You first plow in the dynamical plane and
then harvest in the parameter plane.
Adrien Douady

DYNAMICS OF QUADRATIC POLYNOMIALS, III PARAPUZZLE AND SBR MEASURES.

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1. Introduction

This is a continuation of notes on dynamics of quadratic polynomials. In this part we transfer the geometric result of [L3] to the parameter plane. To any parameter value $c \in M$ in the Mandelbrot set (which lies outside of the main cardioid and little Mandelbrot sets attached to it) we associate a "principal nest of parapuzzle pieces" $D^0(c) \supset D^1(c) \supset \dots$ corresponding to the generalized renormalization type of c. Then we prove:

Theorem A. *The moduli of the parameter annuli* $\mod (D^l(c) \setminus D^{l+1}(c))$ *grow at least linearly (see §4 for a more precise formulation).*

This result was announced at the Colloquium in honor of Adrien Douady (July 1995), and in the survey [L5], Theorem 4.8. The main motivation for this work was to prove the following:

Theorem B (joint with Martens and Nowicki). *Lebesgue almost every real quadratic* $P_c : z \mapsto z^2 + c$ *which is non-hyperbolic and at most finitely renormalizable has a finite absolutely continuous invariant measure.*

More specifically, Martens and Nowicki [MN] have given a geometric criterion for existence of a finite absolutely continuous invariant measure (acim) in terms of the "scaling factors". Together with the result of [L2] on the exponential decay of the scaling factors in the quasi-quadratic case this yields existence of the acim once "the principal nest is eventually free from the central cascades". Theorem A above implies that this condition is satisfied for almost all real quadratics which are non-hyperbolic and at most finitely renormalizable (see Theorem 5.1). Note that Theorem A also implies that this condition is satisfied on a set of positive measure, which yields a new proof of Jacobson's Theorem [J] (see also Benedicks & Carleson [BC]).

A measure μ will be called SBR (Sinai-Bowen-Ruelle) if

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \to \mu \tag{1.1}$$

for a set of x of positive *Lebesgue* measure. It is known that if an SBR measure exists for a real quadratic map $f = P_c$, $c \in [-2, 1/4]$, on its invariant interval I_c, then it is unique and

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(1.1) is satisfied for Lebesgue almost all $x \in I_c$ (see Introduction of [MN] for a more detailed discussion). Theorem B yields

Corollary. *For almost all $c \in [-2, 1/4]$, the quadratic polynomial P_c has a unique SBR measure on its invariant interval I_c.*

Let us now take a closer look at Theorem A. It nicely fits to the general philosophy of correspondence between the dynamical and parameter plane. This philosophy was introduced to holomorphic dynamics by Douady and Hubbard [DH1]. Since then, there have been many beautiful results in this spirit, see Tan Lei [TL], Rees [R], Shishikura [Sh], Branner-Hubbard [BH], Yoccoz (see [H]).

In the last work, special tilings into "parapuzzle pieces" of the parameter plane are introduced. Its main geometric result is that the tiles around at most finitely renormalizable points shrink. It was done by transferring, in an ingenious way, the corresponding dynamical information into the parameter plane.

In [L3] we studied the rate at which the dynamical tiles shrink. Our main geometric result is that the moduli of the principal nest of dynamical annuli grow at least linearly. Let us note that the way we transfer this result to the parameter plane (Theorem A) is substantially different from that of Yoccoz. Our main tool is provided by holomorphic motions whose transversal quasi-conformality yields comparison between the dynamical and parameter moduli.

The properties of holomorphic motions are discussed in §2. In §3 we describe the principal parameter tilings according to the generalized renormalization types of the maps. In §4 we prove Theorem A. In the last section, §5, we derive the consequence for the real quadratic family (Theorem B).

Let us finally draw the reader's attention to the work of LeRoy Wenstrom [W] which studies in detail parapuzzle geometry near the Fibonacci parameter value.

2. Background

2.1. **Notations and terminology.** $\mathbb{D}_r = \{z : |z| < r\}$, $\mathbb{D} \equiv \mathbb{D}_1$, $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{A}(r, R) = \{r < |z| < R\}$. The closed and semi-closed annuli are denoted accordingly: $\mathbb{A}[r, R]$, $\mathbb{A}(r, R]$, $\mathbb{A}[r, R]$.

By a *topological disc* we will mean a simply connected domain $D \subset \mathbb{C}$ whose boundary is a Jordan curve.

Let π_1 and π_2 denote the coordinate projections $\mathbb{C}^2 \to \mathbb{C}$. Given a set $\mathbb{X} \subset \mathbb{C}^2$, we denote by $X_\lambda = \pi_1^{-1}\{\lambda\}$ its vertical cross-section through λ (the "fiber" over λ). Vice versa, given a family of sets $X_\lambda \subset \mathbb{C}$, $\lambda \in D$, we will use the notation: $\mathbb{X} = \cup_{\lambda \in D} X_\lambda = \{(\lambda, z) \in \mathbb{C}^2 : \lambda \in D, z \in X_\lambda\}$.

Let us have a discs fibration $\pi_1 : \mathbb{U} \to D$ over a topological disc $D \subset \mathbb{C}$ (so that the sections U_λ are topological discs, and the closure of U in $D \times \mathbb{C}$ is homeomorphic to $D \times \bar{\mathbb{D}}$ over D). In this situation we call $\mathbb{U}$ an (open) *topological bidisc* over D. We say that this fibration admits an extension to the boundary ∂D if the closure $\bar{U}$ of $\mathbb{U}$ in $\mathbb{C}^2$ is homeomorphic over $\bar{D}$ to $\bar{D} \times \bar{\mathbb{D}}$. The set $\bar{\mathbb{U}}$ is called a (closed) bidisc. We keep the notation $\mathbb{U}$ for the fibration of *open* discs over the closed disc $\bar{D}$ (it will be clear from the context over which set the fibration is considered).

If $U_\lambda \ni 0$, $\lambda \in D$, we denote by $\mathbf{0}$ the zero section of the fibration.

If the fibration π_1 admits an extension over the boundary ∂D, we define the *frame* $\delta\mathbb{U}$ as the topological torus $\cup_{\lambda\in\partial D}\partial U_\lambda$. A section $\Phi : D \to \mathbb{U}$ is called *proper* if it is continuous up to the boundary and $\Phi(\partial D) \subset \delta U$.

We assume that the reader is familiar with the theory of quasi-conformal maps (see e.g., [A]). We will use a common abbreviation K-qc for "K-quasi-conformal". Notation $a_n \asymp b_n$ means, as usual, the the ratio a_n/b_n is positive, and bounded away from 0 and ∞.

2.2. **Holomorphic motions.** Given a domain $D \subset \mathbb{C}$ with a base point $*$ and a set $X_* \subset \mathbb{C}$, a *holomorphic motion* $\mathbf{h}$ of X_* over D is a family of injections $h_\lambda : X_* \to \mathbb{C}$, $\lambda \in D$, such that $h_* = \text{id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_*$. We denote $X_\lambda = h_\lambda X_*$. Let us summarize fundamental properties of holomorphic motions which are usually referred to as the λ-*lemma*. It consists of two parts: extension of the motion and transversal quasi-conformality, which will be stated separately.

