Teichmüller distance for some polynomial-like maps

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1 Introduction

In order to prove the renormalization conjecture for infinitely renormalizable real polynomials of bounded combinatorics, Sullivan in [Sul92] introduced a space of analytic maps where the renormalization operator is defined: the space of polynomial-like maps of degree two and bounded combinatorics modulo holomorphic conjugacies (see [dMvS93]). In this space it is possible to define a distance, $\mathrm{d_T}$ called the Teichmüller distance. This distance measures how far two polynomial-like maps are from being holomorphically conjugate (see Definition 2.5).

It is obvious from the definition that the Teichmüller distance is a pseudo-distance. It is not obvious that this pseudo-distance is actually a distance. To prove this is a distance it is necessary to show that if two polynomial-like maps f and g are such that $\mathrm{d_T}(f,g)=0$ then they are holomorphically conjugate (this can be viewed as a rigidity problem). Sullivan showed in [Sul92] that for real polynomials with connected Julia set this is true. He makes use of external classes of polynomial-like maps (as defined in [DH85]) to reduce the original rigidity problem to a rigidity problem of expanding maps of the circle, previously studied in [SS85]. The last result concerning expanding maps of the circle depends on the theory of Thermodynamical formalism.

In this work we will show that the Teichmüller distance for all elements of a certain class of generalized polynomial-like maps (the class of *off-critically hyperbolic generalized polynomial-like maps*, see Definition 2.1) is actually a distance, as in the case Sullivan studied. This class contains several important classes of generalized polynomial-like maps, namely: Yoccoz,

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Lyubich, Sullivan and Fibonacci.

The initial motivation for this work was to carry on Sullivan's renormalization theory for higher degree Fibonacci maps; the renormalization scheme and *a priori* bounds for these maps were given in [Lyu93] and [LM93]. The only place which needed adjustment after that was exactly the issue of the Teichmüller distance.

In our proof we can not use external arguments (like external classes). Instead we use hyperbolic sets inside the Julia sets of our maps. Those hyperbolic sets will allow us to use our main analytic tool, namely Sullivan's rigidity Theorem for non-linear analytic hyperbolic systems stated in Section 4.

Let us denote by m the probability measure of maximal entropy for the system $f : J(f) \to J(f)$. In [Lyu83] Lyubich constructed a maximal entropy measure m for $f : J(f) \to J(f)$ for any rational function f. Zdunik classified in [Zdu90] exactly when $\mathrm{HD}(m) = \mathrm{HD}(J(f))$. We show that the strict inequality holds if f is off-critically hyperbolic, except for Chebyshev polynomials. This result is a particular case of Zdunik's result if we consider f as a polynomial. It is however an extension of Zdunik's result if f is a generalized polynomial-like map. The proof follows from the non-existence of invariant affine structure proved in Section 6.

The structure of the paper is as follows: in Section 2 we give the definitions necessary to state the main Theorem. We also give the precise definition of the class of polynomial-like maps that we will be working with. In Section 3 we introduce notations and results concerning Thermodynamical formalism which will be used later. As it was mentioned before, in Section 4 we give the statement of Sullivan's rigidity Theorem for non-linear analytic hyperbolic systems. In Section 5 we show that we can apply our main Theorem to several classes of polynomial-like maps, namely: Sullivan, Yoccoz and Lyubich polynomial-like maps and Fibonacci generalized polynomial-like maps. We also in this section construct special hyperbolic sets inside $J(f)$. In Section 6 we prove that the hypothesis of Sullivan's rigidity Theorem is satisfied for the hyperbolic sets constructed in Section 5. In Section 7 we present the proof of the Main Theorem. We finish the paper with Section 8 where we derive some other consequences from the result in Section 6.

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2 Statement of the result

Definition 2.1 Let U and U_i be open topological discs, $i = 0, 1, ..., n$. Suppose that $cl(U_i) \subset U$ and $U_i \bigcap U_j = \emptyset$ if i is different than j. A generalized polynomial-like map is a map $f : \bigcup U_i \to U$ such that the restriction $f|U_i$ is a branched covering of degree $d_i, d_i \geq 1$.

We will not use the above Definition in full generality. From now on, all generalized polynomial-like maps in this work will have just one critical point. We will fix our notation as follows: $f|U_0$ is a branched covering of degree d onto U (with zero being the only critical point) and $f|U_i$ is an isomorphism onto U, if $i = 1, ..., n$.

The *filled in Julia set* of f, denoted by $K(f)$, is defined, as usually, as $K(f) = \bigcap f^{-n}(\bigcup U_i)$. The *Julia set* of f, denoted by $J(f)$, is defined as $J(f) = \partial(K(f))$. Douady and Hubbard introduced in [DH85] the notion of a polynomial-like map. Their definition coincides with the previous one when the domain of f has just one component (the critical one). They also showed that a polynomial-like map of degree d is hybrid conjugate to a polynomial of the same degree (in some neighborhoods of the Julia sets of the polynomial and the polynomial-like maps). The above definition was given in [Lyu91]. It was also showed that a generalized polynomial-like map is hybrid conjugate to a polynomial (generally of higher degree but with only one non-escaping critical point). We should keep in mind that a polynomial map is a particular case of a generalized polynomial-like map.

Definition 2.2 A generalized polynomial-like map f is said to be off-critically hyperbolic if for any neighborhood of the critical point, the set of points of $J(f)$ which avoid this neighborhood under the dynamics is hyperbolic. We also ask f to have its critical point in its Julia set.

