# ASYMPTOTIC RIGIDITY OF SCALING RATIOS FOR CRITICAL CIRCLE MAPPINGS

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**Abstract.** Let $f$ be a smooth homeomorphism of the circle having one cubic-exponent critical point and irrational rotation number of bounded combinatorial type. Using certain pull-back and quasi-conformal surgery techniques, we prove that the scaling ratios of $f$ about the critical point are asymptotically independent of $f$. This settles in particular the *golden mean universality conjecture*. We introduce the notion of *holomorphic commuting pair*, a complex dynamical system that, in the analytic case, represents an extension of $f$ to the complex plane and behaves somewhat as a quadratic-like mapping. We define a suitable renormalization operator that acts on such objects. Through careful analysis of the family of entire mappings given by $z \mapsto z+\theta-\frac{1}{2\pi}\, sin\, 2\pi z$, $\theta$ real, we construct examples of holomorphic commuting pairs, from which certain necessary limit set pre-rigidity results are extracted. The rigidity problem for $f$ is thereby reduced to one of renormalization convergence. We handle this last problem by means of Teichmüller extremal methods made available through the recent work of Sullivan on Riemann surface laminations and renormalization of unimodal mappings.

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## Introduction

The problem of describing the fine scale geometric structure of one-dimensional dynamical systems has been the subject of intense investigation in recent years. A fairly complete theory has emerged through the work of Herman [$H_1$] in the case of smooth diffeomorphisms of the circle. Under reasonable smoothness assumptions, Herman showed that any diffeomorphism $f : S^1 \to S^1$ with diophantine rotation number is differentiably conjugate to a rotation. In particular, the scaling structure of the orbits of $f$ is asymptotically rigid, and completely determined by its rotation number. Herman's results were subsequently sharpened by Yoccoz [$Yo_1$] and Katznelson & Ornstein [KO], and his proofs simplified in some cases with the help of renormalization methods, as in the works of Stark [St], Khanin & Sinai [KS] and Rand [$Ra_3$].

No smooth classification theory as complete as this one exists yet for other non-expanding one-dimensional dynamical systems. When critical points are present, the classical Denjoy estimates used by Herman are no longer sufficient to control the non-linearity of iterates, and even simple bounds on the geometry of orbits seem to require these techniques to be used in conjunction with the cross-ratio distortion tools introduced by Yoccoz, de Melo & van Strien, Świątek and Sullivan, among others (see [MS] for a historical account). In a recent *tour-de-force* by Sullivan, the asymptotic scaling structure of the critical orbit of an infinitely renormalizable, quadratic-like unimodal mapping of the interval was shown to be a universal function of its kneading invariant, in the cases where such invariant is of bounded type (cf. [$S_1$],[MS]).

In this work we study the scaling problem for the simplest smooth, non-expanding dynamical systems on the circle besides diffeomorphisms, namely smooth homeomorphisms with exactly one critical point. These are called *critical circle mappings.* The prototypical examples are the mappings in the Arnold family,

$$x \mapsto x + \theta - \frac{1}{2\pi} \sin 2\pi x \pmod 1. \tag{1}$$

The topological classification of such mappings is just as interesting as that of diffeomorphisms. As Hall showed in [Ha], Denjoy examples exist among critical circle mappings with flat critical points. If the critical point is non-flat, however, then a topological conjugacy to the corresponding rotation always exists [$Yo_2$], provided the rotation number is irrational. This conjugacy can of course never be smooth. Herman and Świątek have shown that it is quasisymmetric if and only if the rotation number is an irrational of bounded combinatorial type (this is still unpublished, but see [$Sw_1$], [$H_2$]). On the other hand, Khanin proved in [Kh] that in the unbounded type case the conjugacy is always purely singular with respect to Lebesgue measure. These facts reinforce the idea that critical circle mappings should be compared to each other, not with rotations, and motivate the following intrinsic rigidity question: *Are any two topologically conjugate smooth critical circle mappings always smoothly conjugate?* To give a precise meaning to this question, let us agree from this point on that a map is *smooth* if it is differentiable of class at least $C^3$ away from critical points. Let us also say that the critical point of a critical circle mapping $f$ *has type* $s > 1$ if $f$ is locally $C^3$-conjugate to $x \mapsto x|x|^{s-1} + a$, for some $a$, in a neighborhood of the critical point. It is clear that having critical points of the same type, if their types are defined at all, is a necessary condition for two smooth critical circle mappings to be smoothly conjugate. The following is supported by numerical observations and analogy with the unimodal case.

**Conjecture.** *Any two smooth critical circle mappings with the same irrational rotation number and the same type of critical point are $C^{1+\beta}$-conjugate for some $0 \le \beta < 1$.*

In this paper we take a step towards proving this conjecture for rotation numbers of bounded combinatorial type and critical points of cubic type ($s = 3$). Further steps are taken in [dFM$_1$] and [dFM$_2$]. Our methods can be adapted to cover all odd exponents $s = 2k + 1$, $k \ge 1$, as well.

We need a few definitions before we can state our results. Let $f : S^1 \to S^1$ be a critical circle mapping with critical point $c$ and let

$$\rho(f) \;=\; [r_0, r_1, \cdots] \;=\; \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{\cdots}}} \;.$$

be the continued-fraction development of its rotation number. We say that $\rho(f)$ is a number of *bounded combinatorial type* if $\max r_n < \infty$. Let $\{q_n\}_{n\ge 0}$ be the sequence of return times of the forward orbit of c to itself, and for each $n \ge 0$ let $J_n$ be the closed interval on the circle with endpoints $f^{q_n}(c)$ and $f^{q_{n+1}}(c)$ that contains $c$. Then $c$ divides $J_n$ into two intervals, $I_n$ with endpoint $f^{q_n}(c)$ and $I_{n+1}$ with endpoint $f^{q_{n+1}}(c)$. The ratio of lengths $s_n(f) = |I_{n+1}|/|I_n|$ is called the *$n$-th scaling ratio* of $f$.

Our main theorem is the analogue for critical circle mappings of the Coullet-Tresser rigidity of infinitely-renormalizable Cantor attractors of unimodal mappings proved by Sullivan in [S$_1$]. We give here two equivalent versions of this result.

**Theorem A.** *If $f$ and $g$ are smooth critical circle mappings with the same irrational rotation number of bounded combinatorial type, then they have asymptotically the same scaling ratios, i.e.*

$$\lim_{n\to\infty} \frac{s_n(f)}{s_n(g)} \;=\; 1 \;. \qquad \square$$

This theorem improves upon the so-called real a-priori bounds for critical circle mappings, according to which the ratios $s_n(f)/s_n(g)$ are eventually bounded by a constant depending only on the common rotation number of both maps. These bounds were proved by Świątek (cf. [Sw$_1$], [Sw$_2$]), Herman [H$_2$] and Yoccoz (unpublished); the proofs assume $C^3$-smoothness and a negative Schwarzian property near critical points. An important corollary to Theorem A is the so-called *golden mean universality conjecture.*

**Corollary.** *If $f$ is a smooth critical circle mapping and $\rho(f) = \frac{\sqrt{5}-1}{2} = [1, 1, 1, \ldots]$ (the golden mean), then the scaling ratios of $f$ converge to a universal constant.* $\square$

Computer-assisted work by Shenker [Sh] (cf. [Ra$_2$]) shows that the value of this universal constant is $0.7760513\ldots$ . We emphasize the golden mean case here owing to its historical significance, yet a more general corollary is the universality of scaling ratios for mappings whose rotation number is a quadratic algebraic number, i.e. has an eventually periodic continued fraction development.

A proper formulation of the second version involves the notion of quasi-symmetry. If $h$ is a homeomorphism of the line (or circle, with its linear coordinate), we let the *quasi-symmetric distortion* of $h$ at scale $t > 0$ be the number

$$k(h,t) \;=\; \sup_{0<s\leq t}\ \sup_{x}\ \frac{h(x+s)-h(x)}{h(x)-h(x-s)}\ .$$

If $k(h,t) \leq k < \infty$ for all $t > 0$ then $h$ is a $k$-*quasisymmetric* mapping. If moreover $k(h,t) \to 1$ as $t \to 0$, then $h$ is said to be *symmetric*.

**Theorem B.** *Any two smooth critical circle mappings with the same rotation number of bounded combinatorial type are conjugate by a symmetric homeomorphism.* □

Both theorems will follow from certain *renormalization convergence results*. Just as in the unimodal case, one can define a renormalization scheme for critical circle mappings, thanks to the fundamental notion of *commuting pair* developed by Lanford and Rand (cf. [$L_1$], [$Ra_1$]). Commuting pairs represent whole conjugacy classes of circle mappings, and so each of them has a rotation number of its own. The first return map to $J_n$, consisting of $f^{q_n}$ restricted to $I_{n+1}$ and $f^{q_{n+1}}$ restricted to $I_n$, is the principal example of a commuting pair, in this case the *n-th renormalization* of $f$ (see section I). This renormalization scheme acts as the *Gauss map* on rotation numbers. As first observed by Ostlund, Rand, Sethna & Siggia in [ORSS], and also by Feigenbaum, Kadanoff & Shenker in [FKS], renormalization can be viewed as an operator acting on an infinite-dimensional space of commuting pairs. In both works, the same claim was made that a hyperbolic fixed-point for this renormalization operator exists, corresponding to an analytic critical circle mapping with golden-mean rotation number. This claim can be generalized in an obvious way to cover all rotation numbers that are periodic under the Gauss map. A computer assisted proof of the existence and hyperbolicity of a golden-mean fixed-point, along the lines of Lanford's proof for the Feigenbaum case, was given by Mestel in [Me]. Later, Epstein and Eckmann proved the existence without essential help from the computer [EE]. Their proof uses Schauder's theorem, and therefore guarantees neither uniqueness nor hyperbolicity of the fixed-point. Taking a broader perspective, and inspired by his own computer-assisted work on unimodal mappings, Lanford conjectured that the renormalization operator is *globally hyperbolic* and possesses an infinite-dimensional *horseshoe-like attractor*.

Although in this paper we don't go so far as proving Lanford's conjectures in full, we do prove the existence and *global* uniqueness of the golden-mean fixed-point, as well as of all other fixed or periodic points of the renormalization operator, and describe their (codimension-one) stable sets. We prove that the successive renormalizations of any two commuting pairs representing critical circle mappings with the same rotation number of bounded type converge together in the $C^0$-topology. Indeed, a stronger form of convergence, implying $C^k$-convergence for all $k < \infty$, takes place if both pairs are real-analytic of a special kind (see section IX). Our methods don't give any rate of convergence, however, which is unfortunate since an exponential rate would yield the Conjecture in the cubic case.

Our approach is based on the deep holomorphic and quasiconformal ideas of Sullivan presented in [$S_1$], [$S_5$], and detailed in [MS, Ch. VI]. Here is a brief outline of the paper. In section I we define a special class of real-analytic commuting pairs, the *Epstein class*, which contains all limits of renormalization due to the real a-priori bounds. In section II, we introduce certain complex-analytic dynamical systems called *holomorphic commuting pairs*. These objects restrict to real-analytic commuting pairs on the line, and resemble quadratic-like mappings in many ways. For instance, they have annular fundamental domains and Julia sets, just as quadratic-like mappings do. A holomorphic commuting pair can be renormalized, and the result is again an object of the same type. In section III, we prove a pull-back theorem for holomorphic commuting pairs. This permits us to assign a quasiconformal distance between topologically equivalent objects of this type. In the resulting metric spaces, any two points can be joined by special paths, whose elements are quasiconformal deformations of the endpoints, called *Beltrami paths* (cf. section V). Renormalization carries Beltrami paths to Beltrami paths. We say that a Beltrami path is *efficient* if the distance between its endpoints is not much smaller than its length. In section VI, we show how to factor the long compositions representing high renormalizations of commuting pairs in the Epstein class so that the factors satisfy the hypotheses of Sullivan's *sector theorem*. This is the point where we have to assume that the rotation number is of bounded combinatorial type. The factoring combined with Sullivan's *sector inequality* proves, as stated in section VII, that any sufficiently high renormalization of a commuting pair in the Epstein class can be extended to a holomorphic commuting pair, whose fundamental domain is a definitely thick annulus. In particular, renormalizing a very long but efficient Beltrami path sufficiently many times, we see using the pull-back theorem that its endpoints are brought within a fixed distance. It is a beautiful discovery by Sullivan that in this situation the image Beltrami path necessarily *coils*, i.e. it cannot be efficient. Therefore the distances between points along the path are contracted, and this implies strong renormalization convergence, as we show in section IX.

This outline overlooks several important points. Thus, since the boundaries of domain and range of holomorphic commuting pairs are quite arbitrary, it is necessary to work with the *germs* of such objects around their Julia sets, with a germ version of the qc-distance called the *Julia-Teichmüller* distance, and also with the infinitesimal form of that distance. Using examples of holomorphic commuting pairs built from the complexified Arnold family, we show in section IV that the Julia sets of these objects carry no invariant line fields. Therefore all quasiconformal deformations of a fixed germ are supported in the *external class* of that germ, which is the Cantor repeller constructed in section VIII. The space of backward-orbits of this Cantor repeller is a compact *Riemann-surface lamination* in the sense of Sullivan. This gives us a space to which Sullivan's coiling idea can be applied. The qc-structures that are invariant for the repeller can be lifted to the lamination. By Sullivan's *almost geodesic principle*, if a structure of this kind comes from an optimal qc-conjugacy between two germs, then it can be used to generate a very long but efficient Beltrami path of structures on the lamination. The coiling lemma used to prove contraction is a partial converse to this fact.

The results in this paper have a number of interesting applications. The basic theory of holomorphic pairs introduced here has been used recently by McMullen [McM] in his elegant study of self-similarity properties of Siegel disks. We mention one further application, connected with the scalings of frequency-locking intervals of one-parameter families of circle mappings. In the Arnold family (1), the values of $\theta$ for which the corresponding map has irrational rotation number form a Cantor set. The gaps of this Cantor set have been examined numerically by Cvitanovic & Söderberg in [CS], and its Hausdorff dimension estimated at about 0.87. Świątek gave a rigorous proof that this Cantor set has zero Lebesgue measure in [$\mathrm{Sw}_1$]. Later, in [GrS], he and Graczyk proved that the Hausdorff dimension is less than 1 but not smaller than $\frac{1}{3}$. Our results can be combined with a careful analysis of the unstable manifolds of the renormalization operator to establish the universality of the Hausdorff dimension among cubic families. The analysis will be carried out in a forthcoming paper.

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## I. Renormalization of real commuting pairs and the Epstein class

Let $f : S^1 \to S^1$ be a smooth, orientation-preserving homeomorphism having exactly one critical point $c \in S^1$ of cubic type. That is, let $f$ be such that we can represent it in the form $h \circ f_\theta \circ H$, where $h$ and $H$ are smooth, orientation-preserving diffeomorphisms, $\theta$ is a real number and $f_\theta$ is the mapping whose lift $E_\theta$ to the real line is given by

$$E_\theta(x) = x + \theta - \frac{1}{2\pi} \sin 2\pi x \ .$$

We call $f$ a *critical circle mapping*. We also refer to $f = h \circ f_\theta \circ H$ as an $hQ$-decomposition of $f$ (cf. [$\mathrm{S}_1$]). Our standing assumption in this paper is that $f$ has no periodic points, i.e. that its rotation number $\rho(f)$ is irrational. Thus $f$ is topologically conjugate to the corresponding irrational rotation, after a well-known theorem of Yoccoz (cf. [$\mathrm{Yo}_2$ ]). We write the rotation number of $f$ as an infinite continued fraction $\rho(f) = [r_0, r_1, \ldots, r_n, \ldots]$ and let $(q_n)_{n\geq 0}$ denote the successive closest return times given recursively by $q_0 = 1$, $q_1 = r_0$ and

$$q_{n+1} = r_n q_n + q_{n-1} \ , \tag{1}$$

for all $n \geq 1$. Recall that each of these numbers appears as denominator in the truncated expansion of order $n$ of $\rho(f)$ in its irreducible form

$$\frac{p_n}{q_n} \ = \ [r_0, r_1, \ldots, r_{n-1}] \ \ ,$$

where $p_0 = 0$, $p_1 = 1$ and $p_{n+1} = r_n p_n + p_{n-1}$ for all $n \geq 1$. It is also convenient to set $q_{-1} = 0$. We denote by $I_n(c)$ the closed interval in $S^1$ with endpoints $c$ and $f^{q_n}(c)$

containing $f^{q_{n+2}}(c)$. The dynamical first return map to the interval $I_n(c) \cup I_{n+1}(c)$ is given by $f^{q_{n+1}}$ on $I_n(c)$ and by $f^{q_n}$ on $I_{n+1}(c)$. Each pair $(f^{q_n}, f^{q_{n+1}})$ yields an example of what one calls *weakly commuting pair* or simply *commuting pair*, after Lanford and Rand (cf. [L$_1$], [L$_2$], [Ra$_1$], [Ra$_2$]). Here is the abstract definition.

*Definition 1.* A commuting pair $\zeta = (\xi, \eta)$ consists of two orientation preserving smooth homeomorphisms $\xi : I_\xi \to \xi(I_\xi), \eta : I_\eta \to \eta(I_\eta)$ into the reals where

(a) $I_\xi = [\eta(0), 0] \subseteq \mathbb{R}$, $I_\eta = [0, \xi(0)] \subseteq \mathbb{R}$;

(b) Both $\xi$ and $\eta$ have homeomorphic extensions, with the same degree of smoothness, to interval neighborhoods of their corresponding domains, and such extensions commute, i.e. $\xi \circ \eta = \eta \circ \xi$, wherever both sides are defined;

(c) $\xi \circ \eta(0)$ belongs to $I_\eta$;

(d) We have $\xi'(x) \neq 0 \neq \eta'(y)$, for all $x$ in $I_\xi \setminus \{0\}$ and all $y$ in $I_\eta \setminus \{0\}$.

A *critical commuting pair* is a commuting pair that has $hQ$-decompositions $\xi = h_\xi \circ Q \circ H_\xi$ and $\eta = h_\eta \circ Q \circ H_\eta$ where $h_\xi$, $h_\eta$, $H_\xi$, $H_\eta$ are smooth diffeomorphisms and $Q$ is the map $z \mapsto z^3$.

An object which is either a commuting pair or obtained from a commuting pair by conjugating $\xi$ and $\eta$ by $x \mapsto -x$ (resp. by $x \mapsto \lambda x$, $\lambda \neq 0$) is called a *commuting pair up to orientation* (resp. *up to linear rescaling*). A critical circle mapping $f$ gives rise to a sequence of critical commuting pairs in the following way. Let $\overline{f}$ be a lift of $f$ to the real line satisfying $\overline{f}'(0) = 0$ and $0 < \overline{f}(0) < 1$. For each $n \geq 0$, let $J_n \subseteq \mathbb{R}$ be the closed interval adjacent to zero that projects down homeomorphically onto $I_n(c)$ via the exponential mapping. Let $T : \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x + 1$ and let $\xi_{f,n} : J_{n+1} \to \mathbb{R}$ be given by $\xi_{f,n}(x) = T^{-p_n} \circ \overline{f}^{q_n}(x)$; similarly, let $\eta_{f,n} : J_n \to \mathbb{R}$ be given by $\eta_{f,n}(x) = T^{-p_{n+1}} \circ \overline{f}^{q_{n+1}}(x)$ . Then $\zeta_{f,n} = (\xi_{f,n}, \eta_{f,n})$ is a critical commuting pair up to orientation.

Conversely, regarding $I = [\eta(0), \xi(0)]$ as the circle (identifying $\eta(0)$ and $\xi(0)$) and letting $f_\zeta : I \to I$ be given by

$$f_\zeta(x) = \begin{cases} \xi(x), & \text{if } \eta(0) \leq x < 0 \\ \\ \eta(x), & \text{if } 0 \leq x \leq \xi(0) \end{cases} , \tag{3}$$

we recover a plethora of critical circle mappings from a critical commuting pair $\zeta = (\xi, \eta)$. We perform the glueing of $\eta(0)$ to $\xi(0)$ via the mapping $\eta\xi^{-1}$, which by conditions (b) and (d) above maps a small neighborhood of $\xi(0)$ diffeomorphically onto a small neighborhood of $\eta(0)$. We obtain a smooth, closed one-manifold $M$ as the quotient space, and $f_\zeta$ projects down to a smooth homeomorphism $F_\zeta : M \to M$. Each identifying diffeomorphism $\varphi : M \to S^1$ gives rise to a critical circle mapping $f^\varphi = \varphi \circ F_\zeta \circ \varphi^{-1}$. Although there is no canonical choice for $\varphi$, any two choices are such that the corresponding $f^\varphi$ 's differ by a diffeomorphism. Therefore we recover a whole *smooth conjugacy class* of critical circle mappings (see [dFM$_1$] for a detailed exposition of the glueing procedure, first introduced by Lanford). We will abuse language henceforth and call $f_\zeta$ *the* critical circle mapping of $\zeta$. We let $I_n \subseteq I$ be the closed interval that corresponds to $I_n(c)$ for any representative $f^\varphi$, for each $n \geq 0$. The endpoints of $I_n$ are 0 and $f_\zeta^{q_n}(0)$, where $\{q_n\}$ is the sequence of return times of any such representative.

Letting $\rho(\zeta) = \rho(f_\zeta)$ be the rotation number of $\zeta$, we are ready to define the renormalization operator for commuting pairs. If $\rho(\zeta) = [r+1, r_1, r_2, \ldots, r_n, \ldots]$, then $\eta^{r+1}\xi(0) < 0 < \eta^r\xi(0)$ and one verifies that the mappings $\eta|[0, \eta^r\xi(0)]$ and $\eta^r \circ \xi|I_\xi$ constitute a commuting pair up to orientation.

*Definition 2.* The commuting pair

$$\mathcal{R}\zeta = (\eta^\#, (\eta^r \circ \xi)^\#) ,$$

where $^\#$ denotes linear rescaling by the factor $\lambda = \xi(0)/\eta(0) < 0$, is called the *first renormalization of* $\zeta$. We also refer to $(\eta, \eta^r \circ \xi)$ as the first renormalization of $\zeta$ without rescaling.

Thus, in the notation introduced above, we have ${\zeta_{f,n}}^\# = \mathcal{R}({\zeta_{f,n-1}}^\#)$ for all $n \geq 1$. These may therefore be regarded as the successive renormalizations of $f$. It is easy to see that $\rho(\mathcal{R}\zeta) = [r_1 + 1, r_2, \ldots, r_n, \ldots]$. Thus, renormalization acts essentially as the Gauss map on rotation numbers.

Now we define a class of commuting pairs containing the *attractor* of renormalization.

