

Hofer's L^∞ -geometry: energy and stability of Hamiltonian flows, part I

François Lalonde*
Université du Québec à Montréal
(flalonde @math.uqam.ca)

Dusa McDuff†
State University of New York at Stony Brook
(dusa @math.sunysb.edu)

Abstract

Consider the group $\text{Ham}^c(M)$ of compactly supported Hamiltonian symplectomorphisms of the symplectic manifold (M, ω) with the Hofer L^∞ -norm. A path in $\text{Ham}^c(M)$ will be called a geodesic if all sufficiently short pieces of it are local minima for the Hofer length functional \mathcal{L} . In this paper, we give a necessary condition for a path γ to be a geodesic. We also develop a necessary condition for a geodesic to be stable, that is, a local minimum for \mathcal{L} . This condition is related to the existence of periodic orbits for the linearization of the path, and so extends Ustilovsky's work on the second variation formula. Using it, we construct a symplectomorphism of S^2 which cannot be reached from the identity by a shortest path. In later papers in this series, we will use holomorphic methods to prove the sufficiency of the condition given here for the characterisation of geodesics as well as the sufficiency of the condition for the stability of geodesics. We will also investigate conditions under which geodesics are absolutely length-minimizing.

1 Introduction

Let (M, ω) be a symplectic manifold without boundary, and let $\text{Ham}^c(M)$ be the group of all compactly supported Hamiltonian symplectomorphisms of (M, ω) . This is an infinite dimensional Lie group, whose tangent spaces equal the space of compactly supported Hamiltonian vector fields on M , or, equivalently, the space

$$C_0^\infty(M; \mathbf{R})/\{\text{constants}\}$$

of compactly supported functions on M , modulo constants. In [6], Hofer considered the Finsler pseudo-metric arising from the norm

$$\|H\| = \text{Totvar } H = \sup_{x \in M} H(x) - \inf_{x \in M} H(x)$$

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on this Lie algebra. He assigned to each C^∞ -path $\{\phi_t\}_{t \in [a,b]}$ in $\text{Ham}^c(M)$ with $\phi_0 = \mathbb{1}$ the length

$$\mathcal{L}(\phi_t) = \int_a^b \text{Totvar } H_t dt,$$

where $H_t \in C^\infty(M; \mathbf{R})$ is its generating Hamiltonian.¹ Further, he defined the pseudo-norm $\|\phi\|$ to be the infimum of $\mathcal{L}(\phi_t)$ over all C^∞ paths $\{\phi_t\}_{t \in [0,1]}$ from $\mathbb{1}$ to ϕ . (This norm is often called the **energy** of ϕ .) Setting the distance $\rho(\psi, \phi)$ between two arbitrary points equal to $\|\phi \circ \psi^{-1}\|$, he obtained a bi-invariant pseudo-metric ρ on $\text{Ham}^c(M)$.

Hofer showed that ρ is indeed a non-degenerate metric when M is Euclidean space \mathbf{R}^{2n} with its standard symplectic structure. In addition, he showed that the flow $\{\phi_t^H\}_{t \geq 0}$ of an autonomous Hamiltonian H on \mathbf{R}^{2n} is a geodesic with respect to this norm, in the sense that all sufficiently short pieces $\{\phi_t^H\}_{s-\varepsilon \leq t \leq s+\varepsilon}$ minimize length. In fact, the path $\{\phi_t^H\}_{t \in [a,b]}$ minimizes length provided that none of the symplectomorphisms $\phi_t^H \circ (\phi_a^H)^{-1}$, $t \in [a, b]$, have non-trivial fixed points. An appropriate version of this result was recently generalised to more general flows by Siburg in [11]. Bialy and Polterovich in [2] improved that result by a careful analysis of the bifurcations of the action spectrum. These proofs use variational methods which exploit the linear structure of Euclidean space at infinity. Thus, other methods are needed in order to extend these results to more general manifolds.

In a previous paper [7], we used global embedding techniques and J -holomorphic curves to show that ρ is a non-degenerate metric for all M . In this paper and its sequels [8, 9], we will apply these and other techniques to investigate the properties of geodesics in $\text{Ham}^c(M)$ for arbitrary M , giving in particular a full characterization of geodesics and of their stability, sufficient conditions for geodesics to be absolutely length minimizing, and other related results. We define geodesics as paths which are *local*¹ minima for \mathcal{L} at each moment. In this paper we present those of our results which were inspired by a variational approach and are proved by a variety of *ad hoc* techniques. In particular, we establish various necessary conditions for a path to be a geodesic by developing several direct ways in which to reduce the length of a given path. We also construct a symplectomorphism of S^2 which cannot be reached from the identity by a shortest path. On the other hand, any result which asserts that a given path is a local or global minimum for \mathcal{L} requires one to measure some associated capacity which cannot be reduced. Our results in this direction require new versions of the non-squeezing theorem which we develop in [8, 9] using holomorphic methods. These will allow us to give conditions under which a path is length-minimizing, and to establish the sufficiency of the necessary conditions presented here for a path to be a geodesic and to be stable. These results generalize those obtained for the case $M = \mathbf{R}^{2n}$ by Bialy-Polterovich in [2] and by Siburg in [11].

1.1 Geodesics

Given points $\phi_0, \phi_1 \in \text{Ham}^c(M)$, let $\mathcal{P} = \mathcal{P}(\phi_0, \phi_1)$ be the space of all C^∞ paths $\gamma = \{\phi_t\}_{t \in [0,1]}$ from ϕ_0 to ϕ_1 with the C^∞ -topology. (Thus two paths γ and γ' are close if the associated maps $M \times [0, 1] \rightarrow M$ are C^∞ -close.) For each $\gamma \in \mathcal{P}(\phi_0, \phi_1)$ let \mathcal{P}_γ be the path-connected component of

¹Note that this norm is L^1 with respect to time t and L^∞ with respect to space. Eliashberg and Polterovich show in [5] that, although one gets an equivalent norm if one varies the norm in the t -direction, the norm becomes degenerate and essentially trivial if L^∞ is changed to L^p .

¹Throughout this paper, we use the word “local” to mean local in the path space, not local with respect to time. A property which holds locally with respect to time will be said to hold “at each moment”.

$\mathcal{P}(\phi_0, \phi_1)$ containing γ . A path $\gamma = \{\phi_t\}_{t \in [a, b]}$ is said to be **regular** if its tangent vector $\dot{\phi}_t$ is non-zero for all $t \in [a, b]$. Further, γ is said to be a **local minimum** of \mathcal{L} if it has a neighbourhood $\mathcal{N}(\gamma)$ in \mathcal{P} such that

$$\mathcal{L}(\gamma) \leq \mathcal{L}(\gamma'), \quad \text{for all } \gamma' \in \mathcal{N}(\gamma).$$

Definition 1.1 Given an interval $I \subset \mathbf{R}$, we will say that a path $\{\phi_t\}_{t \in I}$ is a **geodesic** if it is regular and if every $s \in I$ has a closed neighbourhood $\mathcal{N}(s) = [a_s, b_s]$ in I such that the path $\{\phi_{\beta(t)}\}_{t \in \mathcal{N}(s)}$ is a local minimum of \mathcal{L} , where $\beta : \mathcal{N}(s) \rightarrow [0, 1]$ is the linear reparametrization $\beta(t) = (t - a_s)/(b_s - a_s)$. Such a path will be said to be locally length-minimizing at each moment. (Thus “moments” have some duration.) A geodesic $\{\phi_t\}_{t \in [0, 1]}$ is said to be **stable** if it is a local minimum for \mathcal{L} . Note that the notion of stability depends on the given endpoints of the path, but not the definition of geodesics.

