

DYNAMICAL STABILITY IN LAGRANGIAN SYSTEMS

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1. Introduction

The notion of stability in Dynamical Systems refers to dynamical behavior that persists under perturbation.¹ By altering the nature of the persistence and the class of perturbations one obtains various forms of stability. These various forms of stability have proved to be extremely important throughout the history of dynamics. In perhaps the best known cases, KAM theory and structural stability, the dynamical stability involves small perturbations. However, dynamical persistence under large perturbations (in a restricted class) is often studied and has proved to be quite powerful. Large perturbation theories usually have a strong topological component. This is because behavior that persists under large perturbations must be very fundamental to the system, and the most fundamental aspect of a dynamical system is the topology of the underlying manifold.

In applying stability results, one often begins with a model system whose dynamics are understood and then perturbs it. The stability theorems indicate which dynamics of the model system must be present in the perturbed system. This strategy often yields a great deal of information about the perturbed system that could not be otherwise obtained. In particular, it frequently provides a framework for the investigation the other dynamics present in the perturbed system. Since the dynamics of the model system must be present in all perturbed systems the model system may also be viewed as a dynamically minimal element in the allowed class.

This paper surveys various results concerning stability for the dynamics of Lagrangian (or Hamiltonian) systems on compact manifolds. The main, positive results state, roughly, that if the configuration manifold carries a hyperbolic metric, *i.e.* a metric of constant negative curvature, then the dynamics of the geodesic flow persists in the Euler-Lagrange flows of a large class of time-periodic Lagrangian systems. This class contains all time-periodic mechanical systems on such manifolds. These results are given in Theorem 1 and Theorem 3 in Section 3. Complete proofs appear in [10]. Many of the results on Lagrangian systems also hold for twist maps on the cotangent bundle of hyperbolic manifolds.

We also present a new stability result for autonomous Lagrangian systems on the two torus (see Theorem 5, Section 5) which shows, among other things, that there are

¹In other contexts “stability” may mean that small perturbations of initial conditions in (some region of) the domain of a fixed system give rise to small changes in asymptotic behavior. We will always use stability to refer to persistence of the *global* dynamics under perturbations of the *entire* system.

minimizers of all rotation directions. However, in contrast to the previously known [19] case of just a metric, the result allows the possibility of gaps in the speed spectrum of minimizers. Our negative result is an example of an autonomous *mechanical* Lagrangian system on the two-torus in which this gap actually occurs. The same system also gives us an example of a Euler-Lagrange minimizer which is not a Jacobi minimizer on its energy level.

Our results generalize several theories that contain what may be viewed as stability results. The first is the Aubry-Mather theory. This theory shows that an area preserving monotone twist map of the annulus always has nicely behaved invariant sets with each rotation number. These invariant sets can be viewed as the remnants of the invariant circles of a minimal model, the rigid twist, or equivalently, the time one map of the full geodesic flow of the Euclidean metric on the circle. The Aubry-Mather theory is closely related to Hedlund's work on geodesics on the two torus ([19], *cf* [5]). Hedlund showed that for any Riemannian metric there are geodesics with all rotation directions, and thus the model system in this case is the Euclidean metric on the torus. In related work, closely connected to the hyperbolic manifold results here, Morse [32] showed that any metric on a higher genus surface has a collection of geodesics that "shadow" in the universal cover the geodesics of the hyperbolic metric. There are also generalizations to any dimensions and improvements of Morse's results due to Klingenberg [22], Gromov [17], and MacKay and Denvir [30]. Because we allow time dependent Lagrangians, these results do not imply ours.

All these theories share the property that the orbits of the dynamical system under consideration correspond to extremals of a variational problem defined in the universal cover of the configuration space. The orbits that correspond to minima of the variational problem have special properties; they behave approximately like the solutions to the variational problem associated with the model system. This enables one to take limits of minimizing orbit segments or minimizing periodic orbits in order to construct a large set of minimizing orbits on which the dynamics is similar to that of the unperturbed system. It is natural to study minimizers in the perturbed systems because all orbits of the unperturbed system are minimizers.

There is a simple heuristic connection between these different theories. In the Aubry-Mather theory, Aubry's Fundamental Lemma [4], [29], states that minimizers for a twist map are ordered like orbits under a rigid rotation (*i.e.* like orbits of the time-1 map of the Euclidean geodesic flow on the circle). This is easily seen to imply that such orbits have a rotation number and the rotation number of a limit of such orbits is the limit of the rotation numbers. If $\{x_k\}_{k \in \mathbf{Z}}$ is such an orbit and it has a rotation number $\omega = \lim x_k/k$, then one checks that $|x_k - x_l - (k-l)\omega| \leq 1$ for all $k, l \in \mathbf{Z}$. This immediately implies that

$$|\omega||k-l| - 1 \leq |x_k - x_l| \leq |\omega||k-l| + 1.$$

This, in Gromov's language, says that the well ordered orbits are *quasi-geodesics*. It turns out that most of the old or more recent Riemannian geometry results on the stability of the hyperbolic geodesic flow can be proved using the fact that minimal geodesics for any metric are quasi-geodesics for the hyperbolic metric. In our work on hyperbolic manifolds [10] that we survey here, we also use a limiting argument, and our central Proposition 2 (which is rather trivial in the autonomous case) states that *Euler-Lagrange minimizing segments of a given average speed are quasi-geodesics*. The proof of this proposition uses techniques that go back to Aubry's proof of his Fundamental Lemma, which have a parallel

in Riemannian geometry under the guise of *curve shortening* arguments. We should remark here that the property of being a quasi-geodesic is a rather weak regularity property and even though it is satisfied by all minimizers for our large class of systems on *any* manifolds, it is not sufficiently strong to make the limiting argument work on manifolds that do not support a hyperbolic metric (except in cases of very low dimension). We illustrate this in Section 3.3 with the Hedlund example on the three torus.

We were greatly influenced by and used many techniques of the recent work of Mather [27], [28] (see also [25]), which attempts, among other things, to generalize the Aubry-Mather theory to higher dimensions. Moser showed that convex, time-periodic Lagrangian systems on compact manifolds are a natural generalization of twist maps in that twist maps are always the time-1 maps of such Lagrangians on the tangent space of the circle ([33]). Denzler [12] followed through this philosophy and gave a proof of the Aubry-Mather Theorem in the larger context of time-periodic, convex Lagrangians on the circle.

The class of Lagrangians we consider is almost the same as Mather's. One fundamental difference, however, is that Mather uses minimizers not in the universal cover as is done here in the hyperbolic case, but rather in the universal free Abelian cover (the cover with deck group $H(M; \mathbb{Z})/torsion$). If the fundamental group of the configuration space is torsion-free Abelian (eg. a torus) this cover is the universal cover, but in general, it is much smaller than the universal cover. Whereas Mather's theory works on any compact manifold, in the special case of hyperbolic manifolds, much information is lost. Indeed, many orbits of the hyperbolic geodesic flow are not minimizers in the sense of Mather (see Example 5). Hence there is no chance in finding corresponding orbits in a perturbed system by looking for such minimizers. On the other hand, in our results on the torus presented here, we use the full strength of Mather's results (which we review for the reader's convenience), in particular the Lipschitz graph property of his generalized Mather sets.

It is important to remark, especially in the context of this conference, that Mather's stated goal in his recent work is not so much to prove stability results for their own sake but to use his Mather sets (the remnants he finds of the dynamics of the unperturbed system) as a stairway to (generic) diffusion. He accomplished this program in [26] for twist maps (which are time-1 maps of time periodic Lagrangians on \mathbb{S}^1), and gets partial results for the general case in [28]. We hope that our work might help to reach this important goal.

This paper is organized as follows. Section 2 introduces the Lagrangian setting and several fundamental lemmas are given. In Section 3, we present and outline the sketch of our two theorems on time periodic Lagrangians on hyperbolic manifolds. The first (Theorem 1 in Section 3) states that you can shadow any geodesic of the hyperbolic metric by a minimizer of the Lagrangian at uniform bounded distance. This theorem also holds for symplectic twist maps in T^*M . The second (Theorem 3) concerns large invariant sets for the Euler Lagrange flow which are built from these shadowing minimizers, and shows that the dynamics on the sets is semiconjugate to the hyperbolic geodesic flow. At the end of the section we remark on why this scheme of proof does not work on the three torus.

In Section 4, we briefly review Mather's recent work on minimal measures, and illustrate it by examples. In Section 5, we apply Mather's theory to autonomous systems on the two torus, and give a fairly complete description of the rotation set of minimizers in this case (Theorem 5, Section 5). In particular, we find orbits of all rotation directions, and infinitely many of different average speeds in each direction.

In Section 6, we show that the gaps in speed that are allowed in Theorem 5 actually occur in natural, mechanical systems.

2. Preliminaries

In this section we introduce notation and recall some basic results needed in the sequel. For a thorough discussion of Lagrangian systems and minimizers the reader is urged to consult Mather [27] and Mañé [25]. We also indicate how to translate the different notions to the setting of symplectic twist maps.

2.1. LAGRANGIAN SYSTEMS.

The main objects in the Lagrangian formulation of mechanics are a configuration manifold M and a real valued function called a *Lagrangian* defined on the tangent bundle TM . The configuration spaces of interest here are closed manifolds M with a fixed Riemannian metric g . The induced norm on the tangent bundle is denoted $\|v\|$. We consider time-periodic systems determined by a C^2 -Lagrangian $L : TM \times \mathbb{S}^1 \rightarrow \mathbb{R}$. The basic variational problem is to find curves $\gamma : [a, b] \rightarrow M$ that are extremal for the *action*

$$A(\gamma) = \int_a^b L(\gamma, \dot{\gamma}, t) dt$$

among all absolutely continuous curves $\beta : [a, b] \rightarrow M$ that have the same endpoints $\beta(a) = \gamma(a), \beta(b) = \gamma(b)$.

Under appropriate hypothesis (eg. γ is C^1), such a γ satisfies the Euler-Lagrange second order differential equations

$$\frac{d}{dt} \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t) - \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t), t) = 0.$$

Using local coordinates these equations yield a first order time-periodic differential equation on TM , and thus in the standard way, a vector field on $TM \times \mathbb{S}^1$. Since $TM \times \mathbb{S}^1$ is not compact it is possible that trajectories of this vector field are not defined for all time in \mathbb{R} and thus do not fit together to give a global flow (i.e. an \mathbb{R} -action). When the flow does exist, it is called the *Euler-Lagrange or E-L flow*.

We will require that Lagrangians satisfy certain hypotheses.

Hypothesis:

L is a C^2 function $L : TM \times \mathbb{S}^1 \rightarrow \mathbb{R}$ that satisfies:

(a) *Convexity:* $\frac{\partial^2 L}{\partial v^2}$ is positive definite.

(b) *Completeness:* The Euler-Lagrange flow determined by L exists.

(c) *Superquadratic:* There exists a $C > 0$ so that $L(x, v, t) \geq C\|v\|^2$.

or

(c') *Superlinear:* $\frac{L(x, v, t)}{\|v\|} \rightarrow \infty$ when $\|v\| \rightarrow +\infty$.

