant linear forms. If so, these forms can be chosen in a special *canonical* way. Namely, one can consider the cone C of those $f \in J$ which are *nonnegative* in the usual sense: all $a_i \geq 0$ in (1.7). This is a closed polyhedral cone; $s \in \mathrm{Int}C$, so C generates the space J. In (1.8) $f_1, ..., f_r$ can be taken from the extremal rays of C, one for each ray. In such a way let

$$f_l(x) = \sum_{j=1}^{n} \pi_{lj} x_j \quad (1 \leq l \leq r) \tag{1.9}$$

with

$$\max_j \pi_{lj} = 1 \quad (1 \leq l \leq r) \tag{1.10}$$

Then we say that f_l are *canonical.*

Obviously, $r \geq \dim J$ and $r = \dim J$ iff $f_1, ..., f_r$ are linearly independent. In the last case we say that V has an *elementary gene structure (e.g.s).* Then the coefficients $c_{ik,j}$

in (1.8) are uniquely determined and $c_{ik,j} \geq 0$ ([7]; [17], Corollary 4.3.4 and Lemma 4.3.5). For *nonelementary gene structure (n.e.g.s.)* $c_{ik,j}$ *are not uniquely determined but there exists a set of* $c_{ik,j} \geq 0$ *in (1.8) with canonical* f_1, ..., f_r ([12]; [17], Theorem 4.4.1).

A simplest example of n.e.g.s came from [4]. This is the *quadrille mapping* $V : \Delta^3 \to \Delta^3$,

$$x_1^{'} = p_1 q_1, \quad x_2^{'} = p_2 q_2, \quad x_3^{'} = p_1 q_2, \quad x_4^{'} = p_2 q_1 \tag{1.11}$$

with

$$p_1 = x_1 + x_3, \quad p_2 = x_2 + x_4, \quad q_1 = x_1 + x_4, \quad q_2 = x_2 + x_3. \tag{1.12}$$

A relevant genetical mechanism was suggested in [7].

Like the Hardy-Weinberg case we have $V^2 = V$ *for any s.g.s.* The converse is not true ([17], p.172).

A *restricted Bernstein problem* posed in [7] *is to explicitly describle all regular stochastic quadratic mappings.* In the case of e.g.s. this problem was solved in [7]. Later the general case was solved in [8], [11], [12] (see also [17], Chapter 4). However, it turned out that a satisfactory genetical interpretation reguires an additional property of *normality.*

A stochastic quadratic mapping $V : \Delta^{n-1} \to \Delta^{n-1}$ is called *normal* if in (1.2)

1) all $x_j^{'} \not\equiv 0$ *(the nondegeneracy)*;

2) every two $x_{j_1}^{'}, x_{j_2}^{'}$ $(j_1 \neq j_2)$ are not proportional (*the external irreducibility)*;

3) there is no pair i, k $(i \neq k)$ such that all $x_j^{'}$ only depend on $x_i + x_k$ and x_l $(l \neq i,\ l \neq k)$ (*the internal irreducibility*).

A constant mapping is normal in the only case $n = 1$. The unit mapping is normal in all dimensions n.

If V is not normal one can reduce it to a normal one by a standard procedure of *normalization* ([11]; [17], Section 3.9). Under the normalization the dimension n decreases but this process preserves the regularity, moreover, r and $\dim J$ are invariant.

The explicit description of all regular normal evolutionary operators is contained in [17], Theorem 4.3.9 (for e.g.s.) and Theorem 4.6.1 (for n. e.g.s.). We also explain this below (Section 3) in a more apparent algebraic form remarkably corresponding to some genetical mechanisms (cf. [17], p.189,207). The point is that the types $\{1, ... n\}$ in any normal s.g.s can be identified with some pairs of genes $\mathbf{A}_1$, , , , $\mathbf{A}_r$; the probabilities of these genotypes at a state x are the canonical $f_l(x)$. If a s.g.s. is not normal (but nondegenerate) then there are some different types whose formal genotypes are the same, so some of the types are redundant ([17], Section 4.2).

The degeneracy means that some of types disappear from the population after mating. Such types can not be considered as hereditarily significant ones.

Thus, in the Bernstein problem the only case of normal s.g.s. has a genetical sense. However, in this context the S.P. is not an axiom, it is a consequence of s.g.s. If we wish to preserve the S.P. as an axiom then we should add something else to get s.g.s. as a consequence providing a natural genetical interpretation. In such a way some relevant conjectures were suggested in [16] (see also [17], Section 5.7). A proof of one of them is the subject of the present paper.

Definition. *A stochastic quadratic mapping V is called ultranormal if its restrictions to all invariant faces of the simplex Δ^{n-1} are normal.*

The ultranormality is a natural axiom in addition to the S.P. since we recognize the normality as a necessary property of *all* evolutionary operators, in particular, of the restrictions of V to all invariant faces.

Main Theorem. *Every ultranormal stochastic Bernstein mapping V is regular.*

Our above mentioned works contain a proof of this theorem in the case $\dim(\mathrm{Im}V) \leq 2$ or $\geq n-2$, in particular, for $n \leq 5$.

The Main Theorem combining with our explicit description of the regular normal mappings completely resolves the Bernstein problem in the ultranormal case. Note that *every normal s.g.s. is ultranormal* as directly follows from its explicit form.

Is every *normal* stochastic Bernstein mapping regular? This is an open question for $n \geq 5$ (cf. [17], Section 5.7). An affirmative answer would be a key to the general Bernstein problem by normalization.

Our approach to the Main Theorem is basically algebraical and partly topological one. In Section 3.4 some relevant means are prepared. The corresponding key words are "Bernstein algebra","regular algebras", "stochastic algebras" and their "offspring subalgebras".

The proof of the Main Theorem is given in Section 5. Actually, we prove that *every ultranormal stochastic Bernstein algebra is regular* or, equivalently, admits the above mentioned explicit form.

The result of this paper was announced at the 9th Haifa Matrix Theory Conference on June 1, 1995.

Aknowledgemnt. This paper was partly prepared during the author's visit to the IMS at Stony Brook in the summer 1995. The author thanks the IMS for their hospitality

2 Bernstein algebras.

For any evolutionary mapping V one can consider an algebra $\mathcal{A}_V$ in $\mathbb{R}^n$ whose structure constants at the canonical basis $\{ e_j \}_1^n$ are $p_{ik,j}$, so that we have the multiplicative table

$$e_i e_k = \sum_{j=1}^{n} p_{ik,j} e_j. \tag{2.1}$$

The algebra $\mathcal{A}_V$ is commutative but, as rule, it is not associative.

In a biological interpretation, the types $\{1, ..., n\}$ have to be identified with the corresponding basis vectors $\{ e_1, ..., e_n \}$. With parental types e_i, e_k an offspring is of type e_j with probability $p_{ik,j}$.

The *evolutionary algebra* is *stochastic* in the sense that the symplex Δ^{n-1} is invariant with respect to the multiplication. Indeed, $x \geq 0 \,\&\, y \geq 0 \Rightarrow xy \geq 0$ and $s(x) = 1 \,\&\, s(y) = 1 \Rightarrow s(xy) = 1$ because s is a multiplicative linear functional (a *weight)*:

$$s(xy) = s(x)s(y). \tag{2.2}$$

This means that the pair $(\mathcal{A}_V, s)$ is a real *baric* algebra (see [14]; [17],Sections 3.3, 3.8).

Note that $Vx = x^2$ $(x \in \Delta^{n-1})$ so x^2 is a (unique) quadratic extension $\tilde{V}$ of V from Δ^{n-1} to the whole space $\mathbb{R}^n$. It is very fruitfull to reformulate the Bernstein problem algebraically. We systematically used this approach earlier starting with the following

Lemma 2.1. *A stochastic quadratic mapping V is Bernstein iff the baric algebra $(\mathcal{A}_V, s)$ is Bernstein in the sense*

$$(x^2)^2 = s^2(x)x^2. \tag{2.3}$$

This lemma appeared first in [7] being written in the form $\tilde{V}^2x = s^2(x)\tilde{V}x$, but in [11] we already wrote (2.3). Later Holgate [6] and author [13] considered the Bernstein algebras by itself. (The term *Bernstein algebra* was introduced in [13].) In [17] (Sections 3.3 and 3.4) a part of the Bernstein algebras theory (over the field $\mathbb{R}$) is presented in a form adapted to the Bernstein problem. Below we partially reproduce it with addition of some new facts we need here.

In an arbitrary Bernstein algebra $(\mathcal{A}, \sigma)$ we have

$$\sigma(xy) = \sigma(x)\sigma(y); \quad (x^2)^2 = \sigma^2(x)x^2 \tag{2.4}$$

by definition. The first of these identities shows that the subspace $\mathcal{B} = \mathrm{Ker}\sigma = \{\, x : x \in \mathcal{A},\ \sigma(x) = 0 \,\}$ is an ideal (so-called *barideal*) in $\mathcal{A}$. The second one yields a construction of the idempotents in $\mathcal{A}:$ if $\sigma(x) = 1$ then $e = x^2$ is an idempotent and $\sigma(e) = 1$, so $e \neq 0$. (Conversely, if $e = e^2$ and $e \neq 0$ then $\sigma(e) = 1$.)

Given an idempotent $e \neq 0$, the linear operator $L_e y = 2ey$ is a projection in $\mathcal{B}$ hence, $\mathcal{B} = U \oplus W$ where $U = \mathrm{Im}L_e$, $W = \mathrm{Ker}L_e$. Respectively,

$$\mathcal{A} = E \oplus U \oplus W \tag{2.5}$$

where $E = \mathrm{Lin}\{e\}$, the linear span of $\{e\}$. The subspaces E, U, W depend on e but the dimensions $m-1$ and δ are invariant. The pair (m, δ) is called the *type* of $\mathcal{A}$. Moreover $m = \mathrm{rk}\mathcal{A}$ is called the *rank* of $\mathcal{A}$, and $\delta = \mathrm{def}A$ is called the *defect* of $\mathcal{A}$. Obviously, $m + \delta = n = \dim\mathcal{A}$.

The algebraic structure is reflected in (2.5) by the system of inclusions:

$$U^2 \subset W, \;\; UW \subset U, \;\; W^2 \subset U. \tag{2.6}$$

Moreover, there is a series of identities connecting the variables $u \in U$ and $w \in W$ but we do not need this here.

If according to (2.5)

$$x = \sigma e \oplus u \oplus w \tag{2.7}$$

then $\sigma = \sigma(x)$ and the corresponding decomposition of x^2 is

$$x^2 = \sigma^2 e \oplus (\sigma u + 2uw + w^2) \oplus u^2 \tag{2.8}$$

because of (2.6) and $2eu = L_e u = u, \ 2ew = L_e w = 0$.

The simplest Bernstein algebras are the *constant algebras (c.a.)*, $x^2 = c\sigma^2(x)$ with $c \in \mathcal{A}$. All these algebras are of type $(1, n-1)$ and conversely, *every Bernstein algebra of type* $(1, n-1)$ *is constant.* An opposite simple example is the *unit algebra (u.a.),* $x^2 = \sigma(x)x$ which is the only Bernstein algebra of type $(n, 0)$. *With* $n \leq 2$ *every Bernstein algebra is a c.a. or u.a.*

The evolutionary algebra $\mathcal{A}_V$ is a c.a. (or u.a.) iff V is constant (or identity) mapping.

The multiplication table for a c.a. is

$$e_i e_k = c \quad (1 \leq i, k \leq n) \tag{2.9}$$

and $xy = c\sigma(x)\sigma(y)$ for all x, y.

For the u.a. we have

$$e_i e_k = \frac{e_i + e_k}{2} \quad (1 \leq i, k \leq n) \tag{2.10}$$

and

$$xy = \frac{\sigma(y)x + \sigma(x)y}{2} \tag{2.11}$$

for all x, y.