Remark 1.2 (i) We have restricted to regular paths to make it impossible for a geodesic to stop and then change direction. However, this restriction is not essential: see Remark 4.6. Of course, any regular path may be parametrized by a multiple of its arc-length without changing its length.

(ii) The above definition has the virtue that geodesics exist on all manifolds and have a simple characterization: see Theorem 1.3. One might define geodesics in a stronger sense, requiring that they be absolutely length-minimizing at each moment, instead of locally length-minimizing at each moment. Both definitions have their appeal (and they agree in ordinary Riemannian geometry). Our choice was in the end dictated by the fact that we were unable to establish that geodesics in the stronger sense exist on all M , though they do exist when $M = \mathbf{R}^{2n}$ by the work of Hofer and Bialy–Polterovich (or ours, see [8]). Another possibility would be to use a variational definition. Ustilovsky’s work [13] shows that this works very nicely if one restricts attention to paths which satisfy a certain non-degeneracy condition but, as we shall see below, it is somewhat cumbersome otherwise.

Because Hofer’s norm only takes account of the maximum and minimum values of H_t , it is not surprising that the sets on which H_t assumes these values are important. For each $t \in I$, we write

$$\begin{aligned} \text{minset } H_t &= \{x \in M : H_t(x) = \min H_t\}, \\ \text{maxset } H_t &= \{x \in M : H_t(x) = \max H_t\}. \end{aligned}$$

A point q which belongs to

$$\bigcap_t \text{minset } H_t \quad \text{or} \quad \bigcap_t \text{maxset } H_t$$

will be called a *fixed* extremum of the Hamiltonian H_t over the interval I and of the corresponding path ϕ_t .

Sometimes it is convenient to consider paths $\{\phi_t\}_{t \in [a, b]}$ which do not start at the identity. The Hamiltonian corresponding to such a path is defined by the requirement that

$$\frac{d}{dt} \phi_t(x) = X_{H_t}(\phi_t(x)) \quad \text{for all } t,$$

where X_{H_t} is the vector field such that

$$i(X_{H_t})\omega = \omega(X_{H_t}, \cdot) = dH_t.$$

Thus it coincides with the Hamiltonian which generates the path $\{\phi_t \circ \phi_a^{-1}\}$.

Our first theorem characterizes geodesics.

Theorem 1.3 *A path $\{\phi_t\}_{t \in I}$ is a geodesic if and only if its generating Hamiltonian has at least one fixed minimum and one fixed maximum at each moment. Thus, each $s \in I$ has a neighbourhood $\mathcal{N}_s \subset I$ such that the Hamiltonian which generates the path $\phi_t, t \in \mathcal{N}_s$, has at least one fixed minimum and one fixed maximum.*

We prove here that this condition is necessary, postponing to [8] its sufficiency. In fact, in §2 we describe a simple procedure which shortens every path which does not have a fixed minimum and maximum. The proof that the given condition is sufficient is more delicate, and relies on a local version of the non-squeezing theorem for J -holomorphic curves. This result is already known for the case $M = \mathbf{R}^{2n}$ by the work of Bialy–Polterovich [2].² It is also proved by Ustilovsky [13] for paths on an arbitrary manifold under the hypothesis that there is only one fixed minimum p and one fixed maximum P and that the Hamiltonian is non-degenerate at these points p, P at all times.

This characterization of geodesics implies that they are not at all unique: if $\{\phi_t\}$ is a stable geodesic, any path of the form $\{\psi_t \circ \phi_t\}$ will also be a geodesic of the same length, provided that the support of $\{\psi_t\}$ is disjoint from at least one pair of fixed extrema $\{p, P\}$, and that $\mathcal{L}(\psi_t)$ is sufficiently small. Thus we have:

Corollary 1.4 *Given any isotopy $\phi_t, 0 \leq t \leq 1$, there exist an infinite number of non trivial deformations having the same length. More precisely, there exists an infinite number of smooth 1-parameter deformations $\phi_{t,s}$ such that*

- (1) $\phi_{t,0} = \phi_t$
- (2) $\phi_{0,s} = \mathbb{1}$ and $\phi_{1,s} = \phi_1$ for all s
- (3) for at least one s , the isotopy $\phi_{t \in [0,1],s}$ is distinct from $\phi_{t \in [0,1]}$ and
- (4) for all s , $\phi_{t \in [0,1],s}$ has same length as $\phi_{t \in [0,1]}$.

In particular, a shortest path or a stable geodesic is never unique.

Remark 1.5 As Weinstein points out, such non-uniqueness occurs on a Finsler manifold whenever the unit ball in the tangent space has flat pieces in its boundary. A good example to consider is \mathbf{R}^2 with the metric whose unit ball is the unit square $\{(x, y) : |x|, |y| \leq 1\}$. Here, *any* smooth path $(x(t), y(t))$ from $(0, 0)$ to $(1, 0)$ such that

$$x'(t) > |y'(t)|$$

is a geodesic.

²They use rather different terminology, calling paths with at least one fixed minimum and one fixed maximum “quasi-autonomous” and paths with a fixed minimum and maximum at each moment are called “locally quasi-autonomous”.

1.2 Stability: necessary conditions

Consider a path $\gamma = \{\phi_t\}_{t \in [0,1]}$ with $\phi_0 = \mathbb{1}$. Suppose that q is a fixed extremum of the Hamiltonian $\{H_t\}_{t \in [0,1]}$ and consider the linearizations

$$L_t = d\phi_t(q) : T_q(M) \rightarrow T_q(M)$$

of the ϕ_t at q . Clearly, this is the symplectic isotopy generated by the Hessian of H_t at q . What turns out to be crucial for the stability of γ is the time at which non-trivial closed orbits of the L_t appear. If, for every $x \in T_q(M)$ and every $t' \in (0, T)$, the only trajectories $\alpha(t) = L_t(x)$, $0 \leq t \leq t'$, with $x = L_0(x) = L_{t'}(x)$ are single points, we will say that the linearized flow at q has no non-trivial closed trajectories in the time interval $(0, T)$.

We first state a necessary condition for stability.

Theorem 1.6 *Suppose that γ is a stable geodesic. Then it has at least one fixed maximum and one fixed minimum. Further, if $\dim(M) = 2$, there is at least one fixed maximum and one fixed minimum at which the linearized flow has no non-trivial closed trajectory in the open interval $(0, 1)$; a similar statement holds for arbitrary M provided that the set of fixed extrema of γ is finite.*

The first statement follows immediately from the curve-shortening procedure of Proposition 2.1 which reduces the length of every path which does not have at least one fixed maximum and minimum. The second statement is proved by an explicit construction which shows how to use a closed trajectory α of the linearized flow at q to shorten γ . To do this, one composes γ with a scrubbing motion which moves the points in M lying near q around (the exponential of) the loop α . Intuitively, in the presence of a closed trajectory α at q it costs extra energy to keep q fixed, and one can reduce the energy needed to get to the endpoint ϕ_1 by following α . The details are in § 4.

In fact, this was already proved by Ustilovsky in [13] under the nondegeneracy assumptions mentioned before, and our proof uses essentially the same method, but involves more delicate estimates. The point is that this nondegeneracy hypothesis on γ ensures that the second variation of \mathcal{L} at γ is a well-behaved functional, and Ustilovsky uses it to prove not only the necessity of the above condition, but also its sufficiency. We establish the sufficiency of this condition in the general case in [9].

The above necessary condition places severe restrictions on symplectomorphisms which are the endpoints of stable geodesics from the identity. Combining this with calculations of the Calabi invariant of various related symplectomorphisms, we show:

Proposition 1.7 *There is a symplectomorphism ϕ of S^2 which is not the endpoint of any stable geodesic from the identity. A fortiori, there is no shortest path from the identity to ϕ .*

This map ϕ is generated by a Hamiltonian of the form $H(x, y, z) = h(z)$, and so rotates the parallels of the sphere by varying amounts.