The consecutively improving versions of the Extension Lemma appeared in [L1] and [MSS], [ST],[BR],[Sl]. The final result, which will be actually exploited below, is due to Slodkowsky:

Extension Lemma. *A holomorphic motion $h_\lambda : X_* \to X_\lambda$ of a set $X_* \subset \mathbb{C}$ over a topological disc D admits an extension to a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole complex plane over D.*

Quasi-Conformality Lemma [MSS]. *Let $h_\lambda : U_* \to U_\lambda$ be a holomorphic motion of a domain $U_* \subset \mathbb{C}$ over a hyperbolic domain $D \subset \mathbb{C}$. Then the maps h_λ are $K(r)$-quasi-conformal, where r is the hyperbolic distance between $*$ and λ in D.*

A holomorphic motion $h_\lambda : U_* \to U_\lambda$ over D can be viewed as a complex one-dimensional foliation of the domain $\mathbb{U} = \cup_{\lambda\in D} U_\lambda$, whose leaves are graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in U_*$. A *transversal* to the motion is a complex one dimensional submanifold of $\mathbb{C}^2$ which transversally intersects every leaf at one point (so that "transversal" means *global* transversal). Given two transversals X and Y, we thus have a well-defined holonomy map $H : X \to Y$, $H(p) = q$ iff p and q belong to the same leaf.

Corollary 2.1. *Holomorphic motions are locally transversally quasi-conformal. More precisely, for any two transversals X and Y, the holonomy map $H : X \to Y$ is locally quasi-conformal. If $H(p) = q$ then the local dilatation of H near p depends only on the hyperbolic distance between the $\pi_1(p)$ and $\pi_1(q)$ in D.*

Proof. Let $p = (\lambda, \alpha)$, $q = (\mu, \beta)$. By the λ-Lemma, the map $G = h_\mu \circ h_\lambda^{-1} : U_\lambda \to U_\mu$ is quasi-conformal, with dilatation depending only on the hyperbolic distance between λ and μ in D. Hence a little disc $D(\alpha, \epsilon) \subset U_\lambda$ is mapped by G onto an ellipse $Q_\epsilon \subset U_\mu$ with bounded eccentricity about β (where the bound depends only on the hyperbolic distance between α and β).

But the holonomy $U_\lambda \to X$ is asymptotically conformal near p. To see this, let us select a holomorphic coordinates (θ, z) near p in such a way that $p = 0$ and the leaf via p becomes the parameter axis. Let $z = \psi(\theta) = \epsilon + \dots$ parametrizes a nearby leaf of the foliation, while $\theta = g(z) = bz + \dots$ parametrizes the transversal X.

Let us do the rescaling $z = \epsilon\zeta, \theta = \epsilon\nu$. In these new coordinates, the above leaf is parametrized by the function $\Psi(\nu) = \epsilon^{-1}\psi(\epsilon\nu)$, $|\nu| < R$, where R is any fixed parameter. Then $\Psi'(\nu) = \psi'(\epsilon\nu)$ and $\Psi''(\nu) = \epsilon\psi''(\epsilon\nu)$. Since the family of functions $\{\psi(\nu)\}$ is normal, $\Psi''(\nu) = O(\epsilon)$. Moreover, ψ uniformly goes to 0 as $\psi(0) \to 0$. Hence $\Psi'(0) = \psi'(0) \le \delta_0(\epsilon)$, where $\delta_0(\epsilon) \to 0$ as $\epsilon \to 0$. Thus $\Psi'(\nu) = \delta_0(\epsilon) + O(\epsilon) \le \delta(\epsilon) \to 0$ as $\epsilon \to 0$ uniformly for all $|\nu| < R$. It follows that $\Psi(\nu) = 1 + O(\delta(\epsilon)) = 1 + o(1)$ as $\epsilon \to 0$.

On the other hand, the the manifold X is parametrized in the rescaled coordinates by a function $\nu = b\zeta + 0(1)$. Since the transversal intersection persists, X intersects the leaf at the point $(\nu_0, \zeta_0) = (1, b)(1 + o(1))$ (so that R should be selected bigger than b). In the old coordinates the intersection point is $(\theta_0, z_0) = (\epsilon, b\epsilon)(1 + o(1))$.

Thus the holonomy from U_λ to X transforms the disc of radius $|\epsilon|$ to an ellipse with small eccentricity, which means that this holonomy is asymptotically conformal. As the holonomy from $U(\mu)$ to Y is also asymptotically conformal, the conclusion follows. ☐

Remark. The above argument also shows that if the motion is K-qc over the whole domain D (i.e, all maps h_λ are K-qc, $\lambda \in D$) then it is transversally K-qc.

2.3. **Winding number.** Given two curves $\psi_1, \psi_2 : \partial D \to \mathbb{C}$ such that $\psi_1(\lambda) \ne \psi_2(\lambda)$, $\lambda \in \partial D$, we can define the winding number of the former about the latter as the increment of $\frac{1}{2\pi} \arg(\psi_1(\lambda) - \psi_2(\lambda))$ as λ wraps once around ∂D.

Let us have a bidisc $\mathbb{U}$ over $\bar{D}$. Given a proper section $\Phi : D \to \mathbb{U}$ let us define its *winding number* as follows. Let us mark the torus $\delta\mathbb{U}$ with the homology basis $\{[\partial D], [\partial \mathbb{U}_*]\}$. Then the winding number $w(\Phi)$ is the second coordinate of the curve $\Phi : \partial D \to \delta\mathbb{U}$ with respect to this basis.

Argument Principle. *Let us have a bidisc $\mathbb{U}$ over $\bar{D}$ and a proper holomorphic section $\Phi : D \to \mathbb{U}$, $\phi = \pi_2 \circ \Phi$. Let $\Psi : \bar{D} \to \mathbb{U}$ be another continuous section holomorphic in D, $\psi = \pi_2 \circ \Psi$. Then the number of solutions of the equation $\phi(\lambda) = \psi(\lambda)$ counted with multiplicity is equal to the winding number $w(\Phi)$.*

Proof. Indeed, $w(\Phi)$ is equal to the winding number of ϕ around ψ, which is equal, by the standard Argument Principle, to the number of roots of the equation

$$\phi(\lambda) = \psi(\lambda).$$

3. Parapuzzle combinatorics

3.1. **Holomorphic families of generalized quadratic-like maps.** Let us consider a topological disc $D \subset \mathbb{C}$ with a base point $* \in D$, and a family of topological bidiscs $\mathbb{V}_i \subset \mathbb{U} \subset \mathbb{C}^2$ over D (*tubes*), such that the $\mathbb{V}_i$ are pairwise disjoint. We assume that $V_{0,\lambda} \ni 0$.

Let

$$\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U} \tag{3.1}$$

be a fiberwise map, which admits a holomorphic extension to some neighborhoods of the $\mathbb{V}_i$ (warning: these extensions don't fit), and whose fiber restrictions

$$\mathbf{g}(\lambda, \cdot) \equiv g_\lambda : \bigcup_i V_{i,\lambda} \to U_\lambda, \quad \lambda \in D,$$

are generalized quadratic-like maps with the critical point at $0 \in V_\lambda \equiv V_{0,\lambda}$ (see [L3], §3.7 for the definition). We will also assume that the discs U_λ and $V_{i,\lambda}$ are bounded by quasi-circles.

Let us also assume that there is a holomorphic motion $\mathbf{h}$,

$$h_\lambda : (\partial U_*, \bigcup_i \partial V_{i,*}) \to (\partial U_\lambda, \bigcup_i \partial V_{i,\lambda}), \tag{3.2}$$

with a base point at $* \in D$, which respects action of $\mathbf{g}$:

$$h_\lambda \circ g_*(z) = g_\lambda \circ h_\lambda(z), \quad for \quad z \in \cup \partial V_{i,*}. \tag{3.3}$$

A *holomorphic family* $(\mathbf{g}, \mathbf{h})$ *of (generalized) quadratic-like maps over* D is a map (3.1) together with a holomorphic motion (3.2) satisfying (3.3). We will sometimes reduce the notation to $\mathbf{g}$. In case when the domain of $\mathbf{g}$ consists of only one tube $\mathbb{V}_0$, we refer to $\mathbf{g}$ as *DH quadratic-like family* (for "Douady and Hubbard", compare [DH2]).