The evolutionary algebra corresponding to the Hardy-Weinberg mapping is called the *Mendel algebra (M.a.)* This is a Bernstein algebra of type (2,1) with the multiplication table

$$\begin{cases} e_1^2 = e_1, \ e_2^2 = e_2, \ e_1 e_2 = e_3, & (2.12) \\ e_1 e_3 = \frac{1}{2}(e_1 + e_3), \ e_2 e_3 = \frac{1}{2}(e_2 + e_3), & (2.13) \\ e_3^2 = \frac{1}{4}e_1 + \frac{1}{4}e_2 + \frac{1}{2}e_3. & (2.14) \end{cases}$$

Actually, (2.13) and (2.14) follow from (2.12) by the Bernstein property.

Proposition 2.2. ([17], p.104) *Let* z_1 *and* z_2 *be nonzero idempotents in a Bernstein algebra* $\mathcal{A}$ *and let* $z_3 = z_1 z_2$. *Then*

$$z_1 z_3 = \frac{z_1 + z_3}{2}, \quad z_2 z_3 = \frac{z_2 + z_3}{2} \tag{2.15}$$

and

$$z_3^2 \;=\; \frac{1}{4}z_1 \;+\; \frac{1}{4}z_2 \;+\; \frac{1}{2}z_3. \tag{2.16}$$

Therefore $Z = \mathrm{Lin}\{\, z_1, z_2, z_3 \,\}$ is a Bernstein subalgebra which is isomorphic to the M.a. if $\dim Z = 3$, i.e. if z_1, z_2, z_3 are linearly independent. If $\dim Z = 2$ (i.e. $z_1 \neq z_2$ and $z_3 \in \mathrm{Lin}\{\, z_1, z_2 \,\}$) then Z is the u.a.

It is very useful for our purposes to introduce a new commutative multiplication,

$$R(x,y) \;=\; 2xy \;-\; \sigma(y)x \;-\; \sigma(x)y \tag{2.17}$$

in a Bernstein algebra $\mathcal{A}$. Letting $x \circ y$ for the unit multiplication (2.11) we obtain

$$R(x,y) \;=\; 2(xy \;-\; x \circ y) \tag{2.18}$$

So R measures a deviation of the given algebra from the u.a.,

$$R(x,y) \;=\; 0 \;\Leftrightarrow\; xy \;=\; x \circ y \tag{2.19}$$

Obviously, $R(x,x) = 0$ iff $x^2 = \sigma(x)x$, in particular, $R(x,x) = 0$ for all idempotents x.

Note that the subalgebras are the same for $R(x,y)$ and xy (including the non-Bernstein ones in $(\mathcal{A},\sigma)$, i.e. the subalgebras of the barideal $\mathcal{B}$).

Lemma 2.3. *Any four idempotents* z_1, z_2. z_3, z_4 *in a Bernstein algebra satisfy the relation*

$$R(z_1z_2, z_3z_4) \;+\; R(z_1z_3, z_2z_4) \;+\; R(z_1z_4, z_2z_3) \;=\; 0. \tag{2.20}$$

Proof. It is trivial if one of z_i is zero. If all of them are nonzero we insert

$$x \;=\; \sum_{i=1}^{4} \xi_i z_i, \quad \sigma(x) \;=\; \sum_{i=1}^{4} \xi_i$$

into the identity $(x^2)^2 = \sigma^2(x)x^2$ and then compare the coefficients at the monomial $\xi_1\xi_2\xi_3\xi_4$. This immediately leads to (2.20). □

Corolary 2.4. *For any three idempotents* z_1, z_2, z_3

$$2R(z_1z_2, z_1z_3) \;+\; R(z_1, z_2z_3) \;=\; 0 \tag{2.21}$$

Proof. Take $z_4 = z_1$ in (2.20). □

We also come back to (2.15) and (2.16) setting $z_3 = z_1$ or $z_3 = z_2$ in (2.21).

For some further constructions we need

Corollary 2.5. *If* z_1, z_2, w_1, w_2 *are nonzero idempotents such that*

$$R(z_i, w_j) \;=\; 0 \quad (i,j \;=\; 1,2) \tag{2.22}$$

then

$$R(z_1z_2, w_1w_2) \;=\; -\frac{R(z_1, z_2) \;+\; R(w_1, w_2)}{2}. \tag{2.23}$$

Proof. It follows from (2.20) and (2.19) (by assumption (2.22)) that

$$R(z_1z_2, w_1w_2) \;=\; -R(z_1w_1, z_2w_2) \;-\; R(z_1w_2, z_2w_1) \;=$$

$$-R(z_1 \circ w_1, z_2 \circ w_2) \;-\; R(z_1 \circ w_2, z_2 \circ w_1) \;=$$

$$= -\frac{1}{4}[R(z_1 \;+\; w_1,\; z_2 \;+\; w_2) \;+\; R(z_1 \;+\; w_2,\; z_2 \;+\; w_1)]$$

which can be reduced to (2.23) by (2.22). □

Corollary 2.6. *If* z_1, z_2, w *are nonzero idempotents such that*

$$R(z_i, w) \;=\; 0 \quad (i \;=\; 1, 2) \tag{2.24}$$

then

$$R(z_1z_2, w) \;=\; -\frac{1}{2}R(z_1, z_2). \tag{2.25}$$

Therefore $R(z_1z_2, w)$ is independent of w.

Proof. Take $w_1 \;=\; w_2 \;=\; w$ in (2.23). □

In our context the most important baric algebras are *regular* ones. By one of many equivalent definitions, the *regularity* of a baric algebra $(\mathcal{A}, \sigma)$ means that xy only depends on values $f(x)$ and $f(y)$ where f runs over all *invariant* linear forms. The *invariance* of f means that

$$\sigma(x) \;=\; 1 \;\Rightarrow\; f(x^2) \;=\; f(x), \tag{2.26}$$

or equivalently,

$$f(xy) \;=\; \frac{\sigma(y)f(x) \;+\; \sigma(x)f(y)}{2} \tag{2.27}$$

which in turn can be written as

$$f(R(x, y)) \;=\; 0. \tag{2.28}$$

For the evolutionary algebras $\mathcal{A}_V$ the invariance of f is the same as for V, i.e. $f(Vx) \;=\; f(x)$. Therefore, for any $(\mathcal{A}, \sigma)$ we can use the notation J for the space of all linear invariant forms. Obviosly, $\sigma \;\in\; J$, so $\dim J \;\geq\; 1$.

An evolutionary operator V *is regular iff the algebra* $\mathcal{A}_V$ *is regular.*

Note that the invariant faces of Δ^{n-1} are just such that their linear spans are subalgebras in $\mathcal{A}_V$, i.e. they are coordinate subalgebras. Let us say that $\mathcal{A}_V$ is *normal* if V is so (see [17], Section 3.9 for a more algebraic treat of this notion). Respectively, $\mathcal{A}_V$ is said to be *ultranormal* if V is so, i.e. all coordinate subalgebras are normal. Thus, we are going to prove

The Main Theorem. *Every ultranormal stochactic Bernstein algebra is regular.*

For this goal we need some regularity criteria for the Bernstein algebras.

Certainly, any regular algebra is Bernstein. This easily follows from definitions or from the identity

$$x^2 y = \sigma(x) x y \tag{2.29}$$

characterizing the regularity ([13]; [17], Theorem 3.3.6).(By the way, such a characterization shows that all subalgebras of a regular algebra are regular).

Note that $\dim J \le m$ for any Bernstein of rank m.

Theorem 2.7. *For any Bernstein algebra of rank m the following conditions are equivalent:*

1) *the algebra is regular;*
2) $\dim J = m$;
3) $UW + W^2 = 0$;
4) $UW = 0$ *and* $W^2 = 0$, *so that (2.8) takes the form*

$$x^2 = \sigma^2 e \oplus \sigma u \oplus u^2. \tag{2.30}$$

(see [7]; [17], Theorems 3.3.4, 3.4.15 and 3.4.17).

A very important consequence of this criterion is

Corollary 2.8. *A Bernstein algebra is regular if (2.29) holds for all of x and for a fixed idempotent $y = e \neq 0$.*

Proof. Insert x and x^2 from (2.7) and (2.8) into (2.29). We get

$$\sigma^2 e \oplus \frac{1}{2}(\sigma u + 2uw + w^2) = \sigma^2 e \oplus \frac{1}{2}\sigma u$$

because of $e^2 = e$, $eu = \frac{1}{2}u$, $ew = 0$. Hence $uw + \frac{1}{2}w^2 = 0$. The algebra is regular by part 3) of Theorem 2.7. □

It is conveniente to formulate Corollary 2.8 in a coordinate from.

Corollary 2.9. *Let a Bernstein algebra $\mathcal{A}$ is a linear span of a system of vectors $\{ v_i \}_1^l$, $\sigma(v_i) = 1$ $(1 \le i \le l)$. If there exists an idempotent $e \neq 0$ such that*

$$(v_i v_k) e = \frac{v_i e + v_k e}{2} \quad (1 \le i, k \le l) \tag{2.31}$$

then the algebra is regular.

Proof. Any vector $x \in \mathcal{A}$ is $x = \sum \xi_i v_i$. Then (2.31) implies $x^2 e = \sigma(x) x e$ since $\sigma(x) = \sum \xi_i$. □

Note that (2.31) can be rewriten as

$$R(v_i, v_k) e = 0 \quad (1 \le i, k \le l). \tag{2.32}$$

Similarly, (2.29) can be rewritten as $R(x,x)y = 0$. This identity is equivalent to $R(x,z)y = 0$ which is formally a more general one.

The concrete examples of regular algebras are c.a., u.a., M.a. By Proposition 2.2 any pair of idempotents z_1, z_2 $(z_1 \neq z_2)$ generates either the (2-dimensional) u.a. or the M.a. Using more idempotents one can inductively construct some other regular subalgebras in a Bernstein algebra.

Proposition 2.10. *Let* $\{ z_i \}_1^\nu$ *be a family of idempotents such that the subspace* $L = \mathrm{Lin}\{ z_i z_k \}_{i,k=1}^\nu$ *is a regular subalgebra. Then*

1) *if* w *is an idempotent such that*

$$R(z_j, w) = 0 \quad (1 \le j \le \nu) \tag{2.33}$$

then $L[w] = L \oplus \mathrm{Lin}\{ w \}$ *is a regular subalgebra;*

2) *if* w_1 *and* w_2 *are idempotents such that*

$$R(z_i, w_j) = 0 \quad (1 \le i \le \nu;\ j = 1,2) \tag{2.34}$$

and

$$R(w_1, w_2) z_1 = 0 \tag{2.35}$$

then $L[w_1, w_2] = L \oplus \mathrm{Lin}\{ w_1, w_2, w_1 w_2 \}$ *is a regular subalgebra.*

Proof. 1) By Corollary 2.6 all $R(z_i z_k, w) \in L$, so $(z_i z_k) w \in L[w]$ hence, $L[w]$ is a subalgebra. This is the linear span of w and all of $z_i z_k$ $(1 \le i, k \le \nu)$, z_1 among them. By Corollary 2.9 with $e = z_1$ $L[w]$ is regular. Indeed, by Corollary 2.6

$$z_1 R(z_i z_k, w) = -\frac{1}{2} z_1 R(z_i, z_k) \ \ (1 \le i, k \le \nu)$$

which equals zero because L is regular.

2) We already know that $L[w_1]$, $L[w_2]$ and $\mathrm{Lin}\{ w_1, w_2, w_1 w_2 \}$ are regular subalgebras. Furthermore, all $R(z_i z_k, w_1 w_2) \in L[w_1, w_2]$ by Corollary 2.5. So, $(z_i z_k)(w_1 w_2) \in L[w_1, w_2]$. We see that $L[w_1, w_2]$ is a subalgebra. It is regular by Corollary 2.5 and the regularity of L imply

$$z_1 R(z_i z_k, w_1 w_2) = -\frac{1}{2} z_1 R(w_1, w_2) = 0 \ \ (1 \le i, k \le \nu),$$

and, moreover,

$$R(w_j, w_1 w_2) = 0 \ \ (j = 1,2), \quad R(w_1 w_2, w_1 w_2) = -\frac{1}{2} R(w_1, w_2)$$

by Proposition 2.2. □

In conclusion we formulate a sufficient regularity condition in terms of type (m, δ) ([10]; [17], Corollaries 3.4.28 and 3.4.29).