*Definition 3.* A real-analytic commuting pair $\zeta = (\xi, \eta)$ is said to be in the Epstein class $\mathcal{E}$ if, for $\gamma = \xi, \eta$, there exists a decomposition $\gamma = h_\gamma \circ Q$, where as before $Q : z \mapsto z^3$, such that

(a) $h_\gamma : Q(I_\gamma) \to \gamma(I_\gamma)$ is an orientation preserving diffeomorphism;

(b) $h_\gamma^{-1}$ extends to a schlicht mapping $\mathbf{C}(\widetilde{I}_\gamma) \to \mathbf{C}$, where $\widetilde{I}_\gamma \supseteq \gamma(I_\gamma)$ is some open interval (here $\mathbf{C}(I) = \mathbf{C} \setminus (\mathbb{R} \setminus \mathrm{I})$).

For each $s > 0$, let us write, taking into account condition (b) above,

$$\mathcal{E}_s = \{\zeta \in \mathcal{E} : \widetilde{I}_\gamma \supseteq I_\gamma^s \,, \gamma = \xi, \eta\} ,$$

where $I^s$ denotes the interval centered at the midpoint of $I$ whose length is $(1+s)$-times the length of $I$ (cf. section VI). We also refer to each $\mathcal{E}_s$ as an Epstein class.

Now we have the following fact.

**Lemma I.1.** *The Epstein class $\mathcal{E}$ is invariant under renormalization.* □

The following lemma describes how the successive renormalizations without rescaling of a commuting pair $\zeta$ are nested inside $\zeta$. It will be also extremely useful in section VI, in the breaking-up of long renormalization compositions leading to the complex bounds.

**Lemma I.2.** *Let $\zeta = (\xi, \eta)$ be a critical commuting pair and let $(\xi_n, \eta_n)$ be the sequence of renormalizations of $\zeta$ without rescaling. Then $I_{\xi_n} = I_n$ and $I_{\eta_n} = I_{n-1}$ for all $n \geq 1$ and we have the following hybrid representations*

$$\begin{cases} n \quad \text{even} \quad \Rightarrow \begin{cases} \xi_n(x) = f_\zeta^{q_{n-1}-1} \circ \xi(x) & \text{for all } x \text{ in } I_{\xi_n} = I_n \\ \eta_n(x) = f_\zeta^{q_n - 1} \circ \eta(x) & \text{for all } x \text{ in } I_{\eta_n} = I_{n-1} \end{cases} \\ n \quad \text{odd} \quad \Rightarrow \begin{cases} \xi_n(x) = f_\zeta^{q_{n-1}-1} \circ \eta(x) & \text{for all } x \text{ in } I_{\xi_n} = I_n \\ \eta_n(x) = f_\zeta^{q_n - 1} \circ \xi(x) & \text{for all } x \text{ in } I_{\eta_n} = I_{n-1} \end{cases} \end{cases} \tag{4}$$

*Proof.* The first assertion follows easily by induction on $n$, using the recurrence relations (1). The hybrid expressions in (4) are clear if we observe that $I_n$ is contained in the domain of $\xi$ when $n$ is even, and in the domain of $\eta$ when $n$ is odd. ◻

Because it relates the dynamics of $\zeta$ with that of $f_\zeta$, Lemma I.2 can be used to transfer certain well-known a-priori bounds for critical circle mappings to corresponding ones for critical commuting pairs. Let $s_n(\zeta) = |I_{n+1}|/|I_n|$ be the $n$-th scaling ratio of $\zeta$. Also, if $\zeta_1$ and $\zeta_2$ have the same rotation number, let their *quasi-symmetric* distance be the number

$$d_{QS}(\zeta_1, \zeta_2) = \log k(h) \ ,$$

where $h : [\eta_1(0), \xi_1(0)] \to [\eta_2(0), \xi_2(0)]$ is the conjugacy between both pairs and $k(h)$ is the quasi-symmetric distortion of $h$. Then, we can use Lemma I.2 to re-state the well-known results of Herman [H$_2$], Yoccoz (unpublished), Świątek [Sw$_2$] and Graczyk & Świątek [GrS] in the following combined form.

**Theorem I.3.** *Given $0 < \alpha < 1$ irrational, there exist constants $K_1 > 1$ and $K_2, K_3 > 0$ depending only on $\alpha$ such that the following statements hold.*

*(a) If $\zeta$ is a critical commuting pair with rotation number $\alpha$, then for all sufficiently large $n$ we have $K_1^{-1}|I_n| \leq |I_{n+1}| \leq K_1|I_n|$; in other words the scaling ratios of $\zeta$ are bounded away from zero and infinity;*

*(b) If $\zeta_1$ and $\zeta_2$ are critical commuting pairs with rotation number $\alpha$, then for all sufficiently large $n$ we have*

$$|\frac{s_n(\zeta_1)}{s_n(\zeta_2)} - 1| \leq K_2 \ ,$$

*and moreover $d_{QS}(\mathcal{R}^n\zeta_1, \mathcal{R}^n\zeta_2) \leq K_3$.* ◻

Notice in particular that any two critical commuting pairs with the same irrational rotation number are quasi-symmetrically conjugate. Using Theorem I.3 and the bounded geometry results and techniques of Sullivan [S$_1$, §4], one obtains the following fundamental compactness result, which is essentially the *pure singularity property* of Świątek [Sw$_2$]. A complete, detailed proof of this theorem (and much more) can be found in [dFM$_1$].

**Theorem I.4.** *Let $\zeta$ be a critical commuting pair of class $C^r$ $(r \geq 3)$ with irrational rotation number $\rho(\zeta)$, and consider the hQ-decompositions of its successive renormalizations $\xi_n^\# = h_{\xi,n} \circ Q \circ H_{\xi,n}$ and $\eta_n^\# = h_{\eta,n} \circ Q \circ H_{\eta,n}$. Then the families $\{\xi_n^\#\}_{n\geq 0}$ and $\{\eta_n^\#\}_{n\geq 0}$ are precompact in the sense that, for $\gamma = \xi, \eta$, the following conditions hold.*

*(a) The critical values of $\gamma_n^\#$ are bounded away from zero;*

*(b) There exist $s > 0$ depending only on the rotation number of $\zeta$, fixed intervals $\mathcal{I}_\gamma$, $\mathcal{J}_\gamma$ and a positive integer $N$ such that, for all $n \geq N$, $h_{\gamma,n}^{-1}$ is well-defined on $\mathcal{I}_\gamma$ and $\mathcal{I}_\gamma \supseteq (\gamma_n^\# I_{\gamma_n}^\#)^s$, and $H_{\gamma,n}$ is well-defined on $\mathcal{J}_\gamma$ and $\mathcal{J}_\gamma \supseteq (\gamma_n^\#)^{-1}(\mathcal{I}_\gamma)$;*

*(c) The family $\{h_{\gamma,n}^{-1}|\mathcal{I}_\gamma\}_{n\geq N}$ has compact closure in the $C^r$-topology on diffeomorphisms;*

*(d) The sequence $(H_{\gamma,n}|\mathcal{J}_\gamma)_{n\geq N}$ converges to the identity in the $C^r$-topology on diffeomorphisms.*

*Moreover, every $C^r$-limit of $\zeta_n^\# = (\xi_n^\#, \eta_n^\#)$ is a critical commuting pair in the Epstein class $\mathcal{E}_s$.* □

*Remark 1.* The facts stated in Theorems I.3 and I.4 are collectively known as the *real a-priori bounds* for critical circle mappings.

*Remark 2.* As it turns out, proving the asymptotic rigidity statements of Theorems A and B in the Introduction is tantamount to showing that $d_{QS}(\mathcal{R}^n\zeta_1, \mathcal{R}^n\zeta_2) \to 0$ as $n \to \infty$. An exponential rate of convergence would yield the Conjecture in the cubic case, see [dFM$_1$], [dFM$_2$].

## II. Holomorphic commuting pairs

Now we introduce special complex-analytic extensions of critical commuting pairs, akin to quadratic-like mappings. We need an auxiliary definition. Let us say that a configuration of simply connected domains $(\mathcal{D}, \mathcal{O}_\xi, \mathcal{O}_\eta, \mathcal{O}_\nu)$ in the plane is a *bow-tie* if the following conditions are satisfied: (a) all four domains are symmetric about the real axis; (b) each $\mathcal{O}_\gamma$ is a Jordan domain whose closure is contained in $\mathcal{D}$; (c) $\overline{\mathcal{O}}_\xi \cap \overline{\mathcal{O}}_\eta = \{0\} \subseteq \mathcal{O}_\nu$; (d) the differences $\mathcal{O}_\gamma \setminus \mathcal{O}_\nu$ and $\mathcal{O}_\nu \setminus \mathcal{O}_\gamma$ are non-empty connected sets for $\gamma = \xi, \eta$; (e) the interval $\mathcal{O}_\xi \cap \mathbb{R}$ lies to the left of zero. Let $J_\gamma$ denote the open intervals $\mathcal{O}_\gamma \cap \mathbb{R}$ for $\gamma = \xi, \eta, \nu$. Then $J_\xi$ and $J_\eta$ share an endpoint at the origin, and $J_\xi$ lies in the negative real axis. Also, $J_\nu$ contains the origin and is contained in $\overline{J}_\xi \cup J_\eta$. Moreover, due to condition (d) we know that $\mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{O}_\nu$, as well as $\mathcal{O}_\gamma \cap \mathcal{O}_\nu$ ($\gamma = \xi, \eta$) are Jordan domains. A sketch of the situation we have in mind is shown in Figure 1.

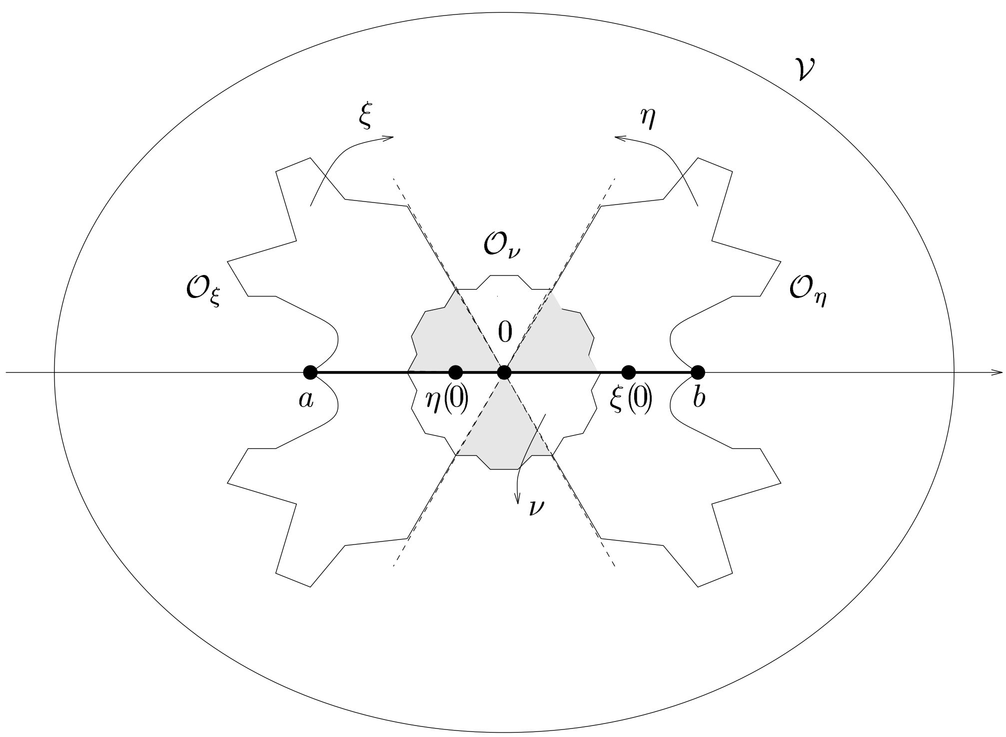


*Figure 1*

*Definition 4.* A *holomorphic commuting pair* consists of a bow-tie $(\mathcal{D}, \mathcal{O}_\xi, \mathcal{O}_\eta, \mathcal{O}_\nu)$ together with complex analytic mappings $\xi, \eta, \nu$ having $\mathcal{O}_\xi, \mathcal{O}_\eta, \mathcal{O}_\nu$ respectively as their domains and a positive integer $m$ satisfying the following conditions:

[$\mathrm{H}_1$] All three mappings commute with complex conjugation.

[$\mathrm{H}_2$] $\xi$ and $\eta$ are schlicht mappings onto $\mathcal{D} \cap \mathbf{C}(\xi(J_\xi))$ and $\mathcal{D} \cap \mathbf{C}(\eta(J_\eta))$ respectively.

[$\mathrm{H}_3$] $\nu$ is a 3-fold branched covering onto $\mathcal{D} \cap \mathbf{C}(\nu(J_\nu))$, with a unique critical point at zero.

[$\mathrm{H}_4$] $\xi$ and $\eta$ have analytic extensions to a certain neighborhood of zero where both $\xi \circ \eta$ and $\eta \circ \xi$ are defined, and we have $\xi \circ \eta(z) = \eta \circ \xi(z) = \nu(z)$ for all $z$ in that neighborhood.

[$\mathrm{H}_5$] If $x \in J_\xi$ then $\xi(x) > x$, whereas if $x \in J_\eta$ then $\eta(x) < x$; moreover, $\xi(0), \nu(0) \in J_\eta$ and $\eta(0) \in J_\xi$.

[$\mathrm{H}_6$] If $a$ is the left endpoint of $J_\xi$ and $b$ is the right endpoint of $J_\eta$ then both $\xi^m(a)$ and $\eta(b)$ are well-defined as boundary values, and we have $\xi^m(a) = \eta(0)$ and $\eta(b) = \xi(0)$.

Holomorphic commuting pairs will be denoted by $\Gamma$. Explicit examples will be constructed in section IV. If $\Gamma$ is given, it is clear from $\mathrm{H}_2$, $\mathrm{H}_3$, $\mathrm{H}_4$ and $\mathrm{H}_5$ that the restrictions $\xi|[\eta(0), 0]$ and $\eta|[0, \xi(0)]$ constitute a real analytic critical commuting pair. Thus, we define the *rotation number* of $\Gamma$ to be the rotation number of its real commuting pair. Note that the intervals $J = \overline{J}_\xi \cup J_\eta$ and $I = [\eta(0), \xi(0)]$ are both forward invariant under the dynamical system generated by $\xi$ and $\eta$. We call them the *large* and *small* dynamical intervals of $\Gamma$, respectively. We also say that the integer $m$ in condition $\mathrm{H}_6$ is the *height* of $\Gamma$. The following proposition is fundamental.

**Proposition II.1.** *In any holomorphic commuting pair, the mappings $\xi$ and $\eta$ have analytic extensions to $\mathcal{O}_\xi \cup \mathcal{O}_\nu$ and $\mathcal{O}_\eta \cup \mathcal{O}_\nu$ respectively. Moreover, the restrictions $\xi_* = \xi|\mathcal{O}_\nu$ and $\eta_* = \eta|\mathcal{O}_\nu$ are 3-fold branched covering maps onto $\mathcal{O}_\eta$ and $\mathcal{O}_\xi \cap \mathbf{C}([\xi^{-1} \circ \eta(0), 0])$ respectively, and we have $\eta \circ \xi_* = \xi \circ \eta_* = \nu$.*

*Proof.* We use the 3-fold symmetry of $\mathcal{O}_\nu$ coming from $\nu$ in order to extend $\xi$ and $\eta$ by Schwarz reflection in the following way.

By $\mathrm{H}_2$, the composition $\eta \circ \xi$ is a well-defined schlicht mapping of $V = \xi^{-1}(\mathcal{O}_\eta)$ onto $\mathcal{D} \cap \mathbf{C}([\eta(0), \eta\xi(0)])$. Let $Y = \nu^{-1}([\nu(0), +\infty))$. Then, using conditions $\mathrm{H}_1$ and $\mathrm{H}_3$ we readily see that $\mathcal{O}_\nu \setminus Y$ has exactly 3 connected components, one of which, call it $W$, is symmetric about the real axis. We claim that $V = W$. Since $V$ is also symmetric about the real axis, it is enough to show that $V^+ = W^+$. Now, $\nu$ maps $W^+$ injectively onto $\mathcal{D}^+$; likewise, $\eta \circ \xi$ maps $V^+$ onto $\mathcal{D}^+$ injectively. Hence the composition $\phi = \nu^{-1} \circ (\eta \circ \xi)$ is well-defined in $V^+$ and maps it onto $W^+$. Since by $\mathrm{H}_4$ we have $\eta \circ \xi \equiv \nu$ on some neighborhood $\mathcal{O}$ of zero, we deduce that $\phi(z) = z$ for all $z \in \mathcal{O} \cap V^+$. Therefore $\phi$ must be the identity map, which settles the claim.

From this, it follows that $\xi^{-1}(0)$ is the left endpoint of $J_\nu$, and since $\nu$ agrees with $\eta \circ \xi$ over all of $W$, we see that $\nu(\xi^{-1}(0)) = \eta(0)$. Switching the roles of $\xi$ and $\eta$ in this argument, we deduce that $\eta^{-1}(0)$ is the right endpoint of $J_\nu$ and that $\nu(\eta^{-1}(0)) = \xi(0)$. Therefore by $\mathrm{H}_3$ the image of $\mathcal{O}_\nu$ under $\nu$ is $\mathcal{D} \cap \mathbf{C}([\eta(0), \xi(0)])$, which by $\mathrm{H}_2$ and the last equality in $\mathrm{H}_6$ is the image of $\mathcal{O}_\eta$ under $\eta$. This shows that $\xi_* = \eta^{-1} \circ \nu : \mathcal{O}_\nu \to \mathcal{O}_\eta$ is well-defined. It is clearly a 3-fold branched covering onto $\mathcal{O}_\eta$, and since $\nu$ agrees with $\eta\circ\xi$ over $W$, we have $\xi_* \equiv \xi$ there. The proof for $\eta$ is similar. □

Our next proposition introduces a renormalization operator for holomorphic commuting pairs which is compatible with the real renormalization operator of section I. In the proof we shall use the following elementary set-theoretic remark.

**Lemma II.2.** *Let $\phi : A \to B$ be one-to-one and onto, and let $(B_n)_{n\geq 0}$ be the sequence of subsets of $B$ defined by $B_0 = B$ and $B_{n+1} = \phi(A \cap B_n)$. Then for each $n \geq 1$ the $n$-th iterate $\phi^n$ is well-defined over $A_{n-1} = A \setminus \bigcup_{i=0}^{n-1} \phi^{-i}(B\setminus A)$, and maps $A_{n-1}$ bijectively onto $B_{n-1}$.* □

**Proposition II.3.** *Let $\Gamma$ be a holomorphic commuting pair. Then there exists a holomorphic commuting pair $\mathcal{R}(\Gamma)$ whose underlying real commuting pair is the first renormalization of the real commuting pair of $\Gamma$.*

*Proof.* Let $\rho(\Gamma) = [r, r_1, r_2, \cdots, r_n, \cdots]$ be the rotation number of $\Gamma$. Recall that the first renormalization of $(\xi, \eta)$ is the pair $(\eta, \eta^r \circ \xi)$ up to the linear rescaling given by $x \mapsto \lambda x$, where $\lambda = \xi(0)/\eta(0) < 0$. We will obtain the desired $\mathcal{R}(\Gamma)$ up to such rescaling by constructing domains $\mathcal{O}_{\widehat{\xi}}, \mathcal{O}_{\widehat{\eta}}, \mathcal{O}_{\widehat{\nu}}$ and corresponding maps $\widehat{\xi}, \widehat{\eta}, \widehat{\nu}$. From $\rho(\Gamma) = (r, r_1, \cdots, r_n, \cdots)$, we also know that

$$\eta^r\xi(0) > 0 > \eta^{r+1}\xi(0)\ ,$$

and therefore $\eta(0) < \eta^{k-1}\xi(0)$ provided $1 \leq k \leq r+1$.

First we take $\mathcal{O}_{\widehat{\xi}} = \mathcal{O}_\eta$ and set $\widehat{\xi} = \eta$. Let $A = \mathcal{D} \cap \mathbf{C}([\eta(0), \xi(0)])$ be the image of $B = \mathcal{O}_\eta$ under $\eta$ and set $\phi = \eta^{-1} : A \to B$. For each $k \geq 1$ we know by Lemma II.2 that $\phi^k$ is a well-defined one-to-one map of

$$A_{k-1} = A \setminus \bigcup_{i=0}^{k=1} \phi^{-i}(B \setminus A) = \mathcal{D} \cap \mathbf{C}([\eta(0), \eta^{k-1}\xi(0)])$$

onto $B_{k-1} \subseteq \mathcal{O}_\eta$. It follows that, for $1 \leq k \leq r+1$, the domain $A_{k-1}$ is symmetric and simply-connected, and since $\phi^k$ is schlicht, the same holds for $B_{k-1}$. One can verify that $A \cap B_{k-1}$ equals $B_{k-1}$ minus the slit $[\xi(0), b)$, which opens-up when $\phi = \eta^{-1}$ is applied. Since $B_k = \phi(A \cap B_{k-1})$ and $B_0 = \mathcal{O}_\eta$ is a Jordan domain, an inductive argument shows that $B_{k-1}$ is a Jordan domain for $1 \leq k \leq r+1$. In particular, $B_{r-1} \subseteq \mathcal{O}_\eta$ is a Jordan domain. Let $\xi_* : \mathcal{O}_\nu \to \mathcal{O}_\eta$ be the 3-fold branched covering given by Proposition II.1, put $U = \xi_*^{-1}(B_{r-1})$ and define $\widehat{\eta}_* : U \to A_{r-1}$ by $\widehat{\eta}_* = \eta^r \circ \xi_*$. Since the critical value of $\xi_*$ belongs to $B_{r-1}$, we see that $U$ is a Jordan domain and that $\widehat{\eta}_*$ is a 3-fold branched covering onto its image. We then take $\mathcal{O}_{\widehat{\eta}} = U \cap \mathcal{O}_\xi$ and set $\widehat{\eta} = \widehat{\eta}_* | \mathcal{O}_{\widehat{\eta}}$. As $U \cap \mathcal{O}_\xi = \xi_*^{-1}(B_{r-1}) \cap \mathcal{O}_\xi$ is mapped bijectively by $\widehat{\eta}_*$ onto $B_{r-1} \setminus [\xi(0), b)$, we deduce that $\mathcal{O}_{\widehat{\eta}}$ is a Jordan domain and that $\widehat{\eta}$ is schlicht over $\mathcal{O}_{\widehat{\eta}}$. On the other hand, $B_r \subseteq \mathcal{O}_\eta$ is also a Jordan domain containing $\xi(0)$, the critical value of $\xi_*$. Hence, if we let $\mathcal{O}_{\widehat{\nu}} = \xi_*^{-1}(B_r) \subseteq \mathcal{O}_\nu$ and put $\widehat{\nu} = \eta^{r+1} \circ \xi_* : \mathcal{O}_{\widehat{\nu}} \to A_r$, we see at once that $\mathcal{O}_{\widehat{\nu}}$ is a Jordan domain and that $\widehat{\nu}$ is a 3-fold branched covering onto its image. Moreover, $\widehat{\nu} = \eta \circ (\eta^r \circ \xi_*) = \widehat{\xi} \circ \widehat{\eta}_*$.