There are several important examples of generalized polynomial-like maps which are off-critically hyperbolic. Some example are the following (see Lemma 5.1): Sullivan polynomials (see [dMvS93] and [Sul92]), Yoccoz polynomials (see [Mil91]), Lyubich polynomials (see [Lyu93]) and their respective analog classes of polynomial-like maps. Fibonacci generalized polynomial-like maps of even degree (see [LM93]) are off-critically hyperbolic too. All those classes just mentioned can be put together inside one bigger class: the class of generalized polynomial like maps which have *a priori* bounds on infinitely many generalized renormalization levels as described in [Lyu93] (which include the usual renormalization levels).

Notice that there exist examples of polynomials which are not off-critically hyperbolic. That would be the case for f having a neutral fixed point inside its Julia set (a Cremer polynomial, for example).

Definition 2.3 We say that a generalized polynomial-like map is Chebyshev if its domain is connected (i.e., it is a polynomial-like map in the sense of Douady and Hubbard) and the second iteration of its critical point is a fixed point.

Definition 2.4 We say that two generalized polynomial-like maps are in the same conformal class if they are holomorphically conjugate in some neighborhoods of their Julia sets. The conformal class of f will be denoted by $[f]$.

Definition 2.5 Let $[f]$ and $[g]$ be, as defined above, two conformal classes of generalized polynomial-like maps. Let h_0 be a homeomorphism conjugating f and g inside their respective Julia sets. Suppose that there exist U and V neighborhoods of the Julia sets of f and g and $h : U \to V$ a conjugacy between f and g. Assume that h is quasi-conformal with dilatation K_h and that it is an extension of h_0. Then we define the Teichmüller distance between $[f]$ and $[g]$ as $\mathrm{d_T}([f],[g]) = \inf_h \log K_h$, where the infimum is taken over all conjugacies h as described.

Notice that $\mathrm{d_T}([f],[g]) \geq 0$ and $\mathrm{d_T}([f],[g]) \leq \mathrm{d_T}([f],[t]) + \mathrm{d_T}([t],[g])$, where f, g and t are polynomial-like maps. In order to say that "$\mathrm{d_T}$" is a distance we need to show that if $\mathrm{d_T}([f],[g]) = 0$ then $[f] = [g]$. We prove the following Theorem:

Theorem 1 *Let f and g be two generalized polynomial-like maps which are off-critically hyperbolic, but not Chebyshev. Suppose that $\mathrm{d_T}([f],[g]) = 0$. Then f and g are conformally conjugate on a neighborhood of their Julia sets.*

We would like to point out that the above result is true even if f is Chebyshev. That would not follow from our proof. It follows from the work of Sullivan [Sul92].

3 Elements of thermodynamical formalism

We refer the reader to [Bow75] for a detailed introduction to the classical theory of thermodynamical formalism. See also [PU] for a more modern exposition of the subject. The goal of this Section is to introduce notations and classical facts.

Definition 3.1 Let f be any conformal map. In what follows by a hyperbolic or expanding set for f we understand, as usual, a closed set X such that $f(X) \subset X$ and $|D(f^n)(x)| \geq c\kappa^n$, for any x in X and for $n \geq 0$, where $c > 0$ and $\kappa > 1$.

Definition 3.2 We say that $f : X \to X$ is topologically transitive if there exists a dense orbit in X.

Suppose that the system $f : X \to X$ is hyperbolic. Then transitivity is equivalent to the following: for every non-empty set $V \subset X$ open in X, there exists $n \geq 0$ such that $\bigcup_{k \leq n} f^k(V) = X$. That is due to the existence of Markov partition for $f : X \to X$.

Throughout this section the system $f : X \to X$ will be conformal (i.e., f will be defined and conformal in a neighborhood of X) and hyperbolic. If $\phi : X \to \mathrm{R}$ is a Hölder continuous function, we say that the probability measure μ_ϕ is a *Gibbs measure* associated to ϕ if:

$$\sup_\nu\{\mathrm{h}_\nu(f) + \int_X \phi d\nu\} = \mathrm{h}_{\mu_\phi}(f) + \int_X \phi d\mu_\phi$$

where $\mathrm{h}_\nu(f)$ is the entropy of f with respect to the measure ν and the supremum is taken over all ergodic probability measures ν of the system $f : X \to X$. In this context we call ϕ a *potential function*. The *pressure* v of the potential ϕ is denoted $\mathrm{P}(\phi)$ and defined as $\mathrm{P}(\phi) = \sup_\nu\{\mathrm{h}_\nu(f) + \int_X \phi d\nu\}$, the supremum is taken over all ergodic probability measures.

The following Theorem assures us the existence of Gibbs measures.

Theorem 3.3 (Ruelle-Sinai) *Given $f : X \to X$ hyperbolic and a Hölder continuous potential $\phi : X \to \mathrm{R}$, there exists a unique Gibbs measure μ_ϕ associated to this potential.*

One needs to know when two potentials generate the same Gibbs measure. We have the following definition and Theorem to take care of that:

Definition 3.4 We say that two real valued functions $\phi, \psi : X \to \mathrm{R}$ are cohomologous (with respect to the system $f : X \to X$) if there exists a continuous function $s : X \to \mathrm{R}$ such that $\phi(x) = \psi(x) + s(f(x)) - s(x)$.