1.3 Variational definition of geodesic

Another approach to defining geodesics is to use a variational definition, looking at paths which are critical points of the length functional \mathcal{L} . In this section we discuss the relationship between the definition which we have chosen and the variational one.

Observe first that the tangent space, $T_\gamma\mathcal{P}$, to the path space \mathcal{P} at $\gamma = \{\phi_t\}_{t \in [0,1]}$ consists of smooth families of functions $G_t, 0 \leq t \leq 1$, such that $G_0 = G_1 = 0$.¹ Further, the tangent vector $\{G_t\}$ exponentiates to the path $\gamma_\varepsilon, |\varepsilon| \leq \varepsilon_0$, in $\text{Ham}^c(M)$ given by

$$\gamma_\varepsilon = \{\phi_{\varepsilon G_t} \circ \phi_t\}_{t \in [0,1]},$$

Here, for each fixed t , $\phi_{\varepsilon G_t}$ is the time-1 flow of the function εG_t , and \circ denotes the usual composition of maps.

The following definition takes into account the fact that \mathcal{L} is not differentiable everywhere. Observe that we do not make a statement about arbitrary deformations, but only those which arise from exponentiating a vector field as described above.

Definition 1.8 A path $\gamma = \{\phi_t\}_{0 \leq t \leq 1}$ generated by a Hamiltonian H_t is said to be **\mathcal{L} -critical** if, for every tangent vector field $\{G_t\}$, the (not necessarily smooth) real valued function $\mathcal{L}(\gamma_\varepsilon)$ of the variable ε is bounded below on some neighbourhood of $\varepsilon = 0$ by a smooth function whose value at $\varepsilon = 0$ is $\mathcal{L}(\gamma)$ and first derivative at $\varepsilon = 0$ vanishes. Further, γ is said to be a smooth point if, for all tangent vector fields $\{G_t\}$, the function $\varepsilon \mapsto \mathcal{L}(\gamma_\varepsilon)$ is differentiable at $\varepsilon = 0$.

Theorem 1.9 *A path $\phi_t, 0 \leq t \leq 1$, is \mathcal{L} -critical if and only if its generating Hamiltonian has at least one fixed minimum and one fixed maximum.*

Comparing this with Theorem 1.3, we see that any \mathcal{L} -critical path is a geodesic, and that, although a geodesic need not be an \mathcal{L} -critical path, it is an \mathcal{L} -critical path at each moment. This is in marked contrast with the situation in Riemannian geometry, where the variational notion of geodesic does not depend on the interval of time considered. A path γ is a Riemannian geodesic exactly when its covariant derivative vanishes at each time, which implies, of course, that the restriction of the path to any subinterval, no matter how long, is also a critical point of the length functional (on the space of paths with fixed endpoints).

The next result gives a necessary condition for a \mathcal{L} -critical path to be a smooth point of \mathcal{L} .

Proposition 1.10 *An isotopy ϕ_t generated by $H_t, 0 \leq t \leq 1$, with at least one fixed minimum and maximum, is a smooth point of the length functional \mathcal{L} only if there exist a fixed minimum p and fixed maximum P such that*

$$\bigcap_{t \in [0,1]} \text{minset } H_t = \{p\} \quad \text{and} \quad \bigcap_{t \in [0,1]} \text{maxset } H_t = \{P\}$$

and such that $\text{minset } H_t = \{p\}$ and $\text{maxset } H_t = \{P\}$ holds for all $t \in [0,1]$ except on a subset of measure 0.

At smooth points (or more generally at points which satisfy the hypothesis of continuity, see §3), Theorem 1.9 follows directly from the first variation formula of Ustilovsky. We give the general proof in §3.2. Since a direct consequence of Ustilovsky's work is that, conversely, a path which satisfies the conditions in Proposition 1.10 is a smooth point of \mathcal{L} provided that each H_t is non-degenerate at both p and P , one sees that the above proposition is close to being sharp.

¹Note that when M is non-compact each tangent vector in $T_\parallel \text{Ham}^c(M)$ has a unique representation by a function G . To recover this uniqueness in the compact case, we normalise the function G by requiring that $\int_M G \omega^n = 0$.

1.4 Organization of the paper

This paper is organized as follows. In §2 we discuss various curve-shortening techniques and use them to prove the necessary condition in Theorem 1.3. §3 discusses the first variation formula for \mathcal{L} and proves Theorem 1.9. §4 starts with a discussion of the second variation formula and then proves the necessary condition for stability in Theorem 1.6. The proof involves a considerable amount of calculation. In §5.1 we apply this theorem to construct a symplectomorphism of S^2 which cannot be reached by a shortest path from the identity. The ideas in §2 and §5 are elementary, and the proofs can be read independently of everything else in the paper.

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2 Curve-shortening procedures

The main aim of this section is to prove the necessity of the condition stated in Theorem 1.3 for a path to be a geodesic. Thus we have to prove that a path $\{\phi_t\}_{t \in I}$ is a local minimum for \mathcal{L} at each moment only if its generating Hamiltonian has at least one fixed maximum and one fixed minimum at each moment. Clearly, this is an immediate consequence of the next proposition.

Proposition 2.1 *Suppose that the generating Hamiltonian for the path $\gamma = \{\phi_t\}_{0 \leq t \leq 1}$ does not have at least one fixed minimum and one fixed maximum. Then there is a deformation $\gamma_s, s \geq 0$, of $\gamma = \gamma_0$ in $\mathcal{P} = \mathcal{P}(\phi_0, \phi_1)$ such that*

$$\mathcal{L}(\gamma_s) < \mathcal{L}(\gamma),$$

for all $s > 0$. In particular, γ is not a local minimum for \mathcal{L} .

Proof: By compactness there is a finite set of t , say $t_0 < t_1 < \dots < t_k$ such that

$$\bigcap_j \text{maxset } H_{t_j} = \emptyset.$$

Write $X_j = \text{maxset } H_{t_j}$. Thus, for some $\nu > 0$

$$N_{2\nu}(X_0) \subset \bigcup_{j \geq 1} (M - X_j),$$

where $N_\nu(X)$ denotes the ν -neighbourhood of $X \subset M$ with respect to some Riemannian metric on M . Let $\{\beta_j\}$ be a partition of unity subordinate to the covering

$$M - N_\nu(X_0), M - X_1, \dots, M - X_k,$$

and choose $\delta > 0$ so that

$$X_0 \subset \bigcup_{j \geq 1} (\beta_j^{-1}([\delta, 1])).$$

For $j \geq 1$, let K_j be a function with support in $\beta_j^{-1}([\delta/2, 1])$ such that

- $K_j \leq 0$,
- K_j is constant and < 0 on $(\beta_j^{-1}([\delta, 1]))$,
- $\text{supp}(K_j) \subset \text{supp}(\beta_j)$.

Now let ψ_t^j be the time- t flow of K_j , and, given $\varepsilon > 0$, define Ψ_t^ε as follows:

- (i) $\Psi_t^\varepsilon = \mathbb{1}$ for $t < t_0 - \varepsilon$ and then flows along $\psi_s^1 \circ \dots \circ \psi_s^k$, where $s = t - t_0$, until $t = t_0 + \varepsilon$.
- (ii) Ψ_t^ε remains unchanged (its time derivative is 0) when $|t - t_j| > \varepsilon$ for all j .
- (iii) When $|t - t_j| \leq \varepsilon$, Ψ_t^ε has the form

$$(\psi_s^j)^{-1} \Psi_{t_j - \varepsilon}^\varepsilon, \quad \text{where } s = t - t_j.$$

Thus as one passes t_j one undoes the j th perturbation.