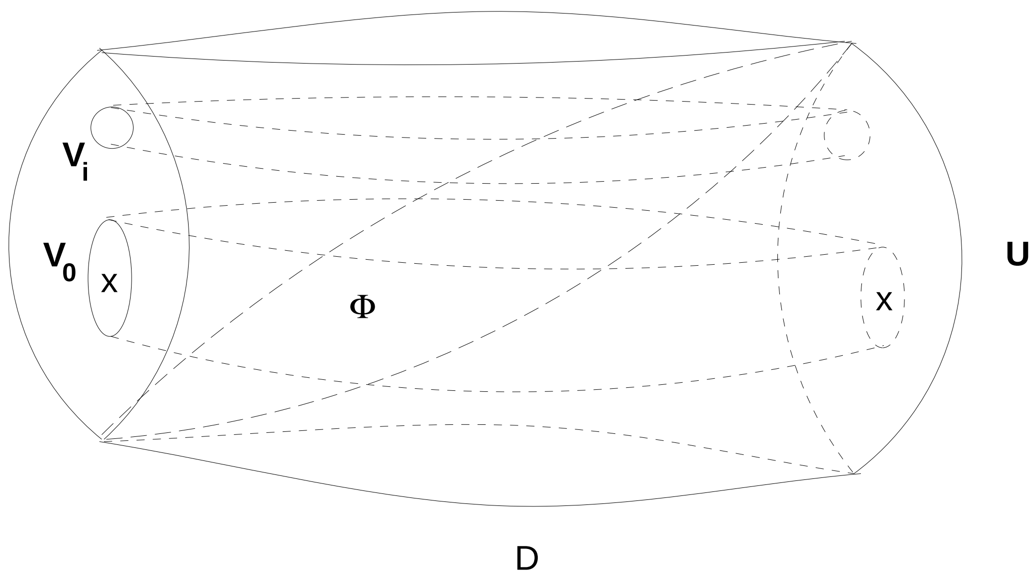

Figure 1: Generalized quadratic-like family.

Note that by the Extension Lemma we can always assume that the holomorphic motion (3.2) is extended to the whole disc U_* (though without respect of dynamics outside $\cup \partial V_{i,*}$).

Let us now consider the critical value function $\phi(\lambda) = g_\lambda(0)$, $\Phi(\lambda) = \mathbf{g}(\lambda, 0) \equiv (\lambda, \phi(\lambda))$. Let us say that $\mathbf{g}$ is a *proper* (or *full*) holomorphic family if the fibration $\pi_1 : \mathbb{U} \to D$ admits an extension to the boundary $\bar{D}$, $\bar{V}_i \subset U$, and $\Phi : D \to \mathbb{U}$ is a proper section. Note that the fibration $\pi_1 : V_0 \to D$ cannot be extended to $\bar{D}$, as the domains $V_{\lambda,0}$ pinch to figure eights as $\lambda \to \partial D$.

Given a proper holomorphic family $\mathbf{g}$ of generalized quadratic-like maps, let us define its *winding number* $w(\mathbf{g})$ as the winding number of the critical value $\phi(\lambda)$ about the critical point 0. By the Argument Principle, it is equal to the winding number of the critical value about any section $\bar{D} \to \mathbb{U}$.

We will also face the situation when $\mathbf{g}$ does not map every tube $\mathbb{V}_i$ onto the whole tube $\mathbb{U}$ but rather on some other tube $\mathbb{V}_j$, while all the rest properties listed above are still valid (see §3.3). Then we call $\mathbf{g}$ a holomorphic family of Markov maps.

Let $\mod(\mathbf{g})$ stand for the $\inf_{\lambda \in D} \mod(U_\lambda \setminus V_{0,\lambda})$.

3.2. **Douady & Hubbard quadratic-like families.** Let us have a proper holomorphic family $\mathbf{f} : \mathbb{V} \to \mathbb{U}$ of DH quadratic-like maps, with winding number 1. The Mandelbrot set $M(\mathbf{f})$ is defined as the set of $\lambda \in D$ such that the Julia set $J(f_\lambda)$ is connected.

Since the U_λ and $V_{i,\lambda}$ are bounded by quasi-circles, there is a qc straightening $\omega_* : \mathrm{cl}(U_* \setminus V_*) \to \mathbb{A}[2, 4]$ conjugating $f : \partial V \to \partial U$ to $z \mapsto z^2$ on $\mathbb{T}_2$. The holomorphic motion $\mathbf{h}$ (extended to the whole "condensator" $\mathbb{U} \setminus \mathbb{V}$) spreads this straightening over the whole parameter region D. We obtain a family of quasi-conformal homeomorphisms

$$\omega_\lambda : \mathrm{cl}(U_\lambda \setminus V_\lambda) \to \mathbb{A}[2, 4] \tag{3.4}$$

conjugating $f_\lambda|U_\lambda$ to $z \mapsto z^2$ on $\mathbb{T}_2$. Pulling them back, we obtain for every f_λ the straightening $\omega_\lambda : V_\lambda \smallsetminus \Omega_\lambda \to \mathbb{A}(\rho_\lambda, 4)$ well-defined up to the critical point level $\rho_\lambda = |\omega_\lambda(0)|$ (so that for $\lambda \in M(\mathbf{f})$ it is well-defined on the whole complement of the Julia set).

Remark. The motion $\mathbf{h}$ can usually be selected uniformly qc (actually smooth) on $\mathbb{U} \smallsetminus \mathbb{V}$ over D. Then the straightenings ω_λ are also uniformly qc.

Let us now define a map $\xi : D \smallsetminus M(\mathbf{f}) \to \mathbb{A}(1, 4)$ in the following way:

$$\xi(\lambda) = \omega_\lambda(f_\lambda 0). \tag{3.5}$$

Lemma 3.1. *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. Then formula 3.5 determines a homeomorphism* $\xi : D \smallsetminus M(\mathbf{f}) \to \mathbb{A}(1, 4)$*. If the holomorphic motion* $\mathbf{h}$ *is selected uniformly qc on the annulus tube* $\mathbb{U} \smallsetminus \mathbb{V}$ *then* ψ *is quasi-conformal.*

Proof. Without loss of generality we can assume that $* \in M(\mathbf{f})$. Let $V_*^n = f_*^{-n} U_*,\ n = 0, 1, \dots$.

Let us consider the critical value graph $C = \Phi(\lambda) \equiv \{(\lambda, f_\lambda 0),\ \lambda \in D\}$. By the Argument Principle, it intersects at a singe point each leaf of the holomorphic motion $\mathbf{h}$ on $\mathbb{U} \smallsetminus \mathbb{V}$, so that the holonomy $\gamma : U_* \smallsetminus V_* \to X$ is a homeomorphism onto the image R_1. Hence $A_1 \equiv \pi_1 R_1 \subset D$ is a topological annulus, and the map

$$\xi^{-1} = \pi_1 \circ \gamma \circ \omega_*^{-1} : \mathbb{A}[2, 4) \to A_1$$

is a homeomorphism.