Theorem 3.5 (Livshitz) *Given $f : X \to X$ hyperbolic and two Hölder continuous functions $\phi, \psi : X \to \mathrm{R}$, the following are equivalent:*

(i) $\mu_\phi = \mu_\psi$;

(ii) $\phi - \psi$ *is cohomologous to a constant;*

(iii) For any periodic point x of $f : X \to X$ we have:

$$\sum_{i=0}^{n-1} \phi(f^i(x)) - \sum_{i=0}^{n-1} \psi(f^i(x)) = n(\mathrm{P}(\phi) - \mathrm{P}(\psi))$$

where n is the period of x.

Of special interest is the one parameter family of potential functions given by $\phi_t(x) = -t\log(|Df(x)|)$. Notice that by the definition of hyperbolic set, the functions ϕ_t are Hölder continuous. One can study the pressure function $\mathrm{P}(t) = \mathrm{P}(\phi_t)$. Here are some properties of this function:

(i) $\mathrm{P}(t)$ is a convex function;

(ii) $\mathrm{P}(t)$ is a decreasing function;

(iii) $\mathrm{P}(t)$ has only one zero exactly at $t = \mathrm{HD}(X)$;

(iv) $\mathrm{P}(0) = \mathrm{h}(f) =$ topological entropy of $f : X \to X$.

One can show that if $f : X \to X$ is hyperbolic, as we are assuming, then the Hausdorff measure of X is finite and positive. That is because one can show that the Gibbs measure associated to the potential given by $\phi_{\mathrm{HD}(X)} = -\mathrm{HD}(X) \cdot \log(|Df|)$ is equivalent to the Hausdorff measure of X. Notice that $\mathrm{P}(\phi_{\mathrm{HD}(X)}) = 0$. The Gibbs measure $\mu_{\phi_0 - \mathrm{P}(\phi_0)}$ associated to the potential $\phi_0 - \mathrm{P}(\phi_0) \equiv -\mathrm{P}(\phi_0) = const$ is the measure of maximal entropy for the system $f : X \to X$. Instead of denoting this measure by $\mu_{\phi_0 - \mathrm{P}(\phi_0)}$ we will simply write μ_{const}.

Let us denote $m = \mu_{const}$ and $\nu = \mu_{\phi_{\mathrm{HD}(X)}}$. The following is a consequence of the previous paragraph and Theorem 3.5.

Corollary 3.6 *Let $f : X \to X$ be hyperbolic. The measures m and ν are equal if and only if there exists a number λ such that for any periodic point x of $f : X \to X$ we have $|Df^n(x)| = \lambda^n$, where n is the period of x.*

4 Sullivan's rigidity Theorem

We refer the reader to [Sul86] and [PU] for the proofs of the results in this Section. In this section, the system $f : X \to X$ is assumed to be conformal (in a neighborhood of X) and hyperbolic.

Definition 4.1 An invariant affine structure for the system $f : X \to X$ is an atlas $\{(\sigma_i, U_i)\}_{i \in I}$ such that $\sigma_i : U_i \to \mathrm{C}$ is a conformal injection for each i where $X \subset \bigcup_i U_i$ and all the maps $\sigma_i \sigma_s^{-1}$ and $\sigma_i f \sigma_s^{-1}$ are affine (whenever they are defined).

Lemma 4.2 (Sullivan) *Let $f : X \to X$ be a conformal transitive hyperbolic system. The potential* $\log(|Df|)$ *is cohomologous to a locally constant function if and only if $f : X \to X$ admits an invariant affine structure.*

We call $f : X \to X$ a *non-linear system* if it does not admit an invariant affine structure. Let $g : Y \to Y$ be another system and let $h : X \to Y$ be a conjugacy between f and g. Then we say that h *preserves multipliers* if for every f-periodic point of period n we have $|Df^n(x)| = |Dg^n(h(x))|$.

Theorem 4.3 (Sullivan) *Let $f : X \to X$ and $g : Y \to Y$ be two conformal non-linear transitive hyperbolic systems. Suppose that f and g are conjugate by a homeomorphism $h : X \to Y$ preserving multipliers. Then h can be extended to an analytic isomorphism from a neighborhood of X onto a neighborhood of Y.*

5 Hyperbolic sets inside the Julia set

Let $f : \bigcup U_i \to U$ be any off-critically hyperbolic generalized polynomial-like map. Let N be any neighborhood of the critical point. We define:

$$A_N = \{z \in J(f) : f^j(z) \notin N, \, \forall j \geq 0\}$$

Notice that the set A_N is forward f-invariant. As f is off-critically hyperbolic, we know that $f : A_N \to A_N$ is hyperbolic.

The next Lemma will show us that several important examples of generalized polynomial-like maps are off-critically hyperbolic.

Lemma 5.1 *The map f restricted to A_N is hyperbolic if f is either a Yoccoz polynomial-like map or a Lyubich polynomial-like map or a Sullivan polynomial-like map or a Fibonacci generalized polynomial-like map of even degree.*

Proof. This Lemma is true because we can construct puzzle pieces for the set A_N if f belongs to one of the classes mentioned in the statement of this Lemma. We will describe how to do that.

Let f be a polynomial-like map (the generalized polynomial-like map case is identical, as we will see later). We can find neighborhoods N_n of the critical point such that their boundaries are made out of pieces of equipotentials and external rays landing at appropriate pre-images of periodic points of f. Moreover the diameter of N_n tends to zero as n grows (reference for this fact: [Hub] or [Mil91] if f is Yoccoz, [Lyu93] if f is Lyubich, [HJ] if f is Sullivan, [LvS95] if f is Fibonacci). Let us fix N_{n_0} such that $N_{n_0} \subset N$.