Theorem 2.11. *Let a Bernstein algebra* $\mathcal{A}$ *of type* (m, δ) *be nuclear, i.e.* $\mathcal{A}^2 = \mathcal{A}$. *Then it is regular if* $m \le 3$ *or* $\delta \le 1$, *or*

$$\delta \ge \frac{(m-1)(m-2)}{2} + 1. \tag{2.36}$$

Corollary 2.12. *Every nuclear Bernstein algebra of dimension* $n \leq 5$ *is regular.*

There exists a nonregular nuclear Bernstein algebra of type (4,2) (see [17], p.102). The question whether *every stochastic nuclear Bernstein algebra is regular* (the Conjecture 5.7.16 from [17]) is still open. However, we have

Theorem 2.13. *With* $m \leq 2$ *or* $\delta \leq 1$ *(in particular, with* $n \leq 4$ *) every normal stochastic Bernstein algebra is regular.*

Finally, we have such a part of the Main Theorem.

Theorem 2.14 ([16]; [17], Section 5.7). *With* $m \leq 3$ *or* $\delta \leq 1$ *(with* $n \leq 5$, *in particular) every ultranormal stochastic Bernstein algebra* $\mathcal{A}$ *is regular.*

Its proof is based on a combinatorial topology structure which is induced by invariant faces on the set ImV where V is the corresponding quadratic mapping, $\mathcal{A}_V = \mathcal{A}$. We develop this approach in Section 4.

3 Normal stochastic regular algebras

The complete solution of the Bernstein problem for the normal regular algebras is given by the following theorem which is an algebraic reformulation of Theorems 4.3.9 and 4.6.1 from [17].

Theorem 3.1. *Every normal stochastic regular algebra* $\mathcal{A}$ *of type* (m, δ) *is one of two following ones.*

1) *Up to enumeration of the canonical basis* $\{e_i\}_1^n$ *the vectors* $e_1, \ldots, e_m$ *are idempotents. Their products are*

$$e_{i_j} e_{k_j} = \alpha_j e_{i_j} + \beta_j e_{k_j} + \gamma_j e_{m+j} \tag{3.1}$$

for some distinct pairs (i_j, k_j) *with* $1 \leq i_j < k_j \leq m, \ 1 \leq j \leq \delta$, *and*

$$e_i e_k = \frac{e_i + e_k}{2} \tag{3.2}$$

for all remaining pairs (i, k) $(1 \leq i < k \leq m, \ i \neq i_j$ *or* $k \neq k_j)$. *In (3.1)* $\alpha_j \geq 0, \ \beta_j \geq 0, \ \alpha_j + \beta_j + \gamma_j = 1$.

Furthermore, for $1 \leq j \leq \delta$ *and* $1 \leq i \leq m$

$$e_i e_{m+j} = c_j e_i e_{i_j} + \overline{c_j} e_i e k_j \tag{3.3}$$

where $0 < c_j < 1; \ \overline{c_j} = 1 - c_j$ *and*

$$\alpha_j + c_j \gamma_j = \beta_j + \overline{c_j} \gamma_j = \frac{1}{2}. \tag{3.4}$$

Finally,

$$e_{m+j} e_{m+l} = c_j c_l e_{i_j} e_{i_l} + c_j \overline{c_l} e_{i_j} e_{k_l} + \overline{c_j} c_l e_{k_j} e_{i_l} + \overline{c_j}\, \overline{c_l} e_{k_j} e_{k_l} \tag{3.5}$$

for $1 \leq j, l \leq \delta$.

2)*All basis vectors* $\{e_i\}_1^n$ *are idempotents. The number* $n = \dim\mathcal{A}$ *is composite,* $n = \nu\overline{\nu}$ *with* $\nu \geq 2$, $\overline{\nu} \geq 2$. *The basis can be enumerated as* $\{e_{ik} : 1 \leq i \leq \nu,\ 1 \leq k \leq \overline{\nu}\}$ *in a way such that*

$$e_{gh}e_{ik} = \frac{e_{gk} + e_{hi}}{2} \tag{3.6}$$

for all pairs (g,h) *and* (i,k).

The first case is the *elementary gene structure (e.g.s.),* the second one is the *nonelementary gene structure (n.e.g.s.).* Both of them allow a natural genetical interpretation (see [17], p.p. 189, 207). E.g.s. is *continual,* i.e. multiparametric. The independent parameters are α_j, β_j $(1 \leq j \leq \delta)$ so that the manifold of all these algebras is 2δ-dimensional. N.e.g.s. is *discrete,* i.e. 0-dimensional.

The unit algebra (u.a.) has e.g.s. In this case $\delta = 0$ and there is no e_{m+j}, no pairs (i_j, k_j), so that (3.2) is the complete multiplication table. It is the only case with e.g.s. when all e_i $(1 \leq i \leq n)$ are idempotents.

The constant algebra (c.a.) is normal in the only case $n = 1$ but then it is also u.a.

The simplest nontrivial situation is 3-dimensional.

Example 3.2. For $m = 2$ and $\delta = 1$ $(n = 3)$ we have e.g.s.

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1e_2 = \alpha e_1 + \beta e_2 + \gamma e_3 \tag{3.7}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma > 0$, $\alpha + \beta + \gamma = 1$. Furthermore,

$$e_3e_1 = ce_1^2 + \overline{c}e_1e_2,$$

so that

$$e_3e_1 = (c + \overline{c}\alpha)e_1 + \overline{c}\beta e_2 + \overline{c}\gamma e_3 \tag{3.8}$$

and similarly,

$$e_3e_2 = ce_1e_2 + \overline{c}e_2^2 = c\alpha e_1 + (c\beta + \overline{c})e_2 + c\gamma e_3 \tag{3.9}$$

with $0 < c < 1$, $\overline{c} = 1 - c$ and

$$\alpha + \gamma c = \beta + \gamma\overline{c} = \frac{1}{2}. \tag{3.10}$$

Finally,

$$e_3^2 = c^2e_1^2 + 2c\overline{c}e_1e_2 + \overline{c}^2e_2^2,$$

so that

$$e_3^2 = (c^2 + 2c\overline{c}\alpha)e_1 + (2c\overline{c}\beta + \overline{c}^2)e_2 + 2c\overline{c}\gamma e_3. \tag{3.11}$$

The Mendel algebra (M.a.) is just the case $\alpha = \beta = 0$, $\gamma = 1$, $c = \frac{1}{2}$, so $\overline{c} = \frac{1}{2}$. M.a. is the point $M(0,0)$ of the 2-dimensional manifold of algebras $M(\alpha,\beta)$ given by (3.7)-(3.11). Therefore $M(\alpha,\beta)$ can be called an *extended Mendel algebra (e.M.a.).*

Let us emphasize that all the algebras $M(\alpha,\beta)$ are isomorphic to the M.a. $M(0,0)$ since (3.7) with $\gamma > 0$ allows us to change the basis $\{e_i\}_1^3$ for $\{e_1, e_2, e_1e_2\}$ and then the algebra turns into the M.a. Proposition 2.2. However, we must distinguish the algebras $M(\alpha,\beta)$ at the fixed basis $\{ e_i \}_1^3$. Biologically, it is necessary because e_i are the types themselves but their linear combinations have no such interpretation; respectively, the coefficients α, β, γ, etc. are the probabilities of types in the offsprings generation.

The term *e.M.a.* can be also used for any algebra of type (m,δ) given by (3.1)-(3.5). The corresponding evolutionary operator is an *extended Hardy-Weibnerg mapping.* Actually, it is

$$x_i^{'} = p_i^2 + 2\sum_{k\neq i}\theta_{ik}p_ip_k \quad (1 \leq i \leq m) \tag{3.12}$$

and

$$x_{m+j}^{'} = 2\gamma_j p_{i_j}p_{k_j} \quad (1 \leq j \leq \delta) \tag{3.13}$$

where

$$p_i = x_i + \sum_{j=1}^{\delta}\pi_{ij}x_{m+j}, \tag{3.14}$$

$\theta_{i_jk_j} = \alpha_j$, $\theta_{k_ji_j} = \beta_j$ and all remaining $\theta_{ik} = \frac{1}{2}$; $\pi_{i_jj} = c_j$; $\pi_{k_jj} = \overline{c_j}$ and all remaining $\pi_{ij} = 0$ (cf. [17], Theorem 4.3.9).

The set $G = \{ p_i \}_1^m$ is just the canonical basis of the cone C of all nonnegative invariant linear forms. Obviously, this set is linearly independent and

$$\sum_{i=1}^{m} p_i = s \tag{3.15}$$

The cone C is minihedral in the case of e.g.s.

Example 3.3. For $\nu = \overline{\nu} = 2$ $(n = 4)$ we have the symplest n.e.g.s. which is actually the *quadrille algebra (q.a.)* corresponding to the quadrille mapping (1.9)-(1.10). A more natural labeling in this case is $x_1 \equiv x_{11}$, $x_2 \equiv x_{22}$, $x_3 \equiv x_{12}$ and $x_4 \equiv x_{21}$. As a result

$$x_{ik}^{'} = p_iq_k \tag{3.16}$$

where p_i are the sums over rows of the matrix $X \equiv (x_{ik})$ and q_k are the sums over columns,

$$p_i = \sum_k x_{ik}, \quad q_k = \sum_i x_{ik}. \tag{3.17}$$

In a matrix form (3.6) is

$$X^{'} = p \otimes q \tag{3.18}$$

where p is the column (p_i) and q is the row (q_k).

The same formulae (3.16)-(3.18) take place in general, i.e. for any $n = \nu\overline{\nu}$ with $X = (x_{ik} : 1 \le i \le \nu,\ 1 \le k \le \overline{\nu})$. Any such a mapping is called an *extended quadrill mapping* and the corresponding algebra is an *extended quadrille algebra (e.q.a.)*.

The set $G = \{ p_i \}_1^{\nu} \bigcup \{ q_k \}_1^{\overline{\nu}}$ is the canonical basis of the cone C in this case. Now this set is linearly dependent since

$$\sum_{i=1}^{\nu} p_i = \sum_{k=1}^{\overline{\nu}} q_k \quad (= s). \tag{3.19}$$

Thus, the cone C is not minihedral in the case of n.e.g.s.

Besides (3.19), there is no linear dependence in G. Therefore the type (m,δ) of the e.q.a. is

$$m = \nu + \overline{\nu} - 1, \quad \delta = (\nu - 1)(\overline{\nu} - 1). \tag{3.20}$$

In particular, the q.a. is of type (3,1).

Corollary 3.4. *Every coordinate subalgebra of e.M.a. is also e.M.a. Every coordinate subalgebra of e.q.a is also e.q.a. or unit.*

Proof. If follows from (3.5) with $l = j$ that every coordinate subalgebra of an e.M.a. containing e_{m+j} $(1 \le j \le \delta)$ must contain both of the idempotents e_{i_j} and e_{k_j}. The converse is also true by (3.1). Thus, any coordinate subalgebra of the e.M.a. is the linear span of the union of the subset $F \subset \{ e_i \}_1^m$ with all of $\{ e_{m+j} : e_{i_j}, e_{k_j} \in F \}$. Obviously, it is an e.M.a. as well.