Let Γ_1 be the inner boundary of A_1, and D_1 be the topological disc bounded by Γ_1. Since the critical value $f_\lambda(0)$, $\lambda \in D_1$, does not land at the leaves of holomorphic motion $\mathbf{h}|D_1$, it can be lifted by $\mathbf{f}$ to a holomorphic motion $\mathbf{h}_1$ of the annulus $V_*^1 \smallsetminus V_*^2$ over D_1. Since the graph C intersects every leaf belonging to $\partial \mathbb{V}^1$ at a single point, the family $(\mathbf{f} : \mathbb{V}^2 \to \mathbb{V}^1, \mathbf{h})$ is proper over D_1 and has winding number 1. Let $A_2 = \Phi^{-1}(\mathbb{V}^1 \smallsetminus \mathbb{V}^2)$. Then the same argument as above shows that the map $\xi^{-1} : \mathbb{A}[\sqrt{2}, 2] \to A_2$ is also a homeomorphism.

Continuing in the same way, we will inductively construct a sequence of holomorphic motions $\mathbf{h}_n$ over nested discs D_n, and a nest of adjoint annuli $A_n = D_{n-1} \smallsetminus D_n$ which are homeomorphically mapped by ξ onto the round annuli $\mathbb{A}[2^{1/(n-1)}, 2^{1/(n-2)}]$. Altogether this shows that ξ is a homeomorphism.

Finally, assume $\mathbf{h}$ is K-qc. Since all further motions $\mathbf{h}_n$ are holomorphic lifts of $\mathbf{h}$ over D_n by $\mathbf{f}^n$, they are K-qc over their domains of definition as well. By Corollary 2.1 (and the Remark afterwards), they are transversally K-qc. Moreover, the straightening $\omega_* : U_* \smallsetminus J(f_*) \to \mathbb{A}(1, 4)$ is qc, while the projection $\pi_1 : X \to D$ is conformal. Since ξ is the composition of the straightening, the holonomy and the projection, it is quasi-conformal. □

Example (see [DH1]). Let us consider the Mandelbrot set M of the quadratic family $P_c : z \mapsto z^2 + c$. Let $R : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \mathbb{D}$ be the Riemann mapping. Recall that parameter equipotentials and external rays are defined as the R-preimages of the round circles and radial rays. Let Ω_r be the topological disc bounded by the equipotential $R^{-1}\{re^{i\theta} : 0 \le \theta \le 2\pi\}$ of level $r > 1$.

For every $c \in \Omega_4$, let us consider the quadratic-like map $P_c : V_c \to U_c$ where V_c and U_c are topological discs bounded by the dynamical equipotentials of level 2 and 4 correspondingly. Then the conformal map $\omega_C : U_c \smallsetminus V_c \to \mathbb{A}(2, 4)$ conjugates $P_c|\partial V_c$ to $z \mapsto z^2$ on $\mathbb{T}_2$, so that it can serve as a straightening (3.4). With this choice of straightening, the parameter map $\xi : D \smallsetminus M \to \mathbb{A}(1,4)$ constructed in Lemma 3.1 just coincides with the Riemann map R. □

With Lemma 3.1, we can extend the the notion of parameter rays and equipotentials to quadratic-like families as the ξ-preimages of the polar coordinate curves in $\mathbb{A}(1,4)$. If $\xi(\lambda) = re^{i\theta}$ then r and θ are called the *equipotential level* and the *external angle* of the parameter value λ. Note that ∂D becomes the equipotential of level 4.

3.3. **Wakes and initial Markov families.** Recall that every quadratic-like map $f : V \to U$ is hybrid equivalent to a quadratic polynomial $P_c : z \mapsto z^2 + c$ (The Straightening Theorem [DH2]). It is constructed by gluing $f|U$ to $z \mapsto z^2$ on $\mathbb{C} \smallsetminus \mathbb{D}_2$, and pulling the standard conformal structure on $\mathbb{C} \smallsetminus \mathbb{D}$ back to $U \smallsetminus K(f)$ by iterates of f. The construction depends on the choice of a qc straightening $\omega : \mathrm{cl}(U \smallsetminus V) \to \mathbb{A}[2,4]$ conjugating $f|\partial V$ to $z \mapsto z^2$ on $\mathbb{T}_2$. However, if the Julia set $J(f)$ is connected, the parameter value $c \equiv \chi(f)$ is determined uniquely.

Given a quadratic-like family $f_\lambda : V_\lambda \to U_\lambda$ over D with winding number 1, let us consider a family of straightenings (3.4) and the corresponding family of quadratic polynomials $P_{\chi(\lambda)} : z \mapsto z^2 + \chi(\lambda)$. The following statement follows from the main result of [DH2]:

Lemma 3.2. *Under the circumstances just described, the straightening $\chi : (D, M(\mathbf{f})) \to (\Omega_4, M)$ is a homeomorphism of the disc D onto a neighborhood Ω_4 of the Mandelbrot set M bounded by the parameter equipotential of level 4.*

Note that the above explicit description of the image of χ follows from the following formula:

$$\xi = R \circ \chi, \tag{3.6}$$

where ξ is defined in (3.5), and R is the Riemann mapping on the complement of the Mandelbrot set. This formula, in turn, follows from the definitions of ξ and χ and the description of R given (see the Example in the end of §3.2).

Lemma 3.2 shows that the landing properties of the parameter rays in a quadratic-like family coincide with the corresponding properties in the quadratic family. This allows us to extend the notions of the parabolic and Misiurewicz wakes from the quadratic to the quadratic-like case. Namely, the *q/p-parabolic wake* $P_{q/p} = P_{q/p}(\mathbf{f})$ is the parameter region in D bounded by the external rays landing at the q/p-bifurcation point $b_{q/p}$ on the main cardioid of $M(\mathbf{f})$ and the appropriate arc of ∂D. Dynamically it is specified by the property that for λ in this wake there are p rays landing at the α-fixed point α_λ of $J(f_\lambda)$, and they form a cycle with rotation number q/p.

The maps

$$f_\lambda^p : V_\lambda \to U_\lambda \tag{3.7}$$

restricted to appropriate domains form a quadratic-like family over the wake (see [D], [L3], §§2.5, 3.2). (The domain V_λ is a thickening of the puzzle piece $Y_\lambda^{(1+p)}$ bounded by two pairs of rays landing at the α-fixed and co-fixed points and two equipotential arcs. The domain U_λ is a thickening of the first puzzle piece $Y_\lambda^{(0)}$ bounded by two rays landing at the α-fixed point and an equipotential arc.) Note however that this family fails to be proper as the domains U_λ don't admit continuous extension at the root.

Proposition 3.3 (see [D]). *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. Then the winding number of the critical value* $\lambda \mapsto f_\lambda^p(0)$ *about 0 when* λ *wraps once about the boundary of the parabolic wake* $\partial P_{q/p}$ *is also equal to 1.*

By [DH2, D], the quadratic-like family 3.7 generates a homeomorphic copy $M_{q/p} = M_{q/p}(\mathbf{f})$ of the Mandelbrot set attached to the bifurcation point $b_{q/p}$. Its complement $M \smallsetminus M_{q/p}$ consists of a component containing the main cardioid and infinitely many *decorations* (using terminology of Dierk Schleicher) $D_{q/p}^{\sigma,i}$, where σ is a dyadic sequence of length $|\sigma| = n-1$, $n = 1, 2, \dots$, $i = 1, \dots, p-1$. A decoration $D_{q/p}^{\sigma,i}$ touches $M_{q/p}$ at a Misiurewicz point $\mu = \mu_{q/p}^{\sigma}$ for which

$$f_\mu^{pk}(0) \in Y_\mu^{(1+p)},\ k = 0, \dots, n-1, \ \text{ while } \ f_\mu^{pn}(0) = \alpha'_\mu,$$

where α'_μ is the α-co-fixed point (i.e., the f_μ-preimage of the fixed point α_μ). (Such Misiurewicz points are naturally labelled by the dyadic sequences).