We claim that $\phi_t^\varepsilon = \Psi_t^\varepsilon \circ \phi_t$ satisfies the requirements. Firstly, it is easy to check that $\Psi_1^\varepsilon = \mathbb{1}$ for all ε . Further, by (i) and the choice of the K_j , the maximum value of the Hamiltonian for ϕ_t^ε is definitely less than that of ϕ_t when $|t - t_0| < \varepsilon$ and ε is sufficiently small. To see this, note that the Hamiltonian for the composite $\Psi_t^\varepsilon \circ \phi_t$ is not the sum $H_\Psi + H_\phi$ of the Hamiltonian for each component but rather is

$$H_\psi * H_\phi = H_\Psi + H_\phi \circ (\Psi_t^\varepsilon)^{-1}.$$

However, for small ε , this shifting of the support of H_ϕ is irrelevant in our situation: $H_\phi \circ (\Psi_t^\varepsilon)^{-1}$ takes its maximum on $\Psi_t^\varepsilon(\text{maxset } H_t)$ which is contained in $\{x : H_\Psi(x) < 0\}$ when $|t - t_0| \leq \varepsilon$ and ε is sufficiently small. Thus, there is a constant c which is independent of ε such that

$$\max(H_{\phi_t^\varepsilon}) < \max(H_{\phi_t}) - c$$

when $|t - t_0| < \varepsilon$ and ε is sufficiently small. Similarly, because the support of ψ_s^j is disjoint from X_j the maximum of the Hamiltonian will remain unchanged by the perturbations described in (iii) for small ε . It follows that

$$\mathcal{L}(\{\phi_t^\varepsilon\}) < \mathcal{L}(\{\phi_t\}) - 2c\varepsilon$$

as required. □

Note that, as in the above proof, we can compose H_t during the time interval $[t - \varepsilon, t + \varepsilon]$ with functions G_j having support disjoint from the extrema of H_t . This proves the non-uniqueness result stated in Corollary 1.4.

Our next result is a curve-shortening procedure, similar to Sikorav's trick [12], which applies to paths with fixed extrema at which a lot of energy is concentrated. It gives conditions under which $\mathcal{L}(\gamma)$ is not minimal. Recall that the displacement energy (or disjunction energy) $e(Z)$ of a subset Z of M is defined by

$$e(Z) = \inf\{\|\phi\| : \phi(Z) \cap Z = \emptyset\}.$$

Proposition 2.2 *Let ϕ_t be a path from $\mathbb{1}$ to ϕ generated by the Hamiltonian H_t normalised so that $\min H_t = 0$ for all t , and suppose that there is $c > 0$ such that the displacement energy of the set*

$$Z = Z_c = \{x : H_t(x) \leq c, \text{ for some } t \in [0, 1]\}$$

is less than $c/4$. Assume further that

$$\max_M H_t > c/2 + \max_Z H_t \quad \text{for all } t.$$

Then the path $\phi_{t \in [0, 1]}$ is not length-minimizing.

Proof: Let $F : M \rightarrow [0, c/2]$ be an autonomous nonnegative Hamiltonian which equals $c/2$ on $Z_{c/2}$ and has support in Z_c . More precisely, one may take a set Z' in the interior of $Z_c - Z_{c/2}$, with $\partial Z'$ smooth, such that a collar neighbourhood $\partial Z' \times [-1, 1]$ embeds in $\text{Int}(Z_c - Z_{c/2})$. Let s be the normal coordinate of the collar chosen so that the points where $s = -1$ are closest to $Z_{c/2}$, and define $F(x) = f(s)$ where f decreases from $c/2$ to 0 in the interval $[-1/2, 1/2]$.

By hypothesis, there exists a symplectic diffeomorphism τ of M of norm less than $c/4$ which disjoins Z_c from itself. Let α_t be the Hamiltonian isotopy generated by $-F$, and β_t the one generated by $(H_t \circ \alpha_t) + F$. Set $\alpha = \alpha_1$ and $\beta = \beta_1$. Then the path $\alpha_t \circ \beta_t$, $t \in [0, 1]$, where the composition is timewise, is generated by

$$-F + ((H_t \circ \alpha_t) + F) \circ \alpha_t^{-1} = H_t.$$

Thus

$$\begin{aligned} \|\phi\| &= \|\alpha \circ \beta\| \\ &= \|\tau^{-1}(\tau\alpha\tau^{-1})\tau\beta\| \\ &= \|\tau^{-1}((\tau\alpha\tau^{-1})\beta)\tau[\tau^{-1}, \beta^{-1}]\| \\ &\leq \|\tau^{-1}, \beta^{-1}\| + \|(\tau\alpha\tau^{-1})\beta\| \\ &< c/2 + \|(\tau\alpha\tau^{-1})\beta\|. \end{aligned}$$

The last inequality holds because

$$\|\tau^{-1}\beta^{-1}\tau\beta\| \leq \|\tau^{-1}\| + \|\beta^{-1}\tau\beta\| = 2\|\tau\| < c/2$$

since the norm is invariant under conjugation. Now the statement of the theorem follows at once if we show that

$$\|(\tau\alpha\tau^{-1})\beta\| \leq \mathcal{L}(\phi_t) - c/2.$$

But $(\tau\alpha\tau^{-1})\beta$ is generated by the Hamiltonian

$$G_t = -F \circ \tau^{-1} + (H_t \circ \alpha_t + F) \circ \tau\alpha_t^{-1}\tau^{-1} = -F \circ \tau^{-1} + F + H_t \circ [\alpha_t, \tau].$$

Now over Z_c each function G_t has minimum at least $c/2$: this is obvious over $Z_{c/2}$, and it holds over $Z_c - Z_{c/2}$ too because each function H_t is bounded below by $c/2$ there. Because each H_t is bounded below by c on $M - Z_c$ and since τ disjoins Z_c from itself and F has support inside Z_c with values in $[0, c/2]$, it is easy to check that the minimum of each G_t on $M - Z_c$ is also bounded below by $c/2$. Thus

$$\min G_t \geq c/2 \quad \text{for all } t.$$

The same reasons, and the hypothesis that each H_t reaches its maximum outside Z_c and satisfies $\max_M H_t > c/2 + \max_{Z_c} H_t$, imply easily that G_t has the same maximum value as H_t . This concludes the proof. \square

Note that the shorter path from $\mathbb{1}$ to ϕ constructed in the above proof is not C^∞ -close to the path ϕ_t . The proof only shows that the path ϕ_t is not length-minimizing, though it might be a local minimum of the Hofer length \mathcal{L} in the path space. We will discuss the local minima of \mathcal{L} in § 4.

Here is an elementary corollary. Recall that the displacement energy of a ball of radius r in Euclidean space is πr^2 . It follows from [7] that this is essentially true for balls in any manifold M .

Corollary 2.3 *Suppose that H is an autonomous Hamiltonian which takes its minimum value at the single point p , and suppose that $H(x) - H(p) > 4\pi r^2$ for all x outside a symplectically embedded ball B of radius r and center p . Suppose further that the displacement energy of B in M is πr^2 . Then, provided that $\|H\| > 8\pi r^2$, the flow at time 1 of H is not length-minimizing.*

The above hypothesis will be satisfied if the Hessian of H at p is large, but we are still quite far from an optimal result. For example, in \mathbf{R}^2 the function πr^2 has closed trajectories at time 1, and it is easy to see that a function which equals $\lambda\pi r^2$ near 0 will not generate a minimal geodesic for any $\lambda \geq 1$. But our result only applies when $\lambda > 4$.