By construction, $\partial N_{n_0} \bigcap J(f)$ is a finite set of pre-images of periodic points. So there exists l_0 such that $\bigcup_{i=0}^{l_0} f^i(\partial N_{n_0} \bigcap J(f))$ is a forward invariant set under f. The same happens with the set of external rays landing at points in $\partial N_{n_0} \bigcap J(f)$. So if R is the set $\partial N_{n_0} \bigcap J(f)$ together with the rays landing at $\partial N_{n_0} \bigcap J(f)$, then there exists l_1 such that $I = \bigcup_{i=0}^{l_1} f^i(R)$ is an invariant set under the dynamics of f.

Each connected component of the complex plane minus the set I, intersecting A_N and bounded by a fixed equipotential of $J(f)$ is by definition a puzzle piece of level zero for A_N. So we have a Markov partition of our set A_N. We define the puzzle-pieces of level k as being the connected components of the k^{th} pre-image of the puzzle pieces of level zero that intersect A_N. We will denote by $Y_n(z)$ the puzzle piece of level n containing z.

Thickening the puzzle pieces of level zero as described in [Mil91] we will obtain open topological disks $V_0, ..., V_n$ covering A_N. We will make use of the Poincaré metric on V_i, $0 \le i \le n$. We do this thickening procedure in such a way to end up with exactly two branches of f^{-1} on each V_i, $0 \le i \le n$. Each one of those inverse branches is a holomorphic map carrying some V_i isomorphically into a proper subset of some V_j. For each puzzle piece of level zero, it follows that the inverse branches of f shrink the Poincaré distance by a factor $\lambda < 1$. That is because each puzzle piece of level zero is compactly contained in some thickened puzzle piece. Now, as $Y_n(z)$ is a connected component of the pre-image of some puzzle piece of level zero under f^n, it is easy to see that $\mathrm{diam}(Y^n(z))$ tends to zero, as n grows. So, $\bigcap_{n\ge 0} Y^n(z) = \{z\}$, for any z in A_N.

Now one can show the hyperbolicity of $f : A_N \to A_N$. Let z be any element of A_N. We can take $Y_n(z)$ with arbitrarily small diameter. Then the map $f^n : Y_n(z) \to Y_0(f^n(z))$ is an isomorphism. If V_i contains $Y_0(f^n(z))$, then the appropriate inverse branch $f^{-n} : Y_0(f^n(z)) \to Y_n(z)$ can be extended to V_i. As $Y_0(f^n(z))$ is compactly contained inside V_i, by Koebe Theorem we conclude that the map $f^n : Y_n(z) \to Y_0(f^n(z))$ has bounded distortion (not depending on n). So we have a map that maps isomorphically a set of arbitrarily small diameter to a set of definite diameter with bounded distortion. Those observations together with the fact that A_N is compact yield hyperbolicity.

The same type of argument can be carried out if f is a Fibonacci generalized polynomial-like map (the only case with disconnected domain that we are considering). If this is the case, then the domain of f has more than one component. In that case the puzzle pieces of level zero are the connected components of the domain of the map f. The puzzle pieces of higher levels are the pre-images of the puzzle pieces of level zero. It follows from [LM93] (in the degree two case) and [LvS95] (in the even degree greater then two case) that the puzzle pieces shrink to points. The rest of the proof is identical to the previous case. $\square$

We would like to point out that the above proof works for any quadratic infinitely renormalizable generalized polynomial-like map with *a priori* bounds. The proof would be exactly the same. We would use results from [Jia] for the construction of the small neighborhoods of the critical points containing just pre-periodic points and external rays on its boundaries (see also Theorem I in [Lyu95]).

Let f be any off-critically hyperbolic generalized polynomial-like map. We will now construct a sequence of sets that we will call B_n. As the sets A_N, the sets B_n will also be f-invariant. The systems $f : B_n \to B_n$ will be hyperbolic and transitive. This is the main reason why we will need this new family of sets.

Let us select one periodic point p_i from every non post-critical periodic orbit of f inside $J(f)$.

Lemma 5.2 *Let f be an off-critically hyperbolic generalized polynomial-like map. Then it is possible to construct a sequence of sets $B_n \subset J(f)$ such that:*

(i) Each set B_n is f-forward invariant, compact and hyperbolic;

(ii) For any $i = 1, 2, ..., n$, the set of pre-images of p_i belonging to B_n is dense in B_n;

(iii) $f|B_n : B_n \to B_n$ is topologically transitive;

(iv) $\bigcup_n B_n$ is dense inside $J(f)$.

Proof. Let us start the construction of the sets B_n. Let the period of p_i be n_i. We denote the orbit of p_i by $\mathcal{O}(p_i)$.

We define the set B_1 simply as being $\mathcal{O}(p_1)$. We will now define the set B_2. Let M_i be a small neighborhood of p_i such that $f^{-n_i}(M_i) \subset M_i$ $i = 1, 2$. Here $f^{-n_i}(M_i)$ stands for the connected component of the pre-image of M_i under f^{-n_i} containing p_i. There exists a pre-image y_1 of p_1 (suppose that $f^{s_1}(y_1) = p_1$) inside M_2 and a pre-image y_2 of p_2 (suppose that $f^{s_2}(y_2) = p_2$) inside M_1. The orbit $y_i, f(y_i), ...f^{s_i}(y_i) = p_i$ will be called a bridge from $\mathcal{O}(p_i)$ to $\mathcal{O}(p_j)$, for $i \neq j$. There exists a small neighborhood $\widetilde{M_i} \subset M_i$ containing p_i such that $y_i \in f^{-s_i}(\widetilde{M_i}) \subset M_j$, $i \neq j$. Notice that we are not using post-critical periodic points in our construction. That implies that all the pre-images we are taking are at a positive distance from the critical point.