Now, if we linearly rescale $\mathcal{D}, \mathcal{O}_{\widehat{\xi}}, \mathcal{O}_{\widehat{\eta}}, \mathcal{O}_{\widehat{\nu}}, \widehat{\xi}, \widehat{\eta}, \widehat{\nu}$ by the map $z \to \lambda z$, we get the desired holomorphic commuting pair $\mathcal{R}(\Gamma)$. Indeed, $(\mathcal{D}, \mathcal{O}_{\widehat{\xi}}, \mathcal{O}_{\widehat{\eta}}, \mathcal{O}_{\widehat{\nu}})$ is a bow-tie up to linear rescaling. Moreover, conditions $\mathrm{H}_1$ through $\mathrm{H}_4$ hold by construction. Condition $\mathrm{H}_5$ is not satisfied until we do the rescaling (which reverses orientation on the line), when it becomes clear. Finally, since we have

$$\widehat{\xi}^{r+1}(b) = \eta^{r+1}(b) = \eta^r \xi(0) = \widehat{\eta}(0)$$

and

$$\widehat{\eta}(c) = \eta^r \circ \xi(c) = \eta^r(\eta^{-r+1}(0)) = \eta(0) = \widehat{\xi}(0) \ ,$$

where $c$ is the left endpoint of $\mathcal{O}_{\widehat{\eta}} \cap \mathbb{R}$, condition $\mathrm{H}_6$ is satisfied if we take $m = r+1$. □

We are interested in the dynamical system generated by the mappings $\xi|\mathcal{O}_\xi, \eta|\mathcal{O}_\eta$ and $\nu|\mathcal{O}_\nu$. We shall identify $\Gamma$ itself with this dynamical system. It will be very important to know that the $\Gamma$-orbits can be encoded by a single (discontinuous, piecewise holomorphic) transformation. This will be made precise in the proposition below. Let $F : (\mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{O}_\nu) \to \mathcal{D}$ be given by

$$F(z) = \begin{cases} \xi(z) & \text{if } z \in \mathcal{O}_\xi \\ \eta(z) & \text{if } z \in \mathcal{O}_\eta \\ \nu(z) & \text{if } z \in \mathcal{O}_\nu \setminus (\mathcal{O}_\xi \cup \mathcal{O}_\eta) \end{cases}$$

We call $F$ the *shadow* of the holomorphic commuting pair $\Gamma$.

**Proposition II.4.** *Given a holomorphic commuting pair* $\Gamma$*, consider its shadow* $F$ *and let* $\mathcal{U} = \mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{O}_\nu$ *and* $X = J \cup F^{-1}(J)$*, where* $J$ *is the large dynamical interval of* $\Gamma$*. Then*

*(a) The restriction of* $F$ *to* $\mathcal{U} \setminus X$ *is a regular 3-fold covering mapping onto* $\mathcal{D}^+ \cup \mathcal{D}^-$*;*

*(b)* $F$ *and* $\Gamma$ *share the same orbits as sets.*

*Proof.* Since $\mathcal{U} \setminus X$ consists of six connected components, each of which is mapped bijectively onto either $\mathcal{D}^+$ or $\mathcal{D}^-$, part (a) follows. In order to prove (b), it suffices to show that the $\Gamma$-orbit of any point of $\mathcal{U}$ is contained in the corresponding $F$-orbit. Thus, let $z \in \mathcal{U}$ and let $\omega$ be any finite admissible word in the alphabet $\{\xi, \eta, \nu\}$. If the letter $\nu$ does not occur in $\omega$ then $\omega(z) = F^{|\omega|}(z)$, where $|\omega|$ = length of $\omega$. Otherwise we write $\omega = \omega_L \nu \omega_R$ for some other words $\omega_L, \omega_R$ in the same alphabet (possibly empty); setting $x = \omega_R(z)$, we have three possibilities

(i) $x \in \mathcal{O}_\nu \setminus (\mathcal{O}_\xi \cup \mathcal{O}_\eta)$: in this case $\nu(x) = F(x)$ by definition so we may replace $\nu$ by $F$ in $\omega$.

(ii) $x \in \mathcal{O}_\xi \cap \mathcal{O}_\nu$: here we may write, using Proposition II.1, $\nu(x) = \eta\xi(x) = \eta F(x)$; since $\xi(\mathcal{O}_\xi \cap \mathcal{O}_\nu) \subseteq \mathcal{O}_\eta$ by that same proposition, we deduce that $F(x)$ is in $\mathcal{O}_\eta$, and so $\eta F(x) = F \circ F(x) = F^2(x)$. Hence in this case we may replace $\nu$ by $F^2$ in $\omega$.

(iii) $x \in \mathcal{O}_\eta \cap \mathcal{O}_\nu$: same as (ii).

This substitution process applied to all occurences of $\nu$ in $\omega$ shows that $\omega(z) = F^n(z)$ for some $n \geq |\omega|$, and so part (b) is proved also. □

## III. The pull-back theorem

The principal reason why holomorphic commuting pairs are useful is the fact that any quasi-symmetric conjugacy between the restrictions of two such objects to the reals can be promoted to a global quasi-conformal conjugacy between them. This is the contents of the *pull-back theorem* below.

Given a domain $\mathcal{O} \subseteq \mathbf{C}$ symmetric about the real axis, we say that a homeomorphism $\psi : \mathcal{O} \to \psi(\mathcal{O}) \subseteq \mathbf{C}$ is *symmetric* if it commutes with complex conjugation and satisfies $\psi(\mathcal{O}^+) = \psi(\mathcal{O})^+$ (where $A^+ = A \cap \{\operatorname{Im} z > 0\}$). A given holomorphic commuting pair $\Gamma$ is said to have *geometric boundaries* if its bow-tie $(\mathcal{D}, \mathcal{O}_\xi, \mathcal{O}_\eta, \mathcal{O}_\nu)$ is such that $\partial\mathcal{D}$ and $\partial\mathcal{U}$ are $K$-quasicircles for some $K \geq 1$, where $\mathcal{U} = \mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{O}_\nu$. The smallest such $K$ together with the number $\operatorname{mod}(\mathcal{D} \setminus D)$ are the *geometric parameters* of $\Gamma$.

**Theorem III.1.** *Let $\Gamma_0, \Gamma_1$ be holomorphic commuting pairs with the same irrational rotation number and the same height, assume they have geometric boundaries, and suppose $h : J_0 \to J_1$ is a $k$-quasisymmetric conjugacy between the restrictions of $\Gamma_0, \Gamma_1$ to their respective large dynamical intervals. Then there exists a quasiconformal conjugacy $H : \mathcal{D}_0 \to \mathcal{D}_1$ between $\Gamma_0$ and $\Gamma_1$ which extends $h$ and whose maximal dilatation depends only on $k$ and on the geometric parameters of both pairs.*

One essential difference from Sullivan's original pull-back theorem must be observed. The straightening theorem of Douady-Hubbard asserts that every quadratic-like mapping is qc-conjugate to a quadratic polynomial [DH]. In particular, quadratic-like mappings have no wandering domains, after another well-known theorem of Sullivan [$S_4$], and the pull-back argument runs smoothly for them. By contrast, we are only able to rule out wandering domains *a posteriori*, see Theorem IV.2. The technical tool we use to deal with them in the proof of Theorem III.1 is the following *qc-sewing lemma* due to L. Bers (cf. [B], [Ric]).

**Lemma III.2.** *Let $\phi : \mathcal{O} \to \phi(\mathcal{O}) \subseteq \widehat{\mathbf{C}}$ be a homeomorphism of an open set $\mathcal{O} \subseteq \widehat{\mathbf{C}}$ onto its image, let $\Lambda \subseteq \mathcal{O}$ be closed in $\widehat{\mathbf{C}}$ and assume that: (a) $\phi|\Lambda$ agrees with the restriction to $\Lambda$ of a $K_1$-quasiconformal homeo defined on some neighborhood of $\Lambda$; (b) $\phi|(\mathcal{O} \setminus \Lambda)$ is $K_2$-quasiconformal. Then $\phi$ is $K$-quasiconformal with $K \leq \max\{K_1, K_2\}$.* □

We need a few other geometric facts. Let $A \subseteq \mathbb{D}$ be a ring domain having $\partial\mathbb{D}$ as its outer boundary. Set $\delta = \inf\{d(z, \partial\mathbb{D}) : z \in \mathbb{D} \setminus A\}$ and suppose $A$ does not contain the origin. Then we have the following inequalities due to Teichmüller

$$\frac{1}{2\pi} \log\left(\frac{1}{1-\delta}\right) \leq \operatorname{mod} A \leq \frac{1}{2\pi} \log\, \Psi\left(\frac{\delta}{1-\delta}\right) , \tag{9}$$

where $\Psi$ is a universal monotone increasing function (cf. [$A_1$]). These inequalities may be combined with Köbe's distortion theorem and the Ahlfors-Beurling extension to yield proofs of the following three lemmas. Recall that a *$K$-quasidisk* is the image of a round disk in the extended complex plane under a global $K$-quasiconformal mapping.

**Lemma III.3.** *Let $Q_0, Q_1 \subseteq \mathbf{C}$ be $K$-quasidisks, symmetric with respect to the real axis and satisfying $\overline{Q}_0 \subseteq Q_1$. Then the Jordan regions $(Q_1 \setminus Q_0)^\pm$, $Q_0 \cup Q_1^\pm$ are all $K'$-quasidisks with $K'$ depending only on $K$ and* $\operatorname{mod}(Q_1 \setminus Q_0)$. □

**Lemma III.4.** *Let $I_0, I_1 \subseteq \mathbb{D} \cap \mathbb{R}$ be closed intervals and let $\phi : I_0 \to I_1$ be a $k$-quasisymmetric homeomorphism. Then $\phi$ has a $K$-quasiconformal extension to a self-mapping of $\mathbb{D}$ which is symmetric about the real axis and whose maximal dilatation $K$ depends only on $k$,* $\operatorname{mod}(\mathbb{D} \setminus I_0)$ *and* $\operatorname{mod}(\mathbb{D} \setminus I_1)$. □

**Lemma III.5.** *Let $A_0, A_1$ be disjoint closed arcs in $\partial\mathbb{D}$ and let $Q$ be the oriented conformal quadrilateral determined by $\mathbb{D}$, $A_0$ and $A_1$. If $h : \partial\mathbb{D} \to \partial\mathbb{D}$ is a homeomorphism such that $h|(\partial\mathbb{D} \setminus A_i)$ is $k$-quasisymmetric for $i = 0, 1$ then $h$ is $k'$-quasisymmetric with $k'$ depending only on $k$ and* $\operatorname{mod} Q$. □

One more result is needed before we move on to the proof of Theorem III.1. Let $\Gamma_0$ and $\Gamma_1$ be holomorphic commuting pairs and $F_0, F_1$ be their corresponding shadows (section II), and suppose $h : J_0 \to J_1$ is a conjugacy between the restrictions $F_i|J_i$.

**Lemma III.6.** *Let $\psi : \mathcal{D}_0 \to \mathcal{D}_1$ be any symmetric homeomorphic extension of $h$. Then there exists a symmetric homeomorphism $\widetilde{\psi} : \mathcal{U}_0 \to \mathcal{U}_1$ such that $F_1 \circ \widetilde{\psi} = \psi \circ F_0$ which is still an extension of $h$.*

*Proof.* Writing $X_i = J_i \cup F_i^{-1}(J_i)$, we know by Proposition II.4 that $F_i|(\mathcal{U}_i \setminus X_i)$ is a regular 3-fold covering map onto $\mathcal{D}_i^+ \cup \mathcal{D}_i^-$. Thus we can lift the restriction $\psi|\mathcal{D}_0^+ \cup \mathcal{D}_0^-$ through the $F_i$'s to get a homeomorphism $\widehat{\psi} : \mathcal{U}_0 \setminus X_0 \to \mathcal{U}_1 \setminus X_1$; such lift is uniquely determined if we require in addition that it be symmetric. Since $F_i(X_i) \subseteq J_i$, and using the fact that $F_i$ is schlicht when restricted to each of the six components of $\mathcal{U}_i \setminus X_i$, we deduce that $\widehat{\psi}$ extends to a symmetric homeomorphism $\widetilde{\psi} : \mathcal{U}_0 \to \mathcal{U}_1$, satisfying $F_1 \circ \widetilde{\psi} = \psi_0 \circ F_0$ everywhere by continuity. As $\psi|J_0 \equiv h$ and since $J_i$ is $F_i$-forward invariant, we conclude that $\widetilde{\psi}|J_0 \equiv h$ also. □

*Proof of Theorem III.1.* By Lemma III.4 and the Riemann mapping theorem, there exists a symmetric quasiconformal homeomorphism $G : \mathcal{D}_0 \to \mathcal{D}_1$ extending $h$ and whose maximal dilatation depends only on $k$ and the geometric parameters. Applying Lemma III.6 to $\psi = G$ yields a symmetric quasiconformal lift $\widetilde{G} : \mathcal{U}_0 \to \mathcal{U}_1$ with $K(\widetilde{G}) = K(G)$, still satisfying $\widetilde{G}|J_0 \equiv h$. By Lemma III.3, the Jordan domains $(\mathcal{D}_i \setminus \mathcal{U}_i)^+$ and $\mathcal{U}_i \cup \mathcal{D}_i^-$ are $K'$-quasidisks with $K'$ depending only on the geometric parameters. Let $\tau_0$, $\tau_1$ be $K'$-quasiconformal mappings of the plane such that $\tau_i(\mathbb{D}) = (\mathcal{D}_\mathrm{i} \setminus \mathcal{U})^+$. Let $\beta : \partial\mathbb{D} \to \partial\mathbb{D}$ be given by

$$\beta(z) = \begin{cases} \tau_1^{-1} \circ \widetilde{G} \circ \tau_0(z) & \text{if } z \in \tau_0^{-1}(\partial(\mathcal{U}_0 \cup \mathcal{D}_0)) \\ \\ \tau_1^{-1} \circ G \circ \tau_0(z) & \text{if } z \in \tau_0^{-1}((\partial D)^+) \end{cases} .$$

By Lemma III.5, $\beta$ is quasisymmetric (with $k(\beta)$ depending only on the geometric parameters). Let $B : \mathbb{D} \to \mathbb{D}$ be the Ahlfors-Beurling extension of $\beta$ and let $\widehat{G} = \tau_1 \circ B \circ \tau_0^{-1} : (\mathcal{D}_0 \setminus \mathcal{U}_0)^+ \to (\mathcal{D}_1 \setminus \mathcal{U}_1)^+$. We have $\widehat{G} \equiv \widetilde{G}$ over $(\partial\mathcal{U}_0)^+$ and $\widehat{G} \equiv G$ over the remaining part of the boundary of $(\mathcal{D}_0 \setminus \mathcal{U}_0)^+$. Then, let $H_1 : \mathcal{D}_0 \to \mathcal{D}_1$ be given by

$$H_1(z) = \begin{cases} \widetilde{G}(z) & \text{if } z \in \mathcal{U}_0 \\ \\ \widehat{G}(z) & \text{if } z \in \overline{(\mathcal{D}_0 \setminus \mathcal{U}_0)^+} \\ \\ \sigma(\widehat{G}(\sigma z)) & \text{if } z \in (\mathcal{D}_0 \setminus \mathcal{U}_0)^- \end{cases} ,$$

where $\sigma$ denotes complex conjugation. This map is a quasiconformal homeomorphism with $K(H_1) = \max\{K(\widehat{G}), K(\widetilde{G})\}$, and $H_1|J_0 \equiv h$.

Now we may start the *pull-back routine*. From $H_1$, we define inductively a sequence of homeomorphisms $H_n : \mathcal{D}_0 \to \mathcal{D}_1$ by

$$H_n(z) = \begin{cases} H_1(z) & \text{if } z \in (\mathcal{D}_0 \setminus \mathcal{U}_0) \\ \\ \widetilde{H}_{n-1}(z) & \text{if } z \in \mathcal{U}_0 \end{cases}$$

where $\widetilde{H}_{n-1} : \mathcal{U}_0 \to \mathcal{U}_1$ is the lift we obtain applying Lemma III.6 to $\psi = H_{n-1}$. The map $H_1$ has been constructed so that $\widetilde{H}_1(z) = H_1(z)$ for each $z \in \partial\mathcal{U}_0$; it follows inductively from Lemma III.6 that each $H_n$ is a symmetric quasiconformal homeomorphism with $K(H_n) = K(H_1)$, and that $H_n|J_0 \equiv h$ for all $n$. By the compactness principle for qc-mappings (cf. [$A_1$]), this sequence has a limit $H_\infty : \mathcal{D}_0 \to \mathcal{D}_1$. We have $K(H_\infty) \leq K(H_1)$ and $H_\infty|J_0 \equiv h$ as well.

Notice that $\{H_n\}$ has the following *stabilization property* : if $z \in \mathcal{U}_0$ then $H_n \circ F_0(z) = F_1 \circ H_n(z)$ if and only if $H_{n+1}(z) = H_n(z)$. Let $E$ be the set of all $z \in \mathcal{U}_0$ which iterated finitely many times by $F_0$ either land outside $\mathcal{U}_0$, where $H_n \equiv H_1$ for all $n$, or land on $J_0$, which is forward invariant and where $H_n \equiv h$ for all $n$. Then for every $z \in E$ the sequence $\{H_n(z)\}$ is eventually constant, hence eventually equal to $H_\infty(z)$. The stabilization property gives us $H_\infty \circ F_0(z) = F_1 \circ H_\infty(z)$ for all $z \in E$. Since $X_0 \subseteq E$, we have $(\mathcal{U}_0 \cap \overline{E}) \setminus E \subseteq \mathcal{U}_0 \setminus X_0$, where $F_0$ is continuous, and so for all $z \in \mathcal{U}_0 \cap \overline{E}$ we have $H_\infty \circ F_0(z) = F_1 \circ H_\infty(z)$ also.

If $\Omega \subseteq \mathcal{U}_0 \setminus \overline{E}$ is a connected component then the restriction $F_0|\Omega$ is schlicht. Since $E$ is backward invariant, it follows by induction that $F_0^n(\Omega) \subseteq \mathcal{U}_0 \setminus \overline{E}$ is a connected component also, for all $n > 0$. Observe that $\partial\Omega \cap X_0$ consists of at most one point. For if $a, b \in \partial\Omega \cap X_0$ are two distinct points, then mapping $\Omega$ forward if necessary we may assume that $a, b \in J_0$ and choose $n > 0$ so that the points $F_0^n(a)$, $F_0^n(b)$ lie in opposite sides of zero inside $J_0$. Since $\Omega$ is connected, this forces $F_0^n(\Omega) \cap X_0 \neq \emptyset$, a contradiction.

Next, suppose $F_0^n(\Omega) = \Omega$ for some $n > 0$; there is no loss of generality in assuming that $\Omega \subseteq \mathcal{D}_0^+$. Then there exists an inverse branch $\Phi : \mathcal{D}_0^+ \to \mathcal{D}_0^+$ of $F_0^n$ such that $\Phi(\Omega) = \Omega$. Since $\mathcal{D}_0^+$ is a Jordan domain, we known by the Denjoy-Wolff theorem (cf. [Mil], [$S_3$]) that either there exists $z \in \mathcal{D}_0^+$ such that $\Phi(z) = z$, necessarily attracting because $\Phi(\mathcal{D}_0^+) \subseteq \mathcal{U}_0^+ \neq \mathcal{D}_0^+$, or there exists $z \in \partial\mathcal{D}_0^+$ such that $\Phi(z) = z$ ($\Phi$ extends continuously to $\partial\mathcal{D}_0^+$). The first possibility is incompatible with $\Phi(\Omega) = \Omega$, while the second implies that $z$ is in the large dynamical interval of $\Gamma_0$, which is impossible because $F_0$ has no periodic points there. We deduce that each connected component of $\mathcal{U}_0 \setminus \overline{E}$ is a *wandering domain*, i.e. its forward images are pairwise disjoint. Therefore $H_\infty$ conjugates $F_0$ and $F_1$ everywhere except along the grand-orbits of wandering domains.