Remark 2.4 (i) Proposition 2.2 is relevant to the optical Hamiltonian flows considered by Bialy–Polterovich in [2]. They are interested in particular Hamiltonians which take their minimum on an n -dimensional section Z of a cotangent bundle TX . When Z is Lagrangian they show that the corresponding path is always a minimal geodesic. The above result makes clear that the Lagrangian condition is essential. For if Z is not Lagrangian, it can always be displaced (in fact the displacement energy is 0 by Polterovich [10]), and so if H grows sharply enough near Z the path will not be a minimal geodesic.

(ii) This proposition can also be improved in various ways. For example, it is clearly unnecessary to assume that the set Z in the statement of the proposition contains $\{H_t^{-1}([0, c])\}$ for all t – if it contains this set for $t \in [a, b]$ then we should only “turn on” the flow of F for these t as in Proposition 2.1, and make corresponding adjustments to the estimates of energy saved.

3 \mathcal{L} -critical paths

The main aim of this section is to prove Theorem 1.9 which characterizes \mathcal{L} -critical paths. In order to show the logical development of ideas, we will begin by discussing the first variation formula. This has also been derived in a slightly more restricted context by Ustilovsky [13]. Since this formula does not apply to all paths, but only to those which satisfy the Hypothesis of Continuity stated below, it is not essential to any of our proofs. However, its form is very suggestive.

3.1 The first variation formula

Given $\gamma = \phi_{t \in [0,1]}$, let $G_{t \in [0,1]}$ be a tangent vector field along γ vanishing at both ends $t = 0, 1$. For any $\varepsilon \in \mathbf{R}$, set

$$\gamma_\varepsilon(t) = \phi_{\varepsilon G_t} \circ \phi_t$$

which is a 1-parameter family of paths with the given ends, where $\phi_{\varepsilon G_t}$ is the time 1 flow of εG_t . We wish to compute

$$\left. \frac{d^k}{d\varepsilon^k} \right|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) = \int_0^1 \left. \frac{d^k}{d\varepsilon^k} \right|_{\varepsilon=0} \left\| \frac{d}{dt} \gamma_\varepsilon \right\|_H dt$$

for $k = 1, 2$.

Proposition 3.1 *The Taylor expansion of the vector field $\frac{d}{dt} \gamma_\varepsilon$ in powers of ε up to order 2 is*

$$\frac{d}{dt} \gamma_\varepsilon = \text{symplectic gradient of } [H_t + \varepsilon (G'_t + \{-G_t, H_t\})$$

$$+\frac{\varepsilon^2}{2} (\{-G_t, G'_t\} + \{-G_t, \{-G_t, H_t\}\}) + o(\varepsilon^2)].$$

Here the notation $o(\varepsilon^k)$ denotes a term R which decreases faster than ε^k :

$$\lim_{\varepsilon \rightarrow 0} \frac{R(\varepsilon)}{\varepsilon^k} = 0$$

uniformly with respect to other variables.

Proof: Let $y \in M$ be any point and put $\bar{y} = \phi_{\varepsilon G_{t_0}}^{-1}(y)$, where $\phi_{\varepsilon G_{t_0}}$ is the time-1-map of the autonomous Hamiltonian εG_{t_0} . We write $\phi_t^{t_0}$ for the flow at time t of the non autonomous Hamiltonian $\{H_{t_0+t}\}$ (that is: we look at the flow of the Hamiltonian H_t starting at time t_0 .) The vector $\frac{d}{dt}|_{t_0} \gamma_\varepsilon(y)$ is the derivative at $t = 0$ of the composition

$$[0, \delta] \xrightarrow{\alpha} M \times [0, \delta] \xrightarrow{F} M$$

where $\alpha(t) = (\phi_t^{t_0}(\bar{y}), t)$ and $F(x, t) = \phi_{\varepsilon G_{t_0+t}}(x)$. Now $\alpha'(0) = (X_{H_{t_0}}(\bar{y}), 1)$ where X denotes the symplectic gradient. Thus

$$\begin{aligned} \frac{d}{dt}|_{t_0} \gamma_\varepsilon(y) &= dF|_{(\bar{y}, 0)}(X_{H_{t_0}}(\bar{y}), 1) \\ &= dF|_{(\bar{y}, 0)}(X_{H_{t_0}}(\bar{y}) + e_t) \end{aligned}$$

where e_t is the unit tangent vector on the real line. Hence

$$\frac{d}{dt}|_{t_0} \gamma_\varepsilon(y) = d\phi_{\varepsilon G_{t_0}}|_{\bar{y}}(X_{H_{t_0}}(\bar{y})) + \frac{d}{dt}|_{t=0} \phi_{\varepsilon G_{t_0+t}}(\bar{y}).$$

Now the first term of the right hand side is equal to

$$X_{H_{t_0}}(\bar{y}) + \varepsilon[X_{-G_{t_0}}, X_{H_{t_0}}] + o(\varepsilon),$$

while the second is

$$\frac{d}{dt} X_{\varepsilon G_{t_0+t}} + o(\varepsilon) = \varepsilon X_{G'_{t_0}} + o(\varepsilon).$$

Thus finally

$$\frac{d}{dt} \gamma_\varepsilon = \text{symplectic gradient of } H_t + \varepsilon(G'_t + \{-G_t, H_t\}).$$

A similar but more elaborate calculation shows that the next term in the Taylor expansion is

$$\frac{\varepsilon^2}{2} (\{-G_t, G'_t\} + \{-G_t, \{-G_t, H_t\}\}).$$

□

Let

$$K_{\varepsilon, t} = H_t + \varepsilon(G'_t + \{-G_t, H_t\}) + o(\varepsilon)$$

be the function appearing in the above Taylor expansion. To derive the variation formula, we first make the following assumption.

Hypothesis of continuity The path $\phi_{t \in [0,1]}$, satisfies the hypothesis of continuity if, given any tangent vector field $G_{t \in [0,1]}$, it is possible to make a choice $p_t(\varepsilon) \in M$ of a point at which the minimum value of $K_{\varepsilon,t}$ is reached in such a way that $p_t(\varepsilon)$ is a smooth path for small values of ε and all $0 \leq t \leq 1$. In this case, $p_t(0) = p_t$ where p_t is a minimal point of H_t . We assume that the same holds for $P_t(\varepsilon)$ with maximum instead of minimum values.

One way to decide when this condition is satisfied is to use the following lemma.

Lemma 3.2 *Let $H_t, 0 \leq t \leq 1$, be any Hamiltonian which has a non-degenerate minimum at p for all t . Then given any smooth functions F_t of the form $f_t + o(\varepsilon^0)$ defined on some neighbourhood N of p , there is for some $\varepsilon' > 0$ a smooth map $p(t, \varepsilon) : [0, 1] \times [-\varepsilon', \varepsilon'] \rightarrow N$ such that $p(t, \varepsilon)$ is the unique minimum of $(H_t + \varepsilon F_t)|_N$. Further,*

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \min(H_t + \varepsilon F_t) = f_t(p).$$

A similar statement holds near a non-degenerate maximum P .

The proof of this lemma is easy, based on ordinary smooth analysis. It immediately implies:

Corollary 3.3 *Suppose that H_t is non-degenerate in the sense that there exist two points p, P such that for all t*

- (i) $\text{minset } H_t = \{p\}$ and $\text{maxset } H_t = \{P\}$ and
- (ii) p, P are non-degenerate extrema of H_t .

Then the path γ which it generates satisfies the hypothesis of continuity.