In what follows $i \in \{1, 2\}$. Consider the pull back of the set M_i along the periodic orbit $p_i, f(p_i), ..., f^{n_i}(p_i) = p_i$: $M_i = M_i^0, M_i^{-1}, ..., \widetilde{M_i^{-n_i+1}}, M_i^{-n_i}$. Here $M_i^{-k} = f^{-k}(M_i)$ for $k = 0, 1, ..., n_i$. Consider also the pull back of the set $\widetilde{M_i}$ along the orbit $y_i, f(y_i), ..., f^{s_i}(y_i) = p_i$: $\widetilde{M_i} = \widetilde{M_i}^0, \widetilde{M_i}^{-1}, ...\widetilde{M_i}^{-s_i}$. Here $\widetilde{M_i}^{-k} = f^{-k}(\widetilde{M_i})$ for $k = 0, 1, ..., s_i$. We have the following collections of inverse branches of f: the first collection is $f^{-1} : M_i^{-l} \to M_i^{-l-1}$, for $l = 0, 1, ..., n_i - 1$ and the second is $f^{-1} : \widetilde{M_i}^{-l} \to \widetilde{M_i}^{-l-1}$ for $l = 0, 1, ..., s_i - 1$ (remember that $i \in \{1, 2\}$).

The union of the two collections of inverse branches of f described in the previous paragraph will be called our "selection" of branches of f^{-1} for B_2 (notice that we are specifying the domain and image of each one of the branches of f^{-1} in our "selection"). Consider now the set of all possible pre-images of p_i, $i = 1, 2$ under composition of branches of f^{-1} in our "selection" of branches. We define the set B_2 as being the closure of the set of all such pre-images.

We define B_n in a similar fashion: instead of letting i in last paragraphs to be just in $\{1, 2\}$, we let i to be in $\{1, 2, ..., n\}$. For each p_i, M_i is as before a small neighborhood around p_i, $i = 1, 2, ..., n$. There exist $y_{i,j}$ pre-image of p_i (suppose that $f^{s_{i,j}}(y_{i,j}) = p_i$) contained inside M_j (those points define the bridges between any two distinct orbits). There exists a small neighborhood $\widetilde{M_{i,j}}$ of p_i contained in M_i such that $y_{i,j} \in f^{-s_{i,j}}(\widetilde{M_{i,j}}) \subset M_j$, $i \neq j$. As for B_2 now we can define the suitable "selection" of branches of f^{-1} for B_n in a similar way. Consider now the set of all possible pre-images of p_i, $i = 1, 2, ..., n$, under composition of branches of

f^{-1} in our "selection" of branches. We define the set B_n as being the closure of the set of all pre-images just described. Notice that we can carry on our construction such that we have $B_{n-1} \subset B_n$. This finishes the construction of the sets B_n. Let us prove their properties.

The invariance and compactness are true by construction. Hyperbolicity follows because we are excluding from our construction post-critical periodic points. That implies that the distance from the set B_n to the critical point of f is strictly positive (depending on n). Hyperbolicity follows now, as f is off-critically hyperbolic.

Let us show the second property. It is clear that the pre-images of the set $\{p_1, p_2, ..., p_n\}$ under the system $f : B_n \to B_n$ is dense inside B_n. So, in order to show that the pre-images of some p_i are dense inside B_n, we just need to show that given any $1 \leq j \leq n$, there exist pre-images of p_i arbitrarily close to p_j. That is true because there exists a pre-image $y_{i,j}$ of p_i inside M_j, the neighborhood of p_j used in the construction of B_n. If we take all pre-images of $y_{i,j}$ along the periodic orbit of of p_j we will find pre-images of p_i arbitrarily close to p_j (remember that all periodic points are repelling).

Let us show (iii). By (ii), inside any open set $V \neq \emptyset$, there exists a pre-image of p_i, for each $i = 1, 2, ..., n$. Then, for some m_i, $f^{m_i}(V)$ is a neighborhood of p_i, for each p_i. Let x be any point in B_n. By the construction of B_n, there exist a j and a positive k such that $f^{-k}(x) \in M_j$. Pulling $f^{-k}(x)$ back along the orbit of p_j sufficiently many times we will find a pre-image of x inside $f^{m_j}(V)$. That implies that for some positive s, $x \in f^s(V)$. So we conclude that $B_n \subset \bigcup_{k \geq 0} f^k(V)$.

The last property is obvious because $\bigcup_n B_n$ contains all the periodic points inside $J(f)$ with the exception of at most finitely many (in the case that the critical point is pre-periodic). $\square$

6 Non-existence of an affine structure

In this Section we will show that if f is off-critically hyperbolic, but not Chebyshev, then $f : B_n \to B_n$ does not admit an invariant affine structure for $n > n_0$, for some n_0 depending on f.