We will refer to (a), (b) and (c) as *Our Hypotheses* and (a), (b) and (c') as *Mather's Hypothesis*. Note that our hypotheses are a little stronger than Mather's (adding a constant to the Lagrangian doesn't change the E-L flow).

Example 1 : Mechanical Lagrangians. As pointed out by Mañé, Mather's Hypothesis (and hence ours) are satisfied for mechanical Lagrangians, i.e. those of the form

$$L(x, v, t) = \frac{1}{2}\|v\|^2 - V(x, t),$$

where the norm is taken with respect to any Riemannian metric on the manifold. (In fact, one may allow the norm to vary with time, under some conditions, see [25], page 44).

2.2. MINIMIZERS

Of particular interest here are extremals of the variational problem that minimize in the following sense. If \tilde{M} is a cover of M , L lifts to a real valued function (also called L) defined on $T\tilde{M} \times \mathbb{S}^1$. A curve segment $\gamma : [a, b] \rightarrow \tilde{M}$ is called a \tilde{M} -*minimizing segment* or an \tilde{M} -*minimizer* if it minimizes the action among all absolutely continuous curves $\beta : [a, b] \rightarrow \tilde{M}$ which have the same endpoints.

A fundamental theorem of Tonelli implies that if L satisfies Mather's Hypotheses, then given $a < b$ and two distinct points $x_a, x_b \in \tilde{M}$ there is always a minimizer γ with $\gamma(a) = x_a$ and $\gamma(b) = x_b$. Moreover such a γ is automatically C^2 and satisfies the Euler-Lagrange equations (this uses the completeness of the E-L flow). Hence its *differential* $d\gamma(t) = (\gamma(t), \dot{\gamma}(t))$ yields a solution $(d\gamma(t), t)$ of the E-L flow. A curve $\gamma : \mathbb{R} \rightarrow \tilde{M}$ is called a *minimizer* if $\gamma|_{[a,b]}$ is a minimizer for all $[a, b] \subset \mathbb{R}$. When the domain of definition of a curve is not explicitly given it is assumed to be \mathbb{R} .

As noted in the introduction, Mather [27] and Mañé [25] use \overline{M} -minimizers where \overline{M} is the universal free Abelian cover. The universal cover (which we denote \tilde{M} from now on) is used here. If γ is an \tilde{M} -minimizer, we will simply say it is a minimizer.

Our main task is to get control of the speed and geometry of \tilde{M} -minimizers. Given a smooth curve $\gamma : [c, d] \rightarrow \tilde{M}$ and a segment $[a, b] \subset [c, d]$, the *average displacement* in the cover over the interval $[a, b]$ is measured by

$$\delta(\gamma; a, b) = \frac{d(\gamma(a), \gamma(b))}{b - a}$$

where d is the topological metric on \tilde{M} constructed from the lift of the given Riemannian metric g .

The fact that L is superlinear or superquadratic implies some very useful simple estimates on the average action of minimizers. These estimates are essentially in Mañé [25] and Mather [27].

Lemma 1 *Given a Lagrangian L satisfying Mather's Hypothesis, there are functions $K \mapsto C_K^{min}, K \mapsto C_K^{max}$ both increasing to infinity such that if γ is a minimizer and $\delta(\gamma; a, b) = K$, then*

$$C_K^{min} K \leq \frac{1}{b - a} \int_a^b L(\gamma, \dot{\gamma}, t) dt \leq C_K^{max} K.$$

In particular, if $L = \frac{1}{2} \|\dot{x}\|^2 - V(x, t)$, we have:

$$\frac{K^2}{2} - V_{max} \leq \frac{1}{b - a} \int_a^b L(\gamma, \dot{\gamma}, t) dt \leq \frac{K^2}{2} - V_{min}.$$

Proof Referring the reader to [25] or [10] for a proof of the first, general statement of the lemma, we give a proof of the mechanical case. We first estimate the lower bound:

$$\begin{aligned} & \frac{1}{b - a} \int_a^b L(\gamma, \dot{\gamma}, t) dt \\ & \geq \frac{1}{b - a} \int_a^b \frac{\|\dot{x}\|^2}{2} - V_{max} dt \geq \frac{1}{2(b - a)^2} \left(\int_a^b \|\dot{x}\| dt \right)^2 - V_{max} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2(b-a)^2} \left(\text{length}(\gamma|_{[a,b]}) \right)^2 - V_{max} \geq \frac{1}{2(b-a)^2} (\text{dist}(\gamma(a), \gamma(b)))^2 - V_{max} \\
&= \frac{K^2}{2} - V_{max}.
\end{aligned}$$

The inequality on the second line follows from Cauchy-Schwarz: $\int f^2 \int g^2 \geq (\int fg)^2$, setting $f = \|\dot{x}\|$ and $g = 1$.

To get the upper bound, let $\Gamma : [a, b] \rightarrow \tilde{M}$ with $\gamma(a) = \Gamma(a)$ and $\gamma(b) = \Gamma(b)$ be a length minimizing *geodesic* segment with respect to the given metric. Then $\|\dot{\Gamma}\| = \rho(\Gamma; a, b) = \delta(\gamma; a, b) = K$. Thus, since γ is a minimizer,

$$A(\gamma|_{[a,b]}) \leq A(\Gamma) \leq \int_a^b \frac{K^2}{2} - V_{min} \leq \left(\frac{K^2}{2} - V_{min} \right) (b-a),$$

yielding the upper bound. \square

2.3. EXACT SYMPLECTIC TWIST MAPS

For more details on symplectic twist maps, the reader is referred to [16] or [23] (see also [7] and [21]). An *exact symplectic twist map* F is a map from a subset U of the cotangent bundle of a manifold N (which we allow to be noncompact) into U , which comes equipped with a *generating function* $S : N \times N \rightarrow \mathbb{R}$ that satisfies

$$F^*(p \, dx) - p \, dx = P \, dX - p \, dx = dS(x, X), \quad (1.2)$$

where (X, P) are the coordinates of $F(x, p)$ (this can also be written in a coordinate free manner).

Because the one-form $P \, dX - p \, dx$ in (1.2) is exact, one says that F is *exact*. Note that taking the exterior differential of (1.2) yields $dP \wedge dX = dp \wedge dx$, and so any exact F is also symplectic, *i.e.* it preserves the standard symplectic form. The fact that S is expressed using the coordinates (x, X) instead of (x, p) is the *twist condition*. Given S , one can retrieve the map (at least implicitly) from $p = -\frac{\partial S}{\partial x}$ and $P = \frac{\partial S}{\partial X}$. This can be done globally (*i.e.* $U = T^*N$) only when N is diffeomorphic to a fiber of T^*N , for example when N is the covering space of the n -torus or of a manifold of constant negative curvature.

The variational problem for Lagrangian systems translates into a discrete variational problem for twist maps: the role of curves in the continuous setting is taken by sequences of points (“integer time curves”), and the action of a finite sequence $\bar{x} = \{x_n, \dots, x_m\}$ is given by $W(\bar{x}) = \sum_n^{m-1} S(x_k, x_{k+1})$. This corresponds closely to the continuous setting when the exact symplectic twist map F is the time-one map of an E-L flow. In this case, $S(x, X) = \int_0^1 L(x, \dot{x}, t) dt$, where $x(t)$ is the minimizer over the interval $[0, 1]$ with endpoints x and X .

In direct correspondence to Lagrangian systems, critical points of W (with fixed time and configuration endpoints) correspond to orbits of F (this is closely related to the method of broken geodesics in Riemannian geometry). Action minimizers are sequences that minimize W over any of their subsegments. The natural growth condition on the generating function

$$S(x, X) \geq C \, \text{dist}^2(x, X),$$

implies the analog of Tonelli’s theorem: minimizers always exist between any two points over any given (integer) interval of time. Moreover, there is an exact analog of Lemma

1.1: the average action of minimizers is bounded below and above by functions of the average displacement. The proof is virtually identical to the continuous time case, replacing geodesics with orbits of the time-one map of the geodesic flow.

Example 2 Let M be \mathbb{T}^n or a closed hyperbolic manifold, and let $N = \tilde{M}$ be the universal cover \mathbb{R}^n or \mathbb{H}^n , respectively. On the covering space, define the *generalized standard map* using its generating function $\tilde{M} \times \tilde{M} \rightarrow \mathbb{R}$,

$$S(x, X) = \frac{1}{2}dist^2(x, X) + V(x)$$

where the distance $dist$ is induced by the Euclidean metric in \mathbb{R}^n , or the hyperbolic metric on \mathbb{H}^n , and $V(x)$ is $\pi_1(M)$ -equivariant, *i.e.* it descends to a function on M . A short argument shows that one can use the relation (1.2) to solve for (X, P) in terms of (x, p) and thus obtain an exact symplectic twist map on $T^*\tilde{M}$ that, in turn, induces a map on T^*M (also called a twist map). For more general examples, *cf* [16].

In certain cases the twist map theory overlaps with the continuous theory. If a twist map f of $T^*\mathbb{T}^n$ has a generating function that is super quadratic in $\|X - x\|$, the mixed partial $\partial_{12}S$ is symmetric, and for some $a > 0$ satisfies the convexity condition

$$\langle \partial_{12}S(x, X).v, v \rangle \leq -a\|v\|^2$$

uniformly in (x, X) , then F is the time-one map of an E-L flow derived from a one-periodic Lagrangian that is superquadratic in the velocity. Moser [33] gives the proof in the case $n = 1$. Bialy and Polterovitch remark in [11] that Moser's proof goes through in the case $n > 1$. This is not quite so, but they subsequently obtained a different proof (personal communication). Note that the generating function for the generalized standard map satisfies these hypothesis.

2.4. JACOBI MINIMIZERS VS. E-L MINIMIZERS

In this section, we consider autonomous mechanical Lagrangian of the form $L(x, \dot{x}) = (1/2)\|\dot{x}\|^2 - V(x)$ where $V \leq 0$ and $\|\dot{x}\|$ comes from a Riemannian metric on the manifold M . In this case there is a geometric way to look for E-L minimizers on a given energy level. Recall that, in the tangent bundle coordinates, the energy (Hamiltonian) is given by $H(x, \dot{x}) = L(x, \dot{x}) + 2V(x) = (1/2)\|\dot{x}\|^2 + V(x)$

The norm on the tangent bundle coming from the *Jacobi metric* with energy $E > V_{max}$ is $\sqrt{E - V(x)}\|\dot{x}\|$. The important fact is that geodesics of the Jacobi metric considered as curves in M are always the projection to M of some solution curve with energy E of the Hamiltonian (or E-L) flow, and conversely, all such projections are geodesics of the Jacobi metric. *However*, the parameterizations of the geodesics and the solutions will usually be different. (eg see Arnol'd [3] or Abraham and Marsden [1])

Put in other language, this means that the extrema of the corresponding integrals coincide in some sense. What is of importance here is whether the minimizers coincide. Let us define the notion of minimizer more carefully.