The case of e.q.a. is similar (even simpler). □

Note that *any e.M.a. is regular and normal* because of (3.12)-(3.14) where the pairs (i_j, k_j) are distinct and the restrictions $0 < c_j < 1,\ \ \gamma_j > 0$ are fulfilled. *Any e.q.a. is also regular and normal* because of (3.16)-(3.17). By Theorem 3.1 and Corollary 3.4 we get

Corollary 3.5. *Every normal stochastic regular algebra is ultranormal.*

In addition, we have

Corollary 3.6. *Every normal stochastic regular algebra is nuclear.*

Proof. In the case of e.M.a.

$$\mathrm{Lin}\{ e_i e_k \}_{i,k=1}^{n} = \mathrm{Lin}\{ e_i \}_1^m \bigcup \mathrm{Lin}\{ e_{i_j} e_{k_j} \}_1^{\delta} = \mathcal{A}$$

because the second set in the union can be changed for $\mathrm{Lin}\{ e_{m+j} \}_1^{\delta}$ using (3.1) with $\gamma_j > 0$. In the case of e.q.a. $\mathcal{A}^2 = \mathcal{A}$ because of $e_i^2 = e_i$ for all $i,\ 1 \le i \le n$. □

4 Offspring subalgebras

Recall that for any vector $x \in \mathbb{R}^n$,

$$x = \sum_{j=1}^{n} x_i e_i,$$

its *support* is defined as

$$\mathrm{supp}x \;=\; \{\, e_i \;:\; x_i \neq 0 \,\}$$

so that

$$x \;=\; \sum \{\, x_i e_i \;:\; e_i \;\in\; \mathrm{supp}x \,\}. \tag{4.1}$$

Obviously, $\mathrm{supp}x \;\neq\; \emptyset$ for $x \;\neq\; 0$ and

$$\mathrm{supp}(\lambda x) \;=\; \mathrm{supp}x \quad (\lambda \neq 0). \tag{4.2}$$

If $x \geq 0$ and $y \geq 0$ then

$$\mathrm{supp}(x+y) \;=\; \mathrm{supp}x \bigcup \mathrm{supp}y. \tag{4.3}$$

We say that an algebra $\mathcal{A}$ with the underlying space $\mathbb{R}^n$ is *nonnegative* if

$$x \;\geq\; 0 \,\&\, y \;\geq\; 0 \;\Rightarrow\; xy \;\geq\; 0 \tag{4.4.}$$

or, equivalently, its structure constants are nonnegative. Every stochastic algebra is so.

Lemma 4.1. *In any nonnegative algebra* $\mathcal{A}$ *for any* $x \geq 0$

$$\mathrm{supp}(x^2) \;=\; \bigcup \{\, \mathrm{supp}(e_i e_k) \;:\; e_i, e_k \;\in\; \mathrm{supp}x \,\}. \tag{4.5}$$

Thus, $\mathrm{supp}(x^2)$ only depends on $\mathrm{supp}x$ $(x \geq 0)$.

Proof. It follows from (4.1) that

$$x^2 \;=\; \sum \{\, x_i x_k e_i e_k \;:\; e_i, e_k \;\in\; \mathrm{supp}x \,\}$$

and then (4.5) follows from (4.2) and (4.3). □

Note that

$$\mathrm{supp}(e_i e_k) \;=\; \{\, e_j \;:\; p_{ik,j} > 0 \,\}.$$

Biologically, $\mathrm{supp}(e_i e_k)$ is the set of all types (characters) really presented in offsprings whose parental types are e_i and e_k. If $e_i^2 \;=\; e_i$ the type e_i is *nonsplitting* in the sense that all its offspring are of the same type.

For any family $F \;\subset\; \{\, e_i \,\}_1^n$ we define its *offspring set*

$$F^{'} \;=\; \bigcup \{\mathrm{supp}(e_i e_k) : \quad e_i, e_k \;\in\; F \,\}.$$

Vice versa F is the *parental set of* $F^{'}$. In the most important case F consists of some idempotents. Then $F^{'} \;\supset\; F$.

Lemma 4.2. $\mathrm{Lin}F$ *is a subalgebra iff* $F^{'} \;\subset\; F$.

Proof. $(\forall e_i, e_k \;\in\; F \;:\; e_i e_k \;\in\; \mathrm{Lin}F) \;\Leftrightarrow\; (\forall e_i, e_k \;\in\; F \;:\; \mathrm{supp}(e_i e_k) \;\subset\; F) \;\Leftrightarrow\; (\,\{\bigcup \mathrm{supp}(e_i e_k) \;:\; e_i, e_k \;\in\; F\,\} \;\subset\; F) \;\Leftrightarrow\; (F^{'} \;\subset\; F)$ by Lemma 4.1.

Corollary 4.3. $\mathrm{Lin}\,\{\, \mathrm{supp}x \,\}$ *is a subalgebra for any idempotent* $x \;\geq\; 0$.

Henceforth we only consider a stochastic Bernstein algebra $(\mathcal{A}, s)$, *so* $\mathcal{A} = \mathcal{A}_V$ *where* $V : \Delta^{n-1} \to \Delta^{n-1}$ *is a stochastic quadratic mapping,* $V^2 = V$.

Lemma 4.4. $F'' = F'$ *for any family* $F \subset \{ e_i \}_1^n$.

In this sense there are no characters coming from the offspring to their offspring but not originating from their parents.

Proof. Let

$$x = \sum \{ e_i : e_i \in F \},$$

so that $\mathrm{supp} x = F$. Then $\mathrm{supp}(x^2) = F'$ and

$$F'' = \mathrm{supp}(x^2)^2 = \mathrm{supp}[s^2(x)x^2] = \mathrm{supp}(x^2) = F'.$$

Corollary 4.5. $\mathrm{Lin}F'$ *is a subalgebra.*

We call this the *offspring subalgebra* of the parental set F. This construction plays a very important role in sequel.

For example, $M(\alpha, \beta)$ is the offspring subalgebra of the set $F = \{ e_1, e_2 \}$ with $e_1^2 = e_1$, $e_2^2 = e_2$. Moreover, e.M.a. end e.q.a. are both the offspring subalgebras of some families consisting of the basis idempotents. Such a family is $F = \{ e_i \}_1^m$ for e.M.a. and $F = \{ e_{g1} \}_{g=1}^{\nu} \bigcup \{ e_{1h} \}_{h=1}^{\overline{\nu}}$ for e.q.a. because $e_{g1}e_{1h} = \frac{1}{2}(e_{gh} + e_{11})$. Those families are minimal; certainly, all their extensions are also parental sets for the same algebra.

Theorem 4.6. *If* $F = \{ e_1, e_2 \}$, $e_1^2 = e_1$, $e_2^2 = e_2$ *then the rank of the corresponding offspring subalgebra does not exceed 3.*

To be prepared for the proof below we consider a special nonnegative projection associated with e_1 by writing

$$Bx = 2e_1x - (2e_1x, e_1)e_1 \tag{4.6}$$

where $(,)$ is the standard inner product at the canonical basis $\{ e_i \}_1^n$. This operator B was introduced in [15] (see also [17], Sections 5.3 and 5.4). Let us recall its properties, basically, without proofs. First of all, by (4.6)

$$Be_1 = 0, \ \ \mathrm{Im}B \perp e_1. \tag{4.7}$$

Lemma 4.7. *In the algebra* $\mathcal{A}$ *of type* (m, δ) *the operator* B *is a nonnegative projection of the form*

$$Bx = \sum_{i=1}^{m-1} (x, b_i^*)b_i \tag{4.8}$$

where

$$b_i \geq 0, \ \ b_i^* \geq 0, \ \ (b_i, b_k^*) = \delta_{ik} \tag{4.9}$$

and

$$\mathrm{supp}b_k \not\subset \bigcup_{i \neq k} \mathrm{supp}b_i \quad (1 \leq k \leq m-1). \tag{4.10}$$

Corollary 4.8. $\text{rk}B = m-1$.

Proof. It follows from (4.8) that $\text{Im}B \subset \text{Lin}\{ b_i \}_1^{m-1}$ and actually these two subspaces coincide since

$$Bb_k = \sum_{i=1}^{m-1}(b_k, b_i^*)b_i = b_k \quad (1 \le k \le m-1).$$

Because of (4.9), b_i^* are linearly indepedent. Thus, they form a basis in $\text{Im}B$. □

¿From now on we assume that b_i are *normalized* in the sense $s(b_i) = 1$ $(1 \le i \le m-1)$. Then we have

Lemma 4.9. *The intersection* $\Delta_B = \text{Im}B \bigcap \Delta^{n-1}$ *coincides with the convex hull of* $\{ b_i \}_1^{m-1}$.

Thus, Δ_B is a symplex.

Proof. Obviously, all convex combinations of $\{b_i\}_1^{m-1}$ belong to Δ_B since all $b_i \in \Delta_B$. Conversely, if $x \in \Delta_B$ then

$$x = Bx = \sum_{i=1}^{m-1} \alpha_i b_i$$

with $\alpha_i = (x, b_i^*) \ge 0$ and $\sum \alpha_i = s(x) = 1$. □

Finally, we have

Lemma 4.10. *The formulas*

$$e_1 b_i = \frac{e_1 + b_i}{2} \quad (1 \le i \le m-1) \tag{4.11}$$

and

$$B(b_i b_k) = \frac{b_i + b_k}{2} \quad (1 \le i, k \le m-1) \tag{4.12}$$

hold.

Proof of Theorem 4.6. Without loss of generality we can assume that $\text{Lin}F' = \mathcal{A}$ and we have to prove that $m \le 3$. Thus,

$$e_1 e_2 = \sum_{k=1}^{n} \pi_k e_k \tag{4.13}$$

where $\sum \pi_k = 1$, $\pi_k \ge 0$ and, moreover, $\pi_k > 0$ for $k \ge 3$ since $\mathcal{A}$ is the offspring algebra of $F = \{ e_1, e_2 \}$. In terms of the projection B

$$Be_2 = 2\sum_{k=2}^{n} \pi_k e_k. \tag{4.14}$$

On the other hand

$$Be_2 = 2\sum_{i=1}^{m-1} \lambda_i b_i \tag{4.15}$$

with $\lambda_i = \frac{1}{2}(Be_2, b_i^*) \geq 0, \quad 1 \leq i \leq m-1.$

Being a projection, $B = B^2$, so (4.14) yields

$$Be_2 = 2\sum_{k=2}^{n} \pi_k Be_k$$

therefore

$$\lambda_i = \sum_{k=2}^{n} \pi_k (Be_k, b_i^*) = 2\pi_2\lambda_i + \sum_{k=3}^{n} \pi_k (Be_k, b_i^*) \tag{4.16}$$

which implies $\lambda_i > 0$ $(1 \leq i \leq m-1)$. Indeed, if there is $\lambda_i = 0$ then $(Be_2, b_i^*) = 0$ and $(Be_k, b_i^*) = 0$ $(3 \leq k \leq n)$ from (4.16) and, finally, $(Be_1, b_i^*) = 0$ from (4.7). As a result, $(Bb_i, b_i^*) = 0$ while $(Bb_i, b_i^*) = (b_i, b_i^*) = 1$.