Lemma 6.1 *Suppose that $f : B_n \to B_n$ and $f : B_{n+1} \to B_{n+1}$ admit invariant affine structures. Then the invariant affine structure in B_{n+1} extends the invariant affine structure in B_n.*

Proof. We will start by taking $n = 1$. Let $\{(\phi_i, V_i)\}$ be a finite atlas of an invariant affine structure for $f : B_1 \to B_1$ and let $\{(\sigma_j, U_j)\}$ be a finite atlas of an invariant affine structure for $f : B_2 \to B_2$. We will show that the collection $\{(\sigma_j, U_j)\} \bigcup \{(\phi_i, V_i)\}$ is an atlas of an invariant affine structure for $f : B_2 \to B_2$. Notice that the invariant affine structure for $f : B_1 \to B_1$ is unique, given by the linearization coordinates of p_1.

Let us suppose that $V_i \bigcap U_j \neq \emptyset$. We will check that the change of coordinates $\sigma_j(\phi_i)^{-1}$ is affine. Suppose that the closure of $V_i \bigcap U_j \bigcap B_2$ is empty. Then if we shrink U_j (to $U_j \setminus V_i$) we can act just as if $V_i \bigcap U_j = \emptyset$. So we can assume that the closure of $B_2 \bigcap V_i \bigcap U_j$ is not empty. Let x be an element belonging to this intersection. We can assume for simplicity that V_i is a chart in B_1 containing p_1 and U_i is a chart in B_2 containing p_1 (remember that p_i is the enumeration of periodic points used to construct the sets B_n and that $B_1 \subset B_2$). As the affine structure for periodic orbits is unique, we conclude that (σ_i, U_i) and (ϕ_i, V_i) are the same (up to an affine map) in a neighborhood of p_1.

We can pull x back by f^{n_1} along the (periodic) orbit of p_1 until we find $y \in B_2$, a pre-image of x under some iterate of f^{n_1}. Notice that y is in fact an element of B_2. That is because the inverse branches of f following the orbit of p_1 are in the "selection" of inverse branches used to construct B_2. Because (ϕ_i, V_i) is a linearization coordinate around the periodic point p_1, we conclude that $\phi_i f^{ln_1}(\phi_i)^{-1}$ is affine from a neighborhood of $\phi_i(y)$ to a neighborhood of $\phi_i(x)$. On the other hand, as $y \in B_2 \bigcap U_i$ and $x = f^{ln_1}(y) \in B_2 \bigcap U_j$, we conclude that $\sigma_j f^{ln_1}(\sigma_i)^{-1}$ is affine from a neighborhood of $\sigma_i(y)$ to a neighborhood of $\sigma_j(x)$. Keeping in mind that σ_i is equal to ϕ_i (up to an affine map), we get that the change of coordinate $\sigma_j(\phi_i)^{-1}$ is affine (see Figure 1). From that follows trivially that any composition of the form $\sigma_j f(\phi_i)^{-1}$ and $\phi_i f(\sigma_j)^{-1}$ is affine, whenever they are defined. So we proved the Lemma in the case $n = 1$.

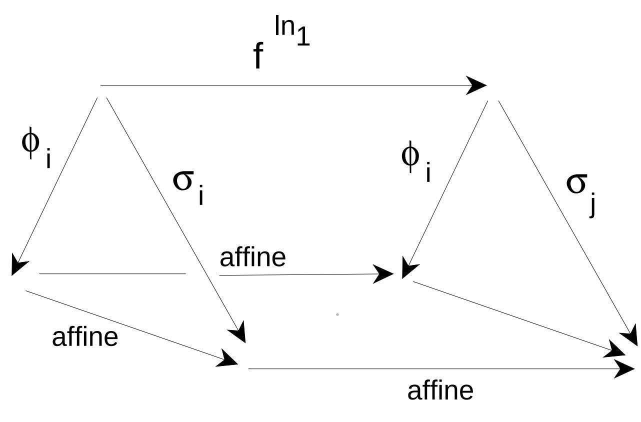

Figure 1: Commutative diagram of charts

Now suppose that we have some invariant affine structure $\{(\phi_i, V_i)\}$ in B_n and $\{(\sigma_j, U_j)\}$ in B_{n+1}. We want to show that $\{(\phi_i, V_i)\} \bigcup \{(\sigma_j, U_j)\}$ is an invariant affine structure in B_{n+1}. Suppose that U_j intersects a chart of one of the periodic points $p_1, p_2, ..., p_n$ in B_n. Then the change of coordinates from U_j to one of those charts is affine (same as the proof for $n = 1$). Now let U_j and V_i be two arbitrary charts with non-empty intersection. We can assume that there exists x in the closure of $B_{n+1} \bigcap U_j \bigcap V_i$ (otherwise we can shrink U_j to $U_j \setminus V_i$). Let V_1 be the chart around p_1 in the affine structure for B_n and U_1 be the chart around p_1 in the affine structure for B_{n+1}. Then $V_1 \bigcap U_1$ is a neighborhood of p_1. Inside B_n we can pull back V_i until we find a pre-image (of V_i with respect to some iterate of $f : B_n \to B_n$) strictly inside $U_1 \bigcap V_1$ (this is possible by property (iii) in Lemma 5.2 and hyperbolicity). So $f^{-l}(V_i) \subset V_1 \bigcap U_1$. Then it is clear that $\phi_i f^l(\phi_1)^{-1}$ is affine in $f^{-l}(V_i)$. On the other hand, $\sigma_j f^l(\sigma_1)^{-1}$ is affine in a subset of $f^{-l}(V_i)$ containing the pre-image y of x via f^{-l} (remember that x is the element in B_{n+1} contained in $U_j \bigcap V_i$). As σ_1 and ϕ_1 are equal up to an affine transformation in a neighborhood of y (because both are linearizing coordinates around a periodic point), we conclude that the change of coordinates $\phi_i(\sigma_j)^{-1}$ is affine (just imagine an appropriate diagram similar to the one in Figure 1). It is trivial to check that the affine structure defined by $\{(\phi_i, V_i)\} \bigcup \{(\sigma_j, U_j)\}$ is invariant under f. $\square$

We would like to point out that with exactly the same demonstration as above we show that if there exists an invariant affine structure for the system $f : B_n \to B_n$, then it is unique.