Every decoration $D_{q/p}^{\sigma,i}$ belongs to the *Misiurewicz wake* $\tilde{O}_{q/p}^{\sigma,i}$ bounded by two parameter rays landing at $\mu_{q/p}^{\sigma}$ (there are p rays landing at this point). Let us truncate such a wake by the equipotential of level $4/(pn-1)$. We will obtain the initial puzzle pieces $O_{\sigma,i}^{q/p}$ which sometimes will also be called "wakes". They can be dynamically specified in terms of the initial puzzle (see [L4],§3.2). Namely, there are $p-1$ puzzle pieces $Z_i^{(1)}$, $n = 1, \dots, p-1$, attached to the co-fixed point α'. Pulling them back by $(n-1)$-st iterate of the double covering $f^p : Y^{(1+p)} \to Y^{(1)}$, we obtain 2^{n-1} puzzle pieces $Z_{\sigma,i}^{1+(n-1)p}$ labelled by the dyadic sequences. The wake $O_{q/p}^{\sigma,i}$ is specified by the property that $f_\lambda^p 0 \in Z_{\sigma,i}^{(1+(n-1)p)}$.

By *tiling* we will mean a family of topological discs with disjoint interiors. Let us consider the initial tiling constructed in [L4], §3.2:

$$Y_\lambda^{(0)} \supset V_\lambda^0 \cup X_{i,\lambda}^k \cup Z_{j,\lambda}^{(1+kp)}. \tag{3.8}$$

Let us recall that $\mathbb{Z}_j^{(1)}$ means $\cup_\lambda Z_{j,\lambda}^{(1)}$.

Lemma 3.4. *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. The initial tiling (3.8) moves holomorphically within the Misiurewicz wake* $O = O_{p/q}^{\sigma,i}$. *The critical value of the return map,* $\Phi : O \to \mathbb{Z}_j^{(1)}$, $\Phi(\lambda) = f_\lambda^{pn} 0$, *is a proper map with winding number 1 (where* $n = |\sigma| + 1$*).*

Proof. Indeed, all puzzle pieces of this initial tiling are the pullbacks of $Z_{j,\lambda}^{(1)}$. As λ ranges over the wake $O \equiv O_{q/p}^{n,i}$, the corresponding iterates of 0 don't cross the boundary of $Z_{j,\lambda}^{(1)}$. It follows that the boundary of the initial tiling moves holomorphically.

Moreover, the torus $\delta\mathbb{Z}_j^{(1)}$ is foliated by the curves with the same external coordinates, and one curve corresponding to the motion of the α-co-fixed point. By the definition of the Misiurewicz wake, the critical value $\Phi(\lambda)$ intersects once every leaf of this foliation when λ wraps once around ∂O. Hence $\Phi : O \to \mathbb{Z}_j^{(1)}$ is a proper map with winding number 1. ☐

Thus we have the initial Markov partition moving holomorphically over the corresponding Misiurewicz wake. The wake $O_{q/p}^{\sigma,i}$ containing a point λ will also be denoted by $O(\lambda)$.

3.4. **First generalized quadratic-like family.** Let us consider a proper DH quadratic-like family $\mathbf{f} = \{f_\lambda\}$ over D with winding number 1. Fix a Misiurewicz wake O of this family. The first generalized quadratic-like map $g_{1,\lambda} : \cup V^1_{i,\lambda} \to V^0_\lambda$ is defined as the first return map to V^0_λ (see [L4], §3.5). The itinerary of the critical point via the elements of the initial tiling (3.8) determines the parameter tiling $\mathcal{D}'$ of a Misiurewicz wake O by the corresponding puzzle pieces. Let $D_0(\lambda)$ stand for such parapuzzle piece containing λ.

More precisely, for any $\lambda \in O$, let us consider the first landing map to $T_\lambda : \cup L_{\bar{i},\lambda} \to V_{0,\lambda}$ (see [L4], §4.4). The puzzle piece $L_{\bar{i},\lambda}$ is specified by its itinerary $\bar{i} = (i_0, \dots, i_{l-1})$ under iterates of f^p_λ through non-central pieces of the initial tiling until the first landing at $V_{0,\lambda}$:

$$L_{\bar{i},\lambda} = \{z : f^{pk}_\lambda z \in V_{i_k},\ k = 0, \dots, l-1,\ \ f^{pl}_\lambda z \in V_0\}.$$

Moreover, T_λ univalently maps $L_{0,\lambda}$ onto $V_{0,\lambda}$. Let $\bar{i}_\lambda$ stand for the itinerary of the critical value $\psi(\lambda) = f^p_\lambda 0$ through the initial tiling, so that $f^p_\lambda 0 \in L_{\bar{i}_\lambda,\lambda} \equiv Q_\lambda$. Then the parapuzzle pieces of the tiling $\mathcal{D}'$ are defined as follows:

$$D_0(\lambda) = \{\mu \in O : Q_\mu = Q_\lambda\}.$$

Since the critical value $\psi(\lambda)$ does not intersect the boundary of the initial tiling as λ ranges over the wake O, the pieces $L_{\bar{i},\lambda}$ form the tubes $\mathbb{L}_{\bar{i}}$ with holomorphically moving boundary. Since the winding number of $\Psi(\lambda) = (\lambda, \psi(\lambda))$ about the tubes of the initial tiling (3.8) over O is equal to 1 (by Lemma 3.4), the function $\Psi : D_0(\lambda) \to \mathbb{Q}$ is proper with winding number 1. Since the first landing map is a fiberwise diffeomorphism of every tube $\mathbb{L}_{\bar{i}}$ onto $\mathbb{V}_0$, the function $\lambda \mapsto g_\lambda(0) = T_\lambda \circ \psi(\lambda)$, $D_0(\lambda) \to \mathbb{V}_0$, is also proper with winding number 1. Thus we have:

Lemma 3.5. *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. Then the first generalized renormalization* $\mathbf{g} = \{g_\lambda\}$ *is a proper family with winding number 1.*

3.5. **Renormalization of holomorphic families.** Let us now have a holomorphic quadratic-like family, $\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}$ over D, see §3.1. Let $\mathcal{I}$ stand for the labeling set of tubes $\mathbb{V}_i$. Remember that $\mathcal{I} \ni \mathbf{0}$ and $\mathbb{V}_0 \ni \mathbf{0}$. Let $\mathcal{I}_\#$ stand for the set of all finite sequences $\bar{i} = (i_0, \dots, i_{l-1})$ of non-zero symbols $i_k \in \mathcal{I} \setminus \{\mathbf{0}\}$. For any $\bar{i} \in \mathcal{I}_\#$, there is a tube $\mathbb{V}_{\bar{i}}$ such that

$$\mathbf{g}^k \mathbb{V}_{\bar{i}} \subset \mathbb{U}_{i_k},\ k = 0, \dots, l-1, \quad \textit{and} \quad g^l \mathbb{V}_{\bar{i}} = \mathbb{U}.$$

We call $l = |\bar{i}|$ the rank of this tube. The map $\mathbf{g}^l : V_{\bar{i}} \to \mathbb{U}$ is a holomorphic diffeomorphism which fibers over id, that is, $g^l_\lambda V_{\bar{i},\lambda} = U_\lambda,\ \lambda \in D$.