Definition 1 1. A curve $\gamma : [a, b] \rightarrow \tilde{M}$ is called an *E-L minimizer* if for all absolutely continuous $\beta : [a, b] \rightarrow \tilde{M}$ with $\beta(a) = \gamma(a)$ and $\beta(b) = \gamma(b)$,

$$\int_a^b L(\gamma, \dot{\gamma}) dt \leq \int_a^b L(\beta, \dot{\beta}) dt$$

2. A curve $\gamma : [a, b] \rightarrow \tilde{M}$ is called a Jacobi minimizer for energy E if for all absolutely continuous $\beta : [a_\beta, b_\beta] \rightarrow \tilde{M}$ with $\beta(a_\beta) = \gamma(a)$ and $\beta(b_\beta) = \gamma(b)$,

$$\int_a^b \sqrt{E - V(\gamma)} \|\dot{\gamma}\| dt \leq \int_{a_\beta}^{b_\beta} \sqrt{E - V(\beta)} \|\dot{\beta}\| dt$$

Note that being an E-L minimizer is a property of both the path and the parameterization. On the other hand, being a Jacobi minimizer just depends on the path since the integral in its definition is 1-homogeneous in the derivative.

The fact that solutions to the E-L equations are extrema of the integral in (1) is usually called Hamilton's Principle. On the other hand, The Principal of Least Action states that if γ is a solution to the E-L equations in \tilde{M} , then among the curves β with $\beta(a_\beta) = \gamma(a)$ and $\beta(b_\beta) = \gamma(b)$ which satisfy the constraint $H(\beta, \dot{\beta}) \equiv E$, the curve γ will extremize the integral $\int \|\dot{\beta}\|^2$. Now if $H(\beta, \dot{\beta}) \equiv E$, then $\|\dot{\beta}\|^2 = \sqrt{2(E - V(\beta))} \|\dot{\beta}\|$, and so the Principle of Least Action is the same as the variational formulation for geodesics with the Jacobi metric as in (2). (The reader is warned that there is a great deal of variance in the literature regarding the meaning of the phrases "Principle of Least Action" and "Hamilton's Principle".)

Proposition 1 *If $\gamma : [a, b] \rightarrow \tilde{M}$ is a Jacobi minimizer with energy $E > V_{max}$, then there exists a re-parameterization γ_1 with $H(\gamma_1, \dot{\gamma}_1) \equiv E$ that is an E-L minimizer.*

Proof Begin by noting that

$$L(\beta, \dot{\beta}) + E \geq \sqrt{2(E - V(\beta))} \|\dot{\beta}\|$$

and equality occurs if $H(\beta, \dot{\beta}) = E$. Indeed, setting $a = \sqrt{E - V}$, $b = \|\dot{\beta}\|/\sqrt{2}$, the above inequality is just $a^2 + b^2 \geq 2ab$, and $a = b$ means $H = E$.

Assume that γ is a Jacobi minimizer on the energy level $\{H = E\}$, and so we may parameterize γ so it is a solution of the E-L equations. Let this parameterization be $\gamma_1 : [a, b] \rightarrow \tilde{M}$. Now let β be a test path as in the definition of E-L minimizer. Using the inequality of the previous paragraph and the fact that γ is a Jacobi minimizer we have

$$\begin{aligned} \int_a^b L(\gamma_1, \dot{\gamma}_1) + E &= \int_a^b \sqrt{2(E - V(\gamma_1))} \|\dot{\gamma}_1\| \\ &\leq \int_a^b \sqrt{2(E - V(\beta))} \|\dot{\beta}\| \\ &\leq \int_a^b L(\beta, \dot{\beta}) + E \end{aligned}$$

and so γ_1 is also a E-L minimizer. \square

Remark 1 Somewhat surprisingly, the converse of the proposition is false; there can be E-L minimizers which are not Jacobi minimizers on their energy levels. What basically happens is that there are two E-L minimizers connecting the same two points in configuration space, *but* they minimize over different time intervals. They have the same energy but different Jacobi lengths, and thus they both cannot be Jacobi minimizers. An example is given at the end of Section 6.

As noted above, The Principle of Least Action is the same as the variational formulation of Jacobi geodesics. Thus there are minimizers of $\int L$ with fixed time- and space-endpoints

that are not minimizers of the action integral with fixed space-endpoint and constant energy constraints.

Since a solution of the E-L equations is an extrema and not necessarily a minimum of the action integral, many authors have adapted the terminology ‘‘Principle of Extreme Action’’. Our example gives another argument for this terminology: while the extrema of the Euler Lagrange and Action problems coincide, their minimizers may not.

3. Stability Results for Lagrangian Systems on Hyperbolic Manifolds

We consider here a closed hyperbolic manifold of arbitrary dimension n , equipped with a metric of constant negative curvature. All such manifolds M have the Poincaré n -disk \mathbb{H}^n as universal covering: $\tilde{M} = \mathbb{H}^n$, with the canonical hyperbolic metric. Geodesics for this metric in \mathbb{H}^n are arcs of (Euclidean) circles perpendicular to the sphere at infinity (the Euclidean unit sphere). The complete proofs of Theorems 1 and 3 appear in [10].

3.1. SHADOWING GEODESICS WITH E-L MINIMIZERS

Our first result gives the first way of formalizing the notion that Lagrangian systems satisfying our hypothesis (see Section 2.1) on hyperbolic manifolds are at least as complicated as the geodesics of a hyperbolic metric. Given a hyperbolic geodesic in the Poincaré Disk \mathbb{H}^n , the theorem asserts that there are minimizers of the Lagrangian system that are a bounded distance away and have a variety of approximate speeds. Recall that $\delta(\gamma; a, b)$ means the average displacement in \mathbb{H}^n over the time interval $[a, b]$, *i.e.* the distance from $\gamma(a)$ to $\gamma(b)$ divided by $b - a$ (see Section 2.1).

Theorem 1 *Let (M, g) be a closed hyperbolic manifold. Given a Lagrangian L which satisfies our hypothesis there are sequences k_i, κ_i, T_i in \mathbb{R}^+ depending only on L , with k_i increasing to infinity, such that, for any hyperbolic geodesic $\Gamma \subset \mathbb{H}^n = \tilde{M}$, there are minimizers $\gamma_i : \mathbb{R} \rightarrow \tilde{M}$ with $\text{dist}(\gamma_i, \Gamma) \leq \kappa_i$, $\gamma_i(\pm\infty) = \Gamma(\pm\infty)$, and $k_i \leq \delta(\gamma_i; c, d) \leq k_{i+1}$ whenever $d - c \geq T_i$.*

Remark 2 This theorem also holds for twist maps of T^*M provided their generating function has super quadratic growth, *eg.* the generalized standard map with generating function $S(x, X) = (1/2)\text{dist}^2(x, X) + V(x)$. The proof is identical to that of the Lagrangian case, using the dictionary given in Section 2.3.

Proof (Sketch) Fix an oriented geodesic with a given parameterization by arclength $\Gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ and a $K > K_0$ with K_0 as in Proposition 2 below. Let $\gamma_N : [-N, N] \rightarrow \mathbb{H}^n$ be a minimizing segment with $\gamma_N(-N) = \Gamma(-KN)$ and $\gamma_N(N) = \Gamma(KN)$, and thus $\delta(\gamma_N; -N, N) = K$. We would like to be able to take a limit of the curves γ_N ’s and prove that the limit is a E-L minimizer with the same endpoints at infinity as Γ . In order to do this, we use Gromov’s theory of quasi-geodesics.

Definition 2 *Given $\lambda \geq 1$ and $\epsilon \geq 0$, a curve $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ or a curve segment $\gamma : [a, b] \rightarrow \mathbb{H}^n$ is called a (λ, ϵ) -quasi-geodesic if*

$$\lambda^{-1}(d - c) - \epsilon \leq d(\gamma(c), \gamma(d)) \leq \lambda(d - c) + \epsilon$$

for all $[c, d]$ in the domain of γ .

The next theorem, often called “Stability of quasi-geodesics”, gives the most important property of quasi-geodesics. It is true in the broader context of what are usually called δ -hyperbolic spaces, but we just state the result in the context needed here. Given two closed subsets $X, Y \subset \mathbb{H}^n$, $d(X, Y)$ denotes their Hausdorff distance as induced by the hyperbolic metric. For a proof and more information see [15]:

Theorem 2 *Given $\lambda \geq 1$ and $\epsilon \geq 0$, there exists a $\kappa > 0$ so that whenever γ is a (λ, ϵ) -quasi-geodesic segment in \mathbb{H}^n and Γ_0 is the geodesic segment connecting the endpoints of γ , then $d(\gamma, \Gamma_0) < \kappa$. If γ is a (λ, ϵ) -quasi-geodesic, then $\gamma(\infty)$ and $\gamma(-\infty)$ exist and further, if Γ is the geodesic connecting $\gamma(\infty)$ and $\gamma(-\infty)$, then $d(\gamma, \Gamma) < \kappa$.*

Proposition 2 *There exists $K_0 > 0$ depending only on L and $\lambda > 1$ and $\epsilon > 0$ depending only on K and L so that whenever $\gamma : [a, b] \rightarrow \tilde{M}$ is a minimizing segment with $\delta(\gamma; a, b) = K > K_0$ and $b - a \in \mathbb{N}$, then γ is a (λ, ϵ) -quasi-geodesic segment.*

Thus, our proposition implies that the γ_N defined in the first paragraph of the proof of Theorem 1 are (λ, ϵ) -quasi-geodesics for the *same* (λ, ϵ) . Theorem 2 implies that the γ_N 's stay at uniformly bounded distance from Γ . This enables us to take a (pointwise) limit of the γ_N 's and get an E-L minimizer γ which has same endpoints at infinity as Γ and stays at bounded distance from Γ .

The proof of Proposition 2 is somewhat technical. We just sketch the essential features. We have that our minimizer γ satisfies $\delta(\gamma; a, b) = K$. Since $\delta(\gamma; a, b) = \frac{\text{dist}(\gamma(a), \gamma(b))}{b-a}$, this translates to $\text{dist}(\gamma(a), \gamma(b)) = K(b - a)$. From this estimate at the endpoints of γ , we want to derive similar estimates for *any* interval $[c, d] \subset [a, b]$. Ideally, we would like to prove that

$$K' \leq \delta(\gamma; c, d) \leq K'' \tag{1}$$

for K', K'' only depending on K . This would immediately prove Proposition 2. The second inequality follows from a results of Mather: $\delta(\gamma; a, b) = K \Rightarrow \|\dot{\gamma}\| \leq K''$ where K'' only depends on K . The first inequality, proved in [10], is only true when $(d - c) > N$, for some N only depending on K . But this restriction only adds an ϵ in the quasi-geodesic estimate. Note that, *in the autonomous case*, these estimates are trivial.