Theorem 3.4 (First variation formula) *Suppose that $\gamma = \phi_{t \in [0,1]}$ satisfies the hypothesis of continuity. Then the first variation is:*

$$\delta^1 \mathcal{L}(\{G_t\}) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) = \int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt.$$

Proof: Let us compute the total variation of

$$K_{\varepsilon,t} = H_t + \varepsilon(G'_t + \{-G_t, H_t\}) + o(\varepsilon)$$

for small ε . Under the hypothesis of continuity we find:

$$\begin{aligned} \text{TotVar}(K_{\varepsilon,t}) &= (H_t + \varepsilon(G'_t + \{H_t, G_t\})) (P_t(\varepsilon)) \\ &\quad - \text{idem}(P_t(\varepsilon) \rightarrow p_t(\varepsilon)) + o(\varepsilon). \end{aligned}$$

We will write $\dot{P}_t(\varepsilon)$ for the derivative of $P_t(\varepsilon)$ with respect to ε . Then

$$\begin{aligned} \frac{d}{d\varepsilon} \text{TotVar}(K_{\varepsilon,t}) &= dH_t(\dot{P}_t(\varepsilon)) + (G'_t + \{H_t, G_t\})(P_t(\varepsilon)) \\ &\quad - \text{idem}(P_t(\varepsilon) \rightarrow p_t(\varepsilon)) + o(\varepsilon^0). \end{aligned}$$

Therefore, because $dH_t = 0$ at $P_t(0)$ we find that

$$\frac{d}{d\varepsilon} \text{Tot Var}(K_{\varepsilon,t})|_{\varepsilon=0} = G'_t(P_t(0)) - G'_t(p_t(0))$$

Integrating over t we get the first variation. Note that $P_t(0)$ is by definition the limit as $\varepsilon \rightarrow 0$ of a point where the function

$$K_{\varepsilon,t} = H_t + \varepsilon (G'_t + \{-G_t, H_t\}) + o(\varepsilon)$$

reaches its maximum and by the hypothesis of continuity belongs to $\text{maxset } H_t$. Since $\{G_t, H_t\}$ vanishes over $\text{maxset } H_t$, $P_t(0)$ must belong to the subset

$$\text{maxset}(G'_t|_{\text{maxset } H_t}) \subset \text{maxset } H_t,$$

and similarly for $p_t(0)$. Therefore the first variation formula becomes:

$$\delta^1 \mathcal{L}(\{G_t\}) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) = \int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt.$$

□

3.2 \mathcal{L} -critical paths

Recall that a path γ is said to be \mathcal{L} -critical if, for every tangent vector field $\{G_t\}$, the (not necessarily differentiable) real valued function $\mathcal{L}(\gamma_\varepsilon)$ of the variable ε is bounded below on some neighbourhood of $\varepsilon = 0$ by a smooth function whose value at 0 is $\mathcal{L}(\gamma)$ and whose first derivative at 0 vanishes.

Proof of Theorem 1.9

We must show that a path ϕ_t , $0 \leq t \leq 1$, is \mathcal{L} -critical if and only if its generating Hamiltonian has at least one fixed minimum and one fixed maximum. Suppose that p, P are fixed minimum and maximum of $\{H_t\}$. Let \bar{H}_t , $0 \leq t \leq 1$, be a 1-parameter family of functions C^∞ -close to H_t , $0 \leq t \leq 1$, which is such that for every t :

- 1) $\bar{H}_t(p) = H_t(p)$ and $\bar{H}_t(P) = H_t(P)$
- 2) $\text{minset } \bar{H}_t = \{p\}$ and $\text{maxset } \bar{H}_t = \{P\}$
- 3) p, P are non-degenerate extrema of \bar{H}_t and, in some symplectic coordinates near p or P , the 2-jet of \bar{H}_t at p is strictly larger than the 2-jet of H_t at p and conversely at P .

As above, for each ε , let $K_{\varepsilon,t}$ be the Hamiltonian which generates the path

$$\gamma_\varepsilon(t) = \phi_{\varepsilon G_t} \circ \phi_t,$$

and set

$$\bar{K}_{\varepsilon,t} = K_{\varepsilon,t} - H_t + \bar{H}_t.$$

Then, at each fixed minimum p , $\min K_{\varepsilon,t} \leq \min \bar{K}_{\varepsilon,t}$. But $\bar{K}_{\varepsilon,t}$ is now the sum of a function \bar{H}_t which is non-degenerate at p and of a smooth function εf_t (plus terms of order $o(\varepsilon)$), where

$$f_t = G'_t + \{-G_t, H_t\}$$

by Proposition 3.1. By Lemma 3.2 above,

$$\frac{d}{d\varepsilon}(\min \bar{K}_{\varepsilon,t})\Big|_{\varepsilon=0} = f_t(p) = G'_t(p) + \{-G_t, H_t\}(p) = G'_t(p).$$

A similar result holds at the fixed maxima P . Hence we get:

$$\mathcal{L}(\gamma_\varepsilon) \geq \int_0^1 \text{Totvar}(\bar{K}_{\varepsilon,t}) dt$$

where the right hand side is a smooth function of ε whose value at 0 is $\|\phi_t\|_H$ and whose first derivative at $\varepsilon = 0$ is therefore

$$\int_0^1 (G'_t(P) - G'_t(p)) dt = 0.$$

Conversely, if the set of fixed minima or the set of fixed maxima is empty, one can easily define a tangent vector field $\{G_t\}$ such that $\mathcal{L}(\gamma_\varepsilon)$ is not bounded from below by a smooth function with vanishing first derivative. The proof is an obvious adaptation of that of Proposition 2.1. Instead of constructing a loop Ψ_t^ε such that $\mathcal{L}(\Psi_t^\varepsilon \circ \phi_t) < \mathcal{L}(\phi_t)$, we now need to find a family of functions G'_t such that

$$\int_0^1 G'_t(x) dt = 0 \quad \text{for all } x,$$

and

$$\int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt < 0.$$

If there are no fixed maxima, for example, there is a finite set of times, say $t_0 < t_1 < \dots < t_k$ such that

$$N_{2\nu}(X_0) \subset \cup_{j \geq 1} (M - X_j),$$

where $X_j = \text{maxset } H_{t_j}$ as before. Then, we can choose a small $\delta > 0$ and functions

$$G_t = \sum_{i=1}^k G_{k,t} \leq 0, \quad \text{for } |t - t_0| \leq \delta,$$

with support in $N_{2\nu}(X_0)$ so that for $i = 1, \dots, k$ and for $|t - t_i| \leq \delta$,

$$\text{maxset}(H_t - G_{i,t+t_i-t_0}) = \text{maxset}(H_t).$$

It is easy to check that these G_t satisfy the required conditions. □

Corollary 3.5 (i) *A Hamiltonian $\{H_t\}_{t \in [0,1]}$ has at least one fixed minimum and one fixed maximum if and only if*

$$\int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt \geq 0$$

for all admissible tangent vector fields $\{G_t\}_{t \in [0,1]}$.

(ii) An isotopy ϕ_t generated by $H_t, 0 \leq t \leq 1$, with at least one fixed minimum and maximum, is a smooth point of the length functional \mathcal{L} only if there exist two fixed extrema p, P such that

$$\bigcap_{t \in [0,1]} \text{minset } H_t = \{p\} \quad \text{and} \quad \bigcap_{t \in [0,1]} \text{maxset } H_t = \{P\}$$

and such that $\text{minset } H_t = \{p\}$ and $\text{maxset } H_t = \{P\}$ holds for all $t \in [0, 1]$ except on a subset of measure 0.