Now we perform a sequence of quasiconformal sewings in order to change $H_\infty$ into a global conjugacy between both pairs. Partitioning the connected components of $\mathcal{U}_0 \setminus \overline{E}$ into grand-orbit equivalence classes and selecting one representative from each class yields countably many domains $\{\Omega_n\}_{n \geq 1}$. First we change $H_\infty$ along the forward orbit of $\Omega_1$. Let $\varphi_0 : \overline{\Omega}_1 \to \overline{F_0(\Omega_1)}$ denote the homeomorphic extension of $F_0$ to the closure of $\Omega_1$. Then $\varphi_0(z) = F_0(z)$ for all $z \in \overline{\Omega}_1$ with at most one exception $z_0 \in \partial\Omega_1$. Similarly, define $\varphi_1 : \overline{H_\infty(\Omega_1)} \to \overline{F_1 H_\infty(\Omega_1)}$ as the homeomorphic extension of $F_1$ to the closure of $H_\infty(\Omega_1)$. Once again $\varphi_1 \equiv F_1$ with at most one exception $z_1 \in \partial H_\infty(\Omega_1)$. Let $\phi : \overline{F_0(\Omega_1)} \to \overline{F_1 H_\infty(\Omega_1)}$ be given by $\phi = \varphi_1 \circ H_\infty \circ \varphi_0^{-1}$. Then $\phi$ is a $K(H_\infty)$-quasiconformal homeomorphism and *a priori* agrees with $H_\infty$ over $\partial F_0(\Omega_1)$ except possibly at one point, so by continuity of both maps $\phi \equiv H_\infty$ everywhere along $\partial F_0(\Omega_1)$. Hence, if we set $\psi^{(1)} \equiv H_\infty$ over $\mathcal{D}_0 \setminus F_0(\Omega_1)$ and $\psi^{(1)} \equiv \phi$ over $F_0(\Omega_1)$ we get by the qc-sewing lemma a $K(H_\infty)$-quasiconformal homeo $\psi^{(1)} : \mathcal{D}_0 \to \mathcal{D}_1$ which satisfies the conjugacy equation $\psi^{(1)} \circ F_0(z) = F_1 \circ \psi^{(1)}(z)$ for all $z \in (\mathcal{U}_0 \cap \overline{E}) \cup \Omega_1$. Repeating this argument with $\psi^{(1)}$ replacing $H_\infty$ and $F_0(\Omega_1)$ replacing $\Omega_1$ and so on inductively, we get a sequence $\psi^{(n)} : \mathcal{D}_0 \to \mathcal{D}_1$ of uniformly ($\leq K(H_\infty)$) quasiconformal mappings. Again by the compactness principle we extract a limit $\psi_1$ of $\{\psi^{(n)}\}$. Feeding $\psi_1$ into our pull-back routine in place of $H_1$ and once again going to a limit yields a quasiconformal homeo $H_{1,\infty} : \mathcal{D}_0 \to \mathcal{D}_1$ with $K(H_{1,\infty}) \leq K(H_\infty)$ that, by the stabilization property, conjugates $F_0$ and $F_1$ not only on $\mathcal{U}_0 \cap \overline{E}$ but also along the full grand-orbit of $\Omega_1$. Proceeding inductively, we take care of the full grand-orbits of $\Omega_2, \Omega_3, \ldots$ through partial quasiconformal conjugacies $H_{2,\infty}, H_{3,\infty}, \ldots$ satisfying $K(H_{n,\infty}) \leq K(H_\infty)$ as well as $H_{n,\infty}|J_0 \equiv h$ for all $n$. Going to a limit one final time we get $H : \mathcal{D}_0 \to \mathcal{D}_1$, a global quasiconformal conjugacy with $K(H) \leq K(H_\infty)$ which is still an extension of $h$. □

*Remark.* Let $\Gamma$ be a holomorphic commuting pair with irrational rotation number. By analogy with the case of quadratic-like mappings, we define the *filled-in limit set* of $\Gamma$ to be

$$\mathcal{K}_\Gamma = \overline{\bigcap_{n \geq 0} F^{-n}(\mathcal{D})} .$$

We also let the *limit* or *Julia set* $J_\Gamma$ be the set $\mathcal{K}_\Gamma$ minus the union of all grand-orbits of wandering domains. As the proof of the pull-back theorem shows, $\mathcal{K}_\Gamma$ has no other stable interior components. In the next section we will show in fact that $\mathcal{K}_\Gamma = J_\Gamma$. It is not difficult to see that $\mathcal{K}_\Gamma \cap \partial\mathcal{U}$ is the disjoint union of six arcs.

## IV. Existence and limit set qc-rigidity of holomorphic commuting pairs

For each $0 \leq \theta < 1$, let $E_\theta : \mathbf{C} \to \mathbf{C}$ be the entire mapping given by $E_\theta(z) = z + \theta - \frac{1}{2\pi}\sin(2\pi z)$. Since $E_\theta \circ T = T \circ E_\theta$, where $T$ is the translation $z \mapsto z+1$, $E_\theta$ is the lift to the complex plane of a holomorphic self-mapping of the cylinder, $f_\theta : \mathbf{C}/\mathbb{Z} \cong \mathbf{C}^* \hookleftarrow$. Moreover, the restriction $E_\theta|\mathbb{R}$ maps the real axis onto itself and satisfies $E_\theta'(x) \geq 0$ for all $x \in \mathbb{R}$, and equality holds iff $x \in \mathbb{Z}$ (these constitute all the critical points of $E_\theta$). Therefore the restriction $f_\theta|S^1$ is a critical circle homeomorphism with rotation number, say, $\rho(\theta)$. It is well-known that $\theta \mapsto \rho(\theta)$ is a continuous, non-decreasing map of $[0,1)$ onto itself such that the interval $\rho^{-1}(t) \subseteq [0,1)$ degenerates to a point whenever $t$ is irrational (see [$\mathrm{H}_1$]).

With the family $\{E_\theta\}$ at hand we shall construct in this section examples of holomorphic commuting pairs with arbitrary rotation number and arbitrary height. More precisely, we shall prove the following theorem.

**Theorem IV.1.** *For each $n \geq 0$ and each $\theta$ such that $\rho(\theta)$ has a continued fraction expansion of length at least $n+1$, the real commuting pair determined by $(f_\theta^{q_n}, f_\theta^{q_{n+1}})$*

*extends to a holomorphic commuting pair $\Gamma_{n,\theta}$ with geometric boundaries. The family $\{\Gamma_{n,\theta}\}$ runs through all possible pairs of combinatorial invariants at least once, and for each $(m,\rho) \in \mathbb{N} \times [0,1)$ with $m \geq 2$ there exist countably many $(n,\theta) \in \mathbb{N} \times [0,1)$ such that $\Gamma_{n,\theta}$ has height $m$ and rotation number $\rho$.*

When combined with the results of section III, this construction yields two crucial properties of holomorphic commuting pairs, which we express as follows.

**Theorem IV.2.** *Let $\Gamma$ be a holomorphic commuting pair with geometric boundaries and irrational rotation number. Then $\Gamma$ has no wandering domains and admits no non-trivial, symmetric, invariant Beltrami differentials entirely supported in its limit set.*

This theorem allows holomorphic commuting pairs to be parametrized by conformal structures supported on the outer annulus of a fixed model, cf. next section.

The main analytic tool to be used in the proof of Theorem IV.1 is the following growth estimate.

**Lemma IV.3.** *There exist a positive constant $C_0$ and a positive monotone non-decreasing function $\varphi(s)$ defined for $s \geq 0$ such that if $|y| \geq \varphi(|x|)$ then $|E_\theta(x+iy)| \geq C_0 \exp(\pi|y|)$.*

*Proof.* When $\theta = 0$, a straightforward computation yields

$$\begin{aligned} \left|E_0(x+iy)\right|^2 \;=\; & \frac{1}{4\pi^2}\cosh^2(2\pi y) + \left[x^2 + y^2 - \frac{1}{4\pi^2}\cos^2(2\pi x)\right] \\ & -\frac{1}{\pi}\left[x\sin(2\pi x)\cosh(2\pi y) + y\cos(2\pi x)\sinh(2\pi y)\right]. \end{aligned}$$

The first expression between brackets is positive as soon as, say, $|y| \geq 1$, while the second is dominated by $(|x| + |y|)\cosh(2\pi y)$. Thus, if $|y| \geq 1$ we have

$$\left|E_0(x+iy)\right|^2 \geq \frac{1}{4\pi^2}\left[\cosh(2\pi y) - 4\pi(|x| + |y|)\right]\cosh(2\pi y)\ . \tag{10}$$

Now, let

$$\varepsilon(t) = \frac{1}{4\pi}\cosh(2\pi t) - t - 1\ .$$

This strictly convex function has a minimum at a certain $t_0 > 0$ such that $\varepsilon(t_0) < 0$. Hence for each $s \geq 0$ there exists a unique $\overline{\varphi}(s) > t_0$ such that $\varepsilon(\overline{\varphi}(s)) = s$. Since $\varepsilon(t)$ is strictly increasing for $t \geq t_0$, so is $\overline{\varphi}(s)$ for $s \geq 0$, and $t \geq \overline{\varphi}(s)$ implies $\varepsilon(t) \geq s$. Setting $\varphi(s) = \max\{1, \overline{\varphi}(s)\}$ and observing that the expression between brackets in (10) is equal to $4\pi[\varepsilon(|y|) + 1 - |x|]$, we deduce that if $|y| \geq \varphi(|x|)$ then

$$\left|E_0(x+iy)\right|^2 \geq \frac{1}{\pi}\cosh(2\pi|y|) \geq \frac{1}{2\pi}\exp(2\pi|y|). \tag{11}$$

On the other hand, when $0 < \theta < 1$ we have $E_\theta(z) = E_0(z) + \theta$, so that $|E_\theta(z)| \geq |1 - |E_0(z)|^{-1}|.|E_0(z)|$. Therefore, if $|y| \geq \varphi(|x|)$, we have by (11)

$$\left|E_\theta(x+iy)\right| \geq \frac{1}{\sqrt{2\pi}}[1 - e^{-\pi}\sqrt{2\pi}]\exp(\pi|y|)\ . \qquad \square$$

We divide the work required to prove Theorem IV.1 into several steps. Let us fix $\theta$ for the time being and write $\rho(\theta) = [r_0, r_1, \ldots, r_n, \ldots]$. We conform with the notation established in the first section, so that, in its irreducible form, $\frac{p_n}{q_n} = [r_0, r_1, \ldots, r_{n-1}]$ satisfies $p_0 = 0$, $q_0 = 1$; $p_1 = 1$, $q_1 = r_0$ and for $n \geq 1$, $p_{n+1} = r_n p_n + p_{n-1}$, $q_{n+1} = r_n q_n + q_{n-1}$.

We need a brief geometric description of the map $E_\theta$. The pre-image of the real axis under $E_\theta$ consists of $\mathbb{R}$ itself together with the family of analytic curves

$$\mathcal{S}_{\pm}^{(k)} : \; x = k \pm \frac{1}{2\pi} \arccos\left[\frac{-2\pi|y|}{\sinh(2\pi y)}\right] ,$$

where $k \in \mathbb{Z}$, arising as solutions to $\operatorname{Im} E_\theta(x+iy) = 0$. For each $k \in \mathbb{Z}$, the curves $\mathcal{S}_+^{(k)}$ and $\mathcal{S}_-^{(k)}$ meet at the critical point $c_k = k$, and are both asymptotic to the vertical lines $x = k \pm \frac{1}{4}$. Notice that each $c_k$ is a critical point of cubic type. In the upper half-plane $\mathbf{C}^+$, let $V_k$ be the simply-connected region bounded by the arcs $\mathcal{S}_+^{(k-1)} \cap \mathbf{C}^+$ and $\mathcal{S}_-^{(k)} \cap \mathbf{C}^+$ and the interval $[k-1, k] \subseteq \mathbb{R}$. Then $E_\theta | V_k$ is schlicht onto $\mathbf{C}^+$; we let $\phi_k : \mathbf{C}^+ \to V_k$ denote the corresponding inverse. Similarly, let $W_k \subseteq \mathbf{C}^+$ be the simply-connected region bounded by $\mathcal{S}_-^{(k)} \cap \overline{\mathbf{C}}^+$ and $\mathcal{S}_+^{(k)} \cap \mathbf{C}^+$, observe that $E_\theta | W_k$ is schlicht onto $\mathbf{C}^-$ and let $\psi_k : \mathbf{C}^- \to W_k$ be the corresponding inverse.

Now let $A_n \subseteq \mathbf{C}^+$ be the unique connected component of $(E_\theta^{q_n})^{-1}(\mathbf{C}^+)$ whose closure contains the point $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(0) \in \mathbb{R}$. Similarly, let $B_n \subseteq \mathbf{C}^+$ be the unique connected component of $(E_\theta^{q_{n+1}})^{-1}(\mathbf{C}^+)$ such that $T^{-p_n} \circ E_\theta^{q_n}(0) \in \overline{B}_n$. We have either $A_n \subseteq V_0$ and $B_n \subseteq V_1$ or $A_n \subseteq V_1$ and $B_n \subseteq V_0$, depending on whether $n$ is even or odd, respectively (Figure 2 illustrates the former case).

**Lemma IV.4.** *For each $n \geq 0$ there exists a unique $q_n$-tuple $(k_1, k_2, \ldots, k_{q_n})$ with $0 = k_1 \leq k_2 \leq \cdots \leq k_{q_n} \leq p_n + 1$ such that $A_n = \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{q_n}}(\mathbf{C}^+)$. A similar statement holds for $B_n$.*

*Proof.* This is an easy consequence of the fact that $0 \leq E_\theta^j(0) < p_n + 1$ for $j = 0, 1, \ldots, q_n$, for all $n \geq 0$, which in turn follows from the very definitions of $p_n, q_n$. □

**Lemma IV.5.** *Let $f$ be a circle homeomorphism with $\rho(f) = [r_0, r_1, \ldots, r_n, \ldots]$, let $c \in S^1$, and for each $n \geq 1$ let $J_n \subseteq S^1$ be the closed interval of endpoints $c$ and $f^{q_{n-1}-q_n}(c)$ containing $f^{q_{n-1}}(c)$. If $j < q_n$ is such that $f^{-j}(c)$ belongs to $J_n$, then $j \leq 0$.* □

Let us use the notation $\langle \alpha, \beta \rangle$ to represent a closed interval on the line with endpoints $\alpha$ and $\beta$, irrespective of order.

**Lemma IV.6.** *For each $n \geq 0$ we have $\overline{A}_n \cap \mathbb{R} = \langle \alpha_n, 0 \rangle$ and $\overline{B}_n \cap \mathbb{R} = \langle 0, \beta_n \rangle$, where $\alpha_0 = -1$, $\beta_0 = \alpha_1$ and for $n \geq 1$ the points $\alpha_n, \beta_n \in \mathbb{R}$ are uniquely determined by the requirements: $T^{-p_n} \circ E_\theta^{q_n}(\alpha_n) = T^{-p_{n-1}} \circ E_\theta^{q_{n-1}}(0)$ and $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(\beta_n) = T^{-p_n} \circ E_\theta^{q_n}(0)$.*

*Proof.* Consider $f = f_\theta$ and take $c$ to be the critical point of $f_\theta$. Then Lemma IV.5 says that there can be no critical points for $f_\theta^{q_n}$ in the interior of $J_n$, for by the chain rule these are precisely the pre-images $f_\theta^{-j}(c)$ with $0 \leq j < q_n$. The result follows. □

Given $R > 0$, let $\mathcal{D}_R = \{z : |z| < R\}$ and let $A_{n,R}$ be the unique connected component of $(T^{-p_n} \circ E_\theta^{q_n})^{-1}(\mathcal{D}_R^+)$ contained in $A_n$. Let $B_{n,R}$ be similarly defined. If $R$ is sufficiently large ($R > p_n + 1$ is good enough) we see that $\overline{A}_{n,R} \cap \mathbb{R} = \overline{A}_n \cap \mathbb{R}$ and $\overline{B}_{n,R} \cap \mathbb{R} = \overline{B}_n \cap \mathbb{R}$ for $n \geq 0$. It is clear that both $A_{n,R}$ and $B_{n,R}$ are Jordan domains, in fact quasidisks, and that they are mapped respectively by $T^{-p_n} \circ E_\theta^{q_n}$ and $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}$ bijectively onto $\mathcal{D}_R^+$.

**Lemma IV.7.** *For every sufficiently large* $R$ *we have* $\overline{A}_{n,R} \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+}$ *and* $\overline{B}_{n,R} \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+}$.

*Proof.* For $s, R$ positive numbers, let

$$\delta(s,R) = \varphi(s) + \frac{1}{\pi} \log^+(C_0^{-1} R) \ ,$$

where $\varphi$ and $C_0$ are given by Lemma IV.3. Then $|y| \geq \delta(|x|, R)$ implies $|E_\theta(x+iy)| \geq R$, which in turn means that $E_\theta(x+iy) \in \mathbf{C} \setminus \mathcal{D}_R$. Therefore, for each $k \in \mathbb{Z}$ we have

$$\phi_k(\overline{\mathcal{D}_R^+}) \subseteq \overline{V}_k \cap \Big\{ x + iy : y \leq \delta(|x|, R) \Big\} \ .$$

Let $V_{k,R}$ denote this last intersection. Since $\delta(s,R)$ has logarithmic growth in $R$, every sufficiently large $R$ satisfies the inequality $R > p_n + 1 + \delta(p_n + 1, 2R)$; for a given $R$ as such, if $0 \leq k \leq p_n + 1$ and $z$ is any point in $V_{k,2R}$ with $z = x + iy$, then

$$|z| \leq |x| + \delta(|x|, 2R) \leq p_n + 1 + \delta(p_n + 1, 2R) < R \ ,$$

and so it follows that $z \in \mathcal{D}_R \cap \overline{\mathbf{C}^+}$. Thus, if $0 \leq k \leq p_n + 1$ then $\phi_k(\overline{\mathcal{D}_{2R}^+}) \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+} \subseteq \overline{\mathcal{D}_{2R}^+}$. Since $T^{p_n}(\overline{\mathcal{D}_R^+}) \subseteq \overline{\mathcal{D}_{2R}^+}$, if we take $(k_1, k_2, \ldots, k_{q_n})$ as in Lemma IV.4 we deduce that

$$\overline{A}_{n,R} = \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{q_n}}(T^{p_n} \overline{\mathcal{D}_R^+}) \subseteq \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{q_n}}(\overline{\mathcal{D}_{2R}^+}) \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+} \ .$$

This proves the first inclusion; the second is proved in similar fashion. □

*Remark.* Observe that if we define $\mathcal{U}_{n,R} = \phi_{k_2} \circ \phi_{k_3} \circ \cdots \circ \phi_{k_{q_n}}(\mathcal{D}_R^+)$ and set $A'_{n,R} = \phi_1(\mathcal{U}_{n,R})$ and $A''_{n,R} = \psi_0 \sigma(\mathcal{U}_{n,R})$, where $\sigma : \mathbf{C} \to \mathbf{C}$ is complex conjugation, then the above argument applies *mutatis mutandis* to yield $\overline{A'_{n,R}} \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+}$, $\overline{A''_{n,R}} \subseteq \mathcal{D}_R \cap \overline{\mathbf{C}^+}$ as well, for every sufficiently large $R$ and all $n \geq 0$.

*Proof of Theorem IV.1.* Given $n \geq 0$, let $R_n > 0$ be large enough for the conclusion of Lemma IV.7 to hold. Let $\xi_n = T^{-p_n} \circ E_\theta^{q_n}$ and $\eta_n = T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}$ and let $\mathcal{O}_{\xi_n}, \mathcal{O}_{\eta_n} \subseteq \mathbf{C}$ be the symmetric Jordan domains (quasidisks) such that $\mathcal{O}_{\xi_n}^+ = A_{n,R_n}, \mathcal{O}_{\eta_n}^+ = B_{n,R_n}$. Then $\xi_n$ and $\eta_n$ commute, and $\overline{\mathcal{O}}_{\xi_n}, \overline{\mathcal{O}}_{\eta_n} \subseteq \mathcal{D}_{R_n}$, by Lemma IV.7. The restrictions $\xi_n | \mathcal{O}_{\xi_n}$ and $\eta_n | \mathcal{O}_{\eta_n}$ are schlicht and onto their images, which by Lemma IV.5 are $\mathcal{D}_{R_n} \cap \mathbf{C}(\langle \xi_n(\alpha_n), \xi_n(0) \rangle)$ and $\mathcal{D}_{R_n} \cap \mathbf{C}(\langle \eta_n(0), \eta_n(\beta_n) \rangle)$, respectively. Also, let $\mathcal{O}_{\nu_n} \subseteq \mathbf{C}$ be the connected component of $\xi_n^{-1}(\mathcal{O}_{\eta_n})$ containing the origin and let $\nu_n = \xi_n \circ \eta_n$. Then the restriction $\nu_n | \mathcal{O}_{\nu_n}$ is a holomorphic 3-fold branched covering map onto its image,

$\nu_n(\mathcal{O}_{\nu_n}) = \mathcal{D}_{R_n} \cap \mathbf{C}(\langle \eta_n(0), \xi_n(0) \rangle)$. Moreover, by the remark following Lemma IV.7, we have

$$\overline{\mathcal{O}^+_{\nu_n}} \subseteq \overline{A}_{n,R} \cup \overline{A'_{n,R}} \cup \overline{A''_{n,R}} \subseteq \mathcal{D}_{R_n} \cap \overline{\mathbf{C}^+} ,$$

and so $\overline{\mathcal{O}}_{\nu_n} \subseteq \mathcal{D}_{R_n}$. It follows at once that $(\mathcal{D}_{R_n}, \mathcal{O}_{\xi_n}, \mathcal{O}_{\eta_n}, \mathcal{O}_{\nu_n})$ is a *bow-tie*.

Now we claim that this bow-tie together with the maps $\xi_n, \eta_n, \nu_n$ determine a holomorphic commuting pair $\Gamma_{n,\theta}$ with geometric boundaries, up to orientation, with rotation number $\rho(\Gamma_{n,\theta}) = [r_{n+1}+1, r_{n+2}, \ldots]$ and height given by $m(\Gamma_{0,\theta}) = r_0$ when $n = 0$, and by $m(\Gamma_{n,\theta}) = r_n + 1$ when $n > 0$. We have indirectly checked all conditions in Definition 4, except perhaps condition $H_6$. We check it for $n > 0$; the case $n = 0$ is just as easy. Using the commutativity of $T$ with $E_\theta$, Lemma IV.4 and the recurrence relations defining $p_{n+1}$ and $q_{n+1}$, we get

$$\xi_n^{r_n+1}(\alpha_n) \;=\; (T^{-p_n} \circ E_\theta^{q_n})^{r_n}(T^{-p_n} \circ E_\theta^{q_n}(\alpha_n)) \;=\; T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(0) = \eta_n(0).$$

Similarly, we have $\eta_n(\beta_n) = \xi_n(0)$. Thus condition $H_6$ is satisfied too, and $m = r_n + 1$ is the height of $\Gamma_{n,\theta}$. The statement on rotation numbers is clear. □

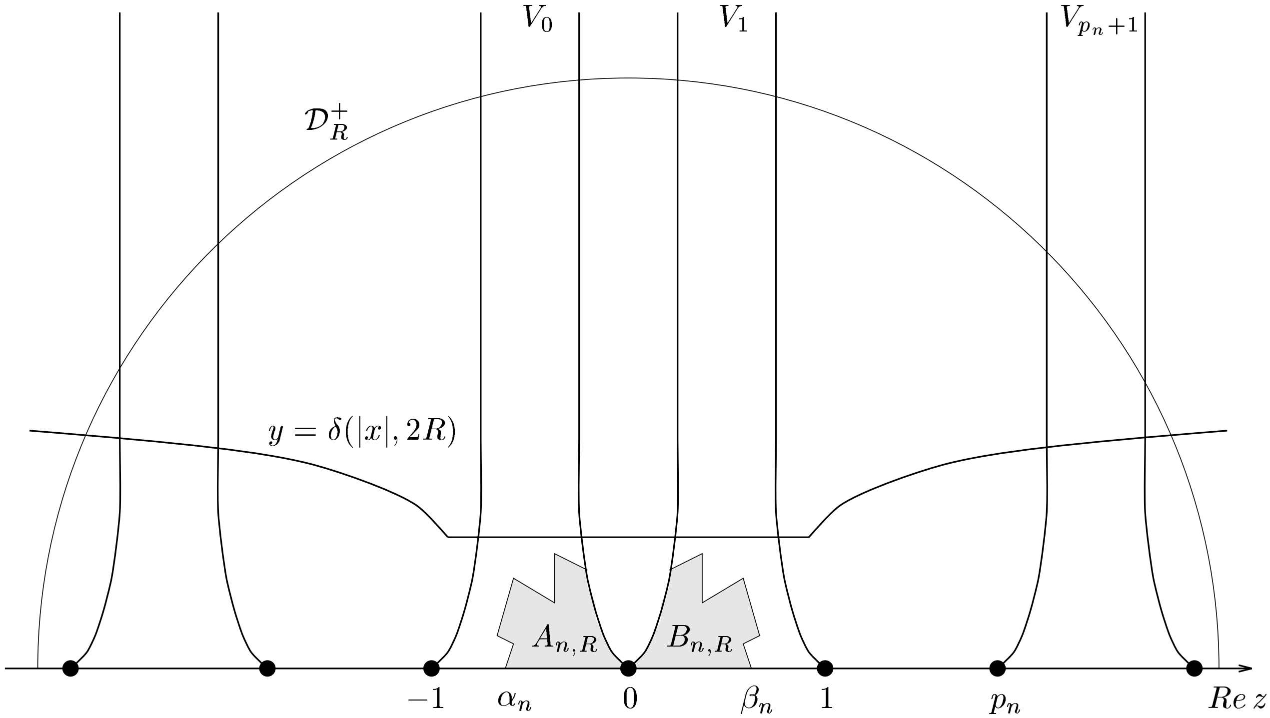


*Figure 2*

*Remark.* Because of Proposition II.3, once $R_0$ is chosen so that the above construction works for $n = 0$, we may take $R_n = R_0$ thereafter. If this is done then, for each $n \geq 0$, $\Gamma_{n+1,\theta}$ becomes the first renormalization of $\Gamma_{n,\theta}$ up to linear rescaling.