To prove the second inequality in Formula 1, Mather uses a surgery argument of a type which is widespread in the Aubry-Mather theory. If γ was going very fast on a subinterval $[c, d]$, then one could cut $\gamma|_{[c, d]}$ out and replace it by a minimizer between the same endpoints but on a greater interval of time, thus reducing the average speed on this interval. To make up for lost time one does the same kind of surgery on another judiciously chosen interval on the curve. The action of this new curve $\gamma^* : [a, b] \rightarrow \tilde{M}$ is then estimated using the average action vs. average speed estimate of Lemma 1 of section 2.2, and shown to be less than that of γ , which is absurd. Our proof of the first inequality uses similar techniques.

To finish the sketch of the proof of Theorem 1, we note that, in our construction of the minimizers γ_N 's, we can vary K . Because of the bounds on the average speed on subintervals we explained above, this implies that, if K_i is chosen to be increasing sufficiently fast to ∞ with i , the corresponding limiting γ_i are distinct and of average speed increasing to infinity.

3.2. SEMICONJUGACY WITH THE GEODESIC FLOW

One way to formulate the fact that the perturbed system is at least as complicated as the model system (*i.e.* the dynamics of the model systems don't go away) is to show that the perturbed system always has a invariant set that carries the dynamics of the minimal model. More precisely, one shows that there is a compact invariant set that is semiconjugate to the minimal model (this strategy is common in more topological theories, see [8]). MacKay and Denvir [30] have recently extended Morse's results to the case with boundary and proved a result giving this semiconjugacy. Also Gromov [17] and others have done this in the case of geodesic flows.

Theorem 3 *Let M be a closed hyperbolic manifold with a hyperbolic metric g with geodesic flow g_t . Given a Lagrangian L which satisfies the Hypotheses of Section 2 with E-L flow ϕ_t , there exists a sequences k_i and T_i with k_i increasing to infinity, and a family of compact, ϕ_t -invariant sets $X_i \subset TM \times \mathbb{S}^1$ so that for all i , (X_i, ϕ_t) is semiconjugate to (T_1M, g_t) and $k_i \leq \delta(\phi_t(x); 0, T) \leq k_{i+1}$, whenever $T \geq T_i$ and $x \in X_i$.*

Note that the geodesic flow of a hyperbolic metric is transitive Anosov and thus is Bernoulli, has positive entropy, etc. Thus Theorem B implies that the E-L flow is always dynamically very complicated.

We first state precisely what we mean by *semiconjugacy* :

Definition 3 *Two flows (X, ϕ_t) and (Y, ψ_t) are said to be semiconjugate (or sometimes orbit semi-equivalent) if there is a continuous surjection $f : X \rightarrow Y$ that takes orbits of ϕ_t to those of ψ_t preserving the direction of the flow, but not necessarily the time parameterization. Note that f is a local homeomorphism when restricted to an orbit of ϕ_t , but f may take many orbits of ϕ_t to the same orbit of ψ_t .*

Given a K and a geodesic Γ , we have constructed a (λ, ϵ) -quasi-geodesic γ_Γ^K which shadows Γ . The differential of such a curve is in $T\tilde{M}$. Project all these differential curves down to TM and take the closure of this set in TM . This gives a compact invariant set Q_K for the E-L flow because the velocities of the γ_Γ^K are uniformly bounded. Finally, define \tilde{Q}_K to be the set of all possible lifts to $T\tilde{M}$ of all the points in Q_K .

We will show that the E-L flow restricted to Q_K is semiconjugate to the geodesic flow. \tilde{Q}_K is a set made of (λ, ϵ) -quasi-geodesics, for some fixed (λ, ϵ) . Each such quasi-geodesic shadows a unique geodesic, by Theorem 2, and each geodesic is shadowed by at least one quasi-geodesic in \tilde{Q}_K , by Theorem 1. Hence we have a well defined application $z \mapsto \Gamma_z$ which, to a point z in \tilde{Q}_K , makes correspond the unique geodesic $\Gamma_z \subset \mathbb{H}^n$ that the quasi-geodesic to which z belongs shadows.

We then project z on $d\Gamma_z$ in the following fashion. Fix a parameterization by arclength for each geodesic Γ . Project $\pi(z) \in \mathbb{H}^n$ on Γ_z via the orthogonal projection (*i.e.* by drawing the unique geodesic through $\pi(z)$ which is perpendicular to Γ_z) and get a point $\Gamma_z(s(z))$ on Γ_z . Define

$$\sigma(z) = \left(\Gamma_z(s(z)), \dot{\Gamma}_z(s(z)) \right).$$

Theorem 1 implies that $\sigma : \tilde{Q}_K \rightarrow T_1\tilde{M}$ is onto. It is not too hard to see that σ is also continuous, and equivariant, *i.e.* it descends to a continuous map $Q_K \rightarrow T_1M$. Unfortunately, it is not necessarily injective when restricted to an orbit of the E-L flow in Q_K . This is remedied using an averaging technique due to Fuller [14].

Given $z = z(0) \in Q_K$ and its orbit $\{z(t), t \in \mathbb{R}\}$ under the E-L flow, let $a(z, t) = s(z(t)) - s(z(0))$, where, $s(z)$ is as above. If $a(z, t)$ were positive for all t , we would be

done: σ would be injective along $z(t)$. Because we cannot assume that, we use the fact that $a(z, t)$ is positive for large t and “average” σ over the interval $[0, t]$. Given $\alpha > 0$ define

$$\bar{\sigma}_\alpha(z) = \sigma(z) + \frac{1}{\alpha} \int_0^\alpha a(z, t) dt.$$

Informally, $\bar{\sigma}_\alpha(z)$ is the average value of σ over the orbit segment $z([0, \alpha])$.

Now since for every $z \in Q_K$ we have that $\omega(z) = \Gamma_z(\infty)$, it follows that for each z there is an α_z so that $a(z, \alpha_z) > 0$. Since Q_K is compact, we may find an α with $a(z, \alpha) > 0$ for all $z \in Q_K$. Let $\bar{\sigma} = \bar{\sigma}_\alpha$. As before, we write $\bar{\sigma}(z) = (\Gamma_z(\bar{\sigma}(z)), \dot{\Gamma}_z(\bar{\sigma}(z)))$. Now $\bar{\sigma}$ is clearly continuous, equivariant, onto and takes orbits to orbits. We then show [10] that it is injective on orbits $z(t)$ in Q_K by showing that for any $t > 0$, $\bar{\sigma}(z(t)) - \bar{\sigma}(z(0)) > 0$. This is done by using the fact that a is an additive cocycle, *i.e.* $a(z, t_1 + t_2) = a(z, t_1) + a(\tilde{\phi}_{t_1}(z), t_2)$, for all t_1, t_2 . By repeating this whole process by setting $K = K_i$, where K_i is as in Theorem 1, we get distinct Q_{K_i} which are semiconjugate to the geodesic flow, and such that the average speed of a E-L orbit on Q_{K_i} goes to ∞ with i . This finishes the sketch of our proof of Theorem 3.

3.3. WHY THE LIMIT ARGUMENT DOES NOT WORK IN \mathbb{T}^3

Quasi-geodesics can obviously be defined on any manifold (in fact, on any metric space). Proposition 2 is valid on any compact manifold, not just those that support a hyperbolic metric. The reason why the proof above does not apply to $M = \mathbb{T}^3$ must therefore be that quasi-geodesics on \mathbb{R}^2 do not satisfy Theorem 2. Thus a sequence of (λ, ϵ) -quasi-geodesic segments with endpoints on a fixed geodesic Γ do not have to stay at bounded distance of Γ .

As a very simple example, one can look at \mathbb{R}^2 with the Euclidean metric and take the geodesic Γ to be the line $y = x$. Take the sequence γ_N of “corner” curves going between the points $(-N, -N)$ and (N, N) of Γ by first following the line $x = -N$ upward at unit speed until it reaches $y = N$, and then follow that line to (N, N) . It is easy to check that the γ_N ’s are quasi-geodesics segment for a (λ, ϵ) uniform in N . But obviously $d(\gamma_N, \Gamma) \rightarrow \infty$. It is not too hard either to see that one can adapt this argument in Hedlund’s example (see Section 4.2) in \mathbb{T}^3 to show how our limiting scheme would fail in finding a minimizer of rotation vector $(1, 1, 0)$ there: in this case, the minimizers γ_N ’s between the points $(-N, -N, 0)$ and $(N, N, 0)$ of the (Euclidean) geodesic Γ given by the line $y = x, z = 0$ of \mathbb{R}^3 would be corner curves that follow the tubes, jumping tubes at the beginning, corner and end of the curve. Again the γ_N are quasi-geodesic, with uniform (λ, ϵ) by Proposition 2, and again their distance to Γ goes to infinity. Note that $\gamma_N \rightarrow \Gamma_x \cup \Gamma_y$ in the Hausdorff topology, where Γ_x, Γ_y are the periodic orbits projection of the tubes parallel to the x and y axes. This is the Mather set for the vector of rotation $(1, 1, 0)$ (see next section).

A very similar phenomenon happens in Example 5 of Section 4.2. In the case considered there the γ_N with “corners” show up in the universal free Abelian cover \bar{M} .

4. A Quick Review of Mather’s Theory of Minimal Measures

4.1. THEORY

For a more detailed exposition the reader is urged to consult Mather [27] or Mañé [25]. Mather also gives a very nice survey of this theory in the beginning of [28]. Given a E-L invariant probability measure with compact support μ on $TM \times \mathbb{S}^1$, one can define its

rotation vector $\rho(\mu)$ as follows: let $\beta_1, \beta_2, \dots, \beta_n$ be a basis of $H^1(M)$ and let $\lambda_1, \dots, \lambda_n$ be closed one-forms with $[\lambda_i] = \beta_i$ in DeRham cohomology.² The reader uncomfortable with homology may read through this section thinking of the case of $M = \mathbb{T}^n$, with angular coordinates (x_1, \dots, x_n) , and taking $[\lambda_i] = [dx_i]$, as a basis for $H^1(\mathbb{T}^n) \simeq H_1(\mathbb{T}^n) \simeq \mathbb{R}^n$. Define the i^{th} component of the rotation vector $\rho(\mu)$ as

$$\rho_i(\mu) = \int \lambda_i d\mu.$$

Note that this integral makes sense when one looks at λ_i as inducing a function from $TM \times \mathbb{S}^1$ to \mathbb{R} by first projecting $TM \times \mathbb{S}^1$ onto TM , and then treating the form as a function on TM that is linear on fibers. The rotation vector does depend on the choice of basis β_i , but because the one forms are closed, $\rho_i(\mu)$ does not depend on the choice of representative λ_i with $[\lambda_i] = \beta_i$. Since the rotation vector is dual to forms, it can be viewed as an element of $H_1(M, \mathbb{R})$. In the case $M = \mathbb{T}^n$, one can check that, for γ a generic point of an ergodic measure μ , the usual rotation vector of γ coincides with that of μ :

$$\rho_i(\gamma) = \lim_{b-a \rightarrow \infty} \frac{\tilde{\gamma}_i(b) - \tilde{\gamma}_i(a)}{b-a} = \lim_{b-a \rightarrow \infty} \frac{1}{b-a} \int_{d\gamma|_{[a,b]}} dx_i = \int dx_i d\mu = \rho_i(\mu)$$

where second equality uses the Ergodic Theorem. (Again, dx_i is seen as a function $TM \times \mathbb{S}^1 \rightarrow \mathbb{R}$). If M is a general compact manifold, one can define the rotation vector of a curve $\gamma : \mathbb{R} \rightarrow M$ by $\rho_i(\gamma) = \lim_{b-a \rightarrow \infty} \frac{1}{b-a} \int_{d\gamma|_{[a,b]}} \lambda_i$, if the limit exists. As before, if $\gamma(0)$ is a generic point for an ergodic measure μ , this rotation vector does exist and coincides with that of μ .