Let us now pull the holomorphic motion $\mathbf{h}$ back to the boundaries $\partial \mathbb{V}_{\bar{i}}$:

$$g^l_\lambda \circ h_{\bar{i},\lambda}(z) = h_\lambda(g^l z),\ z \in \partial V_{\bar{i},*}.$$

This holomorphic motion is an extension of $\mathbf{h}$, as by (3.3) it coincides with $\mathbf{h}$ on $\partial \mathbb{U}_i$. In what follows we will keep notation $\mathbf{h}$ for this extended motion.

Let $\mathbb{L}_{\bar{i}} \subset \mathbb{V}_{\bar{i}}$ be such a tube that $\mathbf{g}^l \mathbb{L}_{\bar{i}} = \mathbb{V}_0$ where $l = l_{\bar{i}} = |\bar{i}|$. Extend the holomorphic motion h_λ to the boundaries of these tubes by pulling it back from $\partial \mathbb{V}_0$ to the $\partial \mathbb{L}_{\bar{i}}$ by $\mathbf{g}^l$.

Let $\bar{i}_\lambda$ be the itinerary of the critical value $\phi(\lambda) = g_\lambda 0$ under iterates of g_λ through the domains $V_{i,\lambda}$, until its first return to $V_{0,\lambda}$. In other words, let $g_\lambda(0) \in V_{\bar{i}_\lambda,\lambda}$ and $g^{l_\lambda}(0) \in V_{0,\lambda}$, where $l_\lambda = |i_\lambda|$. Let $Q_\lambda = L_{i_\lambda,\lambda}$ be the corresponding puzzle piece.

Let us now define a new parameter domain D' as the component of the set

$$\{\lambda : \phi(\lambda) \in V_{i_*,\lambda}\} = \{\lambda : Q_\lambda = Q_*\}$$

containing $*$. For $\lambda \in D'$, the itinerary of the critical value under iterates of g_λ until the first return back to $V_{0,\lambda}$ is the same as for g_* (that is, $\bar{i}_*$). Let us define new tubes $\mathbb{V}'_j \subset \mathbb{V}_0$ as the components of $(\mathbf{g}|\mathbb{V}_0)^{-1}(\mathbb{L}_{\bar{i}} \cap \pi^{-1}D')$. Let

$$\mathbf{g}' : \cup \mathbb{V}'_j \to \mathbb{V}_0 \tag{3.9}$$

be the first return map of the union of these tubes onto $\mathbb{V}_0$.

For $\lambda \in D'$, the critical value $\Phi(\lambda)$ does not intersect the boundaries of the the tubes $\mathbb{L}_{\bar{i}}$. Hence we can pull back the holomorphic motion of $\partial\mathbb{L}_{\bar{i}}$ to a holomorphic motion $\mathbf{h}'$ over D' of $\partial\mathbb{V}'_j$. Then $(\mathbf{g}', \mathbf{h}')$ is a holomorphic family over D' which will be called the *generalized renormalization* of the family $(\mathbf{g}, \mathbf{h})$.

If $\mathbf{g}$ is a proper family then $\mathbf{g}'$ is clearly proper as well. Moreover, $w(\mathbf{g}') = 1$ if $w(\mathbf{g}) = 1$. Indeed, by the Argument Principle the curve $\Phi|D'$ intersects once every leave of $\partial\mathbb{Q}$. Hence it has winding number 1 about this tube. As the map

$$g^{l_\lambda} : \mathbb{Q} \cap \pi^{-1}D' \to \mathbb{V}_0 \cap \pi^{-1}D'$$

is a fiber bundles diffeomorphism, it preserves the winding number.

Let us summarize the above discussion:

Lemma 3.6. *Let $\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}$ be a generalized quadratic-like family over D. Assume it is proper and has winding number 1. Then its generalized renormalization $\mathbf{g}' : \cup \mathbb{V}'_j \to \mathbb{U}'$ over D' is also proper and has winding number 1.*

3.6. **Central paracascades.** In this section we will describe inductively tilings of the parameter plane according to the types of generalized renormalizations. The initial tiling is constructed above in §3.3. Let us assume that we have already constructed a tiling $\mathcal{D}^{\updownarrow}$ of level l. The piece of this tiling containing a point λ is denoted by $D \equiv \Delta^l(\lambda)$. This piece comes together with a holomorphic family $\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}$ of generalized quadratic like maps over D.

We will now subdivide D according to the combinatorics of the central cascades of maps g_λ (see [L3], §§3.1, 3.6). To this end let us first stratify the parameter values according to the length of their central cascade. This yields a nest of parapuzzle pieces

$$D \supset D^{'} \supset \cdots \supset D^{(N)} \supset \ldots$$

For $\lambda \in D^{(N)}$, the map g_λ has a central cascade

$$V_\lambda^{(0)} \equiv U_\lambda \supset V_\lambda \equiv V_\lambda^{(1)} \supset \cdots \supset V_\lambda^{(N)} \tag{3.10}$$

of length N, so that $g_\lambda 0 \in V_\lambda^{(N-1)} \setminus V_\lambda^{(N)}$. Note that the puzzle pieces $V_\lambda^{(k)}$ are organized into the tubes $\mathbb{V}^{(k)}$ over $D^{(k-1)}$ (with the convention that $D^{(-1)} \equiv D$).

The intersection of these puzzle pieces, $\cap D^{(N)}$, is the little Mandelbrot set $M(\mathbf{g})$ "centered" at the superattracting parameter value $c = c(\mathbf{g})$ such that $g_c(0) = 0$. Let us call c the *center* of D.

For a tube $\mathbb{X}$ over D and a domain $\Delta \subset D$, let $\mathbb{X}|\Delta$ stand for the $\mathbb{X} \cap \pi_1^{-1}\Delta$. There is a Bernoulli map

$$\mathbf{G} : \cup \mathbb{W}_j \to \mathbb{U} \tag{3.11}$$

associated with the cascade 3.10 (see [L4], §3.6). Here the tubes $\mathbb{W}_j$ over $D^{(N-1)}$ are the pull-backs of the tubes $\mathbb{V}_i|D^{(k)}$, $i \neq 0$, by the covering maps

$$\mathbf{g}^k : (\mathbb{V}^{(k)} \smallsetminus \mathbb{V}^{(k+1)})|D^{(N-1)} \to (\mathbb{U} \smallsetminus \mathbb{V})|D^{(N-1)}, \quad k = 0, 1 \ldots, N-1. \tag{3.12}$$

In the same way as in §3.5, to any string $\bar{j} = (j_0, \ldots, j_{l-1})$ corresponds the tube over $D^{(N-1)}$,

$$\mathbb{W}_{\bar{j}} = \{p \in \mathbb{U}|D^{(N-1)} : \mathbf{G}^n p \in \mathbb{W}_{j_n},\ n = 0, \ldots, l-1\}.$$