Now we turn our attention to Theorem IV.2. We shall extract the stated rigidity properties of holomorphic commuting pairs from corresponding ones found naturally in the family $\{f_\theta\}$ introduced above.

If $f : \mathbf{C}^* \to \mathbf{C}^*$ is holomorphic, we denote by $S_f$ the set of singular values of $f$, i.e. points in $\mathbf{C}^*$ all neighborhoods $U$ of which are such that $f^{-1}(U) \xrightarrow{f} U$ fails to be a covering map. We also write $X_f = \mathbf{C}^* \setminus S_f$, so that $f^{-1}(X_f) \xrightarrow{f} X_f$ is always a covering map. For example, since $1 \in \partial\mathbb{D}$ is the unique critical point of $f_\theta$, it is easy to see that $S_{f_\theta} = \{f_\theta(1)\}$; in this case $f_\theta^{-1}(X_{f_\theta})$ has an infinite discrete complement in $\mathbf{C}^*$. We let $J_f$ be the Julia set of $f$.

**Lemma IV.8.** *The family $\{f_\theta\}$ is topologically complete, i.e. every symmetric, normalized holomorphic self-map of $\mathbf{C}^*$ which is topologically conjugate to a member of the family is a member also.*

*Proof.* Let $f : \mathbf{C}^* \to \mathbf{C}^*$ be holomorphic and suppose $h : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is an orientation preserving homeo fixing 0 and $\infty$ and satisfying $h \circ f_\theta = f \circ h$. Let $A \in \mathrm{Aut}\,(\widehat{\mathbf{C}})$ be given by $A(z) = \lambda z$, where $\lambda = h \circ f_\theta(1)/f_\theta(1)$. This $A$ is homotopic to $h$ relative to $S_{f_\theta} \cup \{0, \infty\}$, so the covering homotopy theorem yields a holomorphic lift $\widehat{A} : f_\theta^{-1}(X_{f_\theta}) \to f^{-1}(X_f)$, which is then homotopic to $h$ relative to $f_\theta^{-1}(S_{f_\theta}) \cup \{0, \infty\}$. Some easy topology and the removable singularity theorem show that $\widehat{A}$ is Möbius and fixes 0 and $\infty$. In particular, if $f$ is symmetric about $\partial\mathbb{D}$ and is normalized so that its critical point lies at $1 \in \partial\mathbb{D}$, then $\widehat{A}$ is the identity and $|\lambda| = 1$, say $\lambda = e^{2\pi i\alpha}$. Therefore $f = A \circ f_\theta \circ \widehat{A}^{-1} = f_{\theta+\alpha}$. □

**Theorem IV.9.** *The mapping $f_\theta$ has no wandering domains. Moreover, if $\rho(\theta)$ is irrational then $f_\theta$ admits no non-trivial, symmetric, invariant Beltrami differentials entirely supported in its Julia set.*

*Proof.* Since $S_{f_\theta}$ is a finite set, the first assertion follows from a theorem due to L. Keen [K]. Now suppose $\mu$ is an $f_\theta$-invariant Beltrami differential in $\widehat{\mathbf{C}}$ with support in $J_{f_\theta}$; assume also that $\mu$ is symmetric about $\partial\mathbb{D}$. For all sufficiently small real $t$, let $h_t : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be the unique solution to $\overline{\partial} h_t = (t\mu).\partial h_t$ fixing $\{0, 1, \infty\}$ pointwise, and let $f_t = h_t \circ f_\theta \circ h_t^{-1}$. Since $t\mu$ is symmetric and $f_\theta$-invariant, each $f_t$ is symmetric and holomorphic, and has a single critical point at $1 \in \partial\mathbb{D}$. Using Lemma IV.8, we have $f_t = f_{\theta_t}$ for some $\theta_t$. But then $\rho(\theta_t) = \rho(\theta)$ is irrational, so $\theta_t = \theta$ for all $t$, by remark in the first paragraph of this section. Therefore, $h_t$ commutes with $f_\theta$ for all $t$; in particular $h_t$ must permute the elements of $Y_n = f_\theta^{-n}(1)$, which is discrete in $\mathbf{C}^*$, for each $n \geq 0$. Since $h_0 = id_{\widehat{\mathbf{C}}}$ and for each $z \in \widehat{\mathbf{C}}$ the path $t \to h_t(z)$ is continuous by the Ahlfors-Bers theorem [$A_1$], we deduce that $h_t$ fixes $Y_n$ pointwise for all $n \geq 0$, for all $t$. But by Montel's theorem, $J_{f_\theta} \subseteq \overline{\bigcup_{n \geq 0} Y_n}$, so $h_t$ agrees with the identity over $J_{f_\theta}$ for all $t$. Since $h_t$ is conformal off $J_{f_\theta}$, it follows that $h_t \equiv id_{\widehat{\mathbf{C}}}$ for all $t$, and so $\mu \equiv 0$ a.e. □

*Proof of Theorem IV.2.* Combining Theorem III.1 with Theorem IV.1, we know that $\Gamma$ is conjugate to $\Gamma_{0,\theta}$ for some $\theta$ by a qc-homeomorphism $H$. Let $\mu$ be a $\Gamma$-invariant Beltrami differential with support in $J_\Gamma$. Then $\mu' = H^*\mu$ is $\Gamma_{0,\theta}$-invariant. Spreading $\mu'$ through the entire complex plane via the mappings defining $\Gamma_{0,\theta}$ we get a Beltrami differential $\nu$ invariant under both $E_\theta$ and $T^{-1} \circ E_\theta^{r_0}$, and therefore invariant under $T$ also. Thus $\nu$

projects down to a Beltrami differential on the cylinder which is $f_\theta$-invariant and supported in $J_{f_\theta}$. By Theorem IV.9, this Beltrami differential must vanish a.e., and so $\mu \equiv 0$ a.e. also. A similar argument rules out wandering domains. □

## V. Beltrami paths and the Julia-Teichmüller metric

If $\Gamma_1$ and $\Gamma_2$ are holomorphic commuting pairs with the same rotation number and the same height, we define their *quasiconformal distance* to be $d_{QC}(\Gamma_1, \Gamma_2) = \frac{1}{2}\inf_H \log K(H)$, where $H$ ranges over all possible symmetric quasiconformal conjugacies between both pairs. This number is obviously zero if and only if $\Gamma_1$ is holomorphically equivalent to $\Gamma_2$. In order to have a space with an actual metric on it, we proceed just as in the case of Riemann surfaces or Fuchsian groups. Thus, let $\Gamma$ be a fixed holomorphic commuting pair and let $\mathcal{D}$ be the outer disk of its bow-tie. Let $\mathrm{Def}(\Gamma)$ be the class of all holomorphic commuting pairs which are conjugate to $\Gamma$ via a symmetric qc-homeomorphism. In $\mathrm{Def}(\Gamma)$, declare $\Gamma_0$ to be equivalent to $\Gamma_1$ iff there exists a symmetric conformal mapping $\mathcal{D}_0 \to \mathcal{D}_1$ conjugating $\Gamma_0$ to $\Gamma_1$. Then define the *Teichmüller space* of $\Gamma$, say $\mathrm{Teich}(\Gamma)$, to be the quotient of $\mathrm{Def}(\Gamma)$ by the above equivalence relation.

If $[\Gamma_0], [\Gamma_1] \in \mathrm{Teich}(\Gamma)$, set $d_T([\Gamma_0], [\Gamma_1]) = \frac{1}{2}\inf_H \log K(H)$ where $H$ ranges over all possible symmetric qc-conjugacies between any two representatives $\widetilde{\Gamma}_0 \in [\Gamma], \widetilde{\Gamma}_1 \in [\Gamma_1]$, and where as before $K(H)$ denotes the maximal dilatation of $H$. This defines the *Teichmüller metric* on $\mathrm{Teich}(\Gamma)$. As in the case of Fuchsian groups, an alternative description of $\mathrm{Teich}(\Gamma)$ as an orbit space is available. Observe that if $G$ is a group of qc-selfhomeomorphisms of $\mathcal{D}$ and $B^\infty$ is the unit ball of $L^\infty(\mathcal{D}, \mathbf{C})$ then there is a natural action $G \times B^\infty \to B^\infty$,

$$(h, \mu) \mapsto h^*\mu = \frac{\mu_h + (\mu \circ h).\dfrac{\overline{h}_z}{h_z}}{1 + \overline{\mu}_h.(\mu \circ h).\dfrac{\overline{h}_z}{h_z}} \ ,$$

which consists of taking the pull-back under $h$ of $\mu \in B^\infty$ viewed as a Beltrami differential on $\mathcal{D}$. Each $h^* : B^\infty \to B^\infty$ is biholomorphic and $(h^*)^{-1} = (h^{-1})^*$. Call $\mu \in B^\infty$ (a) symmetric, if $\mu$ commutes with complex conjugation, and (b) $\Gamma$-invariant, if $\gamma^*\mu = \mu$ for $\gamma = \xi, \eta, \nu$. If we take $G$ to be the group of all symmetric qc-selfhomeos of $\mathcal{D}$ which commute with $\Gamma$ and let $M(\Gamma) = \{\mu \in B^\infty : \mu \text{ is symmetric and } \Gamma\text{-invariant }\}$, then the above $G$-action on $B^\infty$ restricts to an action $G \times M(\Gamma) \to M(\Gamma)$. Let $\mathrm{Orb}_G(\Gamma)$ be the corresponding orbit space. This space can be given the following metric

$$d([\mu_0], [\mu_1]) = \frac{1}{2} \inf \log K(h^{\widetilde{\mu}_0} \circ (h^{\widetilde{\mu}_1})^{-1}) \ ,$$

where the infimum is taken over all $\widetilde{\mu}_0, \widetilde{\mu}_1 \in M(\Gamma)$ in the $G$-orbits of $\mu_0$ and $\mu_1$, respectively. Here $h^\mu$ denotes the unique symmetric qc-homeo $\mathcal{D} \to \mathcal{D}$ with $h^\mu(0) = 0$ and such that $\mu_{h^\mu} = (h^\mu)^*(0) = \mu$; existence and uniqueness are guaranteed by the measurable Riemann mapping theorem.

Given $\mu \in M(\Gamma)$, let us consider the Jordan domains $\mathcal{D}$ and $\mathcal{O}_{\gamma^\mu} = h^\mu(\mathcal{O}_\gamma)$, and the maps $\gamma^\mu = h^\mu \circ \gamma \circ (h^\mu)^{-1} : \mathcal{O}_{\gamma^\mu} \to \gamma(\mathcal{O}_{\gamma^\mu})$, for $\gamma = \xi, \eta, \nu$, which are holomorphic because $\mu$ is $\Gamma$-invariant.

**Lemma V.1.** *These objects determine a holomorphic commuting pair.*

*Proof.* Of all conditions in the definition given in section II, the only one that is not immediate is $H_4$. We must check that both $\xi^\mu$ and $\eta^\mu$ extend holomorphically across some neighborhood of zero, where they ought to commute. The point is that, by Proposition II.1, the map $\eta^{-1} \circ \nu$ is well-defined over $\mathcal{O}_\nu$ and agrees with $\xi$ over $\mathcal{O}_\xi \cap \mathcal{O}_\gamma$. Therefore $(\eta^\mu)^{-1} \circ \nu^\mu = h^\mu \circ (\eta^{-1} \circ \nu) \circ (h^\mu)^{-1}$ is well-defined over $\mathcal{O}_{\nu^\mu}$ and agrees with $\xi^\mu$ over $\mathcal{O}_{\xi^\mu} \cap \mathcal{O}_{\nu^\mu}$. Similarly, $(\xi^\mu)^{-1} \circ \nu$ is well-defined over $\mathcal{O}_{\nu^\mu}$ and extends $\eta^\mu$. It follows at once that both extensions commute on that common part of their domains. □

If we denote by $\Gamma^\mu$ the resulting holomorphic commuting pair then this lemma gives us the right to write formally $\Gamma^\mu = h^\mu \circ \Gamma \circ (h^\mu)^{-1}$. Therefore, just as with Fuchsian groups (cf. [$A_1$]), we have the following statement.

**Proposition V.2.** *The orbit space* $\mathrm{Orb}_G(\Gamma)$ *with the metric* $d$ *is naturally isomorphic to* $\mathrm{Teich}(\Gamma)$ *with its Teichmüller metric* $d_T$.

*Proof.* Let $\Phi : \mathrm{Orb}_G(\Gamma) \to \mathrm{Teich}(\Gamma)$ be given by $\Phi([\mu]) = [\Gamma^\mu]$. The proof that $\Phi$ is an isometry is standard. □

We see at once that $\mathrm{Teich}\,(\Gamma)$ is a path-connected space. A *Beltrami path* in $M(\Gamma)$ is a path $t \mapsto \mu_t$ such that for almost every $z \in \mathcal{D}$, the path $t \mapsto \mu_t(z)$ is a geodesic in $\mathbb{D}$. This definition is equivariant with respect to the action of the group $G$, so we have Beltrami paths in $\mathrm{Orb}_G(\Gamma)$, and therefore Beltrami pahts in $\mathrm{Teich}(\Gamma)$ also, joining any two points in the space.

We will need a germ version of the qc-distance called the *Julia-Teichmüller distance.* Let $\Gamma|\mathcal{O}$ denote the restriction of (all arrows of) $\Gamma$ to the open set $\mathcal{O} \subseteq \mathcal{U}$. Consider the class of pairs $(\Gamma, \mathcal{O})$ where $\mathcal{O}$ is an open neighborhood of the Julia set of $\Gamma$. Define an equivalence relation on such pairs as follows: $(\Gamma_1, \mathcal{O}_1) \sim (\Gamma_2, \mathcal{O}_2)$ iff $\Gamma_1|(\mathcal{O}_1 \cap \mathcal{O}_2) \equiv \Gamma_2|(\mathcal{O}_1 \cap \mathcal{O}_2)$. The resulting equivalence classes are the germs of holomorphic commuting pairs around their limit sets. The germ of $\Gamma$ *up to conformal equivalence* will be denoted by $\langle \Gamma \rangle$. Now let

$$d_{JT}(\langle \Gamma_1 \rangle, \langle \Gamma_2 \rangle) = \frac{1}{2} \inf_H K(H)$$

where $H : (\Gamma_1', \mathcal{O}_1') \simeq (\Gamma_2', \mathcal{O}_2')$ ranges over all possible quasiconformal conjugacies between all representatives of both germs.

*Definition 5.* $d_{JT}(\langle \Gamma_1 \rangle, \langle \Gamma_2 \rangle)$ is the *Julia-Teichmüller distance* between $\langle \Gamma_1 \rangle$ and $\langle \Gamma_2 \rangle$.

The Julia-Teichmüller distance is clearly a pseudo-metric. However, it is not clear yet that it is a *metric*, cf. section VIII. More importantly, it is weakly contracted by the renormalization operator. We also verify without difficulty that the map from $\mathrm{Teich}(\Gamma)$ to the space of germs up to analytic conjugacy given by $[\Gamma_\mu] \mapsto \langle \Gamma_\mu \rangle$ is distance-nonincreasing.

## VI. The factoring of renormalization compositions

In order to develop complex bounds for renormalization of holomorphic commuting pairs, we shall need a generalization of the so-called *sector theorem* of Sullivan [$S_3$, §5], which we proceed to state.

Given $a, b \in \mathbb{R}$ with $a < b$, let $S(a,b)$ be the class of all schlicht mappings $\phi$ defined on $\mathbf{C}(I_\phi) = \mathbf{C} \setminus (\mathbb{R} \setminus I_\phi)$, where $I_\phi \supseteq (a,b)$ is some open interval, which preserve both half-planes $\mathbf{C}^+, \mathbf{C}^-$ and are such that $\phi((a,b)) = (a,b)$. We refer to $I_\phi$ as the *base* of $\phi \in S(a,b)$: it is the largest interval containing $(a,b)$ restricted to which $\phi$ is a homeomorphism into the reals. An element $A \in S(a,b)$ is a *left $\alpha$-root* (where $0 < \alpha < 1$) if there exists $a_0 \le a$ such that $A(z) = u.(z - a_0)^\alpha + v$, where $u, v \in \mathbb{R}$ and the branch of $z \mapsto (z - a_0)^\alpha$ are uniquely determined by the requirements $A(a) = a$, $A(b) = b$. The point $a_0 \in \mathbb{R}$ is the *pole* of $A$. Right roots are defined similarly. Given a bounded interval $J \subseteq \mathbb{R}$ and some $\lambda > 0$, we denote by $J^\lambda$ the closed interval centered at the midpoint of $J$ whose length is $(1+\lambda)$-times the length of $J$.

**Theorem VI.1.** *Let there be given $A_i, B_i \in S(a,b)$, for $i = 1, 2, \cdots, m$, and constants $\lambda, K, s > 0$ and $0 < \alpha < 1$ satisfying*

- *(a) Each $A_i$ is a left $\alpha_i$-root with $\alpha_i \le \alpha$ and pole at $a_i$, where $a_1 = a$ and $a_i < a$ for all $i \ge 2$;*
- *(b) There exists a finite sequence of stopping times $1 = i_0 < i_1 < \cdots < i_q = m$ with $i_{n+1} - i_n \le s$ such that, setting $d_n = \min\{|a_i - a| : i_n \le i < i_{n+1}\}$, the inequality $\sum_{j \ge n} d_j^{-1} \le K d_n^{-1}$ holds for all $n$;*
- *(c) The following holds for all $i \ge 2$: if $I_i$ is the base of $B_i$ then $B_i(I_i) \supseteq [a_i, b]^\lambda$ and setting $J_i = B_i^{-1}([a_i, b])$ then $J_i^\lambda \subseteq I_i$.*

*Under these assumptions, there exists a positive angle $\theta = \theta(\alpha, s, K, \lambda)$ such that the image of the upper half-plane by the composition $A_m B_m \cdots A_i B_i \cdots A_1 B_1$ is contained in the sector $0 \le \arg(z - a) \le \pi - \theta$.*

A complete proof of this powerful tool is given in [dF$_2$]. Now, considering the long renormalization compositions of a critical commuting pair in the Epstein class, we would like to break them up into factors that will satisfy the hypotheses of Theorem VI.1 after affine rescaling. This is accomplished at the end of this section.

Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with irrational rotation number $\rho(f) = [r_0, r_1, \cdots, r_n, \cdots]$. As in section I, given $x \in S^1$ and $k \ge 0$, let $I_k(x) \subseteq S^1$ be the unique closed interval with endpoints $x$ and $f^{q_k}(x)$ containing $f^{q_{k+2}}(x)$. For a distinguished point $c \in S^1$, we shall write $I_k$ instead of $I_k(c)$. Fix some large $n$, and consider the ordered collection of intervals $\mathcal{B} = \{f^i(I_n) : 1 \le i \le q_{n+1} - 1\}$. These intervals have pairwise disjoint interiors. For $k = 0, 1, \cdots, n+1$, let $j_k$ be the largest $j \ge 1$ such that $f^i(I_n) \cap \mathring{I}_k = \emptyset$ for $1 \le i < j$. Observe that $j_0 = 1$.

**Lemma VI.2.** *For $1 \le k \le n+1$, we have $j_k = q_k$ if $n - k$ is odd, while $j_k = q_k + q_{k+1}$ if $n - k$ is even.*

*Proof.* For all $k$ in that range, either $I_n \subseteq I_k$ or $I_n$ is adjacent to $I_k$, depending on whether $|n - k|$ is even or odd. The lemma follows, then, from the dynamical interpretation of $\{q_i\}_{i \ge 0}$ as a sequence of return times. □

Let us consider the *blocks* $\mathcal{B}_k = \{f^i(I_n) \in \mathcal{B} : j_{k-1} \le i < j_{k+1}\}$, for $k = 1, 2, \cdots, n$. Notice that $\mathcal{B} = \bigcup_{k=1}^n \mathcal{B}_k$ and that $\mathcal{B}_k \cap \mathcal{B}_{k+2} = \emptyset$ for $1 \le k \le n-2$. These blocks correspond roughly to what Sullivan calls *epochs* in [S$_1$]. Let the *scale* of $f^i(I_n) \in \mathcal{B}$ be the largest $k \ge 1$, if any, such that $f^i(I_n) \subseteq I_{k-1} \setminus \mathring{I}_{k+1}$, and let it be equal to zero otherwise.

An element of $\mathcal{B}_k$ is a *$k$-marked interval* if its scale is equal to $k$. We also call an element of $\mathcal{B}_1$ a *0-marked interval* if its scale is zero and it precedes all 1-marked intervals in the forward dynamical order of $\mathcal{B}$. We denote by $\mathcal{M}_k$ the collection of all $k$-marked intervals, for $k = 0, 1, \cdots, n$. It is not difficult to see that $\mathcal{M}_0$ has either 0 or $r_0 - 1$ elements, depending on whether $n$ is odd or even, respectively.