Next we define the *average action* of a E-L invariant probability on $TM \times \mathbb{S}^1$ by

$$A(\mu) = \int L d\mu,$$

which, when μ is ergodic, we can relate to the average action along μ -a.e. orbit γ by

$$A(\mu) = \lim_{b-a \rightarrow \infty} \frac{1}{b-a} \int_a^b L(\gamma, \dot{\gamma}) dt.$$

The set of invariant probability measures, denoted \mathcal{M}_L is a convex set in the vector space of all measures³ and the extreme points of \mathcal{M}_L are the ergodic measures [24]. Now consider the map $\mathcal{M}_L \rightarrow H_1(M) \times \mathbb{R}$ given by:

$$\mu \mapsto (\rho(\mu), A(\mu)).$$

This map is trivially linear and hence maps \mathcal{M}_L to a convex set U_L whose extreme points are images of extreme points of \mathcal{M}_L , *i.e.* images of ergodic measures. Mather shows, by taking limits of measures supported on long minimizers representing rational homology classes, that for each ω , there is μ such that $\rho(\mu) = \omega$ and $A(\mu) < \infty$.⁴ Since L is

²When homology and cohomology coefficients are unspecified they are assumed to be \mathbb{R} , so the notation $H_1(M)$ means $H_1(M; \mathbb{R})$, etc.

³It is also compact for the weak topology if, as Mather does, one compactifies TM .

⁴The impatient reader may be tempted to proclaim, from this fact, the existence of orbits of all rotation vectors. Alas, one can only deduce the rotation vector of orbits from that of a measure when the measure is ergodic...

bounded below, the action coordinate is bounded below on U_L . Hence we can define a map $\beta : H_1(M) \rightarrow \mathbb{R}$ by

$$\beta(\omega) = \inf\{A(\mu) \mid \mu \in \mathcal{M}_L, \rho(\mu) = \omega\},$$

which is bounded below and convex; the graph of β is the boundary of U_L .

We say that a probability measure $\mu \in \mathcal{M}_L$ is a *minimal measure* if the point $(\rho(\mu), A(\mu))$ is on the graph of β . Hence, an extreme point $(\omega, \beta(\omega))$ of $\text{graph}(\beta)$ corresponds to at least one minimal ergodic measure of rotation vector ω . It turns out that if μ is minimal, μ -a.e. orbit lifts to a E-L minimizer in the covering \overline{M} of M whose deck transformation group is $H_1(M; \mathbb{Z})/\text{torsion}$ (i.e. the universal cover when $M = \mathbb{T}^n$). Conversely, if μ is an ergodic probability measure whose support consists of \overline{M} -minimizers, then μ is a minimal measure.

Hence, each time we prove the existence of an extreme point $(\omega, \beta(\omega))$, we find at least one recurrent orbit of rotation vector ω which is a \overline{M} -minimizer.

Another important property of β is that it is superlinear, i.e. $\frac{\beta(x)}{\|x\|} \rightarrow \infty$ when $\|x\| \rightarrow \infty$. Since we will need the estimate later, we motivate this in the simple case where $L = \frac{1}{2}\|\dot{x}\|^2 - V(x)$ and $\|\cdot\|$ comes from the Euclidean metric on the torus. If μ is any invariant probability measure, then

$$\begin{aligned} A(\mu) &= \int L \, d\mu \geq \int \left(\frac{\|\dot{x}\|^2}{2} - V_{max} \right) d\mu \\ &\geq \frac{1}{2} \left| \int \dot{x} \, d\mu \right|^2 - V_{max} \\ &= \frac{1}{2} |\rho(\mu)|^2 - V_{max} \end{aligned} \tag{2}$$

where we used the Cauchy-Schwarz inequality for the second inequality.

The superlinearity of β implies the existence of many extreme points for $\text{graph}(\beta)$ (although in most cases still too few, as we will see at the end of this discussion). Indeed, β 's superlinear growth implies that its graph cannot have flat, or linear domains going to infinity. Any point $(\omega, \beta(\omega))$ is part of at least one linear domain of $\text{graph}(\beta)$, which we call S_c (we suppress the dependence of c on ω). Here, the index c denotes the ‘‘slope’’ of the supporting hyperplane whose intersection with U_L is exactly the convex and flat domain S_c (c can be seen as element of first cohomology). Let X_c be the projection on $H_1(M)$ of S_c . Then the X_c 's are compact and convex domains which ‘‘tile’’ the space $H_1(M)$. Extreme points of X_c are projections of extreme points of S_c . Hence there are infinitely many such extreme points, and infinitely many outside any compact set. Their convex hull is $H_1(M)$, and in particular, they must span $H_1(M)$ as a vector space. Since these extreme points are the rotation vectors of minimal ergodic measures, we have found that *there always exist at least countably many minimal ergodic measures and at least $n = \dim H_1(M)$ of them with distinct rotation directions*. We will see in Hedlund's example that this lower bound can be attained.

Finally, the generalized *Mather sets* are defined to be $M_c = \text{support}(\mathcal{M}_c)$, where \mathcal{M}_c is the set of minimal measures whose rotation vector lies in X_c . Let $\pi : TM \times \mathbb{S}^1 \rightarrow M \times \mathbb{S}^1$ denote the projection. Mather's main result in [27] is the following theorem.

Theorem 4 (Mather's Lipschitz Graph Theorem) *For all $c \in H^1(M)$, M_c is a compact, non-empty subset of $TM \times \mathbb{S}^1$. The restriction of π to M_c is injective. The inverse mapping $\pi^{-1} : \pi(M_c) \rightarrow M_c$ is Lipschitz.*

In the case $M = \mathbb{T}^n$, Mather proves that, when they exist, KAM tori coincide with the sets M_c , and that they are in the closure of these sets (see also [21] for some related results). In a sense, Mather's theorem generalizes Birkhoff's theorem on invariant curves of twist maps, which says that such curves have to be graphs (see [20], [11] for more straightforward generalizations of this theorem). The proof of the Lipschitz Graph Theorem (see [27] or [25]), which is quite involved, uses a curve shortening argument: if curves in $\pi(M_c)$ were too close to crossing transversally, one could "cut corners" and, because of recurrence, construct a closed curve with lesser action than A_{min} .

Remark 3 An important special case is that of *autonomous* systems. In this case, one can discard the time component and view M_c as a compact subset of TM . Then Mather's theorem implies that M_c is a Lipschitz graph for the projection $\pi : TM \rightarrow M$. To see this, suppose that two curves $x(t)$ and $y(t)$ in $\pi(M_c)$ have $x(0) = y(s)$ for some s . Mather's theorem rules out immediately the possibility that s is an integer, unless $x = y$ is a periodic orbit. For a general s , consider the curve $z(t) = y(t - s)$. Then, $\dot{z}(t) = \dot{y}(t - s)$ and, by time-invariance of the Lagrangian, $(z(t), \dot{z}(t))$ is a solution of the E-L flow. It has same average action and rotation vector as (y, \dot{y}) and hence it is also in M_c . But then $z(0) = x(0)$ is impossible, by Mather's theorem, unless $\dot{z}(0) = \dot{y}(s) = \dot{x}(0)$.

In the realm of twist maps, one can also deduce from Mather's theory the existence of many invariant sets that are graphs over the base and are made of minimizers (see also [21]).

4.2. EXAMPLES

One would hope that β is, in general, strictly convex, *i.e.* each point on $graph(\beta)$ is an extreme point. This is true when $M = \mathbb{S}^1$, and Mather shows in [28] how his Lipschitz Graph Theorem implies the classical Aubry-Mather Theorem, by taking a E-L flow that suspends the twist map. The fact that M_c is a graph nicely translates into the fact that orbits in an Aubry-Mather set are well ordered.

The graph of β is also strictly convex when L is a Riemannian metric on \mathbb{T}^2 . This was known by Hedlund [19] in the 30's, albeit in a different language. It will also be an easy consequence of Theorem 5. In Section 6, we will show that this may not be true if one adds a potential term to the metric in the Lagrangian.

Here we briefly describe three other counter-examples to the strict convexity of β .

Example 3 (Mañe [25]) Take $L : T\mathbb{T}^2 \rightarrow \mathbb{R}$, given by $L(x, \dot{x}) = \|\dot{x} - X\|^2$ where X is a vector field on \mathbb{T}^2 . The integral curves x of X are automatically E-L minimizers since $L \equiv 0$ on these curves. Mañe chooses the vector field X to be a (constant) vector field of irrational slope multiplied by a carefully chosen function on the torus which is zero at exactly one point q . The integral flow of X has the rest point $q(t) = q$, and all the other solutions are dense on the torus. The flow of X (and its lift to $T\mathbb{T}^2$ by the differential) has exactly two ergodic measures: one is the Dirac measure supported on $(q, 0)$, with zero rotation vector, the other is equivalent to the Lebesgue measure on \mathbb{T}^2 and has nonzero rotation vector, say ω (see [18], for more details on this \mathbb{T}^2 flow). Mañe checks that $\beta^{-1}(0)$ (trivially always an X_c) is the interval $\{\lambda \omega \mid \lambda \in [0, 1]\}$, and that no ergodic measure has a rotation vectors strictly inside this interval. Thus the Mather set M_0 is the union of the supports of the two measures.

In the context considered here, this example is a little unsatisfactory because it is not a mechanical Lagrangian. We will give an example in Section 6 of a mechanical Lagrangian

on \mathbb{T}^2 which displays a similar phenomena. The next two examples involve metrics, on the three-torus and on the surface of genus two, respectively.