Now it follows from (4.15) that

$$\mathrm{supp}(Be_2) = \bigcup_{i=1}^{m-1} \mathrm{supp} b_i. \tag{4.17}$$

Coming back to (4.13) and (4.14) we get

$$e_1 e_2 = \pi_1 e_1 + \frac{1}{2} Be_2 \tag{4.18}$$

whence,

$$(e_1e_2)^2 = \pi_1^2 e_1 + \pi_1 e_1 (Be_2) + \frac{1}{4}(Be_2)^2.$$

Multiply (4.15) by e_1 and using (4.11) we obtain

$$e_1(Be_2) = e_1 \sum_{i=1}^{m-1} \lambda_i + \sum_{i=1}^{m-1} \lambda_i b_i = e_1 \sum_{i=1}^{m-1} \lambda_i + \frac{1}{2} Be_2$$

However, since $s(b_i) = 1$ $(1 \leq i \leq m-1)$ we have

$$\sum_{i=1}^{m-1} \lambda_i = \frac{1}{2} s(Be_2) = \sum_{k=2}^{m-1} \pi_k = 1 - \pi_1$$

Finally,

$$e_1(Be_2) = (1 - \pi_1) e_1 + \frac{1}{2} Be_2$$

and then

$$(e_1e_2)^2 = \pi_1 e_1 + \frac{1}{2}\pi_1 Be_2 + \frac{1}{4}(Be_2)^2. \tag{4.19}$$

On the other hand,

$$(e_1e_2)^2 \ = \ \frac{1}{4}e_1 \ + \ \frac{1}{4}e_2 \ + \ \frac{1}{2}e_1e_2$$

by Proposition 2.2. With (4.18) this yields

$$(e_1e_2)^2 \ = \ (\frac{1}{4} \ + \ \frac{1}{2}\pi_1)e_1 \ + \ \frac{1}{4}e_2 \ + \ \frac{1}{4}Be_2. \tag{4.20}$$

Let us compare the e_1-coordinates in (4.19) and (4.20). Taking into account that $Be_2 \perp e_1$ (see (4.7)) and $(Be_2)^2 \geq 0$ we get $\frac{1}{4} + \frac{1}{2}\pi_1 \geq \pi_1$, i.e. $\pi_1 \leq \frac{1}{2}$. Now we compare the e_2-coordinates and get

$$\frac{1}{4} \ + \ \frac{1}{2}\pi_2 \ = \ \pi_1\pi_2 \ + \ \frac{1}{4}((Be_2)^2, e_2)$$

whence,

$$\frac{1}{4} \ + \ (\frac{1}{2} - \pi_1)\pi_2 \ = \ \frac{1}{4}((Be_2)^2, e_2) \ = \ \sum_{i,k=1}^{m-1} \beta_{ik}\lambda_i\lambda_k \tag{4.21}$$

where $\beta_{ik} = (b_ib_k, e_2)$ by (4.15). Since all $b_i \geq 0$ we have $\beta_{ik} \geq 0$. Moreover, there exists $\beta_{i_1k_1} > 0$, otherwise all $\beta_{ik} = 0$ which contradicts (4.21) because of $\frac{1}{2} - \pi_1 \geq 0$ and $\pi_2 \geq 0$. Applying B to the inequality $b_{i_1}b_{k_1} \geq \beta_{i_1k_1}e_2$ and using (4.12) we obtain

$$\frac{b_{i_1} \ + \ b_{k_1}}{2} \ \geq \ \beta_{i_1k_1}Be_2.$$

Hence,

$$\mathrm{supp}(Be_2) \ \subset \ \mathrm{supp}b_{i_1} \bigcup \mathrm{supp}b_{k_1}.$$

However, we have (4.17). Therefore

$$\bigcup_{i=1}^{m-1} \mathrm{supp}b_i \ = \ \mathrm{supp}b_{i_1} \bigcup \mathrm{supp}b_{k_1}.$$

In view of (4.10) we get $m = 2$ if $i_1 = k_1$ and $m = 3$ if $i_1 \neq k_1$. □

As a consequence we obtain a very useful

Theorem 4.11. *Let* $F = \{ e_1, e_2 \}$, $e_1^2 = e_1$, $e_2^2 = e_2$. *If the correspondidg offspring subalgebra is normal then it is either 2-dimensional u.a. or 3-dimensional e.M.a., or g.a. (which is 4-dimensional).*

Proof. By Theorem 4.6 the subalgebra is of rank ≤ 3. By Theorem 2.11 this is regular. By Theorem 3.1 this is either e.M.a. or e.q.a.

If a normal e.M.a. is the offspring subalgebra of two parental idempotents then its dimension is 2 or 3. This is an u.a. in the case of dimension 2. If an e.q.a. has rank ≤ 3 then it is q.a. □

We say that a stochastic Bernstein algebra is *grounded* if it is the offspring subalgebra of the set of its basis idempotents. As we know every normal stochastic regular algebra is grounded.

Lemma 4.12. *Let $\mathcal{A}$ be not constant. If every proper coordinate subalgebra of $\mathcal{A}$ is grounded then the greatest offspring subalgebra $\mathcal{A}_0$ is also grounded.*

Obviously, $\mathcal{A}_0$ is the linear span of the set

$$\bigcup_{1\le i\le k\le n} \mathrm{supp}(e_i e_k). \tag{4.22}$$

The corresponding invariant face Δ of the simplex Δ^{n-1} is the convex hull of the same set (4.22). On the other hand, Δ is the smallest face containing $\mathrm{Im}V$, the image of the evolutionary operaror $Vx = x^2$ $(x \in \Delta^{n-1})$. Indeed,

$$x^2 = \sum_{i,k=1}^{n} x_i x_k e_i e_k$$

for

$$x = \sum_{i=1}^{n} x_i e_i.$$

Therefore $x \in \mathrm{Im}V \Rightarrow x = x^2 \in \Delta$ and $x \in \mathrm{Int}\Delta^{n-1} \Rightarrow x^2 \in \mathrm{Im}V \bigcap \mathrm{Int}\Delta$, so that $\mathrm{Im}V \subset \Delta$ and $\mathrm{Im}V \bigcap \mathrm{Int}\Delta \neq \emptyset$.

A face Γ of the simplex Δ^{n-1} is called *essential* if $C_\Gamma \equiv \mathrm{Im}V \bigcap \mathrm{Int}\Gamma \neq \emptyset$. Obviously,

$$\mathrm{Im}V = \bigcup_{\Gamma} C_\Gamma = \bigcup \{\, C_\Gamma \,:\, \Gamma \text{ is essential} \,\}. \tag{4.23}$$

It turns out that this partition is an *elementary cell complex on* $\mathrm{Im}V$ in the following sense (see [14]; [17], Section 5.7).

For any Hausdorff topological space X a subset $C \subset X$ is called a *ν-dimensional elementary cell* if there exists a bounded open set $U \subset \mathbb{R}^\nu$ whose closure $\overline{U}$ is contractible (within itself to a point) and homeomorphic to $\overline{C}$ by a boundary preserved homeomorphism.

A finite partition of X is called an *elementary cell complex on* X if 1) all the parts are elementary cells; 2) the boundary of each of one is a union of some lower dimensional cells; 3) the intersection of the closures of any two cells is contractible.

The maximal cell dimension d is called the *dimension of the complex.* (It is equal to the usual topological $\dim X$).

In our case $X = \mathrm{Im}V$, the cells are C_Γ for essential faces Γ and $\dim C_\Gamma = m_\Gamma - 1$ where $m_\Gamma = \mathrm{rk}(\mathcal{A}_{V|\Gamma})$. If $\Gamma_1 \subset \Gamma_2$ and $\Gamma_1 \neq \Gamma_2$ then $\dim C_{\Gamma_1} < \dim C_{\Gamma_2}$. Thus, the *dimension of this elementary cell complex is* $m-1$ where $m = \mathrm{rk}\mathcal{A}$ as usual. The only $(m-1)$-dimensional cell is $C_\Delta = \mathrm{Im}V \bigcap \mathrm{Int}\Delta$.

Every essential face Γ *is invariant* and

$$\mathrm{Im}(V \mid \Gamma) = \mathrm{Im}V \bigcap \Gamma = \overline{C_\Gamma}. \tag{4.24}$$

The boundary $\partial C_\Gamma = \overline{C_\Gamma} \setminus C_\Gamma$ is actually

$$\partial C_\Gamma = \mathrm{Im}V \bigcap \partial\Gamma. \tag{4.25}$$

Obviously, $\partial C_\Gamma = \emptyset$ iff $\dim C_\Gamma = 0$ which means that $V \mid \Gamma$ is constant.

For a topological reason ([17]), Lemma 5.7.2) in any d-dimensional elementary cell complex the number of 0-dimensional cells is at least $d+1$. Hence *there exists at least* m *constant coordinate subalgebras in any stochastic Bernstein algebra* ([14]; [17], Theorem 5.7.1)

Remark 4.13. At least one constant subalgebra can be obtained in a much more simple way. This follows from Theorem 5.2.1 [17] saying that *the subalgebra corresponding to a minimal invariant face is constant.* (The latter is a generalization of a Bernstein theorem provided in [4] with a very complicated proof. A short proof was found in [9].)

After these preliminaires we can directly pass to

Proof of Lemma 4.12. As aforesaid, there exists a constant coordinate subalgebra in $\mathcal{A}$. This is a proper subalgebra because $\mathcal{A}$ is not constant. Being grounded this subalgebra is 1-dimensional, generated by a basis idempotent. We conclude that the set of all basis idempotents is not empty. Let $\mathcal{A}_1$ be its offspring subalgebra, so $\mathcal{A}_1$ is the greatest grounded subalgebra. Obviously, $\mathcal{A}_1 \subset \mathcal{A}_0$. We have to prove that $\mathcal{A}_1 = \mathcal{A}_0$.

Suppose that $\mathcal{A}_1 \neq \mathcal{A}_0$. Then $\Gamma_1 \neq \Delta$ where Γ_1 is the invariant face corresponding to the algebra $\mathcal{A}_1$. Since $\Gamma_1 \subset \Delta$ and $\Gamma_1 \neq \Delta$, we have $\Gamma_1 \subset \partial\Delta$. By (4.24) and (4.25)

$$\overline{C_{\Gamma_1}} = \mathrm{Im}V \bigcap \Gamma_1 \subset \mathrm{Im}V \bigcap \partial\Delta = \partial C_\Delta. \tag{4.26}$$

However, $\partial\Delta = \bigcap \mathrm{Int}\Gamma$ where Γ runs over all faces $\Gamma \subset \Delta$, $\Delta \neq \Delta^{n-1}$. Hence,

$$\partial C_\Delta = \bigcup \{\, \mathrm{Im}V \bigcap \mathrm{Int}\Gamma \,:\, \Gamma \subset \Delta, \ \ \Gamma \neq \Delta \,\} = \bigcup \{\, C_\Gamma \,:\, \Gamma \subset \Delta, \ \ \Gamma \neq \Delta \,\}.$$

Since ∂C_Δ is closed, we get

$$\partial C_\Delta = \bigcup \{\{\overline{C_\Gamma} \,:\, \Gamma \subset \Delta, \ \Gamma \neq \Delta \,\}. \tag{4.27}$$

All Γ in (4.27) may be supposed to be essential (otherwise $C_\Gamma = \emptyset$). Therefore they are invariant, i.e. they correspond to some coordinate subalgebras. Being proper these subalgebras are grounded hence, they are contained in $\mathcal{A}_1$. Hence, $\Gamma \subset \Gamma_1$ for all essential Γ in (4.27). Respectively, $\overline{C_\Gamma} \subset \overline{C_{\Gamma_1}}$ and we conclude that $\partial C_\Delta \subset \overline{C_{\Gamma_1}}$. Jointly with (4.26) this results in the equality $\partial C_\Delta = \overline{C_{\Gamma_1}}$. But this contradicts a well known topological fact: the boundary of any cell (except for 0-dimensional one) is not contractible. □

Corollary 4.14. *Let* $\mathcal{A}$ *be not constant and nondegenerate. If every proper coordinate subalgebra of* $\mathcal{A}$ *is grounded then* $\mathcal{A}$ *is also grounded.*

Proof. The nondegeneracy means that $\Delta = \Delta^{n-1}$, i.e. $\mathcal{A}_0 = \mathcal{A}$. □

5 Proof of the Main Theorem

Given an ultranormal stochastic Bernstein algebra $\mathcal{A}$. *We have to prove that* $\mathcal{A}$ *is regular.* As usual, (m,δ) denotes the type of $\mathcal{A}$, $\dim\mathcal{A} = n = m+\delta$.

Above all, let us come back to the projection B which is associated with a basis idempotent, say e_1, via (4.6). Such an idempotent does exist because all constant subalgebras of $\mathcal{A}$ are 1-dimensional by ultranormality. As we know, a constant subalgebra does exist (Remark 4.13), moreover, there exist at least m constant subalgebras, so there are at least m basis idempotents in $\mathcal{A}$. In fact, some m basis idempotents can be obtained by one of them using the projection B. This way also yiels an additional useful information.

Lemma 5.1. *In notation of Lemma 4.7, for every vector* b_i $(1 \le i \le m-1)$ *there exists a unique basis idempotent* e_{j_i}, $j_i > 1$, *such that* $Be_{j_i} = \lambda_i b_i$, $\lambda_i > 0$.