Proof: The proof of (i) follows easily from what is said above. As for (ii), if $\phi_t, 0 \leq t \leq 1$, is a smooth point of \mathcal{L} , it has a first derivative which must then be

$$\delta^1 \mathcal{L}(\{G_t\}) = \int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt.$$

In particular, this means that the integral expression above is linear in $\{G_t\}$. If $p \in \bigcap_{t \in [0,1]} \text{minset } H_t$ and $P \in \bigcap_{t \in [0,1]} \text{maxset } H_t$ is any choice, then

$$\int_0^1 \left(\sup_{\text{maxset } H_t} G'_t - \inf_{\text{minset } H_t} G'_t \right) dt \geq \int_0^1 (G'_t(P) - G'_t(p)) dt = 0$$

for all $\{G_t\}$. If $H_t, 0 \leq t \leq 1$, does not satisfy condition (ii), we constructed in the proof of the Proposition tangent vector fields $\{G_t\}$ such that the above inequality is strict: but then the same integral evaluated on $\{-G_t\}$ cannot be negative, and so the left hand side cannot be a linear map (it is a singular non-negative “conic map”). \square

4 Geodesics and stability

We begin this section by discussing the second variation formula. Using this as a guide, we then prove Theorem 1.6 which gives a necessary condition for stability.

4.1 The second variation formula

Let q be a fixed extremum of the path γ at which the Hessian d^2H_t of H_t is non-degenerate for all t , and let $\{G_t\} \in T_\gamma \mathcal{P}$ be a tangent vector to γ . The second variation of γ at q when evaluated on $\{G_t\}$ depends only on the loop $g(t)$ traced out by the gradient $\nabla G_t(q)$ of G_t at q . (Note that $g(0) = g(1) = 0$ because $G_0 = G_1 \equiv 0$.) We will choose symplectic coordinates around q and then identify the tangent space $T_q M$ with \mathbf{R}^{2n} equipped with its standard symplectic form ω_0 and complex structure J . Here $Jx_{2i-1} = x_{2i}$, and $Jx_{2i} = -Jx_{2i-1}$, so that

$$\omega_0(u, v) = (Ju) \cdot v,$$

where \cdot denotes the usual dot product. Then the symplectic area enclosed by a loop g in \mathbf{R}^{2n} is

$$\text{area } g = \int_{D_g} \omega_0 = \frac{1}{2} \int_0^1 (Jg) \cdot g' dt,$$

where D_g is a 2-disc with boundary along g . We will write

$$\langle u, v \rangle_t = ((d^2 H_t)^{-1})u \cdot v$$

for the metric induced on $T_q M$ by the inverse of the Hessian $d^2 H_t$ of H_t at q . The following theorem is proved by Ustilovsky in [13], and may also be derived from the Taylor expansion in Proposition 3.1.

Theorem 4.1 (Second variation formula) *Suppose that $H_{t \in [0,1]}$ has at least one fixed minimum and one fixed maximum. Suppose further that each fixed extremum of $\{H_t\}$ is a non-degenerate critical point of all the functions $H_t, 0 \leq t \leq 1$. Let $G_{t \in [0,1]}$ be a tangent vector field along $\phi_{t \in [0,1]}$, and set $g(t) = \nabla G_t(p)$. Then the contribution of the fixed minimum p of $H_{t \in [0,1]}$ to the second variational formula, is*

$$\delta^2 \mathcal{L}(\{G_t\})(p) = \int_0^1 \langle g', g' \rangle_t dt + 2 \text{area}(g).$$

Similarly, the contribution of the fixed maximum P is

$$\delta^2 \mathcal{L}(\{G_t\})(P) = \int_0^1 \langle g', g' \rangle_t dt - 2 \text{area}(g)$$

where this time $g(t) = \nabla G_t(P)$.

We denote by \mathcal{Q}_q the quadratic functional

$$\mathcal{Q}_q(g) = \int_0^1 (\langle g', g' \rangle_t \pm (Jg) \cdot g') dt$$

on the space of smooth loops g based at the origin in $T_q M = \mathbf{R}^{2n}$ which appears above. The analysis of this functional is an isoperimetric problem relating the area of the loop to its time-dependent energy defined by the varying metric $\langle \cdot, \cdot \rangle_t$. It has been carried out as part of the development of index theory for positive-definite periodic linear Hamiltonian systems (see Ekeland [4]) as well as by Ustilovsky in [13]. The results of the present section show that there is a very close connection between the periodic linear theory and the question of stability of geodesics in Hofer geometry. This will become even more apparent in [9].

Theorem 4.2 (Ustilovsky,[13]) *Let γ have fixed non-degenerate extrema $q = p, P$ as above, and suppose that there is no other fixed extremum. Then, the quadratic functional \mathcal{Q}_q is positive definite if the linearized isotopy $d\phi_t$ at q , generated by the 2-jet $\tilde{H}_t, 0 \leq t \leq 1$ of H_t at q , has no non-constant closed trajectory α in time ≤ 1 . Moreover, if this is the case at both p and P , γ is a stable geodesic, i.e. it is a local minimum for \mathcal{L} on the path space $\mathcal{P}(\gamma)$. Conversely, if such α does exist in time less than 1 at either p or P , then \mathcal{Q}_q has non-vanishing index and the path γ is not a local minimum of \mathcal{L} .*

This theorem. can be proved by looking at the 1-parameter family of functionals

$$\mathcal{Q}_{t'}(g) = \int_0^{t'} (\langle g', g' \rangle_t \pm (Jg) \cdot g') dt, \quad t' \in (0, 1]$$

defined on the space of closed loops $g : [0, t'] \rightarrow T_q M = \mathbf{R}^{2n}$ based at the origin. Note that these functionals are quadratic (and therefore have generically only the zero loop as critical point) and are invariant by translation and multiplication by -1 .¹

¹The symmetry group will be larger if, for instance, all the metrics $\langle \cdot, \cdot \rangle_t, 0 \leq t \leq t'$ are conformally equivalent.

Lemma 4.3 *The loop g belongs to the null space of $\mathcal{Q}_{t'}$ if and only if $-Jg$ is the translate of a closed trajectory α of $d\phi_t$, $0 \leq t \leq t'$.*

Proof: Let us suppose that q is a minimum so that

$$\mathcal{Q}_{t'} = \int_0^{t'} (\langle g', g' \rangle_t + (Jg) \cdot g') dt.$$

Since we may normalise H_t so that its minimum value $H_t(p)$ is 0, its 2-jet \tilde{H}_t may be written in local symplectic coordinates about $p = 0$ as

$$\tilde{H}_t(x) = \frac{1}{2} \sum B_{ij}(t)x_i x_j = \frac{1}{2} x \cdot B_t x,$$

for some symmetric matrix $B_t = B_{ij}(t)$. Then the inner product $\langle u, v \rangle_t = v \cdot (B_t)^{-1}u$, and the linearized flow $L_t = d\phi_t$ is generated by the vector field $-JBx$.²

Recall that the null space of a quadratic form \mathcal{Q} on a vector space V is defined to be

$$\text{null } \mathcal{Q} = \{g : \mathcal{Q}(g, h) = 0 \text{ for all } h \in V\}.$$

Thus $g \in \text{null } \mathcal{Q}$ if and only if g is a critical point of \mathcal{Q} . Now,

$$\begin{aligned} \frac{\partial}{\partial s} \Big|_{s=0} \mathcal{Q}_{t'}(g + s\xi) &= \int_0^{t'} \left(2\xi' \cdot B_t^{-1}g' + J\xi \cdot g' + Jg \cdot \xi' \right) dt \\ &= \int_0^{t'} 2\xi' \cdot (B_t^{-1}g' + Jg) dt. \end{aligned}$$

Hence, g is in the null space of $\mathcal{Q}_{t'}$ if and only if

$$B_t^{-1}g' + Jg = \text{const},$$

or equivalently if

$$g'(t) = B_t(-Jg(t) + c), \quad 0 \leq t \leq t'.$$

It follows that $-Jg(t) + c$ is a closed trajectory of the linearized flow L_t , $0 \leq t \leq t'$. □

Intuitively, the idea above is that when \tilde{H} , or equivalently B , is small, the term $\langle \cdot, \cdot \rangle_t$ dominates \mathcal{Q}_t . When q is a minimum, the closed orbits of the linearized flow L_t enclose negative area, which increases as H does. The two terms exactly balance out when Jg is an orbit of L_t , $0 \leq t \leq 1$. When q is a maximum the area enclosed by the closed orbits of L_t is positive, and similar reasoning applies.