Example 4 (Hedlund-Bangert) Consider in \mathbb{R}^3 the three nonintersecting lines given by the x -axis, the y -axis translated by $(0, 0, 1/2)$ and the z -axis translated by $(1/2, 1/2, 0)$. Construct a \mathbb{Z}^3 -lattice of nonintersecting axes by translating each one of these by all integer vectors. Take a metric in \mathbb{R}^3 which is the Euclidean metric everywhere except in small, nonintersecting tubes around each of the axes in the lattice. In these tubes, multiply the Euclidean metric by a function λ which is 1 on the boundary and attains its (arbitrarily small) minimum along the points in the center of the tubes, *i.e.* at the axes of the lattice. Because the construction is \mathbb{Z}^3 periodic, this metric induces a Riemannian metric on \mathbb{T}^3 . One can show ([6]), if λ is taken sufficiently small, that a minimal geodesic (which is a E-L minimizer in our context) can make at most three jumps between tubes. In particular, a recurrent E-L minimizer has to be one of the three disjoint periodic orbits which are the projection of the axes of the lattice. Thus there are only three rotation directions that minimizers can take in this example, or six if one counts positive and negative orientations. In terms of Mather's theory, the level sets of the function β are octahedrons with vertices $(\pm a, 0, 0)$, $(0, \pm a, 0)$, $(0, 0, \pm a)$ (we assume here that λ is the same around each of the tubes). Since we are in the case of a metric, one can check that β is quadratic when restricted to a line through the origin (a minimizer of rotation vector $a\omega$ is a reparametrization of a minimizer of rotation ω). Hence a set S_c is either a face, an edge or a vertex of some level set $\{\beta = b\}$, and the corresponding M_c is, respectively, the union of three, two (parameterized at same speed) or one of the minimal periodic orbits one gets by projecting the disjoint axes. Note that, instead of the function β of Mather, Bangert uses the *stable norm*. Mather's function β is a generalization of that norm.

It is important to note that the nonexistence of minimizers of a certain rotation vector ω does not mean that there are no orbits of the E-L flow that have rotation vector ω . For example, Mark Levi has shown the existence of orbits of all rotation vectors in the Hedlund example (personal communication). In addition, in our torus example (Section 6), it is easy to see that there are (nonminimizing) orbits with rotation vectors in the excluded interval. The next example was brought to our attention by A. Fathi.

Example 5 Take the metric of constant negative curvature on the surface of genus 2 (the two-holed torus) which has a thin neck between the two holes (see Figure 1). In this case, the notion of minimizing is just that of least length using the hyperbolic metric. With a and b as shown, the minimal measure for the homology class $a + b$ will be a linear combination of the ergodic measures supported on Γ_a and Γ_b , where Γ_a and Γ_b are the closed geodesics in the homotopy classes of a and b , respectively. This is because any closed curve that crosses the neck will be longer than the sum of the lengths of Γ_a and Γ_b . Hence $(a + b, \beta(a + b))$ cannot be an extreme point of $graph(\beta)$.

This example illustrates the remarks of the introduction, namely, on hyperbolic manifolds the notion of \overline{M} minimizers is not the correct one if one wants to show that all the dynamics of the geodesic flow of a metric of constant negative curvature are preserved under global perturbation. The right notion of minimality comes from the observation that there is a curve of least length in the *homotopy* class of ab . This notion is generalized to an asymptotic invariant on hyperbolic manifolds that associates an ergodic measure of a given dynamical system with its "rotation measure". The rotation measure is the ergodic measure of the hyperbolic geodesic flow whose dynamics best mirrors that of the given ergodic measure (see [9]).

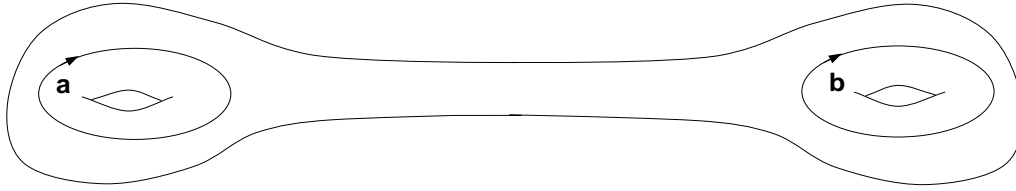


Figure 1. The genus two surface in Example 5

5. Autonomous Lagrangian Systems on the 2-Torus

We now give a concrete application of Mather's theory to autonomous Lagrangians on the two torus. If the Lagrangian is mechanical, then for each $E > V_{max}$ we get a Jacobi metric. Hedlund showed ([19]) that for each Riemannian metric on the two torus there are minimizing geodesics in all directions. Applying this to the Jacobi metric we see that there will be minimizers in all directions on each energy level. However, the parameterization of these minimizers as solutions of the E-L flow is different than as Jacobi geodesics. Thus in applying Hedlund's result we must examine how the minimizers on the various levels fit together in terms of their rotation vectors. This can be done nicely in the framework of Mather's theory which also allows us to consider a wider class of Lagrangians.

Recall that the sets X_c are projections on the rotation vector space of flat domains in $\text{graph}(\beta)$.

Theorem 5 *For any autonomous Lagrangian system on \mathbb{T}^2 which satisfies Mather's hypothesis, a set X_c is either a finite interval in a line through the origin or a point. If X_c is an interval, either it contains 0, or it is supported by a line of rational slope. In addition:*

- (i) *There are minimal ergodic measures (and hence recurrent E-L minimizers) of all rotation directions.*
- (ii) *If v has irrational direction, then there is a positive number $K(v)$ such that $\|v\| > K(v)$ implies that there is a minimal ergodic measure (and hence a recurrent E-L minimizer) with rotation vector v .*
- (iii) *In each rational direction, represented by a vector v , there is a sequence of positive numbers $\lambda_n \rightarrow \infty$ and a sequence γ_n of periodic E-L minimizers such that γ_n has rotation vector $\lambda_n v$.*
- (iv) *If the Lagrangian is of the form $L(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 - V(x)$, where $\|\cdot\|$ is the Euclidean metric,⁵ then the origin is not in the interior of any X_c and the X_c which have the origin as endpoint have length less than $2\sqrt{V_{max} - V_{min}}$. In particular, $K(v)$ in (ii) can be taken to be $2\sqrt{V_{max} - V_{min}}$.*
- (v) *The support of minimal measures can be either a point, a closed curve, the suspension of a Denjoy example (a lamination), or the whole torus.*

Proof By Mather's Lipschitz Graph Theorem and the remark after it, any set M_c projects injectively to \mathbb{T}^2 . It thus yields a flow on compact set $X \subset \mathbb{T}^2$. Most of the theorem is a consequence of the fact that there are severe restrictions on what X and its flow can be. We review some of these well known results. They follow easily from, for example, the techniques of [13] (cf [34]).

⁵Clearly, similar results can be obtained for other metrics

If $X \subset \mathbb{T}^2$ is a compact set and there is a continuous flow ϕ_t defined on X , then the support of a ϕ_t -invariant ergodic measure on X can only be (1) a point, (2) a closed orbit, (3) the whole torus, or (4) homeomorphic to the suspension of the minimal set in a homeomorphism of the circle that is a Denjoy counter example (and it must be embedded in \mathbb{T}^2 in the obvious way).

In cases (1) and (2) there is a unique invariant probability measure on the set and so the rotation vector exists and is equal for all points. Case (1) happens only if the rotation vector is 0. Case (2) can happen only if the rotation vector is zero for homotopically trivial curves and a nonzero $v \in \mathbb{Q}^2$, if the curve is homotopically nontrivial. Cases (3) and (4) are more subtle. The set X can contain a fixed point, but in any event the rotation vectors of ergodic measures supported on X can take on at most two values, zero and a vector with irrational slope.

Of particular importance in what follows are the following three consequences. First, there cannot be two invariant measures in X whose rotation vectors are in different directions, *i.e.* $\frac{v}{\|v\|} \neq \frac{w}{\|w\|}$. Second, if an ergodic measure has a nonzero rotation vector $v \in \mathbb{Q}^2$, then it is supported on a closed orbit. Finally, X cannot support two ergodic measures with nonzero rotation vectors that have the same irrational slope.

To prove the first paragraph of the theorem, recall that by Mather's theory, the endpoints of each X_c corresponds to a minimal ergodic measure with that rotation vector. Next observe that any X_c is a finite interval on a line through the origin or a point, because as noted in the previous paragraph, no M_c can contain orbits with different nonzero rotation directions and each M_c can contain ergodic measures that yield at most two rotation vectors. (The fact that X_c is finite also follows from the superlinearity of β .) If we fix an ω with irrational slope, then if both endpoints of the interval X_c are nonzero, they correspond to at least two distinct ergodic measures with two irrational rotation vectors of same direction but different lengths, another impossibility.

To prove (i), take any $\omega \in \mathbb{R}^2$. Using what we have just proved, any set X_c to which ω belongs is a finite interval with endpoints in the same direction as ω . These endpoints are the rotation vectors of two minimal ergodic measures. For (ii), note that if ω has irrational direction, and if $\|\omega\|$ is large enough, we have just seen that X_c can only be a point. Hence there is a minimal ergodic measure of rotation vector ω . As for (iii), note that the previous paragraph implies, in each rational rotation direction v , the existence of sequences $\lambda_n \rightarrow \infty$ and minimal ergodic measures μ_n with $\rho(\mu_n) = \lambda_n v$. As noted at the beginning of the proof, such an ergodic measure necessarily comes from a periodic orbit.

We now prove part (iv). We first show that $\beta(0)$ is an isolated global minimum, thus proving that 0 cannot be in the interior of any X_c . In the case of autonomous mechanical systems, it is easy to see that a measure that gives the absolute minimum for A is the Dirac measure on a fixed point x at which $V(x) = V_{max}$. This measure has rotation vector 0 and action V_{max} .

Recall, moreover, that from Formula (2) in Section 4, we have the estimate $\beta(\omega) \geq \frac{1}{2}\|\omega\|^2 - V_{max}$ for such systems. This implies that the graph of β above an X_c that contains zero cannot be horizontal, and that $(0, V_{max})$ must be an isolated minimum of β .

Finally, we prove our estimate for the length of X_c 's which have 0 as one of their endpoints. On an energy level $E > V_{max}$ the Jacobi metric $\sqrt{E - V} ds$ is a Riemannian metric on \mathbb{T}^2 . It is not hard to see that the Jacobi minimizers and the E-L minimizers for $L_E = \frac{1}{2}(E - V)\|\dot{x}\|^2$ form the same set of curves on the torus (See [31], page 70). L_E obviously satisfies Mather's hypothesis and so, by Part (i), there are Jacobi minimizers

of all rotation directions (we could also have invoked Hedlund [19], who proved this in the 30's). By Proposition 1, these Jacobi minimizers are also E-L minimizers for $L = \frac{1}{2}\|\dot{x}\|^2 - V(x)$. In terms of minimal ergodic measures, after a time change we have found minimal ergodic measures of all rotation direction in each energy level $E > V_{max}$.

Let μ be such a measure. Using Lemma 1, we have:

$$\frac{1}{2}(\rho(\mu))^2 - V_{max} \leq \int L d\mu \leq \int E - 2V_{min} d\mu = E - 2V_{min}$$

so $\rho(\mu) \leq \sqrt{2(E - 2V_{min} + V_{max})}$. This is true for the measures obtained from energy levels E arbitrarily close to V_{max} . Thus we have shown that $\|\omega\| \leq 2\sqrt{V_{max} - V_{min}}$.

Statement (v) is an immediate consequence of the facts at the beginning of the proof.