Proof. Since b_i^2 is a nonzero idempotent and $b_i^2 \ge 0$, the subspace $L = \mathrm{Lin}(\mathrm{supp} b_i^2)$ is a subalgebra. Being a coordinate subalgebra of the ultranormal algebra $\mathcal{A}$, L is normal. If it is constant then $\dim L = 1$ i.e. $b_i^2 = e_j$ where e_j is a basis vector (recall that $s(b_i)=1$), in fact, e_j is a basis idempotent. By (4.12) $Be_j = b_i$.

Let L be nonconstant. Then $\mathrm{rk} L \ge 2$ hence, L has at least two of basis idempotents. One of them is not e_1, say it is e_2, so that $e_2 \in \mathrm{supp}(b_i^2)$. This means that $\beta_2 > 0$ in the expansion

$$b_i^2 = \sum_{j=1}^{n} \beta_j e_j.$$

Applying B we get by (4.12)

$$b_i = \sum_{j=2}^{n} \beta_j B e_j.$$

However, b_i is an extreme point in the symplex Δ_B (see Lemma 4.9). Hence $Be_2 = \lambda b_i$ with $\lambda \ge 0$. Actually $\lambda > 0$ because $Be_2 = 0$ means that $e_1e_2 = e_1$ and then $\mathrm{Lin}\{e_1, e_2\}$ is a subalgebra but not unit and nonconstant which is impossible.

It remains to prove that e_2 is the only idempotent such that $Be_2 = \lambda b_i$, $\lambda > 0$.

The last equality means that $2e_1e_2 = \alpha e_1 + \lambda b_i$, $\alpha = 1-\lambda$. But, according to Theorem 4.6 we have only three cases: 1) $2e_1e_2 = e_1 + e_2$ (u.a.); 2) $2e_1e_2 = \alpha e_1 + \beta e_2 + \gamma e_3$, $\gamma > 0$ (e.M.a.); 3) $2e_1e_2 = e_3 + e_4$ (q.a.). Respectively, $Be_2 = e_2$ or $Be_2 = \beta e_2 + \gamma e_3$ $(\gamma > 0)$, or $Be_2 = e_3 + e_4$, i.e. $\lambda b_i = e_2$ or $\lambda b_i = \beta e_2 + \gamma e_3$ $(\gamma > 0)$, or $\lambda b_i = e_3 + e_4$.

Since $s(b_i) = 1$ we get such three cases: 1) $b_i = e_2$; 2) $b_i = \varepsilon e_2 + \omega e_3$ $(\omega > 0)$; 3) $b_i = \frac{1}{2}(e_3 + e_4)$. In case 1) b_i coincides with the idempotent e_2. In case 2) e_3 is the only nonidempotent in $\mathrm{supp} b_i$ and e_2 is the only idempotent in $\mathrm{supp} e_3^2$ different from e_1 (see (3.11)). Finally, in case 3) $\mathrm{supp} b_i$ consists of two idempotents, e_3 and e_4 and $e_2 = 4b_i^2 - 2b_i - e_1$. $\square$

It is convenient to denote the algebras in case 1), 2) and 3) by $\{e_1, e_2 \mid \emptyset\}$, $\{e_1, e_2 \mid e_3\}$ and $\{e_1, e_2 \mid e_3, e_4\}$ respectively.

Let the number of idempotents in the canonical basis is ρ, so we can assume that they are e_1, ..., e_ρ. We already know that $\rho \geq m$ (following Lemma 5.1 or the previous topological argumentation). Now we even get

Corollary 5.2. rk $\{ Be_i \}_2^\rho = m - 1$.

Proof. As we know $\{b_i\}_1^{m-1}$ is a basis in ImB. On the other hand, $\{b_i\}_1^{m-1} \subset \text{Lin}\{Be_i\}_2^\rho$ by Lemma 5.1. □

We continue the proof of the Main Theorem in the frameworks of the following alternative: *all basis vectors* e_i $(1 \leq i \leq n)$ *are idempotents, i.e.* $\rho = n$, *or not, i.e.* $\rho < n$. Let us say that the first possibility is the *pure case* and the second one is the *mixed case.*

A). The pure case $(\rho = n)$. Take a pair $\{ e_i, e_k \}$ of basis vectors, $i \neq k$. Its offspring subalgebra is normal as every coordinate subalgebra of $\mathcal{A}$. According to Theorem 4.11 this is either $\{ e_i, e_k \mid \emptyset \}$ or $\{ e_i, e_k \mid e_g, e_h \}$. Using the multiplication R introduced by (2.17) we have $R(e_i, e_k) = 0$ or $R(e_i, e_k) \neq 0$ respectively and in the second case $R(e_i, e_g) = 0$, $R(e_i, e_h) = 0$ and similarly for e_k.

Lemma 5.3 *If* $R(e_i, e_k) \neq 0$ *then there is no* e_j *such that* $R(e_i, e_j) = 0$ *and* $R(e_k, e_j) = 0$ *except for* $e_j = e_g$ *and* $e_j = e_h$.

Proof. The subspace $L = \text{Lin}\{ e_j, e_i, e_k, e_g, e_h \}$ is the offspring subalgebra of the family $\{ e_j, e_i, e_k \}$. The algebra L is nuclear as the linear span of a set of idempotents. By Corollary 2.12 L is regular. Moreover, L is normal. By Theorem 3.1 L must be an e.q.a. Indeed, L is not an u.a. since L contains a q.a. and L is not an e.M.a. since all vectors from the canonical basis of L are idempotents. Under $4 \leq \text{dim}L \leq 5$, actually $\text{dim}L = 4$ because no prime number can be dimension of an e.q.a. Since $e_j \neq e_i$ and $e_j \neq e_k$, we conclude that $e_j = e_g$ or $e_j = e_h$. □

Let us write $e_i R_0 e_k$ in the case $R(e_i, e_k) = 0$, so that R_0 is a binary relation on the set $\{ e_j \}_1^n$. Obviosly, it is reflexive and symmetric. For any e_j we define its pool

$$P(e_j) = \{ e_k : e_j R_0 e_k \} = \{ e_k : R(e_j, e_k) = 0 \} \tag{5.1}$$

We also consider the *punctured pool*

$$P^*(e_j) = P(e_j) \setminus \{ e_j \}.$$

Lemma 5.4. *The equality*

$$\text{card}P^*(e_j) = m - 1 \quad (1 \leq j \leq n) \tag{5.3}$$

holds.

Proof. The projection B_j associated with e_j $(B_1 = B$ in this notation) acts as follows:

$$B_j e_k = e_k \quad (e_k \in P^*(e_j)) \tag{5.4}$$

and

$$B_j e_k = e_g + e_h \tag{5.5}$$

if $e_k \notin P^*(e_j)$ and $\{ e_k, e_j \mid e_g, e_h \}$ is the corresponding q.a. Since e_g and e_h belong to $P^*(e_j)$, it follows from (5.4) and (5.5) that $\mathrm{rk}B_j = \mathrm{card}P^*(e_j)$. On the other hand, $\mathrm{rk}B_j = m - 1$. □

Corollary 5.5. $\mathrm{card}P^*(e_j)$ *is independent of* j.

Coming back to the binary relation R_0 we prove

Lemma 5.6. *The restriction* $R_0 \mid P^*(e_j)$ *is an equivalence relation.*

Proof. We only need to check that R_0 is transitive on $P^*(e_j)$. Let $\{ e_i, e_k, e_l \}$ be a triple from $P^*(e_j)$ such that $R(e_i, e_l) = 0$ and $R(e_k, e_l) = 0$ but $R(e_i, e_k) \neq 0$. Since $R(e_i, e_j) = 0$ and $R(e_k, e_j) = 0$ as well, Lemma 5.3 yields the q.a. $\{ e_i, e_k \mid e_j, e_l \}$ which contradicts $R(e_j, e_l) = 0$. □

Obviously, all classes of this equivalence relation are u.a.

From now on we *assume that* $\mathcal{A}$ *is not unit.* (Otherwise, $\mathcal{A}$ is regular a fortiori.)

Lemma 5.7. *There are exactly two classes of the relation* $R_0 \mid P^*(e_j)$, $1 \leq j \leq n$.

(By the way, we see that R_0 is not transitive on the whole pool $P(e_j)$).

Proof. Let $e_l \notin P(e_j)$, so that $R(e_j, e_l) \neq 0$. Then we have $\{ e_j, e_l \mid e_i, e_k \}$ where e_i and e_k belong to $P^*(e_j)$ and $R(e_i, e_k) \neq 0$, so that e_i and e_k are not equivalent. In such a way either there are at least two classes in $P^*(e_j)$ or $P(e_j) = \mathcal{A}$ and $P^*(e_j)$ is an entire class. But in the last case $P^*(e_j)$ an u.a. and then $\mathcal{A}$ is so as.

Suppose that there are more than two classes in $P^*(e_j)$. Then there is a triple $\{ e_i, e_k, e_l \} \subset P^*(e_j)$ such that $R(e_i, e_k) \neq 0$, $R(e_i, e_l) \neq 0$ and $R(e_k, e_l) \neq 0$. By Lemma 5.3 there are three q.a., namely,

$$\{ e_i, e_k \mid e_j, e_p \}, \quad \{ e_i, e_l \mid e_j, e_q \}, \quad \{ e_k, e_l \mid e_j, e_r \}. \tag{5.6}$$

In (5.6) the seven involved vectors are pairwise distinct. For example, $e_p \neq e_q$ since $2e_pe_j = e_i + e_k$ but $2e_qe_j = e_i + e_l$. Also $e_p \neq e_l$ since $R(e_k, e_l) \neq 0$ but $R(e_k, e_p) = 0$. In situation (5.6) the punctured pool $P^*(e_j)$ is $\{ e_i, e_k, e_l \}$. According to Lemma 5.4 $m = 4$, so card $P^*(e_i) = 3$ by Corollary 5.5. However, $P^*(e_i) \supset \{ e_j, e_p, e_q \}$ hence,

$$P^*(e_i) = \{ e_j, e_p, e_q \}. \tag{5.7}$$

Since $e_r \notin P^*(e_i)$, i.e. $R(e_i, e_r) \neq 0$, we get one more q.a., say $\{ e_i, e_r \mid e_g, e_h \}$, where

$$\{ e_g, e_h \} \subset P^*(e_i) \bigcap P^*(e_r). \tag{5.8}$$

This is a contradiction because $P^*(e_r) \supset \{ e_k, e_l \}$ (see (5.6)) and card $P^*(e_r) = 3$ so the intersection (5.8) can not contain more than one element. □

Let us denote the classes of $R_0 \mid P^*(e_j)$ by C_j and $\overline{C_j}$.

Lemma 5.8. *There exists a bijective mapping from the complement of the pool* $P(e_j)$ *onto the Cartesian product* $C_j \times \overline{C_j}$, $1 \leq j \leq n$.

Proof. For any $e_k \notin P(e_j)$ we have the q.a. $\{ e_j, e_k \mid e_{g_k}, e_{h_k} \}$ where e_{g_k} and e_{h_k} are both from the punctured pool $P^*(e_j)$ and $R(e_{g_k}, e_{h_k}) \neq 0$ which means that e_{g_k} and e_{h_k}

are from different classes, say $e_{g_k} \in C_j$ and $e_{h_k} \in \overline{C_j}$. The mapping defined in such a way is injective since $e_k = 2e_{g_k}e_{h_k} - e_j$. It is also surjective. Indeed, if $e_g \in C_j$ and $e_h \in \overline{C_j}$ then $R(e_g, e_h) \neq 0$, so we have the q.a. $\{ e_g, e_h \mid e_j, e_k \}$ where e_j appears by Lemma 5.3 and then $e_k \notin P(e_j)$. This means that $e_g = e_{g_k}$ and $e_h = e_{h_k}$. □

Corollary 5.9. *Let* $m_j = \text{card}C_j + 1$ and $\overline{m_j} = \text{card}\overline{C_j} + 1$. *Then*

$$m_j + \overline{m_j} = m + 1, \quad m_j\overline{m_j} = n \quad (1 \le j \le n). \tag{5.9}$$

Proof. Since $C_j \bigcup \overline{C_j} = P^*(e_j)$ and $C_j \bigcap \overline{C_j} = \emptyset$ we obtain the first of equalities (5.9) from (5.3). On the other hand, it follows from Lemma 5.8 that

$$n = \text{card}(C_j \times \overline{C_j}) + \text{card}P(e_j) = (m_j - 1)(\overline{m_j} - 1) + m = m_j\overline{m_j}. \qquad \square$$

Corollary 5.10. $\delta = (m_j - 1)(\overline{m_j} - 1)$.