**Lemma VI.3.** *If $1 \leq k \leq n$, then $r_k \leq \operatorname{card}(\mathcal{M}_k) \leq r_k(r_{k+1}+1)$, and moreover $\operatorname{card}(\mathcal{M}_k) = r_k$ whenever $k \equiv n$ (mod 2).*

*Proof.* Observe that the intervals $J_0 = f^{q_{k-1}}(I_k)$, $J_s = f^{sq_k}(J_0)$, $s = 1, 2, \cdots, r_k - 1$, constitute a partition of $I_{k-1} \setminus I_{k+1}$ modulo endpoints. Suppose $i < j$ are such that the intervals $f^i(I_n), f^j(I_n)$ belong to $\mathcal{B}_k$ and are both in the same $J_s$. Then they are $k$-marked by definition, and by Lemma VI.2 either (a) $j-i \leq q_{k+1}-q_{k-1}$ or (b) $j-i \leq r_{k+1}q_{k+1}+r_kq_k$, depending on whether $n-k$ is even or odd. Since $f^{j-i}(J_s) \cap \mathring{J}_s \neq \emptyset$, we have $f^{j-i}(I_k) \cap \mathring{I}_k \neq \emptyset$ as well, which implies $j - i \geq q_{k+1}$. In case (a) this yields a contradiction, and so each $J_s$ contains at most one element of $\mathcal{M}_k$, i.e. $\operatorname{card}(\mathcal{M}_k) \leq r_k$. In case (b) it shows that the number of elements of $\mathcal{M}_k$ in each $J_s$ is at most the smallest integer greater than $(r_{k+1}q_{k+1} + r_kq_k)/q_{k+1}$, which is $r_{k+1} + 1$, and so $\operatorname{card}(\mathcal{M}_k) \leq r_k(r_{k+1}+1)$. In either case we have $J_s \supseteq f^{q_{k-1}+(s+\varepsilon_k)q_k}(I_n)$ for $s = 0, 1, \cdots, r_k - 1$, where $\varepsilon_k$ is the remainder of $n-k$ modulo 2. Since such images of $I_n$ are in $\mathcal{B}_k$ by Lemma VI.2, they are in $\mathcal{M}_k$ too, hence $\operatorname{card}(\mathcal{M}_k) \geq r_k$. □

Now let $f$ be a critical circle mapping, $c$ its critical point, so that the bounded geometry results of section I are valid for $f$. More precisely, we assume the following axioms.

*Axiom 1.* There exists $0 < \sigma < 1$ such that the inequality $|I_{n+1}(f^ic)| \leq \sigma |I_{n-1}(f^ic)|$ holds for all $n \geq 1$ and all $i \in \mathbb{Z}$.

*Axiom 2.* There exists $\lambda > 0$ such that the following holds for all $n \geq 1$: if $0 < i < i+j \leq q_{n+1} - 1$ and $J \supseteq f^i(I_n)$ is the largest interval restricted to which $f^j$ is a diffeo onto its image then $[f^i(I_n)]^\lambda \subseteq J$.

Both axioms are straightforward consequences of Theorem I.3 (interpreted directly for circle mappings). The second axiom is in fact obtained from the Koebe principle for distortion of cross-ratios, cf. [MS, Ch. VI].

Let the *polar-ratio* of a non-degenerate interval $J$ with respect to a point $x$ be the number $P(x, J) = \operatorname{dist}(x, J)/|J|$. Observe that, under a map with bounded cross-ratio distortion, polar-ratios do not decrease by more than a multiplicative factor depending only on the cross-ratio distortion of the map.

**Lemma VI.4.** *There exist constants $C > 0$ and $\mu > 1$, depending only on constant $\sigma$ of Axiom 1, such that $P(c, f^i(I_n)) \geq C\mu^{n-k}$ for each interval $f^i(I_n)$ whose scale is equal to $k$.*

*Proof.* Let $x = f^ic$; all intervals written $[a, b]$ in this proof will be contained in $S^1 \setminus \{x\}$. We assume that $n - k$ is even; the odd case is similar. Since $f$ is topologically conjugate to the corresponding rotation, we have

(a) $f^{-q_j}(x) \in [f^{q_{j-1}}(x), f^{q_{j+1}}(x)]$ for all $j \geq 1$;

(b) if $f^i(x) \in [f^{q_{k-1}}(x), f^{q_{k+1}}(x)]$ then $f^{-i}(x) \in [f^{-q_{k-1}}(x), f^{-q_{k+1}}(x)]$.

Putting these two facts together we get $c = f^{-i}(x) \in I_{k-2}(x) \setminus I_{k+2}(x)$, and in particular $d(c, f^i(I_n)) \geq |I_{k+2}(x)|$. On the other hand, since $n-k$ is even, we have

$$I_n(x) \subseteq I_{n-2}(x) \subseteq \cdots \subseteq I_{k+2}(x) \ ,$$

Applying Axiom 1 for $x = f^i c$ and using a *telescoping* trick, we deduce that

$$P(c, f^i(I_n)) \geq \big(\frac{1}{\sqrt{\sigma}}\big)^{n-k-2} \ ,$$

and this proves the Lemma. □

Now we are ready to exhibit the promised factoring of the $n$-th renormalization of a critical commuting pair $\zeta = (\xi, \eta)$ in the Epstein class, for all sufficiently large $n$. We know that there exist open intervals $\widetilde{I}_\xi \supseteq \xi(I_\xi)$ and $\widetilde{I}_\eta \supseteq \eta(I_\eta)$, as well as symmetric schlicht mappings $h_\xi^{-1} : \mathbf{C}(\widetilde{I}_\xi) \to \mathbf{C}$, $h_\eta^{-1} : \mathbf{C}(\widetilde{I}_\eta) \to \mathbf{C}$ such that $\xi \equiv h_\xi \circ Q$ and $\eta \equiv h_\eta \circ Q$, where $Q$ denotes the cubic polynomial $z \mapsto z^3$. Renormalizing $\zeta$ enough times if necessary, we may assume also, by Theorem I.4, that $\gamma(I_\gamma)$ sits inside $\widetilde{I}_\gamma$ with universal space around it ($\gamma = \xi, \eta$). In particular, each restriction $h_\gamma^{-1} | \gamma(I_\gamma)$ has universally bounded cross-ratio distortion (cf. observation preceding Lemma VI.4).

Consider the successive renormalizations of $(\xi, \eta)$ without rescaling: $(\xi_0, \eta_0) = (\xi, \eta)$ and $(\xi_{n+1}, \eta_{n+1}) = (\eta_n, \eta_n^{r_n} \circ \xi_n)$, for all $n \geq 0$. We concentrate on the case $n$ even (and large), and within it we show how to achieve the desired factoring only for $\xi_n$; the other cases are similarly handled. By Lemma I.2, we have the hybrid representation

$$\xi_n(x) \ = \ f^{q_{n-1}-1} \circ \xi(x)$$

for all $x$ in $I_n$, where $f = f_\zeta : I \to I$ is the circle mapping associated to $\zeta$, and $I_j = I_j(c)$ for $c = 0$, the critical point. We examine here the diffeomorphic part of $\xi_n^{-1}$, namely the composition

$$\widehat{\xi}_n = (f^{q_{n-1}-1})^{-1} : f^{q_{n-1}}(I_n) \to f(I_n) \ . \tag{13}$$

Let (13) be written as a word in $\xi^{-1}$, $\eta^{-1}$. A factor $\gamma^{-1}$ in this composition is called a *left* or a *right* factor, according to whether $\gamma = \eta$ or $\gamma = \xi$. Each such factor, remember, has a further decomposition $\gamma^{-1} \equiv Q^{-1} \circ h_\gamma^{-1}$. A *left root* is the part of a left factor corresponding to $Q^{-1}$; *right roots* are similarly defined. The $h_\gamma^{-1}$ are called *h-factors.* A left root is said to be $k$-marked ($k$ necessarily even) if the domain of its left factor is some $J \in \mathcal{M}_k$. Similarly, a right root is said to be $k$-marked ($k$ necessarily odd) if the domain of its right factor is some $J \in \mathcal{M}_k$. Here $k$ ranges from 0 up to $n-1$. Lemma VI.3 bounds the number of $k$-marked roots for any such $k$ in terms of the combinatorics of the rotation number $\rho(f) = \rho(\zeta)$. Organize all marked left roots in the composition giving $\widehat{\xi}_n$ by their order of appearance from right to left in that composition, and call them successively $\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A}_m$. In this order, first come the $(n-2)$-marked roots, then come the $(n-4)$-marked roots, and so on, and $\widehat{A}_m$ is the very last factor in that composition. We have $m = (r_0 - 1) + r_2 + \cdots + r_{n-2}$, after Lemma VI.3. In (13), let $\widehat{B}_1$ be the sub-composition going from the first factor on the right up to and including the left-most factor before $\widehat{A}_1$, which is precisely the $h$-factor associated to the left-root $\widehat{A}_1$. Also, for $j = 2, 3, \cdots, m$, let $\widehat{B}_j$ be the sub-composition running strictly between $\widehat{A}_{j-1}$ and $\widehat{A}_j$. In this new notation, (13) becomes

$$\widehat{\xi}_n = \widehat{A}_m \circ \widehat{B}_m \circ \cdots \circ \widehat{A}_j \circ \widehat{B}_j \circ \cdots \circ \widehat{A}_1 \circ \widehat{B}_1 \ . \tag{14}$$

Let $T_2 \subset I$ be the largest open interval containing $f^2(I_n)$ restricted to which $f^{q_{n-1}-2}$ is a diffeo onto its image. Remembering that $\eta$ is defined well to the right of $\xi(0)$ (which is the right endpoint of $f(I_n)$), let $(a, b) = T_1 = \eta^{-1}(T_2)$ and put also $T_i = f^{i-2}(T_2)$ for $i = 3, \cdots, q_{n-1}$. Then each $T_i$ contains the corresponding $f^i(I_n)$ plus definite, *beau* space on both sides, after Axiom 2. Let all factors $\widehat{A}_j, \widehat{B}_j$ be rescaled via the affine, orientation preserving maps taking the relevant $T_i$'s back onto $(a, b)$. Call the rescaled mappings $A_j, B_j$, respectively: these are now elements in the class $S(a, b)$. We are ready to state and prove the promised factoring of renormalization compositions.

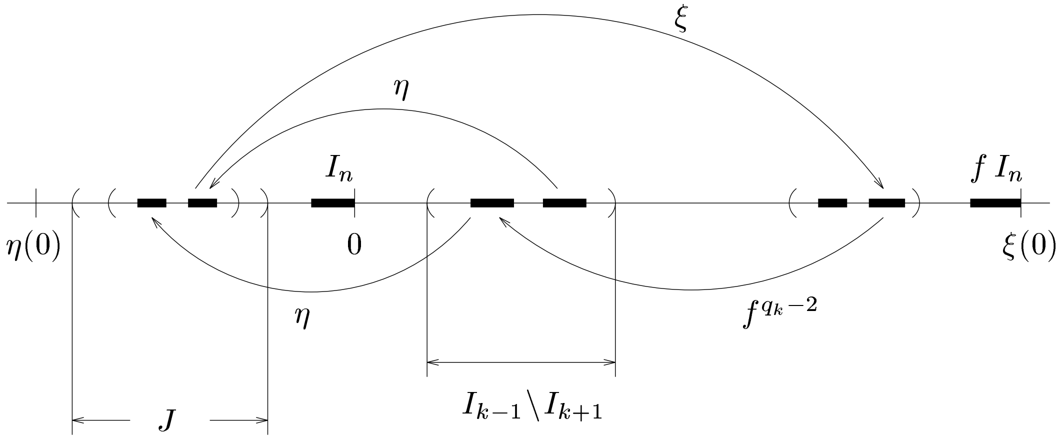


*Figure 3*

**Theorem VI.5.** *If the rotation number $\rho(\zeta)$ is of bounded combinatorial type and $n$ is sufficiently large, then the rescaled composition $A_m \circ B_m \circ \cdots \circ A_j \circ B_j \circ \cdots \circ A_1 \circ B_1$ satisfies all hypotheses of the generalized sector theorem, and the bounds involved are beau.*

*Proof.* Since each $A_j$ is a left $\alpha_j$-root with $\alpha_j = \frac{1}{3}$ (i.e., a cubic root), and the pole of $A_1$ is in dynamical correspondence with $a$, assumption (a) of Theorem VI.1 is satisfied. Since the number of marked left-roots at each scale is uniformly bounded by the hypothesis on $\rho(\zeta)$ and Lemma VI.3, the bounded gap condition of assumption (b) is fulfilled if we group the roots together by scales. Now, let $a_j$ be the pole of $A_j$, and let $i$ be such that $f^i(I_n)$ is the marked left interval corresponding to $A_j$. Combining the observation following Axiom 2 with the remarks preceding the statement of this theorem and the definition of polar ratio, we obtain

$$|a_j - a| \geq C_0 |a - b| P(c, f^i(I_n)) \ ,$$

for a certain *beau* constant $C_0$. Therefore, by Lemma VI.4, $|a_j - a|$ grows exponentially with $n - k$, where $k$ is the scale of $A_j$, and this takes care of the series condition of assumption (b). It remains to check whether assumption (c) holds. For all $j \geq 2$ we may write

$$\widehat{B}_j = h_j \circ (f^{q_k - 1})^{-1} \ ,$$

provided $k$ is the scale of $\widehat{A}_j$, where $h_j$ is the $h$-factor associated to $\widehat{A}_j$. Thus, we have the situation depicted in Figure 3 (there are two cases, depending on whether the scale of $\widehat{A}_{j-1}$, the preceding marked left-root, is equal to $k$ or $k+2$; Figure 3 illustrates the former). By Axiom 2, $f^{q_k-1}$ is a diffeomorphism on an interval $J \supseteq [f(I_{k-1})]^\lambda$. Hence assumption (c) is indeed verified if we take into account that: (i) $J$ is in dynamical correspondence with the base of $B_i$; (ii) $f(0)=\eta(0)$, the left endpoint of $f(I_{k-1})$, is in dynamical correspondence with the pole of $A_j$, and (iii) $h_j$ is a map of *beau* bounded cross-ratio distortion. The case $j=1$ is similarly proved. Since all bounds involved are *beau*, we are done. ◻

## VII. The complex bounds

We recall Sullivan's *sector inequality.* Given real numbers $u<u_*$ and $0<\vartheta\leq\frac{\pi}{3}$, consider in the upper half-plane the truncated sector

$$\mathcal{S}(u,u_*,\vartheta) \;=\; \mathbf{C}^+ \cap \{arg\,(z-u) < \pi-\vartheta\} \cap \{arg\,(z-u_*) > \frac{2\pi}{3}\} \;. \tag{15}$$

Also, let $w'<u<u'<u_*<u'_*<w'_*$ and let $w$, $w_*$ be points in $\mathcal{S}(a,b,\vartheta)$. Assume $C>1$ is a constant such that all non-zero distances between points in $\{u,u_*,w,w_*,u',u'_*,w',w'_*\}$ lie between $C^{-1}$ and $C$. Moreover, denote by $\mathcal{N}_R$ the portion in the upper half-plane of the Poincaré neighborhood of radius $R$ of the geodesic $(w',w'_*)$ in $\mathbf{C}((w',w'_*))$.

**Lemma VII.1.** *There exists $r>0$ depending only on $\vartheta$ and $C$ such that the following holds. For each $R>r$ there exists $R_0>R$ such that, if $\psi:\mathcal{N}_{R_0}\to\mathcal{S}(u,v,\vartheta)$ is univalent and maps $u$, $u_*$, $w$, $w_*$ respectively to $u'$, $u'_*$, $w'$, $w'_*$, then $\psi(\mathcal{N}_R)\subseteq\mathcal{N}_{R/2}$.* ◻

Now, let us define the *conformal type* of a holomorphic commuting pair to be the modulus of the annulus determined by the inner and outer domains of its bow-tie. Let us also say that an irrational number has *combinatorial type bounded by $N>0$* if the convergents of its continued-fraction development are bounded by $N$.

**Theorem VII.2.** *Given a positive integer $N$, there exists $\tau=\tau(N)>0$ with the following property. If $\zeta$ is a critical commuting pair with rotation number of combinatorial type bounded by $N$ and if $\zeta$ either belongs to some Epstein class or extends to a holomorphic commuting pair, then for all $n$ sufficiently large, $\mathcal{R}^n\zeta$ extends to a holomorphic commuting pair $\Gamma_n(\zeta)$ with geometric boundaries and conformal type bounded from below by $\tau$.*

*Proof.* If $\zeta$ extends to a holomorphic commuting pair, then $\zeta$ is analytically equivalent to a critical commuting pair in the Epstein class. Thus one may assume that $\zeta$ is in the Epstein class already. Let $a$, $\theta$ and $n$ be as in Theorem VI.5. To define $\Gamma_n(\zeta)$, one first constructs its bow-tie $(\mathcal{D}_n,\mathcal{O}_{\xi_n},\mathcal{O}_{\eta_n},\mathcal{O}_{\nu_n})$. For $\mathcal{D}_n^+$ one takes a large Poincaré neighborhood $\mathcal{N}_R$ as above, with $w'$ and $w'_*$ to satisfy the conditions below and $R$ given by Lemma VII.1. Then one takes

$$\mathcal{O}_{\xi_n}^+ \;=\; \xi_n^{-1}(\mathcal{D}_n^+) \;=\; (f^{q_{n-1}-1}\circ\xi)^{-1}(\mathcal{D}_n^+) \;.$$

where $f$ is as in section VI (and where $n$ is assumed to be even). By Theorem VI.5, $\xi(\mathcal{O}_{\xi_n}^+)$ is contained in a sector in the upper half-plane with angle $\theta$ on the left. Since $\xi^{-1}$ is a right cubic-root factor, it follows that $\mathcal{O}^+_{\xi_n}$ lies within a truncated sector of type (15), where $\vartheta = \theta/3$, $u_*$ is the origin and $u$ is a point in dynamical correspondence with $a$. Therefore by the sector inequality the closure of $\mathcal{O}_{\xi_n}$ lies well within $\mathcal{D}_n$. One constructs $\mathcal{O}_{\eta_n}$ by a similar procedure, so that its closure is also contained in $\mathcal{D}_n$. One then completes the bow-tie following the method in section IV. Now let $\mathcal{U}_n$ be the inner domain of this bow-tie, and let $J_n = \mathcal{U}_n \cap \mathbb{R}$. The points $w'$ and $w'_*$ had to be chosen from the start so that $[w', w'_*]$ contains the interval $J_n$ with *beau* space on both sides. The boundary of $\mathcal{U}_n$ consists of finitely many analytic arcs meeting at finitely many corners with internal angles $\geq \pi/3$. The total number of corners is bounded in terms of the number $k$ of critical values of $\xi_n$ found in $[w', w'_*]$. If the size of this interval is commensurable with the size of $J_n$, then by the real a-priori bounds (cf. Theorems I.3 and I.4) one has $k$ bounded in terms of $N$ only. Therefore, $\partial\mathcal{U}_n$ is a $K$-quasicircle with $K = K(N)$, by Ahlfors' characterization of quasicircles in [A$_1$]. Finally, the minimum distance between the boundaries of the annulus $\mathcal{D}_n \backslash \mathcal{U}_n$ depends only on the space of $J_n$ inside $[w', w'_*]$. Therefore its modulus is controlled, by the Teichmüller inequalities (9). □

A major consequence of this theorem is the following compactness property enjoyed by renormalization.

**Corollary VII.3.** *Given a positive integer $N$, there exists $B_N > 0$ with the following property. If $\zeta_1$ and $\zeta_2$ are critical commuting pairs with the same irrational rotation number of combinatorial type bounded by $N$, and if each of them either belongs to an Epstein class or extends to a holomorphic commuting pair, then for all sufficiently large $n$ we have, in the notation of Theorem VII.1,*

$$d_{JT}(\langle\Gamma_n(\zeta_1)\rangle, \langle\Gamma_n(\zeta_1)\rangle) \leq d_{QC}\big(\Gamma_n(\zeta_1), \Gamma_n(\zeta_2)\big) \leq B_N \ .$$

*Proof.* Combine Theorem VII.2 with the pull-back argument of Theorem III.1. □

One of the keys to renormalization contraction, this corollary states that if we renormalize a finite Beltrami arc sufficiently many times, then its endpoints, no matter how far apart in the Julia-Teichmüller sense, come within a fixed distance from each other.

*Remark.* In a very recent work, Yampolsky [Ya] proved without using the sector theorem that the modulus of $\mathcal{D}_n \setminus \mathcal{U}_n$ is always bounded from below, *independently of the combinatorics*, thereby obtaining complex bounds for critical circle maps with arbitrary rotation numbers.

## VIII. A Cantor repeller and its Riemann surface lamination

A *Cantor repeller* consists of two collections $\{D_i\}_{0\leq i\leq n}$ and $\{\Delta_j\}_{0\leq j\leq m}$ of topological disks in the plane and a surjective holomorphic map $\phi : \bigcup_{i=0}^n D_i \to \bigcup_{j=0}^m \Delta_j$ such that (a) in each collection, any two disks have pairwise disjoint closures; (b) each disk of the first collection is compactly contained in some disk of the second collection; (c) each $\phi|D_i$ is schlicht onto $\Delta_j$ for some $j$. The invariant limit set $K_\phi = \bigcap_{n=0}^\infty \phi^{-n}(\bigcup_j \Delta_j)$ is a Cantor set, hence the name, and the restriction $\phi|K_\phi$ is topologically conjugate to a certain shift of finite type (cf. [Bo]). We call this shift the *topological type* of our Cantor repeller. Writing

$U$ and $V$ for domain and range of $\phi$, we represent the Cantor repeller by $(U, \phi, V)$. We also say that a Cantor repeller is *linear* if $U$, $V$ and $\phi$ are symmetric about the real line.

Now let $\Gamma$ be a holomorphic commuting pair with irrational rotation number, and let $\mathcal{K}_\Gamma$ be its (connected) limit set. We want to show how to extract from within $\Gamma$ a certain Cantor repeller that, off of its limit set, turns out to be conformally conjugate to $\Gamma$ in the vicinity of its small dynamical interval. More precisely, we have the following result.