Note that $\mathbf{G}^l$ univalently maps each $\mathbb{W}_{\bar{j}}$ onto $\mathbb{U}|D^{(N-1)}$. Thus $\mathbb{W}_{\bar{j}}$ contains a tube $\mathbb{L}_{\bar{j}}$ which is univalently mapped by $\mathbf{G}^l$ onto the central tube $\mathbb{V}^{(N)}$. These maps altogether form the first landing map to $\mathbb{V}^{(N)}$

$$\mathbf{T} : \cup \mathbb{L}_{\bar{j}} \to \mathbb{V}^{(N)}. \tag{3.13}$$

For a $\lambda \in D^{(N-1)} \smallsetminus D^{(N)}$, let us now consider the itinerary $\bar{j}_\lambda$ of the critical value $\phi(\lambda) \equiv g_\lambda(0)$ through the tubes W_j until its first return to $V^{(N)}$, so that $\phi(\lambda) \in L_{\bar{j}_\lambda}$. Let

$$\Delta^{l+1}(\lambda) = \Phi^{-1}\mathbb{L}_{\bar{j}_\lambda}, \quad \mathbb{Q}^{l+1}(\lambda) = \Phi^{-1}\mathbb{W}_{\bar{j}_\lambda}. \tag{3.14}$$

Thus the annuli $D^{(N-1)} \smallsetminus D^{(N)}$ are tiled by the parapuzzle pieces $\Delta(\lambda)$ according as the itinerary of the critical point through the Bernoulli scheme (3.11) until the first return to $V_\lambda^{(N)}$. Altogether these tilings form the desired new subdivision of D. (Note however that the new tiles don't cover the whole domain D: the residual set consists of the Mandelbrot set $M(\mathbf{g})$ and of the Misiurewicz parameter values for which the critical orbit never returns back to $V_\lambda^{(N)}$, $\lambda \in D^{(N-1)} \smallsetminus D^{(N)}$.

The affiliated holomorphic family over $\Delta(\lambda)$ is defined as the first return map to $V_\lambda^{(N)}$. Its domain is obtained by pulling back the tubes $\mathbb{L}_{\bar{j}}$ from (3.13) by the double branched covering $\mathbf{g} : \mathbb{V}^{(N)} \to \mathbb{V}^{(N-1)}|\Delta(\lambda)$, and the return map itself is just $\mathbf{T} \circ \mathbf{g}$.

3.7. **Principal parapuzzle nest.** Let us now summarize the above discussion. Given a quadratic-like family $\mathbf{f}$, we consider the first tiling $\mathcal{D}'$ of a Misiurewicz wake O as described in §3.4. Each tile $\Delta \in \mathcal{D}'$ comes together with a generalized quadratic-like family $(\mathbf{g}_\Delta, \mathbf{h}_\Delta)$ over Δ.

Now assume inductively that we have constructed the tiling $\mathcal{D}^\updownarrow$ of level l. Then the tiling of the next level, $\mathcal{D}^{\updownarrow+\infty}$ is obtained by partitioning each tile $\Delta \in \mathcal{D}^\updownarrow$ as described in §§3.5,3.6.

Let $\Delta^l(\lambda)$ stand for the tile of $\mathcal{D}^\updownarrow$ containing λ, while $\Delta^l(\lambda) \subset Q^l(\lambda) \subset \Delta^{l-1}(\lambda)$ stand for another tile defined in (3.14). Each tile $\Delta = \Delta^l(\lambda)$ contains a *central subtile* $\Pi^l(\lambda) = \Phi^{-1}\mathbb{V}_0$ corresponding to the central return of the critical point (here $\Phi_\Delta = \mathbf{g}_\Delta(\mathbf{0})$).

Let us then consider the sequence of renormalized families $\mathbf{g}_n$ over topological discs $\Delta^l(\lambda)$. We call the nest of topological discs $\Delta^0(\lambda) \supset \Delta^1(\lambda) \supset \Delta^2(\lambda) \supset \ldots$ *the principal parapuzzle nest* of λ. This nest is finite if and only if λ is renormalizable.

Let $c_{n,\lambda} \in \Delta^l(\lambda)$ be the centers of the corresponding parapuzzle pieces. We call them the *principal superattracting approximations* to λ. If λ is not renormalizable, then $c_{l,\lambda} \to \lambda$ as $l \to \infty$, as $\operatorname{diam} \Delta^l(\lambda) \to 0$ (see the next section).

The $\mod (\Delta^l(\lambda) \smallsetminus \Delta^{l+1}(\lambda))$ are called the *principal parameter moduli* of $\lambda \in D$.

4. Parapuzzle geometry

The following is the main geometric result of this paper:

Theorem A. *Let us consider a proper DH quadratic-like family* $\mathbf{f}$ *with winding number 1 over* D*, and a Misiurewicz wake* $O \subset D$*. Then for any* $\lambda \in M(\mathbf{f}) \cap O$*,*

$$\mod (\Delta^l(\lambda) \smallsetminus \Delta^{l+1}(\lambda)) \geq Bl, \quad \textit{and} \qquad \mod (\Delta^l(\lambda) \smallsetminus \Pi^l(\lambda)) \geq Bl,$$

where the constant $B > 0$ *depends only on* O *and* $\mod (\mathbf{f})$.

The rest of this section will be devoted to the proof of this theorem.

4.1. **A priori bound on parameter moduli.** In this section we will show that the parameter annuli have definite moduli. Given a holomorphic motion h_λ and a holomorphic family of affine maps $g_\lambda : z \mapsto a_\lambda z + b_\lambda$, we can consider an "affinely equivalent"motion $g_\lambda \circ h_\lambda$. In this way the motion can be normalized such that any two points $z, \zeta \in U_*$ don't move (that is, $h_\lambda(z) \equiv z$ and $h_\lambda(\zeta) \equiv \zeta$ for $\lambda \in D$). Let us start with a technical lemma:

Lemma 4.1. *Let us have a holomorphic motion* $h : (U_*, V_*) \to (U_\lambda, V_\lambda)$ *of a pair of nested topological discs over a domain* D*. Assume that the maps* $h_\lambda : (\partial U_*, \partial V_*) \to (\partial U_\lambda, \partial V_\lambda)$ *admit* K*-qc extensions* $H_\lambda : (\mathbb{C}, U_*) \to (\mathbb{C}, V_\lambda)$ *(not necessarily holomorphic in* λ *but with uniform dilatation* K*). Then there exists an* $M = M(K)$ *such that if* $\mod (U_* \smallsetminus V_*) > M$ *then after appropriate normalization of the motion, there exists a round cylinder* $D \times \{q < |z| < 2q\}$ *embedded into* $\mathbb{U} \smallsetminus \mathbb{V}$*.*

Proof. Let us normalize the motion in such a way that $0 \in V_*$, and $h_\lambda(0) = 0$. Let z_* be a point on ∂U_* closest to 0. Normalize the motion in such a way that $z_* = 1$, and this point does not move either. With this normalization, $V_* \subset D(0, \epsilon)$ where $\epsilon = \epsilon(m) \to 0$ as $m \equiv \mod (U_* \smallsetminus V_*) \to \infty$.