Since $\mathcal{A}$ is not unit, we have $\delta > 0$. Therefore $m_j \ge 2$ and $\overline{m_j} \ge 2$.

Corollary 5.11. *The numbers* m_j and $\overline{m_j}$ *do not depend on* j.

Therefore one can set $m_j = \nu$, $\overline{m_j} = \overline{\nu}$ for all of j, $1 \le j \le n$.

Now we enumerate the basis $\{ e_j \}$ in a new way:

$$e_1 \equiv e_{11}, \quad C_1 = \{ e_{i1} \}_{i=2}^{\nu}, \quad \overline{C_1} = \{ e_{1k} \}_{k=2}^{\overline{\nu}}. \tag{5.10}$$

By Lemma 5.8 the ordered pairs (e_{i1}, e_{1k}) are in $1-1$ correspondence with the complement of the pool $P(e_1)$. Hence, this complement can be listed as $\{ e_{ik} : 2 \le i \le \nu, \ 2 \le k \le \overline{\nu} \}$ The correspondence is established by the q.a. $\{ e_{11}, e_{ik} \mid e_{i1}, e_{1k} \}$. The whole basis $\{ e_j \}_1^n$ can be written in the matrix form, $E = (e_{ik}) \quad (1 \le i \le \nu, \ \ 1 \le k \le \overline{\nu})$.

Lemma 5.12. *For any element* e_{ik} *its pool* $P(e_{ik})$ *is the union of the i-th row and the k-th column of the matrix* E. *Being punctured at* e_{ik} *these lines are the equivalence classes of the punctured pool* $P^*(e_{ik})$.

Proof. Let us denote the punctured k-th column and i-th row by C_{ik} and $\overline{C_{ik}}$ respectively. In particular, $C_{11} = C_1$ and $\overline{C_{11}} = \overline{C_1}$ by (5.10), so the lemma is true for $P(e_{11})$. Now we consider $P(e_{i1})$, $i > 1$.

The 1-st column is $C_1 \bigcup \{ e_{11} \}$ therefore $R(e_{i1}, e_{g1}) = 0$ for all of g, $1 \le g \le \nu$. Thus, C_{i1} is contained in a class of $P^*(e_{i1})$. Note that $e_{11} \in C_{i1}$. As the q.a. $\{ e_{11}, e_{ik} \mid e_{i1}, e_{1k} \}$ $(i, k \neq 1)$ shows, $R(e_{i1}, e_{ik}) = 0$ for all of k, $1 \le k \le \overline{\nu}$. Thus, $\overline{C_{i1}}$ is contained in $P^*(e_{i1})$. However, $R(e_{11}, e_{ik}) \neq 0$ for $k \neq 1$ hence, $\overline{C_{i1}}$ lies in another class. In fact, C_{i1} and $\overline{C_{i1}}$ must coincide with the corresponding classes because of the same (up to transposition, a priori) cardinalities. The lemma is proved for $P(e_{i1})$. Quite similarly, this is true for $P(e_{1k})$. But then $R(e_{ik}, e_{jk}) = 0$ for all j, $1 \le j \le \nu$ and $R(e_{ik}, e_{il}) = 0$ for all l, $1 \le l \le \overline{\nu}$ which yields the first part of the lemma for $P(e_{ik})$. Moreover, the second part is also true. Indeed, $C_{ik} = (C_{ik} \bigcup \{ e_{1k} \}) \setminus \{ e_{ik} \}$, so C_{ik} is contained in a class of $P^*(e_{ik})$ and, similarly, $\overline{C_{ik}}$ has such a property. It remains to refer to their cardinalities again. □

Now we able to obtain the multiplication table of the algebra $\mathcal{A}$.

First of all we get

$$e_{ik}e_{jk} \;=\; \frac{e_{ik}+e_{jk}}{2}, \quad e_{ik}e_{il} \;=\; \frac{e_{ik}+e_{il}}{2} \tag{5.11}$$

from Lemma 5.12. Now if $i \neq j$ and $k \neq l$ we have $R(e_{ik}, e_{jl}) \neq 0$ from the same lemma which says that $e_{jl} \notin P(e_{ik})$ in this case. Then there is a q.a. $\{\, e_{ik}, e_{jl} \mid e_g, e_h \,\}$ with

$$\{\, e_g, e_h \,\} \subset P(e_{ik}) \bigcap P(e_{jl}) \;=\; \{\, e_{il}, e_{kj} \,\}.$$

This means that

$$e_{ik}e_{jl} \;=\; \frac{e_{il}+e_{kj}}{2} \quad (i \neq j,\ k \neq l). \qquad (5.12).$$

The algebra $\mathcal{A}$ turns out to be an e.q.a. Hence $\mathcal{A}$ is regular. The Main Theorem is proved in the case under consideration.

B) **The mixed case** $(\rho < n)$. In this case we can argue by induction on n. Recall that the Main Theorem is true for $n \leq 5$ by Theorem 2.14.

Given $n \geq 6$, we suppose that the theorem is true in all dimensions less than n, in particular, for all proper coordinate subalgebras. All of them are ultranormal together with $\mathcal{A}$. Therefore they are regular and then each one is either u.a. or e.M.a., or e.q.a. (Theorem 3.1). As a result, all proper coordinate subalgebras are grounded. By Corollary 4.14 $\mathcal{A}$ is also grounded, i.e. $\mathcal{A}$ is the offspring subalgebra of the set of its basis idempotents, say $\{\, e_i \,\}_1^\rho$.

Lemma 5.13 *For every basis vector* e_j *with* $j > \rho$ *there exists a unique pair* $\{e_{i_j}, e_{k_j}\}$ *with* $1 \leq i_j < k_j \leq \rho$ *such that* $\mathrm{Lin}\{\, e_{i_j}, e_{k_j}, e_j \,\}$ *is an e.M.a.,* $\{\, e_{i_j}, e_{k_j} \mid e_j \,\}$.

Proof. Since $\mathcal{A}$ is the offspring subalgebra of the family of idempotents $e_1, ..., e_\rho$, there exists a pair e_{i_j}, e_{k_j} $(1 \leq i_j < k_j \leq \rho)$ such that $e_j \in \mathrm{supp}(e_{i_j}e_{k_j})$. The offspring subalgebra of this pair is neither unit (2-dimensional) nor quadrille (because e_j is not an idempotent). By Theorem 4.11 it is an e.M.a. By (3.11)

$$\{\, e_{i_j}, e_{k_j} \,\} \;=\; \mathrm{supp}(e_j^2) \setminus \{\, e_j \,\},$$

therefore the pair $\{\, e_{i_j}, e_{k_j} \,\}$ is unique. □

We will say that e_j is the *offspring* of the *marked* pair $\{\, e_{i_j}, e_{k_j} \,\}$. We also set $e_{ik} = e_ie_k$ $(1 \leq i \leq k \leq n)$, so that $e_{ii} = e_i$, $1 \leq i \leq \rho$, and $\mathrm{Lin}\{e_{i_j}, e_{k_j}, e_{i_jk_j}\} = \mathrm{Lin}\{e_{i_j}, e_{k_j}, e_j\}$.

Corollary 5.14. *The algebra* $\mathcal{A}$ *is nuclear.*

For any nonmarked pair $\{\, e_i, e_k \,\}$ of the basis idempotents the offspring subalgebra is either u.a. or q.a.

Lemma 5.15. *For every basis idempotent* e_i *there are exactly* $\rho - m$ *idempotents* e_k $(k \neq i)$ *such that the offspring subalgebra of the pair* $\{\, e_i, e_k \,\}$ *is q.a.*

Thus, this number is the same for all e_i.

Proof. Let $i = 1$ for definiteness and let the offspring subalgebra L_k of the pair $\{\, e_1, e_k \,\}$ be an e.M.a., i.e.

$$2e_1e_k \;=\; \alpha_ke_1 \,+\, \beta_ke_k \,+\, \gamma_ke_{\rho+k}, \quad \gamma_k > 0.$$

which means that $Be_k = \beta_k e_k + \gamma_k e_{\rho+k}$ where B is the usual projection associated wich e_1. If now L_k is the u.a. then $Be_k = e_k$. If, finally, L_k is a q.a. then $Be_k = e_j + e_l$ where e_j and e_l correspond to the unit L_j and L_l. We see that rk $\{ Be_k \}_2^\rho$ equals the total number of e.M.a. and u.a. On the other hand, this rank equals $m-1$ by Corollary 5.2. Therefore the number of q.a. coincides with $\rho - m$. □

Lemma 5.15 is a quite preliminary fact because of

Lemma 5.16. *There is no q.a. among the coordinate subalgebras of* $\mathcal{A}$.

Proof. Let we have the e.M.a. $\{ e_1, e_2 \mid e_{\rho+1} \}$ jointly with a q.a. By Lemma 5.15 e_1 is involved in a q.a., say, $\{ e_1, e_3 \mid e_4, e_5 \}$. The offspring set of the triple $\{ e_1, e_2, e_3 \}$ is $\{ e_j \}_1^5 \bigcup \{ e_{\rho+1} \} \bigcup \operatorname{supp}(e_2 e_3)$. The corresponding offspring subalgebra is $\mathcal{A}$ because of the regularity of all proper coordinate subalgebras.

Let the offspring subalgebra of the pair $\{ e_2, e_3 \}$ be u.a. or e.M.a., so that there is no new idempotents in $\operatorname{supp}(e_2 e_3)$. Then $\rho = 5$, $e_{\rho+1} = e_6$ and $\{ e_1, e_3 \mid e_4, e_5 \}$ is the unique q.a. containing e_1. By Lemma 5.15 $\rho - m = 1$ (so that $m = 4$) and then e_2 must be also involved in a q.a. $\{ e_2, e_i \mid e_k, e_j \}$ where $i = 4$ or $i = 5$ since the offspring subalgebras of $\{ e_1, e_2 \}$ and $\{ e_2, e_3 \}$ are not q.a. We get at least two q.a. containing e_4 (or e_5) in contradiction to Lemma 5.15.

Suppose that the offspring subalgebra of $\{ e_2, e_3 \}$ is a q.a., say, $\{ e_2, e_3 \mid e_i, e_k \}$, so that $\mathcal{A} = \operatorname{Lin}(\{ e_j \}_1^5 \bigcup \{ e_i, e_k, e_{\rho+1} \})$. Now e_3 is involved in two q.a. but for e_1 there is no more q.a. than $\{ e_1, e_3 \mid e_4, e_5 \}$. Indeed, such a q.a. must be $\{ e_1, e_i \mid *, * \}$ (up to transposition $e_i \leftrightarrow e_k$). Both of the omitted members must satisfy $R(e_1, *) = 0$ and $R(e_i, *) = 0$. They must be e_4 or e_5 but $\{ e_1, e_i \mid e_4, e_5 \}$ contradicts the pre-existence of $\{ e_1, e_3 \mid e_4, e_5 \}$. □

Lemmas 5.15 and 5.16 immediately imply

Corollary 5.17. $\rho = m$.

Thus, the set of the basis idempotents is $\{ e_i \}_1^m$.