**Theorem VIII.1.** *Let $\Gamma$ be a holomorphic commuting pair having a connected filled-in limit set $\mathcal{K}_\Gamma$. Then there exist an open set $\mathcal{O}$ and a linear Cantor repeller $(U, \phi, V)$ of fixed topological type such that*

*(a) $\phi|U^+$ is conjugate to $\Gamma|(\mathcal{O} \setminus \mathcal{K}_\Gamma)$ by a holomorphic map $H$;*

*(b) For each $n \geq 0$ there exists an open neighborhood $\mathcal{V}_n$ of the small dynamical interval of $\Gamma$ contained in $\mathcal{O}$ such that $H(\phi^{-n}(U^+)) = \mathcal{V}_n \setminus \mathcal{K}_\Gamma$.*

*Proof.* Since $\widehat{\mathbf{C}} \setminus \mathcal{K}_\Gamma$ is simply-connected, let $\Phi : \widehat{\mathbf{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbf{C}} \setminus \mathcal{K}_\Gamma$ be the Riemann-mapping, normalized to be symmetric about the real axis and fixing $\infty$. Consider the simply-connected regions $V_0 = \mathcal{O}_\xi^+ \setminus \mathcal{K}_\Gamma$, $V_1 = \mathcal{O}_\nu^+ \setminus (\mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{K}_\Gamma)$, $V_2 = \mathcal{O}_\eta^+ \setminus \mathcal{K}_\Gamma$, $V_3 = \mathcal{O}_\xi^- \setminus \mathcal{K}_\Gamma$, $V_4 = \mathcal{O}_\nu^- \setminus (\mathcal{O}_\xi \cup \mathcal{O}_\eta \cup \mathcal{K}_\Gamma)$, $V_5 = \mathcal{O}_\eta^- \setminus \mathcal{K}_\Gamma$, and also $W_0 = \Delta^+ \setminus \mathcal{K}_\Gamma$, $W_1 = \Delta^- \setminus \mathcal{K}_\Gamma$. Let $\mathcal{O}_i = \Phi^{-1}(V_i)$ and $\Omega_j = \Phi^{-1}(W_j)$. Now, let $\psi : \bigcup_{i=0}^5 \mathcal{O}_i \to \Omega_0 \cup \Omega_1$ be the mapping $\Phi^{-1} \circ F \circ \Phi$, where $F$ is the shadow of $\Gamma$ (cf. section II). Then each $\psi|\mathcal{O}_i$ is schlicht onto either $\Omega_0$ or $\Omega_1$, depending on whether $i$ is even or odd. The intervals $I_i = \partial\mathcal{O}_i \cap \partial\mathbb{D}$ are pairwise disjoint, and each $\psi|\mathcal{O}_i$ carries the corresponding $I_i$ onto $(\partial\mathbb{D})^+$ if $i$ is even, and onto $(\partial\mathbb{D})^-$ if $i$ is odd. Next, let $\Delta_0 \subseteq \mathbf{C}^+$ be a Jordan domain such that (a) $\Delta_0$ is symmetric about $\partial\mathbb{D}$ under geometric inversion; (b) $\overline{\Delta_0} \setminus \overline{\mathbb{D}} \subseteq \Omega_0$; (c) $\overline{\mathcal{O}}_0 \cup \overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2 \subseteq \Delta_0$. Let $\Delta_1 \subseteq \mathbf{C}^-$ be similarly defined, and consider the inverse mappings $\psi_i = f^{-1} : \Omega_0 \to \mathcal{O}_i$ for $i$ even. Restrict each $\psi_i$ to $\Delta_0 \setminus \overline{\mathbb{D}}$ and then extend the corresponding restriction to $\Delta_0$ by Schwarz's reflection. Continue to denote these extensions by the same names, and let $D_i = \psi_i(\Delta_0)$, $i$ even. Define $D_i$ for $i$ odd in similar fashion, using $\Delta_1$. Then $\psi : D_0 \cup \cdots \cup D_5 \to \Delta_0 \cup \Delta_1$ is a Cantor repeller. Moreover, each of the open sets

$$\mathcal{V}_n = int\Big(\Phi\left(\psi^{-n}\big(\bigcup_{i=0}^{5} D_i \setminus \overline{\mathbb{D}}\big)\right) \cup \mathcal{K}_\Gamma\Big) ,$$

contains the small dynamical interval of $\Gamma$ (cf. remark at the end of section III). Note that $\mathcal{V}_{n+1} \subseteq \mathcal{V}_n$, and take $\mathcal{O} = \mathcal{V}_0$. Finally, let $M$ be a fractional linear transformation taking $\widehat{\mathbf{C}} \setminus \mathbb{D}$ onto $\mathbf{C}^+$ and, say, the point $-1$ to $\infty$. Then let $H = \Phi \circ M^{-1}$, $\phi = M \circ \psi \circ M^{-1}$ and let $U$ and $V$ be domain and range of $\phi$. This puts our Cantor repeller in linear form and proves (a) and (b). ◻

The Cantor repeller given by this theorem is determined only up to holomorphic conjugacy. It is a weak analogue of the Douady-Hubbard external class for holomorphic commuting pairs, even though a straightening theorem is missing.

We define the *germ* of a Cantor repeller around its limit set just as in the case of holomorphic commuting pairs (cf. section V), and write $\langle\phi\rangle$ for the germ of $(U, \phi, V)$ up to holomorphic equivalence. We also define the corresponding Julia-Teichmüller distance between such germs, but call it instead the *germ distance* to avoid confusion, and denote it by $d_G$. Theorem VIII.1 implies that, if $\Gamma_1$ and $\Gamma_2$ are qc-conjugate and $\phi_1$ and $\phi_2$ are the corresponding Cantor repeller maps, then $d_{JT}(\langle\Gamma_1\rangle,\langle\Gamma_2\rangle) \geq d_G(\langle\phi_1\rangle,\langle\phi_2\rangle)$.

**Theorem VIII.2.** *Let $\Gamma_1$ and $\Gamma_2$ be holomorphic commuting pairs and let $\phi_1$ and $\phi_2$ be the corresponding Cantor repeller maps. Consider the following five statements.*

*(a) $\Gamma_1$ and $\Gamma_2$ have the same germ up to holomorphic equivalence;*
*(b) $d_{JT}(\langle\Gamma_1\rangle,\langle\Gamma_2\rangle) = 0$;*
*(c) $d_G(\langle\phi_1\rangle,\langle\phi_2\rangle) = 0$;*
*(d) $\phi_1$ and $\phi_2$ are analytically conjugate on the line.*
*(e) There exists $k \geq 0$ such that $d_{JT}(\langle\mathcal{R}^k\Gamma_1\rangle,\langle\mathcal{R}^k\Gamma_2\rangle) = 0$.*

*Then we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e).*

*Proof.* We only prove that (c) $\Rightarrow$ (d) $\Rightarrow$ (e) and refer to [MS, Ch.VI, §4, Corollary 1] for the other implications. Assume (c) holds. For each $\varepsilon > 0$ we find neighborhoods $U_1^\varepsilon \supseteq K_{\phi_1}$ and $U_2^\varepsilon \supseteq K_{\phi_2}$ and a $(1+\varepsilon)$-quasiconformal map $h_\varepsilon : U_1^\varepsilon \to U_2^\varepsilon$ conjugating $\phi_1$ and $\phi_2$. This conjugacy restricts to a quasi-symmetric map on the line that is Hölder with exponent $1 - O(\varepsilon)$. Hence, the *scaling functions* of $K_{\phi_1}$ and $K_{\phi_2}$ differ on corresponding points of their dual Cantor sets by $O(\varepsilon)$. Letting $\varepsilon \to 0$ we deduce that $K_{\phi_1}$ and $K_{\phi_2}$ have the same scaling function. Therefore $\phi_1$ and $\phi_2$ are analytically conjugate on some real-line neighborhoods of both Cantor sets, by [S$_3$]. Now, if (d) holds, then both repellers are analytically conjugate on neighborhoods of their limit sets in the complex plane. Therefore by Theorem VIII.1(b), $\Gamma_1$ and $\Gamma_2$ are analytically conjugate on neighborhoods of their small dynamical intervals. By the complex bounds, these neighborhoods contain the inner domains of the bow-ties of all sufficiently large renormalizations of both pairs, so (e) follows. □

We note a further relationship between the Julia-Teichmüller distance on holomorphic commuting pairs and the germ distance on Cantor repellers.

**Proposition VIII.3.** *Let $\Gamma$ and $\Gamma'$ be topologically equivalent holomorphic commuting pairs. For each $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$*

$$d_{JT}(\langle\Gamma_n\rangle,\langle\Gamma_n'\rangle) \;\leq\; d_G(\langle\phi\rangle,\langle\phi'\rangle) + \varepsilon \;,$$

*where $\Gamma_n = \mathcal{R}^n\Gamma$ and $\Gamma_n' = \mathcal{R}^n\Gamma'$ (cf. Proposition II.3).*

*Proof.* This follows at once from Theorem VIII.1(b) together with the complex bounds given by Corollary VII.2. □

Now we associate to the germ of a Cantor repeller around its limit set a compact Riemann surface lamination in the sense of Sullivan (cf. [S$_1$]). Roughly, this will be the space of backward branch orbits (or *threads*) of points in any deleted neighborhood of $K_\phi$ factored by the equivalence relation determined by the dynamics of $\phi$ itself. Recall that a *Riemann surface lamination (or RSL -) structure* on a Hausdorff topological space $X$ consists of a maximal atlas $\{(U_\alpha,\varphi_\alpha)\}$ covering $X$ such that (a) each $\varphi_\alpha$ maps the corresponding $U_\alpha$ homeomorphically onto $D_\alpha \times T_\alpha$, where $D_\alpha \subseteq \mathbf{C}$ is a disk and $T_\alpha$ is a Hausdorff space, and (b) each ovelapping homeo $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$

is of the form $(z,t) \mapsto (\psi_t(z), \phi(t))$ with $\psi_t$ holomorphic for each $t$. Provided with such a structure, the space $X$ is called a *Riemann surface lamination.* One defines the leaves of a Riemann surface lamination just as in the case of foliations; leaves come with obvious intrinsic structures making them into Riemann surfaces in a natural way. A lamination (qc-) morphism $X \to Y$ between two Riemann surface laminations is a continuous map that sends leaves of $X$ into leaves of $Y$ and is holomorphic (resp. quasiconformal) on leaves. Every locally trivial bundle over a Riemann surface with totally disconnected fiber above each element of the base has a unique RSL-structure on the total space that makes the projection map into a lamination morphism. This fact yields the following lemma on inverse limits.

**Lemma VIII.4.** *Let $\ldots \to S_n \xrightarrow{g_n} S_{n-1} \to \ldots \xrightarrow{g_1} S_0$ be an inverse system where each $S_n$ is a free union of Riemann surfaces and each $g_n$ is a proper holomorphic covering map, let $S_\infty$ be its topological inverse limit and let $\pi_n : S_\infty \to S_n$, $n \geq 0$ be the canonical projections. Then $S_\infty$ is a locally compact, $2^{nd}$-countable space and has a unique RSL-structure making each $\pi_n$ into a lamination morphism.* □

We have also the following facts.

**Lemma VIII.5.** *Let $X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} \cdots \to X_n \xrightarrow{g_n} \cdots$ be a direct system where each $X_n$ is a Riemann surface lamination and each $g_n$ is an open, injective lamination morphism. Then the direct limit space $X^\infty$ has a unique RSL-structure making the canonical maps $\rho_n : X_n \to X^\infty$, $n \geq 0$, as well as the direct limit map $g^\infty : X^\infty \to X^\infty$, into open, injective lamination morphisms.* □

**Lemma VIII.6.** *Let $X_n = X$ and $g_n = g$ for each $n \geq 0$ in Lemma VIII.5, where $X$ is locally compact and first countable and $g$ acts discontinuously on $X$. Then the orbit space $X^\infty/\langle g^\infty \rangle$ has a unique RSL-structure for which the canonical projection $X^\infty \to X^\infty/\langle g^\infty \rangle$ is a lamination morphism. Moreover, if $\phi$ has a relatively compact fundamental domain in $X$ then $X^\infty/\langle g^\infty \rangle$ is a compact space.* □

Following Sullivan, we say that a Riemann surface lamination is *hyperbolic* if each of its leaves is covered by the disk.

**Theorem VIII.7.** *For each Cantor repeller $(U,\phi,V)$ there exists a compact, hyperbolic Riemann surface lamination $L(U,\phi,V)$ such that*

*(a) If $(U,\phi,V)$ and $(\widetilde{U},\widetilde{\phi},\widetilde{V})$ represent the same germ, then we have a lamination isomorphism $L(U,\phi,V) \cong L(\widetilde{U},\widetilde{\phi},\widetilde{V})$;*

*(b) Every qc-conjugacy $(U_1,\phi_1,V_1) \sim (U_2,\phi_2,V_2)$ induces a qc-isomorphism of laminations $L(U_1,\phi_1,V_1) \cong L(U_2,\phi_2,V_2)$.*

*Proof.* Let $V_0 = V \setminus K_\phi$. Consider the inverse system

$$\cdots \xrightarrow{\phi} \phi^{-n}V_0 \xrightarrow{\phi} \phi^{-(n-1)}V_0 \to \cdots \to \phi^{-1}V_0 \xrightarrow{\phi} V_0 \tag{16}$$

and its sub-system

$$\cdots \xrightarrow{\phi} \phi^{-(n+1)}V_0 \xrightarrow{\phi} \phi^{-n}V_0 \to \cdots \to \phi^{-2}V_0 \xrightarrow{\phi} \phi^{-1}V_0 \tag{17}$$

and let $V_\infty$, $V'_\infty$ be their inverse limit spaces. Both are Riemann surface laminations by Lemma VIII.4, and since (17) is cofinal in (16), we have a lamination isomorphism $\Phi : V_\infty \to V'_\infty$. The inclusions $\phi^{-(n+1)}V_0 \subseteq \phi^{-n}V_0$ yield an open, injective lamination morphism $\Psi : V'_\infty \hookrightarrow V_\infty$, so $g = \Psi \circ \Phi$ is a map with these properties also. Since the multivalued map $\phi^{-1}$ acts discontinuously on $V_0$, $g$ acts discontinuously on $V_\infty$. Applying Lemmas VIII.5 and VIII.6 to the direct system $V_\infty \xrightarrow{g} V_\infty \xrightarrow{g} \cdots \to V_\infty \xrightarrow{g} \cdots$, we get the desired $L(U, \phi, V)$ as the orbit space of the direct limit map $g^\infty$ acting on the direct limit space $V^\infty$. This lamination is compact because $V_0 \setminus \phi^{-1}V_0$ is a relatively compact fundamental domain for the action of $\phi^{-1}$ on $V_0$. All leaves are hyperbolic Riemann surfaces, cf. geometric description of the dyadic pair-of-pants lamination in [S$_5$, Example 3]. Parts (a) and (b) are straightforward. ◻

## IX. Renormalization convergence

We are now in a position to use the Teichmüller theory of Riemann surface laminations, introduced by Sullivan in the appendix to [S$_1$], in order to prove that renormalization contracts the Julia-Teichmüller distance. For details on the unproved assertions in this section, see the book by de Melo and van Strien [MS, Ch. VI].

In [S$_1$], Sullivan defined Beltrami vectors and quadratic differentials on a compact Riemann surface lamination $X$ as cross-sections of suitable tensor bundles over $X$. Thus, a Beltrami vector $\mu$ locally on each flow-box chart $(D_\alpha \times T_\alpha, \psi_\alpha)$ is a Borel measurable function $\mu_\alpha : D_\alpha \times T_\alpha \to \mathbf{C}$ satisfying (a) $\mu_\alpha(\cdot, t) \in L^\infty(D_\alpha)$ for each $t \in T_\alpha$, and the map $t \mapsto \mu_\alpha(\cdot, t)$ is continuous if we provide $L^\infty(D_\alpha)$ with the weak topology; (b) if $\psi_{\alpha\beta}$ denotes the chart transition $\psi_\beta \circ \psi_\alpha^{-1}$ and we write $\psi_{\alpha\beta} = (\psi^z_{\alpha\beta}, \psi^t_{\alpha\beta})$ then we have

$$\mu_\alpha = \frac{\overline{\partial \psi^z_{\alpha\beta}}}{\partial \psi^z_{\alpha\beta}} \mu_\beta \circ \psi_{\alpha\beta} \ .$$

A Beltrami coefficient on $X$ is an essentially bounded Beltrami vector with essential norm less than one. Sullivan also defined quadratic differentials on $X$ as the corresponding dual objects. More precisely, a quadratic differential $\varphi$ on $X$ is an assignment of a $\sigma$-finite measure class $[m_\alpha]$ to the transversal $T_\alpha$ of each flow-box chart satisfying (a) the transversal components $\psi^t_{\alpha\beta}$ of chart transitions are absolutely continuous as maps $(T_\alpha, [m_\alpha]) \to (T_\beta, [m_\beta])$; (b) for each choice of representative $m_\alpha \in [m_\alpha]$ there exists a measurable function $\varphi_\alpha : D_\alpha \times T_\alpha \to \mathbf{C}$ such that, on overlappings

$$\varphi_\alpha = \varphi_\beta \circ \psi_{\alpha\beta} \left[ \frac{\partial \psi^z_{\alpha\beta}}{\partial z} \right]^2 \mathrm{Jac}(\psi^t_{\alpha\beta}) \ , \tag{18}$$

where the Jacobian is measured with respect to the measures $m_\alpha$ and $m_\beta$; (c) each $\varphi_\alpha$ is integrable with respect to the product measure $dz\, d\overline{z}\, dm_\alpha$ on $D_\alpha \times T_\alpha$. It follows from this definition that there exists a well-defined measure $d|\varphi|$ associated to a quadratic differential on $X$. Its expression on a given chart is $|\varphi_\alpha|\, dz\, d\overline{z}\, dm_\alpha$ for each choice of measure $m_\alpha$, and from (18) any two choices differ by the Jacobian of the identity with respect to both transversal measures, i.e. by their Radon-Nikodym derivative. If the total mass

$$|\varphi| = \int_X d|\varphi|$$

is finite, we say that $\varphi$ is an *integrable* quadratic differential, and $|\varphi|$ is the *norm* of $\varphi$. A quadratic differential is said to be *holomorphic* if it is holomorphic on almost all leaves with respect to the transversal measure class that it defines. The *Teichmüller norm* of a Beltrami vector $\omega$ is

$$|\omega|_T = \sup |\int_X \omega\varphi| \ ,$$

where the supremum is taken over all integrable holomorphic quadratic differentials of norm $|\varphi| = 1$. These definitions are set-up so that the natural pairing in the second member is well-defined. Given $\varepsilon \geq 0$, a Beltrami vector $\omega$ on $X$ is called *$\varepsilon$-extremal* if $|\omega|_\infty \leq (1+\varepsilon)|\omega|_T$.

Sullivan used an elegant ergodic argument to prove a *generalized Grötzsch inequality* relating the dilatation of qc-isomorphisms on $X$ which are leafwise isotopic to the identity and the *metric up to a multiple* given by an integrable quadratic differential of norm one on $X$ (cf. [MS, Ch.VI, §7] for details). He then used this inequality to prove the almost geodesic principle below. Let us say that a Beltrami coefficient $\mu$ on $X$ is *dynamical* if, integrating $\mu$ via the MRMT along the leaves of $X$, we get a transversally continuous map $X \to X$ that is qc on leaves, i.e. a lamination qc-morphism. If $\mu$ is dynamical, let $c(\mu)$ be the RSL-structure on $X$ given by $\mu$. A *dynamical* Beltrami vector $\omega$ on $(X, c(\mu))$ is one for which there exists a (unique) path of dynamical Beltrami coefficients $\mu_t$, $t \geq 0$, with $\mu_0 = \mu$ and tangent to $\omega$ at $t = 0$, such that for all $t$ the smallest maximal dilatation of a qc-morphism between $c(\mu)$ and $c(\mu_t)$ is equal to $e^{2t}$. We write $c_t(\omega) = c(\mu_t)$ and call the path $\mu_t$ the *Beltrami ray* of $\omega$ at $\mu$.

*Example.* Let $(U, \phi, V)$ be a Cantor repeller and let $\mathcal{L}_\phi$ be the lamination of Theorem VIII.7. Given any $\phi$-invariant Beltrami vector $\widetilde{\omega}$ on the Riemann surface $V_0 = V \setminus K_\phi$, we pull it back via the natural projection to the inverse limit space $V_\infty$ and then project it down to a $g^\infty$-invariant Beltrami vector on the direct limit space $V^\infty$, thus getting a dynamical Beltrami vector $\omega$ on $\mathcal{L}_\phi = V^\infty/\langle g^\infty\rangle$. By [$S_5$], all dynamical Beltrami vectors on $\mathcal{L}_\phi$ arise in this way (they are precisely the transversally locally constant ones, in Sullivan's terminology). In particular, the Teichmüller norm of $\omega$ can be computed by pairing $\widetilde{\omega}$ on a fundamental domain for $\phi$, such as $V \setminus U$, with holomorphic quadratic differentials there.

Now the *almost geodesic principle* can be stated as follows.

**Theorem IX.1.** *Given $\varepsilon$, $L > 0$, there exists $\delta = \delta(\varepsilon, L) > 0$ such that the following holds. If $\mu$ is a dynamical Beltrami coefficient on a compact hyperbolic Riemann surface lamination $X$, $\omega$ is a $\delta$-extremal dynamical Beltrami vector on $X$ at $\mu$ and $\{\psi_t\}_{0\leq t\leq 1}$ is a leafwise qc-isotopy between $(X, c(\mu))$ and $(X, c_\ell(\omega))$, then we have $L \leq K(1+\varepsilon)$, where $K$ is the maximal dilatation of $\psi_1$ and $\ell = \frac{1}{2}\log L$. Moreover, $\delta \to 0$ as $\varepsilon \to 0$.* □

We need a converse to this theorem within the realm of the example above. Consider the germ $\langle\Gamma\rangle$ of a holomorphic commuting pair and let $\langle\phi\rangle$ be the germ of the corresponding Cantor repeller constructed in Theorem VIII.1. Let $(\Gamma,\mathcal{O})$ be a representative of $\langle\Gamma\rangle$ and let $\mu$ be a Beltrami differential with domain $\mathcal{O}$. We call $\mu$ *admissible* for $\langle\Gamma\rangle$ if $\mu$ is $\Gamma$-invariant and vanishes a.e. on the limit set $\mathcal{K}_\Gamma$. If $\mu$ defined on $\mathcal{O}$ is an admissible Beltrami coefficient for $\langle\Gamma\rangle$, let $h_\mu$ be a suitably normalized qc-mapping with dilatation $\mu$ and let $\Gamma(\mathcal{O},\mu)$ be the holomorphic dynamical system generated by the mappings

$$h_\mu\circ(\gamma|(\mathcal{O}_\gamma\cap\mathcal{O}))\circ h_\mu^{-1}\ ,$$

where $\gamma=\xi,\eta,\nu$. If $\sigma$ is another admissible Beltrami coefficient for $\langle\Gamma\rangle$ defined on $\mathcal{O}'$, we say that $\mu$ and $\sigma$ are *equivalent* if $\Gamma(\mathcal{O}\cap\mathcal{O}',\mu)$ is analytically conjugate to $\Gamma(\mathcal{O}\cap\mathcal{O}',\sigma)$. We then let $|\mu|_{JT}=\inf|\sigma|_\infty$, where $\sigma$ runs through all admissible Beltrami coefficients for $\langle\Gamma\rangle$ that are equivalent to $\mu$ (caution: this is a non-linear norm). We also say that $\mu$ is $\varepsilon$*-efficient* if $|\mu|_\infty\le(1+\varepsilon)|\mu|_{JT}$. Similarly, admissible Beltrami coefficients or vectors for $\langle\phi\rangle$ are those defined in the domain of a representative of the germ of $\phi$ which are $\phi$-invariant and vanish a.e. on the limit set $K_\phi$. The definitions we have just given can be repeated here. We denote by $|\cdot|_G$ the non-linear norm of admissible coefficients for $\langle\phi\rangle$ that corresponds to $|\cdot|_{JT}$. Observe that Theorem VIII.1 sets up a correspondence between admissible Beltrami differentials for $\langle\Gamma\rangle$ and admissible Beltrami coefficients for $\langle\phi\rangle$. Admissible objects for $\langle\phi\rangle$ are precisely those that lift to dynamical objects in the lamination $\mathcal{L}_\phi$. Observe that a globally $\Gamma$-invariant $\widetilde{\mu}$ in the sense of section V is admissible, and the definitions have been arranged so that

$$d_{JT}(\langle\Gamma\rangle,\langle\Gamma^{\widetilde{\mu}}\rangle)\ =\ \frac{1}{2}\,\log\frac{1+|\widetilde{\mu}|_{JT}}{1-|\widetilde{\mu}|_{JT}}\ .$$

Likewise, if $\mu$ is the admissible coefficient for $\langle\phi\rangle$ corresponding to $\widetilde{\mu}$ and $\phi^\mu$ is the corresponding Cantor repeller, we have (cf. section VIII)

$$d_G(\langle\phi\rangle,\langle\phi^\mu\rangle)\ =\ \frac{1}{2}\,\log\frac{1+|\mu|_G}{1-|\mu|_G}\ . \tag{19}$$

Now Sullivan's *coiling lemma* can be stated as follows.