□

6. Speed Defect: an Example in the 2-Torus

6.1. THE MAIN EXAMPLE

In this section, we exhibit an autonomous Lagrangian system on the torus \mathbb{T}^2 which fails to have recurrent minimizers of all rotation vectors in a prescribed rational direction. This is a different kind of counterexample to a generalization of the Aubry–Mather theorem than Hedlund’s example (one can also derive counterexamples in the realm of symplectic twist maps using examples in [2]). In terms of Mather’s theory, we find in our system an X_c in a rational direction which contains an interval, and hence the function β of Mather is not strictly convex in this case, confirming the worst predictions of Theorem 5.

Note that the Lagrangian in such an example could not be a Riemannian metric, since it is easy to show that Mather’s function β for such systems is quadratic along lines through the origin of \mathbb{R}^2 . Hence Theorem 5 above implies that, in the case of metrics, β is strictly convex.

We first outline the features of our example. We will find, in a given energy level ($h = 0$), exactly two closed minimizers γ_1, γ_2 in the homology class $(0, 1) \in \mathbb{Z}^2 = H_1(\mathbb{T}^2; \mathbb{Z})$ but with different rotation vector $(0, \rho_1), (0, \rho_2)$, *i.e.* different average speeds. We then show that any periodic minimizer in that same homology class but in a different energy level goes strictly faster for higher energy, and strictly slower for lower energy than these two minimizers, leaving the segment between $(\rho_1, 0)$ and $(\rho_2, 0)$ empty of recurrent minimizers.

On the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, consider a Lagrangian of the form:

$$L(x, \dot{x}) = \frac{\alpha(x_1)}{2}\|\dot{x}\|^2 - V(x_1) \tag{3}$$

$$V(x_1) = -\frac{2 + \cos 2x_1}{2 + \sin x_1} \tag{4}$$

$$\alpha(x_1) = 2 + \sin x_1, \tag{5}$$

where $\|\dot{x}\|$ denotes the usual, Euclidean norm on \mathbb{T}^2 . The corresponding Hamiltonian is $H(x, p) = \frac{\|p\|^2}{2\alpha(x_1)} + V(x_1)$.

If there is a minimal ergodic measure in the homology direction $(0, 1)$, then as noted at the beginning of the proof of Theorem 5, it must be supported on a periodic orbit. Furthermore, we claim that if γ is a closed curve which is the projection of the support

of an ergodic, minimal measure in the $(0, 1)$ direction, it has to be of the form $x_1 = \text{constant}$. Indeed, since $H(x, p)$ is independent of x_2 , the Hamiltonian flow is invariant under translations in the x_2 direction. Hence γ is actually part of a one parameter family of periodic orbits that are translates of one another in the x_2 direction. Obviously, all the orbits in this family have same rotation vector and action. Since γ is the support of a minimal measure, all its x_2 translates are as well, and they all belong to the same X_c . Because γ is in the class $(0, 1)$, it must have (at least 2) points at which $p_1 = 0$. If it is not so at all points, *i.e.* if x_1 is not constant along γ , it is easy to see that γ will intersect transversally its x_2 translates, contradicting (by Mather's theorem) the fact that these curves belong to the projection of the same X_c .

Our task is now to find all the possible periodic orbits of the Hamiltonian flow that project to curves $x_1 = \text{constant}$ and to compute their rotation vector and action. Such a curve occurs when $0 = \dot{x}_1 = \frac{p_1}{\alpha(x_1)}$ and hence (by setting $p_1(0) = 0$), whenever

$$0 \equiv \dot{p}_1(t) = -\frac{\partial H}{\partial x_1} = \frac{\alpha'}{2\alpha^2} \|p\|^2 - V'$$

Simplifying this equation by fixing an energy level $H = E$ and using $\|p\|^2 = 2\alpha(E - V)$, one gets:

$$0 = \cos x_1(2 + \sin x_1)(4 \sin x_1 - E)$$

which has the solutions:

$$x_1 = \frac{\pi}{2}, \tag{6}$$

$$x_1 = \frac{3\pi}{2}, \tag{7}$$

$$4 \sin x_1 = E, \tag{8}$$

with some restrictions on what the range of E is in each case, which we will deal with later.

We now find the curves that the corresponding ergodic measures trace in the $(\|\omega\| = \rho, A)$ - plane, and show that the third curve is always in the union of the epigraphs of the two first ones, and hence cannot be a minimal measure (the epigraph of a function is the set of points above its graph).

In the energy level $\{H = E\}$, the Lagrangian is given by $L = E - 2V$. Since it is constant along the curves $x_1 = \text{constant}$, the average action of the corresponding ergodic (probability) measure μ is

$$A(\mu) = \int L d\mu = L \int d\mu = L$$

From $L = E - 2V$, we can also get the rotation vector $(0, \rho)$ of an $x_1 = \text{constant}$ curve:

$$L + 2V = \frac{\alpha(x_1)}{2} \|\dot{x}\|^2 + V(x_1) = E \Rightarrow \|\dot{x}\|^2 = \frac{2(E - V(x_1))}{\alpha(x_1)}$$

In particular, the speed at which a curve $x_1 = \text{constant}$ is traversed by the E-L flow is constant. The rotation vector of such a curve is:

$$(0, \rho(E, x_1)) = \left(0, \frac{\|\dot{x}\|}{2\pi}\right) = \left(0, \frac{1}{2\pi} \sqrt{\frac{2(E-V)}{\alpha}}\right)$$

We found it more convenient to first look at the curves (c, A) formed by these measures, where

$$c \stackrel{\text{def}}{=} \frac{(E-V)}{\alpha} = 2\pi^2 \rho^2.$$

We denote by $(c_i(E), A_i(E))$ the curve corresponding to the i^{th} family of solutions, $i = 1, 2, 3$.

- If $x_1 = \frac{\pi}{2}$, $A_1(E) = L = E - 2V(\frac{\pi}{2}) = E + \frac{2}{3}$, whereas $c_1(E) = \frac{E-V(\frac{\pi}{2})}{\alpha(\frac{\pi}{2})} = \frac{1}{3}(E + \frac{1}{3})$.

Hence:

$$A_1(c) = 3c + \frac{1}{3}.$$

One can verify that the curve $x_1 = \frac{\pi}{2}$ is indeed a solution for E in $[-1/3, \infty)$, which gives $c \in [0, \infty)$.

- If $x_1 = \frac{3\pi}{2}$, $A_2(E) = E + 2$, $c_2(E) = E + 1$ and hence

$$A_2(c) = c + 1.$$

One can verify that $x_1 = \frac{3\pi}{2}$ is a solution for E in $[-1, \infty)$, which corresponds to $c \in [0, \infty)$.

- If $E = 4 \sin x_1$, we get, by replacing $\sin x_1 = E/4$ and $\cos 2x_1 = 1 - 2 \sin^2 x_1 = 1 - 2(E/4)^2$:

$$\begin{aligned} c_3(E) &= 2 \frac{E^2 + 16E + 24}{(E+8)^2} \\ A_3(E) &= 8 \frac{(E+3)}{(E+8)}. \end{aligned}$$

We will be content to say that the curves $E = 4 \sin x_1$ correspond to solutions for E in some interval included in $[-2, 4]$ (one can check that $-2 < V_{\min} < -1$), which we do not need to find, as we will see.

Instead of trying to find $A_3(c)$ explicitly, we compute $A_3(E) - (c_3(E) + 1) = \frac{5E^2 - 8E + 80}{(E+8)^2}$ which is positive for all E , thus showing that the curve $(c, A_3(c))$ is above $(c, A_1(c))$. The relation is trivially the same for the corresponding $(\rho, A(\rho))$ curves and hence the measures corresponding to the curves $E = 4 \sin x_1$ can never be minimal.

Replacing c by $2\pi^2 \rho^2$, we get:

$$\begin{aligned} A_1(\rho) &= 6\pi^2 \rho^2 + \frac{1}{3} \\ A_2(\rho) &= 2\pi^2 \rho^2 + 1, \end{aligned}$$

the graphs of which are two parabolas crossing at $\rho = \frac{2\pi^2}{3}$, with $A_1(\rho) \leq A_2(\rho)$ for $\rho \in [0, \frac{2\pi^2}{3}]$, $A_2 < A_1$ after that. Note that the union of the epigraphs of these curves is not convex. Remember that the graph of Mather's function β is the boundary of the convex hull of the points $(\rho(\mu), A(\mu))$ for all possible ergodic measures μ . To form the lower boundary of the convex hull for the epigraphs of $(\rho, A_i(\rho))$, $i = 1, 2$ (which, as we

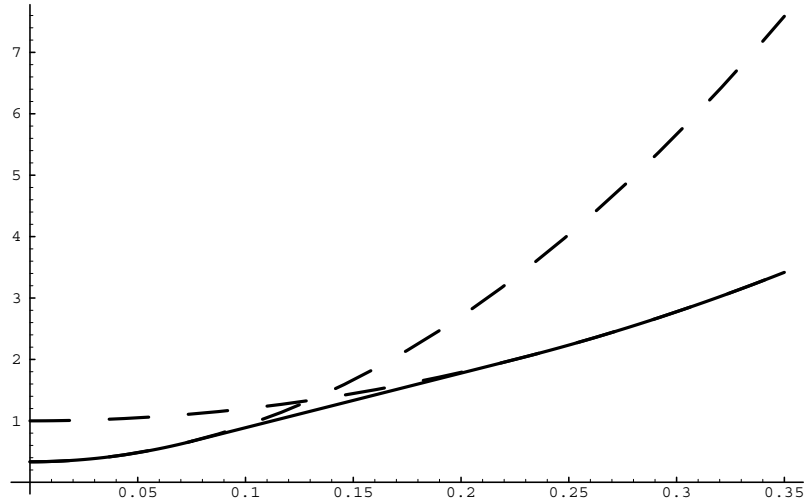


Figure 2. The graph of β in the $(0, 1)$ direction

showed, necessarily contains the epigraph of $(\rho, A_3(\rho))$, we look for the line tangent to both these graphs. It is easy to check that it joins the point $Q_1 = (\frac{1}{3\sqrt{2\pi}}, 2/3)$ of the graph of A_1 to the point $Q_2 = (\frac{1}{\sqrt{2\pi}}, 2)$ of the graph of A_2 . Hence, any measure corresponding to points on the graphs of A_1 or A_2 above the segment of line between Q_1 and Q_2 fails to be minimal. Instead, if μ_i are the measures corresponding to Q_i , $i = 1, 2$, the minimal measure with rotation vector in the interval $I = [\frac{1}{3\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}]$ will be a convex combination of μ_1 and μ_2 . The set X_c for any $\omega = (0, \rho)$ with $\rho \in I$ is thus an interval, and the corresponding Mather set M_c is the union of the support of μ_1 and μ_2 , *i.e.* 2 loops, with rotation vectors $(0, \frac{1}{3\sqrt{2\pi}})$ and $(0, \frac{1}{\sqrt{2\pi}})$. Since any recurrent E-L minimizer with rotation number ω belongs to the Mather set M_c , it follows that there can't be any recurrent minimizer of rotation vector $(0, \rho)$ with ρ in the interior of I . This is the gap we announced.