Lemma 6.2 *If f is off-critically hyperbolic and is not Chebyshev, then there is a positive number n_0 such that $f : B_n \to B_n$ does not admit an invariant affine structure if $n > n_0$ (n_0 depends on f).*

Proof. Suppose that $f : B_n \to B_n$ admits an affine structure, for infinitely many n. Then all those structures coincide when defined in common subsets by Lemma 6.1. This implies that we can define the set $X = \bigcup_n B_n$ and an invariant affine structure for $f : X \to X$ (notice that X is f-invariant and dense inside $J(f)$). Let us denote the elements of the atlas defining such affine structure over X by (σ_i, U_i).

There exists n such that some element of $f^{-n}(0)$, say y_0, belongs to some U_β (here we need to have our map f not conjugate to z^d). There exists m such that some element of $f^{-m}(f^2(0))$, say y_1, which is not a pre-image of the critical point 0 belongs to U_α, for some α (notice that this is not true if f is a Chebyshev generalized polynomial-like map). We can take U_α and U_β small enough such that $f^m : U_\alpha \to f^m(U_\alpha) = U'_\alpha$ and $f^n : U_\beta \to f^n(U_\beta) = U'_\beta$ are isomorphisms (see Figure 2). We can also assume that $f^2(U'_\beta) = U'_\alpha$. We can find $x \in X \bigcap U'_\beta$ because X is dense inside $J(f)$. Notice that we need to have the critical point inside $J(f)$ in order to be able to construct U'_β with small diameter intersecting X. Then $f^2(x) \in U'_\alpha$. We can take charts from the atlas on X, say $(\sigma_\gamma, U_\gamma), U_\gamma \subset U'_\beta$ and $(\sigma_\nu, U_\nu), U_\nu \subset U_{\alpha'}$ containing x and $f^2(x)$ respectively. Let $\sigma'_\beta = \sigma_\beta f^{-n}$ and $\sigma'_\alpha = \sigma_\alpha f^{-m}$, where the inverse branches f^{-n} and f^{-m} are defined according to our previous discussion. Notice that σ'_β and σ'_α are isomorphisms onto their respective images. Let $A = \sigma_\nu f^2 \sigma_\gamma^{-1}$. The map A is affine (because A is the map f^2 viewed from the atlas over X).

Notice that

$$\sigma'_\alpha f^2(\sigma'_\beta)^{-1} = (\sigma_\nu f^m \sigma_\alpha^{-1})^{-1} A (\sigma_\gamma f^n \sigma_\beta^{-1})$$

when we restrict both sides of the equation to the set $\sigma'_\beta(U_\gamma)$.

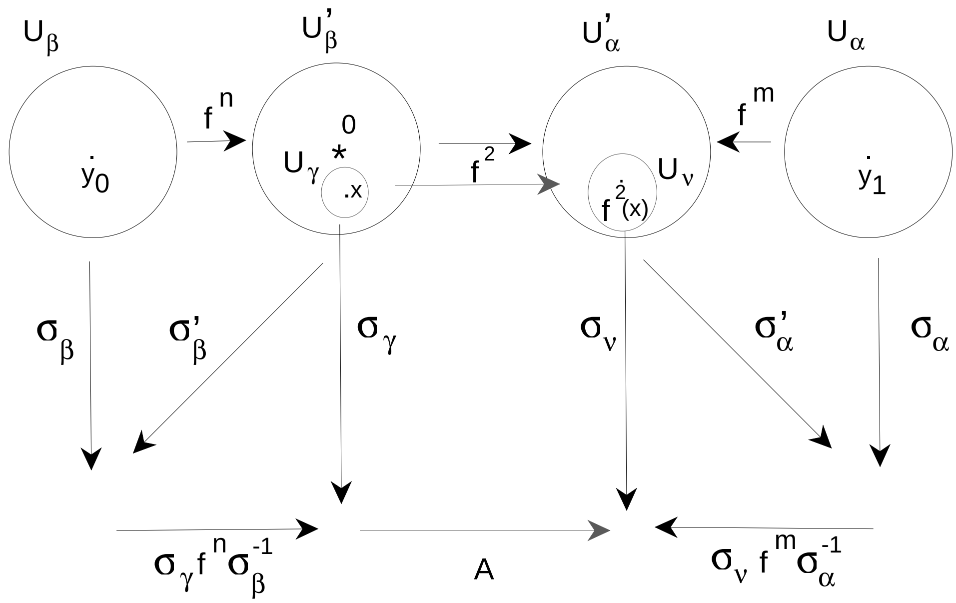

Figure 2: Commutative diagram

The left-hand side of the above formula is a restriction of a degree two branched covering, namely $\sigma_\alpha f^{-m} f^2 f^n (\sigma_\beta)^{-1}$. If we restrict the right-hand side of our equation to $\sigma'_\beta(U_\gamma)$ we get an affine map. Contradiction! The Lemma is proved. $\square$

7 Proof of the Theorem

We will present in this Section the proof of Theorem 1. Let $f : \bigcup U_i \to U$ and $g : \bigcup V_i \to V$ be two off-critically hyperbolic generalized polynomial-like maps, but not Chebyshev. Let us suppose that $\mathrm{d_T}(f, g) = 0$. This implies that there exists a homeomorphism $h : J(f) \to J(g)$ conjugating f and g which is extended by quasi-conformal maps of arbitrarily small distortion. This implies that h preserves multipliers.