Lemma 5.18. *If* $m \le 4$ *then* $\mathcal{A}$ *is regular.*

By Corollary 5.14 and Theorem 2.11 $\mathcal{A}$ is regular for types (m, δ) such that $m \le 3$ or $\delta \le 1$, or $\delta \ge \frac{1}{2}(m-1)(m-2)+1$. Thus, we can assume $m \ge 4$ or $2 \le \delta \le \frac{1}{2}(m-1)(m-2)$, so that

$$6 \le n \le \frac{m(m-1)}{2} + 1. \tag{5.13}$$

Proof. By (5.13) $n = 6$ or $n = 7$. Let $n = 6$, so that type of $\mathcal{A}$ is (4,2). By Corollary 5.17 the basis idempotents are e_1, e_2, e_3, e_4 and there are exactly two marked pairs, $\{ e_{i_1}, e_{k_1} \}$ and $\{ e_{i_2}, e_{i_{k_2}} \}$; e_5 and e_6 are respectively their offsprings. Suppose that those pairs do intersect, say, they are $\{ e_1, e_2 \}$ and $\{ e_1, e_3 \}$. By Lemma 5.16 the offspring subalgebras of all nonmarked pairs $\{ e_i, e_k \}$ are u.a. The offspring subalgebra of the triple $\{ e_1, e_2, e_3 \}$ is

$$L = \operatorname{Lin}\{ e_1, e_2, e_3, e_5, e_6 \} = \operatorname{Lin}\{ e_1, e_2, e_3, e_{12}, e_{13} \}$$

It is regular since $\dim L = 5$. Now $\mathcal{A} = L[e_4]$ is regular by Proposition 2.10, part 1.

If $\{ e_{i_1}, e_{k_1} \} \bigcap \{ e_{i_2}, e_{k_2} \} = \emptyset$ one can assume that those pairs are $\{ e_1, e_2 \}$ and $\{ e_3, e_4 \}$. As before, we have the u.a. $\{ e_i, e_k \mid \emptyset \}$ with $i = 1, 2$ and $k = 3, 4$. The offspring subalgebra

$L = \mathrm{Lin}\{ e_1, e_2, e_5 \}$ is regular being e.M.a., and $\mathcal{A} = L[e_3, e_4]$ is regular by Proposition 2.10, part 2. (Condition (2.35) in the form $R(e_1, e_2)e_3 = 0$ is fulfilled because the offspring subalgebra $\mathrm{Lin}\{ e_1, e_2, e_3, e_5 \}$ is regular being of dimension 4.)

On this stage the Main Theorem is proved for $n \le 6$.

Henceforth $n = 7$, so that $\mathcal{A}$ is of type (4,3). Then there are exactly three marked pairs, $\{ e_{i_1}, e_{k_1} \}$ $\{ e_{i_2}, e_{k_2} \}$, $\{ e_{i_3}, e_{k_3} \}$; their offsprings are e_5, e_6, e_7 respectively. We have the u.a. $\{ e_i, e_k \mid \emptyset \}$ for all nonmarked pairs again.

Suppose the intersection of all marked pairs is not empty. Then they are $\{e_1, e_2\}$, $\{e_1, e_3\}$ and $\{e_1, e_4\}$ (up to enumeration). Every triple $\{e_1, e_{ik}, e_{i'k'}\}$ belongs to an offspring subalgebra of dimension ≤ 5 which is regular a fortiori. Hence, $R(e_{ik}, e_{i'k'})e_1 = 0$ for all i, k, i', k' (some of them may coincide). By Corollary 2.9 $\mathcal{A}$ is regular.

Let the intersection of all marked pairs is empty. However, since all e_{i_j} and e_{k_j} are from $\{ e_l \}_1^4$, there exists a couple of the pairs with nonempty intersection, say $\{ e_1, e_2 \}$ and $\{ e_1, e_3 \}$. Suppose that the third pair does not include e_4. Then this is $\{ e_2, e_3 \}$. The offspring subalgebra $M = \mathrm{Lin}\{ e_1, e_2, e_3, e_5, e_6, e_7 \}$ is regular because of $\dim L = 6$. Since $\mathcal{A} = \mathbf{M}[e_4]$, this is regular by Proposition 2.10, part 1.

Suppose that e_4 is included in the third pair. Then those pairs are $\{ e_1, e_2 \}$, $\{ e_1, e_3 \}$ and $\{ e_3, e_4 \}$ (up to the transposition $e_2 \leftrightarrow e_3$). We are going to use Corollary 2.9 again. In this context we only must consider those triples $\{ e_1, e_{ik}, e_{i'k'} \}$ wich are not located in a proper offspring subalgebra because all of these subalgebras are of dimension ≤ 6. Such "badly located" triples appear iff e_2, e_3 and e_4 are all among $e_i, e_k, e_{i'}, e_{k'}$. Because of the symmetry between e_{ik} and $e_{i'k'}$ one can assume that $\{ i, k \}$ is a lexicographic predecessor of $\{ i', k' \}$. The complete list of couples $\{ e_{ik}, e_{i'k'} \}$ under our conditions is the following:

$$\{e_{12}, e_{34}\},\ \{e_{13}, e_{24}\},\ \{e_{14}, e_{23}\},\ \{e_{22}, e_{34}\},\ \{e_{23}, e_{24}\},\ \{e_{23}, e_{34}\},\ \{e_{24}, e_{33}\},\ \{e_{24}, e_{34}\}.$$

Here $e_{jl} = \frac{1}{2}(e_j + e_l)$ (i.e. $R(e_j, e_l) = 0$) except for the marked ones, i.e. e_{12}, e_{13} and e_{34}. In particular, $e_{14} = \frac{1}{2}(e_1 + e_2)$ and $e_{23} = \frac{1}{2}(e_2 + e_3)$. Hence,

$$R(e_{14}, e_{23}) = \frac{1}{4}\{ R(e_1, e_2) + R(e_1, e_3) + R(e_3, e_4)\} \tag{5.14}$$

because of $R(e_2, e_4) = 0$. Since every of triples $\{ e_1, e_1, e_2 \}$, $\{ e_1, e_1, e_3 \}$ and $\{ e_1, e_3, e_4 \}$ is well located, (5.14) yields

$$R(e_{14}, e_{23})e_1 = 0. \tag{5.15}$$

Similarly, $R(e_{23}, e_{24}) = \frac{1}{4}R(e_3, e_4) = R(e_{24}, e_{33})$ hence,

$$R(e_{23}, e_{24})e_1 = 0, \quad R(e_{24}, e_{33})e_1 = 0. \tag{5.16}$$

Now $R(e_{22}, e_{34}) = R(e_2, e_{34}) = -\frac{1}{2}R(e_3, e_4)$ by (2.25). Hence,

$$R(e_{22}, e_{34})e_1 = 0. \tag{5.17}$$

Since $R(e_{23}, e_{34}) = \frac{1}{2}\{R(e_2, e_{34}) + R(e_3, e_{34})\}$, we obtain from (5.17)

$$R(e_{23}, e_{34})e_1 = \frac{1}{2}R(e_3, e_{34})e_1 = 0 \tag{5.18}$$

because the triple $\{ e_1, e_3, e_{34} \}$ is well located. Similarly,

$$R(e_{24}, e_{34})e_1 = 0. \tag{5.19}$$

Since $R(e_{13}, e_{24}) = \frac{1}{2}\{R(e_{13}, e_2) + R(e_{13}, e_4)\}$ and the triples $\{ e_1, e_{13}, e_2 \}$ and $\{ e_1, e_{13}, e_4 \}$ are well located, we obtain

$$R(e_{13}, e_{24})e_1 = 0 \tag{5.20}$$

Finally, $R(e_{12}, e_{34}) = -\{R(e_{13}, e_{24}) + R(e_{14}, e_{23})\}$ by (2.20) and then

$$R(e_{12}, e_{34})e_1 = 0 \tag{5.21}$$

by (5.15) and (5.20). □

Now we are able to finish the proof of the Main Theorem. Actually, we are going to prove that $\mathcal{A}$ is an e.M.a., which means that (3.1)-(3.5) is valid for the set $\{ e_i \}_1^m$ of the basis idempotents. We already have (3.1) with marked pairs $\{ e_{i_j}, e_{k_j} \}$ coming from Lemma 5.13 (where $\rho = m$ by Corollary 5.17). For nonmarked pairs $\{ e_i, e_k \}$ we have (3.2) because of Theorem 4.11 and Lemma 5.16.

In order to get (3.3) and (3.4) we consider the offspring subalgebra L_{ij} of the set $\{ e_i, e_{i_j}, e_{k_j} \}$. Since $\dim L \leq 6$ (the equality is attained if both of pairs $\{ e_i, e_{i_j} \}$ and $\{ e_i, e_{k_j} \}$ are marked), L is regular. Therefore we have (3.3) and (3.4), moreover, c_j is independent of i by virtue of (3.4) and also $\overline{c_j} = 1 - c_j$. (Recall that $\gamma_j > 0$.)

It remains to get (3.5). For this goal we consider the offspring subalgebra M_{jl} of the set $\{ e_{i_j}, e_{k_j}, e_{i_l}, e_{k_l} \}$. The offspring subalgebras of all six pairs $\{ e_{i_j}, e_{k_j} \}$, $\{ e_{i_j}, e_{i_l} \}$, ..., $\{ e_{i_l}, e_{k_l} \}$ are e.M.a. or u.a. Therefore the idempotents in the canonical basis of M_{jl} are only $e_{i_j}, e_{k_j}, e_{i_l}$ and e_{k_l} i.e. $\rho \leq 4$ for M_{jl}. By Corollary 5.17 $m \leq 4$. By Lemma 5.18 M_{jl} is regular. Hence, (3.5) is also valid.

References

1. S.N. Bernstein. Mathematical problems in modern biology. *Science in the Ukraine* **1** (1922), 14-19 (in Russian).

2. S.N. Bernstein. Demonstration mathematique de la loi d'heredite de Mendel. *C.r.Acad. Sci. Paris* **177** (1923), 528-531.

3. S.N. Bernstein, Principe de stationarite et generalisation de la loi de Mendel. *C.r.Acad. Sci. Paris* **177** (1923), 581-584.

4. S.N. Bernstein. Solution of a mathematical problem related to the theory of inheritance. *Uch. Zap. n.-i. kaf. Ukrainy*, **1** (1924), 83-115 (in Russian).

5. G.H. Hardy. Mendelian proportions in a mixed population. *Science* **28**, 706 (1908), 49-50.

6. P. Holgate. Genetic algebras satisfying Bernstein stationarity principle. *J.London Math. Soc.* **9** (1975), 613-624.

7. Yu.I. Lyubich. Basic concepts and theorems of evolutionary genetics for free populations. *Russian Math. Surveys* **26**, 5 (1971), 51-123.

8. Yu.I. Lyubich. On the mathematical theory of heredity. *Soviet Math. Dokl.* **14**, 2 (1973), 579-581.

9. Yu.I. Lyubich. About a theorem of S.N. Bernstein. *Siberian Math. J.* **14**, 3 (1973), 474.

10. Yu.I. Lyubich. A class of quadratic maps. *Teor. Func.,Func. Anal. and Appl.* **21** (1974), 36-42 (in Russian).

11. Yu.I. Lyubich. Two-level Bernstein populasions. *Math. of the USSR Sbornik* **24**, 1 (1974), 593-615.

12. Yu.I. Lyubich. Proper Bernstein populations. *Probl. Peredachi Inform.* **13**, 3 (1977), 91-100. Transl. in *Probl. Inform.Transmiss.* (1978), Jan., 228-235.

13. Yu.I. Lyubich. Bernstein algebras. *Uspekhi Mat. Nauk* **32**, 6 (1977), 261-263 (in Russian).

14. Yu.I. Lyubich. Stochastic algebras and their applications to mathematical genetics, In *Mathematical methods in biology.* Naukova Dumka, Kiev, (1977), 119-131 (in Russian).

15. Yu.I. Lyubich. Algebraic proof of S.N.Bernstein's theorem on two pure types. *Vestnik Kharkov Univ., Ser. Mat. and Mech.* **177**, 44 (1979), 86-94 (in Russian).

16. Yu.I. Lyubich. A topological approach to a problem in mathematical genetics. *Russian Math. Surveys* **34** (1979), 6, 60-66.

17. Yu.I. Lyubich. *Mathematical Structures in Population Genetics.* Naukova Dumka, Kiev, 1983. English transl. in Springer, 1992.

18. W. Weinbreg. Uber den Nachweis der Verebung beim Menschen. *Jahresber. Ver. vaterl. Naturk. in Wurtemb.* **64** (1908), 368-372.