Remark 4 It is easy to check that both μ_1, μ_2 have their support in $\{H = 0\}$, and that $(\rho, A_1(\rho))$ for $\rho \in [0, \frac{1}{3\sqrt{2\pi}}]$ corresponds to measures supported in $E \leq 0$ whereas $(\rho, A_2(\rho)), \rho \geq \frac{1}{\sqrt{2\pi}}$ corresponds to $E \geq 0$.

It is instructive to think of the example in terms of the Jacobi metric. For $E = 0$, this metric is $\sqrt{2 \cos(2x_1)} \|\dot{x}\|$, which has exactly two minimizers (with respect to the metric) in the homology class of $(1, 0)$, namely, $x_1 = \pi/2$ and $x_1 = 3\pi/2$. Of necessity these closed curves have the same Jacobi length, *but* when they are parameterized as solutions to the E-L equations, they have different speeds and thus different rotation vectors.

As the energy E is swept through 0 there is a “minimality exchange”. Namely, when $E \leq 0$, $x_1 = \pi/2$ is the Jacobi minimizer and corresponds to a minimal measure for the E-L flow. When $E \geq 0$, $x_1 = 3\pi/2$ has these properties. At $E = 0$ they share these properties, but as noted above, they have different rotation vectors for the E-L flow.

6.2. A E-L MINIMIZER WHICH IS NOT A JACOBI MINIMIZER

Using the example above we construct a E-L minimizer that is not a Jacobi minimizer on its energy level, thus showing that the converse of Proposition 1 in Section 2.4 is false.

Let $\tilde{\alpha}_E$ be the parameterization of the curve $x_1 \equiv \pi/2$ with the property that $d\tilde{\alpha}_E$ is a trajectory of the E-L flow with $H = E \in [1/3, \infty)$. Similarly, $\tilde{\beta}_E$ corresponds to the curve $x_1 \equiv 3\pi/2$ for $E \in [-1, \infty)$. We shall use α and β to refer to these curves without a specific parameterization.

If μ_E and η_E are the ergodic invariant probability measures supported on $d\alpha_E$ and $d\beta_E$, respectively, then we have that for $E < 0$ and $E > 0$, respectively, μ_E and η_E are minimal measures. Further, μ_0 and η_0 are the minimal measures corresponding to the rotation vectors $\rho_1 = (0, \frac{1}{3\sqrt{2}\pi})$ and $\rho_2 = (0, \frac{1}{\sqrt{2}\pi})$ and for any $\rho = (0, p)$ with $\frac{1}{3\sqrt{2}\pi} < p < \frac{1}{\sqrt{2}\pi}$, the minimal measure is a linear combination of μ_0 and η_0 . In particular, if $p_3 = \frac{2}{3\sqrt{2}\pi}$ and $\rho_3 = (0, p_3) = \frac{\rho_1 + \rho_2}{2}$, then the minimal measure with rotation vector ρ_3 is $\frac{\mu_0 + \eta_0}{2}$.

Now for each $N \in \mathbb{N}$, let γ_N be the E-L minimizer satisfying $\gamma_N(-N/2) = (\pi, -Np_3/2)$ and $\gamma_N(N/2) = (\pi, Np_3/2)$, and so $\rho(\gamma_N) = \rho_3$. If we let σ_N be the probability measure uniformly distributed with respect to time on $d\gamma_N$, then the results in section 2 of [27] says that as $N \rightarrow \infty$, σ_N will converge weakly to the minimal measure with rotation vector ρ_3 and so $\sigma_N \rightarrow \frac{\mu_0 + \eta_0}{2}$. In particular, by choosing N large, we can make $H(\gamma_N, \dot{\gamma}_N)$ arbitrarily close to zero, and we can insure that γ_N spends about half its time near α_0 and the other half near β_0 . For each N , let ℓ_N be the path that is the union of

$$[(\pi/2, \frac{Np_3}{2}), (\pi, \frac{Np_3}{2})], [(\pi/2, \frac{Np_3}{2}), (\pi/2, \frac{-Np_3}{2})], \text{ and } [(\pi/2, \frac{-Np_3}{2}), (\pi, \frac{-Np_3}{2})],$$

where we use square brackets to indicate the straight segment connecting two points in the plane. Thus ℓ_N goes directly left from the top endpoint of γ_N , then travels down α and then directly across to the bottom endpoint of γ_N .

Now we assume that for all N , γ_N is a Jacobi minimizer for its energy E_N and obtain a contradiction. For N large, we have that $|E_N|$ is small. Now if $E_N \leq 0$, since α_E is the minimizer for those E , it is clear that ℓ_N has lesser Jacobi length than γ_N for energy E_N because γ_N must spend half of its time near β . By considering a curve analogous to ℓ_N but going to the right and then down β , we find that $E_N \geq 0$ is impossible also.

Remark 5 As noted in Section 3.3 and Example 4 in Section 4.2, when one attempts to take the limit of minimizers γ_N without adequate control, it can happen that the γ_N converge to something that is not in the same direction as each γ_N . The example of this section illustrates another mechanism by which the limit argument can fail. It can happen that the γ_N converge to something in the correct direction and the speed of the limiting measures average to the correct speed, but the various ergodic components of the limit measure have either greater or lesser speed than the desired one. This gives rise to the speed gap one has to allow in Theorem 1 and Theorem 5 .

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Note Added in Proof

Just prior to the publication of this paper we became aware of the elegant paper “On minimizing measures of the action of autonomous Lagrangians”, by M. J. Dias Carneiro

(*Nonlinearity*, **8**, 1077–1085, 1995) which contains much that is relevant to Sections 5 and 6 of this paper. In particular, Dias Carneiro proves that in the autonomous case, the function β (defined in Section 4.1) has a derivative in radial directions, calculates the derivative, and as a corollary, shows that Mather sets are always supported on a single energy level. These results simplify the required calculations in the example of Section 6. In addition, Dias Carneiro makes a remark that contains part of Theorem 5 and gives a result connecting the minimal measures of the Jacobi metric with those of the E-L flow.

In addition, R. Iturriaga showed us notes of Mañé which contain 11 beautiful theorems about autonomous Lagrangian systems. His students are providing the proofs of these results.

We dedicate this paper to the memory of Ricardo Mañé.

References

1. Abraham, R. and Marsden, J.E. (1985) *Foundations of Mechanics*, Addison-Wesley
2. Arnaud, M.C. (1989) Sur les points fixes des difféomorphismes exacts symplectiques de $\mathbb{T}^n \times \mathbb{R}^n$, *C.R. Acad. Sci, Paris* **309**, pp. 191-194
3. Arnold, V.I. (1980) *Mathematical methods of Classical mechanics*, Springer-Verlag
4. Aubry, S. and Le Daeron, P.Y. (1983) The discrete Frenkel-Kontorova model and its extensions I: Exact results for the ground states, *Physica* **8D**, pp. 381-422
5. Bangert, V. (1988) Mather sets for twist maps and geodesics on tori, *Dynamics Reported* **1** (eds. Kirchgraber, U. and Walther, H.O.) John Wiley
6. Bangert, V. (1989) Minimal geodesics, *Ergod. Th. & Dynam. Sys.*, **10**, pp. 263-286
7. Bernstein, D. & Katok, A. (1987) Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonian systems, *Invent. Math.* **88**, pp. 225-241
8. Boyland, P. (1994) Topological methods in surface dynamics *Topology and its Applications* **58**, pp. 223-298
9. Boyland, P, Asymptotic dynamical invariants on hyperbolic manifolds, in preparation
10. Boyland, P. and Golé, C. (1995) Lagrangian systems on hyperbolic manifolds, preprint
11. Bialy, M. and Polterovitch, L. (1992) Hamiltonian systems, Lagrangian tori, and Birkhoff's theorem *Math. Ann.* **292**, pp. 619-627
12. Denzler, J. (1987) Mather sets for plane Hamiltonian systems. *J. Appl. Math. Phys. (ZAMP)* **38**, pp. 791-812
13. Frank, J. and Misiurewicz, M. (1990) Rotation sets for toral flows, *Proc. A.M.S.* **109**, pp. 243-249
14. Fuller, F. B. (1965) On the surface of section and periodic trajectories, *Am. J. Math.*, **87**, pp. 473-480
15. Ghys, E. and de la Harpe, P. (eds) (1990) *Sur les groupes hyperboliques d'après Mikhael Gromov*, Birkhäuser
16. Golé, C. (1994) Periodic orbits for Hamiltonians in cotangent bundles *Trans. A.M.S.* **343**, Number 1, pp. 327-347
17. Gromov, M. (1977) Three remarks on geodesic dynamics and fundamental group, preprint, SUNY at Stony Brook
18. Handel, M., Periodic point free homeomorphisms of \mathbb{T}^2 , *Proc. A.M.S.* **107**, pp. 511-516.
19. Hedlund, G.A. (1932) Geodesics on a two-dimensional Riemannian manifold with periodic coefficients. *Ann. of Math* **33**, pp. 719-739
20. Herman, M. (1989) Inégalités "a priori" pour des tores Lagrangiens invariants par des difféomorphismes symplectiques, *Publ. Math., I.H.E.S.* **70**, pp. 47-101
21. Katok, A. (1992) Minimal orbits for small perturbations of completely integrable Hamiltonian systems, *Twist mappings and their applications*, *I.M.A.* **44** (McGehee, R. and Meyer, K. eds) Springer-Verlag
22. Klingenberg, W. (1982) *Riemannian Geometry* Walter de Gruyter
23. MacKay, R.S., Meiss, J.D. and Stark, J. (1989) Converse KAM theory for symplectic twist maps, *Nonlinearity* **2**, pp. 555-570
24. Mañé, R. *Ergodic theory and differentiable dynamics*, Springer-Verlag
25. Mañé, R. (1991) *Global variational methods in conservative dynamics*, 18^o Colóquio Brasileiro de Matemática, IMPA
26. Mather, J. (1991) Variational construction of orbits of twist diffeomorphisms, *Jour. of A.M.S.* **4**, pp. 207-263
27. Mather, J. (1991) Action minimizing invariant measures for positive definite Lagrangian systems,

- Math. Z.* **207**, pp. 169-207
28. Mather, J. (1993) Variational constructions of connecting orbits, *Ann. Inst. Fourier, Grenoble* **43**, 5, pp. 1349-1386
 29. Meiss, J. D. (1992) Symplectic maps, variational principles, and transport, *Rev. Mod. Phys.* **64**, No. 3, pp. 795-848
 30. Denvir, J. and MacKay, R.S. (1995) Consequences of contractible geodesics on surfaces, preprint
 31. Milnor, J. (1969) *Morse theory*, Princeton University Press
 32. Morse, M. (1924) A fundamental class of geodesics on any closed surface of genus greater than one, *Trans. Math. Soc.* **26**, pp. 25-60
 33. Moser, J. (1986) Monotone twist mappings and the calculus of variations, *Erg. Th. & Dynam. Sys.* **8***, pp. 199-214
 34. Walsh, J. (1995) Rotation vectors for toral maps and flows: a tutorial, *Int. Journ. of Bif. and Chaos* **5**, pp. 321-348