**Theorem IX.2.** *Given $\varepsilon'>0$ and $0<d\le 1$, there exists $\theta=\theta(\varepsilon',d)>0$ such that, if $\omega$ is a an admissible Beltrami vector for $\langle\phi\rangle$ and the admissible Beltrami coefficient $\mu_s=s\omega$ is $\theta$-efficient for some $0<s<d|\omega|_\infty^{-1}$, then $\omega$ is $\varepsilon'$-extremal.* □

Renormalization without rescaling acts on admissible Beltrami vectors in a natural way. Thus, if $\widetilde{\mu}$ is admissible for $\langle\Gamma\rangle$ and defined on $\mathcal{O}$, let its $n$-th renormalization $\widetilde{\mu}_n$ be the restriction of $\widetilde{\mu}$ to $\mathcal{U}_n\cap\mathcal{O}$, where $\mathcal{U}_n$ is the inner domain of the bowtie of $\Gamma_n=\mathcal{R}^n(\Gamma)$. Then let $\mu_n$ be an admissible Beltrami coefficient for $\langle\phi_n\rangle$ that corresponds to $\widetilde{\mu}_n$, where $\langle\phi\rangle$ is the Cantor repeller germ associated to $\Gamma_n$. By Corollary VII.3, if $\Gamma$ is of bounded combinatorial type then for every $n$ sufficiently large $\mathcal{U}_n$ is contained in $\mathcal{O}$. Therefore the holomorphic commuting pair $\Gamma_n^{\widetilde{\mu}_n}$ is well-defined (cf. section V), and we have $\Gamma_{n+1}^{\widetilde{\mu}_{n+1}}=\mathcal{R}\,\Gamma_n^{\widetilde{\mu}_n}$, for all sufficiently large $n$.

We have at last the main *renormalization contraction theorem* that follows.

**Theorem IX.3.** *Let $\langle\Gamma\rangle$ and $\langle\Gamma'\rangle$ be germs of holomorphic commuting pairs with the same rotation number of bounded combinatorial type and the same height. Then the distance $d_{JT}(\mathcal{R}^n\langle\Gamma\rangle, \mathcal{R}^n\langle\Gamma'\rangle)$ converges to zero as $n \to \infty$.*

*Proof.* We argue as in the proof of [MS, Ch. VI, Thm. 8.3]. By Proposition VIII.3, it suffices to show that renormalization contracts the germ distance $d_G$ between the germs of corresponding repellers $\langle\phi\rangle$, $\langle\phi'\rangle$. Let $\widetilde{\mu}$ be the Beltrami coefficient of a qc-conjugacy $(\Gamma, \mathcal{O}) \to (\Gamma', \mathcal{O}')$ which is $\varepsilon_1$-efficient for some $\varepsilon_1$ to be specified below. Let $\mu$ be the corresponding admissible Beltrami coefficient for $\langle\phi\rangle$. Then $\mu$ is $\varepsilon_1$-efficient also. Let $\omega = \mu/|\mu|_\infty$ and lift $\omega$ to a dynamical Beltrami vector $\widehat{\omega}$ on the lamination $\mathcal{L}_\phi$ with its standard structure. Note that $\widehat{\omega}$ is $\varepsilon_1$-extremal. For each $t > 0$, let $\mu(t) = (\tanh t)\omega$ and note that the lifted path $\widehat{\mu}(t)$ is the Beltrami ray of $\widehat{\omega}$ at zero.

Let $B$ be the constant in the complex bounds (Corollary VII.3) and fix a constant $M$ so large that $M > B^{-1}\log(1 + 2e^B)$. Take $L = 2e^{MB}$, $\ell = \frac{1}{2}\log L$ and $0 < \varepsilon < 1$ to be specified later, and then choose $\varepsilon_1 = \delta(\varepsilon, L)$ using the almost geodesic principle for $\widehat{\omega}$. We get $K(1+\varepsilon) \geq L$, where $K$ is the smallest dilatation of all qc-morphisms leafwise isotopic to the identity in $\mathcal{L}_\phi$ between the standard structure and $c(\widehat{\mu}(\ell))$. Therefore

$$K \;=\; \frac{1 + |\mu(\ell)|_G}{1 - |\mu(\ell)|_G} \;\geq\; \frac{2e^{MB}}{1+\varepsilon} \;\geq\; e^{MB} \;. \tag{20}$$

But if $n$ is sufficiently large, then by Corollary VII.3 we have

$$d_{JT}(\langle\Gamma_n\rangle, \langle\Gamma_n^{\widetilde{\mu}_n(\ell)}\rangle) \leq B$$

and therefore

$$|\mu_n(\ell)|_G \;\leq\; \frac{e^B - 1}{e^B + 1} \;. \tag{21}$$

Combining (20) and (21), we get

$$|\mu(\ell)|_G \;\geq\; \frac{e^{MB} - 1}{e^{MB} + 1} \;\geq\; \frac{e^B}{e^B - 1}\,|\mu_n(\ell)|_G \;, \tag{22}$$

by our choice of $M$. Now let $k > 1$ be such that $k(1 - e^{-B}) \leq 1 - e^{-2B}$; we can choose $k$ as close to 1 as we like. Then *either* $|\mu|_\infty > k|\mu_n|_\infty$, in which case

$$|\mu_n|_G \;\leq\; \frac{1+\varepsilon_1}{k}\,|\mu|_G \;, \tag{23}$$

*or* $|\mu|_\infty \leq k|\mu_n|_\infty$, in which case (22) gives us $|\mu_n(\ell)|_G \leq (1 - e^{-2B})|\mu_n(\ell)|_\infty$. In this last case, applying the coiling lemma to the admissible Beltrami vector $\mu_n(\ell)$ with $\varepsilon' = e^{-2B}$ and $d = 1$, we see that there exists $0 < \theta < 1$ depending only on $B$ such that, for $0 < \tanh t < 1$, the admissible Beltrami coefficients $\mu_n(t)$ cannot be $\theta$-efficient. In particular, taking $t = \operatorname{arctanh}|\mu|_\infty$, we have

$$|\mu_n|_G \;\leq\; (1-\theta)|\mu_n|_\infty \;\leq\; k(1-\theta)(1+\varepsilon_1)\,|\mu|_G \;. \tag{24}$$

Now choose $k$ first so that $k(1-\theta) < 1$ and then $\varepsilon$ so small that

$$\lambda_1 = \max\{k^{-1}(1+\varepsilon_1),\, k(1-\theta)(1+\varepsilon_1)\} < 1 \ .$$

Then in both (23) and (24) we have $|\mu_n|_G \leq \lambda_1 |\mu|_G$. Using (19), we deduce that

$$d_G(\langle\phi_n\rangle, \langle\phi_n^\mu\rangle) \ \leq \ \lambda_2\, d_G(\langle\phi\rangle, \langle\phi^\mu\rangle) \ ,$$

for some $0 < \lambda_2 < 1$, and this is the desired contraction. □

This contraction of the Julia-Teichmüller distance results in stronger forms of renormalization convergence. Following [MS, Ch. VI, §8], we define strong convergence as follows.

*Definition 6.* A sequence $g_n : W \to \mathbf{C}$, where $W \subseteq \mathbf{C}$ is compact, *converges strongly* to $g : W \to \mathbf{C}$ if there exist an open neighborhood $\mathcal{O}$ of $W$ and holomorphic extensions $G : \mathcal{O} \to \mathbf{C}$ of $g$ and $G_n : \mathcal{O} \to \mathbf{C}$ of $g_n$, for all but finitely many $n$, such that $G_n$ converges to $G$ uniformly in $\mathcal{O}$.

Notice that if $W$ is an interval on the line, say, then strong convergence of $g_n$ to $g$ in $W$ implies $C^k$-convergence for all $k < \infty$. Now let $\mathcal{B}^\omega(N)$ (resp. $\mathcal{B}^3(N)$) be the class of *normalized* real-analytic (resp. $C^3$-smooth) critical commuting pairs with irrational rotation number of combinatorial type bounded by $N$. We say that a sequence $\zeta_n$ in $\mathcal{B}^\omega(N)$ converges strongly to $\zeta$ in $\mathcal{B}^\omega(N)$ if both $\eta_n - \eta$ and $\xi_n - \xi$ converge strongly to zero. This last condition makes sense because the first implies that $\eta_n(0) \to \eta(0)$ and therefore any fixed neighborhood of $[\eta(0), 0]$ contains $[\eta_n(0), 0]$ for all sufficiently large $n$. Strong convergence of a sequence of holomorphic commuting pairs, or of a sequence of Cantor repellers, can be similarly defined. We need the following statement, which is Lemma 8.4 of [MS, Ch. VI, §8].

**Lemma IX.4.** *Given $\varepsilon > 0$ and $R_1 > 1$, there exist $\delta > 0$ and $R_2 > R_1$ with the following property. If $h$ is a $(1+\delta)$-qc homeo that fixes $0$ and $1$ and whose domain and range contain the disk of radius $R_2$ about zero, then $|h(z) - z| < \varepsilon$ for all $|z| < R_1$.* □

Now, given $\zeta \in \mathcal{B}^\omega(N)$, let $\zeta_n = (\xi_n, \eta_n) = \mathcal{R}^n\zeta$ be the *normalized* renormalizations of $\zeta$.

**Theorem IX.5.** *Let $\zeta, \zeta' \in \mathcal{B}^\omega(N)$ be critical commuting pairs of the same combinatorial type which either belong to some Epstein class or extend to holomorphic commuting pairs. Then $\xi_n - \xi_n'$ and $\eta_n - \eta_n'$ converge strongly to zero.*

*Proof.* By Corollary VII.3, if $n$ is sufficiently large then $\zeta_n$ and $\zeta_n'$ extend to normalized holomorphic commuting pairs $\Gamma_n$ and $\Gamma_n'$ with conformal types bounded from below. Let $\mathcal{U}_n$ and $\mathcal{U}_n'$ be the inner domains of the bow-ties of $\Gamma_n$ and $\Gamma_n'$. Also, for each $k > 0$, let $\mathcal{U}_{n,k} \subseteq \mathcal{U}_n$ be the linear copy of $\mathcal{U}_{n+k}$ corresponding to the $k$-th renormalization of $\Gamma_n$ without rescaling, and let $\mathcal{U}_{n,k}' \subseteq \mathcal{U}_n'$ be similarly defined. We have $\mathcal{U}_{n,k+1} \subseteq \mathcal{U}_{n,k}$ for all $k$, and $\mathrm{mod}\,(\mathcal{U}_n \setminus \mathcal{U}_{n,k}) \to \infty$ as $k \to \infty$, by the complex bounds. Likewise, $\mathrm{mod}\,(\mathcal{U}_n' \setminus \mathcal{U}_{n,k}') \to \infty$ as $k \to \infty$. Given $\varepsilon > 0$ and $R_1 > 1$ so large that the disk of radius $R_1$ about the origin contains the small dynamical intervals of $\Gamma_n$ and $\Gamma_n'$ for all $n$, take $\delta$ and $R_2$

as in Lemma IX.4. By Theorem IX.3, there exist $m > 0$ and a $(1+\delta)$-quasiconformal conjugacy $h : \mathcal{U}_m \to \mathcal{U}'_m$ between $\Gamma_m$ and $\Gamma'_m$. Note that the restriction of $h$ to $\mathcal{U}_{m,k}$ is a conjugacy between the $k$-th renormalizations without rescaling of $\Gamma_m$ and $\Gamma'_m$. Let $\Lambda_k$ and $\Lambda'_k$ be the linear maps that perform such rescaling, so that $\Lambda_k(\mathcal{U}_{m,k}) = \mathcal{U}_{m+k}$. Writing $H_k = \Lambda'_k \circ h \circ \Lambda_k^{-1}$, we have $H_k(0) = 0$ and $H_k(1) = 1$, each $H_k$ is $(1+\delta)$-qc, and also

$$\begin{cases} \xi_{m+k} = H_k^{-1} \circ \xi'_{m+k} \circ H_k \\ \\ \eta_{m+k} = H_k^{-1} \circ \eta'_{m+k} \circ H_k \end{cases} . \tag{25}$$

Moreover, if $\mathcal{W}_k = \Lambda_k(\mathcal{U}_m)$ and $\mathcal{W}'_k = \Lambda'_k(\mathcal{U}'_m)$, then

$$\operatorname{mod}(\mathcal{W}_k \setminus [0,1]) \;>\; \operatorname{mod}(\mathcal{W}_k \setminus \mathcal{U}_{m+k}) \;=\; \operatorname{mod}(\mathcal{U}_m \setminus \mathcal{U}_{m,k}) \to \infty$$

as $k \to \infty$, and similarly for $\mathcal{W}'_k$. Therefore there exists $k_0$ such that $\mathcal{W}_k$ and $\mathcal{W}'_k$ contain the disk of radius $R_2$ about the origin for all $k \geq k_0$. By Lemma IX.4, we have $|H_k(z) - z| < \varepsilon$ and $|H_k^{-1}(z) - z| < \varepsilon$ for all $|z| < R_1$, for all $k \geq k_0$. Going back to (25), we get strong convergence as claimed. □

**Lemma IX.6.** *Let $(U_n, \phi_n, V_n)$ and $(U'_n, \phi'_n, V'_n)$, $n \geq 0$, be two sequences of Cantor repellers of the same topological type, and suppose they converge strongly to $(U, \phi, V)$ and $(U', \phi', V')$, respectively. If $d_G(\langle\phi_n\rangle, \langle\phi'_n\rangle) = 0$ for all $n$, then $d_G(\langle\phi\rangle, \langle\phi'\rangle) = 0$ also.*

*Proof.* If $d_G(\langle\phi_n\rangle, \langle\phi'_n\rangle) = 0$ then, by Theorem VIII.2, there exist neighborhoods $\mathcal{O}_n \supseteq K_{\phi_n}$ and $\mathcal{O}'_n \supseteq K_{\phi'_n}$, and an analytic homeo $h_n : \mathcal{O}_n \to \mathcal{O}'_n$ conjugating $\phi_n$ to $\phi'_n$. Take $k > 0$ large enough (depending on $n$) so that

$$\Delta_n \;=\; \phi_n^{-k}(V_n \setminus U_n) \subseteq \mathcal{O}_n \ .$$

Then $\Delta_n$ and $\Delta'_n \;=\; h_n(\Delta_n) \subseteq \mathcal{O}'_n$ are fundamental domains for $\phi_n$ and $\phi'_n$, respectively. Now, let $\varepsilon > 0$. Since $\phi_n$ converges strongly to $\phi$, we can find a fundamental domain $D_n$ for $\phi$ and a homeomorphism $\psi_n : D_n \to \Delta_n$ very close to the identity which conjugates $\phi$ to $\phi_n$ on corresponding boundaries and is $(1+\varepsilon)$-quasiconformal, provided $n$ is sufficiently large. Similarly, we can find a fundamental domain $D'_n$ for $\phi'$ and a $(1+\varepsilon)$-quasiconformal map $\psi'_n : D'_n \to \Delta'_n$ conjugating $\phi'$ to $\phi'_n$, possibly by making $n$ larger still. This gives us a conjugacy $(\psi'_n)^{-1} \circ h_n \circ \psi_n : D_n \to D'_n$ between the fundamental domains of $\phi$ and $\phi'$ which is $(1+\varepsilon)^2$-quasiconformal. By a simple pull-back argument, this map extends to a qc-conjugacy with the same dilatation between $\phi$ and $\phi'$ in full-neighborhoods of their limit sets. Therefore $d_G(\langle\phi\rangle, \langle\phi'\rangle) \;=\; 0$ as claimed. □

**Lemma IX.7.** *Let $\Gamma_n$ and $\Gamma'_n$, $n \geq 0$, be two sequences of holomorphic commuting pairs, and suppose they converge strongly to $\Gamma$ and $\Gamma'$, respectively. If $d_{JT}(\langle\Gamma_n\rangle, \langle\Gamma'_n\rangle) \;=\; 0$ for all $n$, then there exists $k \geq 0$ such that $d_{JT}(\langle\mathcal{R}^k\Gamma\rangle, \langle\mathcal{R}^k\Gamma'\rangle) \;=\; 0$ also.*

*Proof.* This follows at once from Lemma IX.6 and Theorems VIII.1 and VIII.2. □

**Theorem IX.8.** *For each $n \in \mathbb{Z}$, let $\zeta_n$ and $\zeta'_n$ be normalized critical commuting pairs with the same bounded combinatorial type and suppose they extend to holomorphic commuting pairs $\Gamma(\zeta_n)$ and $\Gamma(\zeta'_n)$, respectively, whose conformal types are uniformly bounded from below. If $\Gamma(\zeta_{n+1}) = \mathcal{R}\Gamma(\zeta_n)$ and $\Gamma(\zeta'_{n+1}) = \mathcal{R}\Gamma(\zeta'_n)$ for all $n$, then $\zeta_0 = \zeta'_0$.*

*Proof.* The proof of [MS, Ch. VI, Lemma 8.3] can be reproduced here almost verbatim. Lemma IX.7 replaces the argument on continuity of the Douady-Hubbard external class used in that proof. □

Finally, we give a characterization of the *attractor* of the renormalization operator for critical commuting pairs. Theorems A and B of the Introduction are straightforward consequences of this last theorem, which is the exact analogue of [MS, Ch. VI, Theorem 1.1]. We denote by $W^s(\zeta)$ the *stable set* of $\zeta \in \mathcal{B}^3(N)$, i.e. the set of all $\zeta' \in \mathcal{B}^3(N)$ whose successive renormalizations are $C^0$-asymptotic to those of $\zeta$.

**Theorem IX.9.** *Let $N$ be a positive integer. There exists a renormalization-invariant, strongly compact set $\mathcal{A} \subseteq \mathcal{B}^\omega(N)$ such that*

- *(a) If $\zeta \in \mathcal{B}^3(N)$, then the $C^3$-distance between $\mathcal{R}^n(\zeta)$ and $\mathcal{A}$ converges to zero as $n \to \infty$;*
- *(b) There exist $a > 0$ and $\tau > 0$ such that $\mathcal{A} \subseteq \mathcal{E}_a$ and each element of $\mathcal{A}$ extends to a holomorphic commuting pair with conformal type bounded by $\tau$;*
- *(c) The restriction of $\mathcal{R}$ to $\mathcal{A}$ is a homeomorphism topologically conjugate to the two-sided full-shift on $N$ symbols;*
- *(d) If $\zeta \in \mathcal{A}$ then $W^s(\zeta)$ is the set of critical commuting pairs $\zeta'$ such that $\mathcal{R}^m(\zeta')$ and $\mathcal{R}^m(\zeta)$ have the same bounded combinatorial type for some $m > 0$.*

*Moreover, there exists a strongly compact set $\mathcal{C} \supseteq \mathcal{A}$ such that (i) for any real-analytic $\zeta$ of combinatorial type bounded by $N$ in some Epstein class, there exists $n_0(\zeta) > 0$ such that $\mathcal{R}^n(\zeta) \in \mathcal{C}$ for all $n \geq n_0(\zeta)$, and (ii) if $\zeta, \zeta' \in \mathcal{C}$ have the same bounded combinatorial type then $\xi_n - \xi'_n$ and $\eta_n - \eta'_n$ converge strongly to zero.*

*Proof.* Define $\mathcal{A}$ as the set of all $C^0$-limits of successive renormalizations of critical commuting pairs in $\mathcal{B}^3(N)$. Then (a) follows from Theorem I.4 which, combined with Theorem VII.2 and Corollary VII.3, proves (b) also. Proceeding as in [MS, Ch. VI, Thm. 8.4], one shows that for each bi-infinite sequence $(\ldots, r_{-1}, r_0, r_1, \ldots, r_n, \ldots)$ with $r_n \in \{0, 1, \ldots, N\}$, there exists a bi-infinite sequence $(\ldots, \zeta_{-1}, \zeta_0, \zeta_1, \ldots, \zeta_n, \ldots)$ of critical commuting pairs $\zeta_n \in \mathcal{A}$ such that $\rho(\zeta_n) = [r_n + 1, r_{n+1}, \ldots]$ and $\zeta_{n+1} = \mathcal{R}\zeta_n$ for all $n$, and such sequence is unique by Theorem IX.8. Therefore the map

$$(\ldots, r_{-1}, r_0, r_1, \ldots, r_n, \ldots) \mapsto \zeta_0 \in \mathcal{A}$$

is one-to-one and onto and conjugates the full-shift on $N$ symbols to the restriction of renormalization to $\mathcal{A}$. If $\mathcal{A}$ is given the strong topology of Definition 6, this map is continuous, hence a homeomorphism, and this proves (c). In particular, $\mathcal{A}$ is strongly compact. Finally, let $\mathcal{C} \subseteq \mathcal{B}^\omega(N)$ be the set of critical commuting pairs that can be extended to holomorphic commuting pairs with conformal type bounded from below by $\tau$. Then $\mathcal{C}$ is strongly compact, (i) is Theorem VII.1, and (ii) is Theorem IX.5. □

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