We define the hyperbolic sets $X_n = B_n \subset J(f)$ (as introduced in Section 5) and $Y_n = h(X_n) \subset J(g)$.

The systems $f : X_n \to X_n$ and $g : Y_n \to Y_n$ do not admit invariant affine structures if n is big (see Lemma 6.2). In other words, $f : X_n \to X_n$ and $g : Y_n \to Y_n$ are non-linear systems, for n big. So by Theorem 4.3 we know that there exist open neighborhoods O_n of X_n and O'_n of Y_n and holomorphic isomorphisms $H_n : O_n \to O'_n$ extending h_n. We can assume that $O_n \subset O_{n+1}$ and $O'_n \subset O'_{n+1}$. Notice that by analytic continuation we have that $H_n = H_{n+1}$ on O_n.

We define two open sets, $O = \bigcup_n O_n$ and $O' = \bigcup_n O'_n$. We can define $H : O \to O'$ by the following: for any $z \in O$ there exists some n such that $z \in O_n$. Then we define $H(z) = H_n(z)$. The map H is well defined. The map H is holomorphic because locally it coincides with H_n, for some n. It is also injective. The map H conjugates $f|X$ and $g|Y$, where we define $X = \bigcup X_n$ and $Y = \bigcup Y_n$. The sets X and Y are dense subsets of $J(f)$ and $J(g)$, respectively. So the conjugacy H is defined in a open neighborhood of a dense subset of $J(f)$. Our goal is to extend H to a neighborhood of the whole Julia set.

Suppose that z is a point in $J(f)$ not belonging to O. If z is not the critical value, then there exists n and an element z_{-n} of $f^{-n}(z)$, such that $z_{-n} \in O$, and the iteration f^n restricted to a small ball around z_{-n} is injective. Consider the holomorphic map defined in a small neighborhood W of z by $\phi = g^n H f^{-n}$, where by f^{-n} we understand the branch of f^{-n} that takes z to z_{-n}. If W is sufficiently small, then ϕ is an isomorphism. It is clear that ϕ and h coincide where both are defined. By analytic continuation, that means that ϕ also coincides with H where both are defined. In this way we managed to extend H to an open neighborhood of $J(f) \setminus \{f(0)\}$. We will keep calling this extension H. If z is the critical value, then instead of looking for pre-images of z in order to repeat the previous reasoning, just look for the first image of z. Remember that now the second iterate of the critical point belongs to the domain of H. We can define an isomorphism in a small neighborhood of the critical value given by $\phi = g^{-1} H f$. The same argument as before goes through to show that we have extended H to an open neighborhood of $J(f)$. This proves the Theorem.

8 Other consequences of the non-existence of affine structure

According to Lemma 4.2, the non-existence of affine structure for the system $f : A_N \to A_N$ is equivalent to $\log(|Df)|$ not cohomologous to a locally constant function in A_N. In particular, the non-existence of an affine structure implies that $\log(|Df|)$ is not cohomologous to a constant function inside A_N. This last observation together with Theorem 3.5 implies the following:

Corollary 8.1 *If f is off-critically hyperbolic, then there is no λ such that for any n and any f-periodic point p, $|Df^n(p)| = \lambda^n$.*

Proof. By our previous comments, we conclude that if $\mathrm{diam}(N)$ is small, then there is no λ such that $|Df^n(p)| = \lambda^n$ for any n and any f-periodic point p inside A_N. That implies the Corollary. $\square$

If μ is a Borel probability measure in $J(f)$, then we define the Hausdorff dimension of μ as $\mathrm{HD}(\mu) = \inf \mathrm{HD}(Y)$ where the infimum is taken over all sets $Y \subset J(f)$ with $\mu(Y) = 1$.

Remember that the measure $m = \mu_{const}$ is the measure of maximal entropy for the hyperbolic system $f : X \to X$. Zdunik proved in [Zdu90] that for rational maps $\mathrm{HD}(m) = \mathrm{HD}(J(f))$ if and only if f is $z \mapsto z^d$ or a Chebyshev polynomial. The following is a particular case of Zdunik's result if we consider f as a polynomial. It is however an extension of Zdunik's result if f is a generalized polynomial-like map:

Corollary 8.2 *If f is off-critically hyperbolic and m is the measure of maximal entropy for f, then* $\mathrm{HD}(m) < \mathrm{HD}(J(f))$.

Proof. It was shown in [PUZ89] (see Theorem 6) that it is enough to check that $\log(|Df|)$ is not cohomologous to a constant in $J(f)$. By that we mean the following: there is no real function h which is equal m-a.e. to a continuous function in a small neighborhood of any point in $J(f)$ without the post-critical set and $\log(|Df|) = c + h(f(x)) - h(x)$.

Suppose that $\log(|Df|)$ is cohomologous to a constant, in the sense defined in the previous paragraph. Remember that the sets B_n are at a positive distance from the closure of the critical orbit. So we would conclude that $\log(|Df|)$ is cohomologous to a constant (in the sense of Definition 3.4). Lemma 4.2 and Lemma 6.2 imply that this is impossible. $\square$

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