Since the space of normalized K-qc maps is compact, $|H_\lambda(\epsilon e^{i\theta}|) < \delta$, where $\delta = \delta(\epsilon, K) \to 0$ as $\epsilon \to 0$, K being fixed, and $|H_\lambda(e^{i\theta})| > r$ where $r = r(K) > 0$. It follows that the domain $\mathbb{U}$ contains the round cylinder $D \times \{\delta < |z| < r\}$, and we are done. □

Corollary 4.2. *Under the circumstances of Lemma 4.1, let* $\Phi : D \to \mathbb{U}$ *be a proper analytic map with winding number 1. Let* $D' = \Phi^{-1}\mathbb{V}$*. If* $\mod (U_* \smallsetminus V_*) > M = M(K)$ *then* $\mod (D \smallsetminus D') \geq \log 2$*.*

Proof. By Lemma 4.1, $\mathbb{U} \smallsetminus \mathbb{V} \supset D \times A$ where $A = \{q < |z| < 2q\}$. Let $Q = \Phi^{-1}(D \times A)$. By the Argument Principle, $\phi = \pi_2 \circ \Phi$ univalently maps Q onto A, so that $\mod (D \smallsetminus D') \geq \mod Q = \mod A = \log 2$. □

4.2. **A priori bound on dilatation.** Fix a point $*$ in the Misiurewicz wake O. Let us consider a generalized quadratic-like family $(\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ over $D \equiv \Delta^l(*)$. In what follows we will use the notations of §3.4. Let $* \in D^{(N)}$. Let us lift the holomorphic motion $\mathbf{h}$ from the cylinder $\mathbb{U} \setminus \mathbb{V}_0$ to the cylinders $(\mathbb{V}^k \setminus \mathbb{V}^{k+1})|D^{(N-1)}$ via the coverings (3.12). This provides us with a holomorphic motion $\mathbf{H}$ of $\mathbb{U} \setminus \mathbb{V}^{(N)}$ over $D^{(N-1)}$ with the same dilatation as $\mathbf{h}$.

Lemma 4.3. *The holomorphic motion* $\mathbf{H}$ *of* $\mathbb{U} \setminus \mathbb{V}^{(N)}$ *over* $\Delta^{l+1}(*)$ *(see (3.14)) has a uniformly bounded dilatation, depending only on the choice of the Misiurewicz wake* O *and* $\operatorname{mod}(\mathbf{f})$.

Proof. Let us go back to (3.14). By [L3], $\operatorname{mod}(W_{j_*} \setminus L_{j_*})$ is big for big l. By [L4], for $\lambda \in Q^{l+1}(*)$, there is a K-qc pseudo conjugacy $\psi_\lambda : (U_*, \cup V_{i,*}) \to (U_\lambda, \cup V_{i,\lambda})$, with K depending only on the choice of wake O and $\operatorname{mod}(\mathbf{f})$. Hence Corollary 4.2 can be applied. We conclude that

$$\operatorname{mod}(Q^{l+1}(*) \setminus \Delta^{l+1}(*)) \geq \log 2 \tag{4.1}$$

for l sufficiently big (depending on O and $\operatorname{mod}(\mathbf{f})$).

The lift of $\mathbf{H}$ to the landing tubes (3.13) yields a K-qc holomorphic motion of $(\mathbb{U} \setminus \mathbb{V}^{(N)}, \cup \mathbb{L}_{\bar{j}})$ over $D^{(N-1)}$. By the Extension Lemma, this motion can be extended through $\mathbb{V}^{(N)}$. Let us keep the same notation $\mathbf{H}$ for this motion.

By the Quasi-Conformality Lemma and (4.1), $\mathbf{H}$ is K-qc over $\Delta^{l+1}(*)$, with an absolute K provided l is big enough. But the holomorphic motion $\mathbf{h}_{l+1}$ on $\mathbb{U}^{l+1} \setminus \mathbb{V}_0^{l+1}$ is the lift of $\mathbf{H}$ on $\mathbb{V}^{(N-1)} \setminus \mathbb{L}_{\bar{j}_*}$ over $\Delta^{l+1}(*)$ via the fiberwise analytic double covering

$$\mathbf{g}_l : \mathbb{U}^{l+1} \setminus \mathbb{V}_0^{l+1} \to \mathbb{V}^{(N-1)} \setminus \mathbb{L}_{\bar{j}_*}.$$

Hence $\mathbf{h}_{l+1}$ is also K-qc. □

4.3. **Proof of Theorem A.** We are now prepared to complete the proof:

$$\operatorname{mod}(\Delta^l \setminus \Delta^{l+1}) \geq \operatorname{mod}(W_{\bar{i}_*} \setminus L_{\bar{i}_*}) \geq Bl.$$

The first estimate in the above row follows from Lemma 4.3 and the Remark following Corollary 2.1. The last estimate is the main result of [L3].

For the same reason,

$$\operatorname{mod}(\Delta^l \setminus \Pi^l) \asymp \operatorname{mod}(U_*^l \setminus V_{0,*}^l) \geq Bl.$$

5. Application to the measure problem

In this section we will apply the previous results to the real quadratic family $P_c : z \mapsto z^2 + c$, $c \in \mathbb{R}$. Let $\mathcal{NR}$ stand for the set of non-renormalizable real parameter values $c \in [-2, -3/4)$. Note that all periodic points of the $P_c : z \mapsto z^2 + c$, $c \in \mathcal{NR}$, are repelling. Indeed, the interval $[-3/4, 1/4]$ where P_c has a non-repelling fixed point is excluded, while maps with non-repelling cycles of higher period are renormalizable.

Let $\mathcal{NC}$ stand for the set of parameter values $c \in \mathcal{NR}$ such that the principal nest of P_c constants only finitely many non-trivial (i.e., of length > 1) central cascades.

Theorem 5.1. • *The set* $\mathcal{NR}$ *has positive measure;*

- *The set $\mathcal{NC}$ has full Lebesgue measure in $\mathcal{NR}$.*

Remarks. 1. The former (positive measure) result is known (see [BC], [J]). The latter (full measure) is new.

2. The corresponding statements concerning at most finitely renormalizable parameter values are derived from the above statements by considering quadratic-like families associated with little copies of the Mandelbrot set.

3. By the result of Martens & Nowicki [MN] together with [L2], P_c has an absolutely continuous invariant measure for any $c \in \mathcal{NC}$. Altogether these yield Theorem B stated in the Introduction.

Proof of Theorem 5.1. Let d stand for the real tip of the little Mandelbrot set attached to the main cardioid (i.e. $P_d^3(0) = \alpha$). As all parameter values $c \in [d, -3/4)$ are renormalizable, we can restrict ourselves to the interval $[-2, d) \supset \mathcal{NR}$. This interval belongs to the Misiurewicz wake O attached to d.

Given measurable sets $X, Y \subset \mathbb{R}$, with $\text{length}(Y) > 0$, let $\text{dens}(X|Y)$ stand for the $\text{length}(X \cap Y)/\text{length}(Y)$.

We will now restrict all tilings $\mathcal{D}^\updownarrow$ constructed above to the real line, without change of notations. We will use the same notation, $\mathcal{D}^\updownarrow$, for the union of all pieces of $\mathcal{D}^\updownarrow$. For every $\Delta = \Delta^l(\lambda) \in \mathcal{D}^\updownarrow$, let us consider the central piece $\Pi \subset \Delta$ corresponding to the central return of the critical point. By Theorem A, $\text{dens}(\Pi|\Delta) \leq Cq^l$ for absolute $C > 0$ and $q < 1$. Let Γ^l be the union of these central pieces. Summing up over all $\Delta \in \mathcal{D}^\updownarrow$, we conclude that

$$\text{length}(\Gamma^l) \leq \text{dens}(\Gamma^l|\mathcal{D}^\updownarrow) \leq \mathcal{C}\Pi^\updownarrow \tag{5.1}$$

(the whole interval is normalized so that its length is equal to 1).

It follows that for l sufficiently big,

$$\text{dens}(\bigcup_{k\geq 0} \Gamma^{l+k}|\mathcal{D}^\updownarrow) \leq \mathcal{C}_\infty \Pi^\backslash < \infty,$$

which means that with positive probability central returns will never occur again. This proves the first statement.

To prove the second one just notice that (5.1) together with the Borel-Cantelli Lemma yield that infinite number of central returns occurs with zero probability. □

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