The next step is to show that the values t' where the null space of \mathcal{Q}_t is non-trivial are conjugate values. In other words, for $t < \min t'$, \mathcal{Q}_t is positive definite, and the index of \mathcal{Q}_t increases at the passage of a conjugate value t' by a quantity equal to the (finite) nullity of $\mathcal{Q}_{t'}$. One can do this by a Lagrange multiplier method, or by using the Jacobi sufficient condition: see [3, 13]. This proves the first statement in Theorem 4.2.

The other statements are proved by investigating explicit deformations of $\phi_{t \in [0,1]}$ along the loops g in $T_q M$. Let $\bar{\alpha} : [0, t'] \rightarrow T_p M = \mathbf{R}^{2n}$ be a closed trajectory of L_t , $0 \leq t \leq t'$ and compose it with

²Recall that our convention is that the symplectic gradient X_H satisfies $\omega(X_H, \cdot) = dH$.

some slowing down function $f : [0, 1] \rightarrow [0, t']$ which is the identity on $[0, t' - \varepsilon]$ for $\varepsilon > 0$ sufficiently small and sends $[t' - \varepsilon, 1]$ onto $[t' - \varepsilon, t']$. If

$$\alpha(t) = \bar{\alpha}(f(t))$$

denotes this composition, define the loop g by requiring that

$$g(t) = J(\alpha(t) - \alpha(0)).$$

Thus, $-Jg$ follows a path which is, up to translation, the same as the path of a closed trajectory of the linearised isotopy during the time interval $[0, t']$. Observe that the choice of $\bar{\alpha}$ is not unique: it may be replaced by $\rho\bar{\alpha}$ for any non-zero scaling factor ρ , positive or negative.

Given such g we define G_t to be a vector field supported near p with gradient $\nabla G_t(p) = g(t)$. The corresponding deformation $\phi_{\varepsilon G_t} \circ \phi_t$ is the composition of ϕ_t with the time-1 map of εG_t . Thus, up to order 1 in ε ,

$$\phi_{\varepsilon G_t} \circ \phi_t(p) = -\varepsilon Jg(t) = \varepsilon(\alpha(t) - \alpha(0)).$$

Ustilovsky showed that it is possible to choose the vector field G_t in such a way that the energy of this deformation is the sum of the energy $\mathcal{L}(\phi_t)$ of the original path with $\mathcal{Q}(g)$ (up to terms of order ε^3). Therefore, if $\mathcal{Q}(g) < 0$, one can decrease the length of ϕ_t , while if \mathcal{Q} is positive definite one cannot.

The striking fact here is that the deformation which optimally reduces the length is given by composing the isotopy ϕ_t with a motion that moves p in the *same* direction as does the flow of ϕ_t , $0 \leq t \leq 1$, round p . Thus, if the linearised motion at p has a closed orbit, the path $\{\phi_t\}$ uses extra energy to keep the point p fixed rather than letting it move around p in the direction of this orbit. In the next section we extend the range of validity of this result, getting rid of most of the non-degeneracy hypotheses on the path $\gamma = \{\phi_t\}$.

4.2 Stability of geodesics: necessary condition

We use the preceding results as a guideline to give a rigorous proof of a necessary condition for the stability of geodesics. For simplicity, we first consider the case when M has dimension 2.

Theorem 4.4 *Let $H_{t \in [0,1]}$ be any Hamiltonian defined on a surface S , and $\gamma = \phi_t$, $0 \leq t \leq 1$, the corresponding path in $\text{Ham}^c(S)$. If γ is a stable geodesic, H_t has at least one fixed minimum p and one fixed maximum P at which the differential $d\phi_t$ of the isotopy has no non-trivial closed trajectory in the time interval $(0, 1)$. Indeed, if this condition fails, there is a canonical deformation of the path γ which reduces $\mathcal{L}(\gamma)$.*

Proof: We have already seen that a stable geodesic must have at least a fixed minimum and a fixed maximum. Assuming that at all fixed minima of the family $\{H_t\}$ the differentials have a closed trajectory of period less than 1, we construct a deformation of the path γ which increases the minimum of all H_t without changing the maxima, and hence reduces the Hofer length of γ . A similar argument works for maxima. In the first step of the proof we show how to avoid the worst degeneracies of H_t . The heart of the proof is Steps 2 and 3 which construct and analyse the scrubbing motion which reduces the length of the path, and Lemma 4.9 of Step 4 which handles the degeneracies of H_t at the fixed extremum.

Let p be a fixed minimum where $L_t = \{d\phi_t(p)\}_{t \in [0,1]}$ has a non-trivial closed trajectory in time less than 1. Observe first that this implies that p is isolated among the *fixed* extrema of $\{H_t\}$, since the manifold is a surface. Rescale all functions H_t so that their minimum value $H_t(p)$ is 0. Note that because $\{H_t\}_{t \in [0,1]}$ defines a geodesic, no function H_t can be identically zero. Then let $M = \min_t \max_S H_t > 0$ be the minimax of the family.

Step 1.

Working in local coordinates near $p = 0$, let $A = A(\delta)$ be the annulus $D(4\delta) - D(\delta/2)$ for some small $\delta > 0$, centered at the origin.

Lemma 4.5 *There exists a deformation of $\gamma = \{\phi_t\}$ to a path (with the same end points and same length $\mathcal{L}(\gamma)$) which is generated by a Hamiltonian which is strictly positive on A for all t .*

Proof: Suppose to begin with that, for at least one value t_0 , the 2-jet \tilde{H}_{t_0} of H_{t_0} at p is non-degenerate. We can assume that $t_0 \in (0, 1)$ is an interior value, and that δ, ξ are small enough so that H_t is strictly positive on A for all $t \in (t_0 - 2\xi, t_0 + 2\xi)$. Then let $f : N(A) \rightarrow [0, 1]$ be a S^1 -invariant function defined on a small neighbourhood of A and strictly positive and constant on A . As in Proposition 2.1, we consider the path $\{\Psi_t \circ \phi_t\}$, where Ψ_t is generated by the Hamiltonian $F_t = \lambda(t)f$, $0 \leq t \leq 1$, where

- $\beta : [0, 1] \rightarrow (-\varepsilon, \varepsilon)$ has vanishing integral; and
- β is equal to its minimum on $(t_0 - \xi, t_0 + \xi)$, and to its maximum on $[0, 1] - (t_0 - 2\xi, t_0 + 2\xi)$.

Since $\Psi_0 = \Psi_1 = \mathbb{1}$, the path $\{\Psi_t \circ \phi_t\}$ has the same end points $\mathbb{1}, \phi_1$ as $\{\phi_t\}$. If δ is chosen sufficiently small and ε is smaller than $M/2$, the maximum value of the generating Hamiltonian is unchanged and therefore so is the length \mathcal{L} . The new path is generated by a Hamiltonian, that we still denote H_t , which is the same as before everywhere except on $N(A)$ and is always strictly positive on A .

To obtain the same result when all the 2-jets \tilde{H}_t are degenerate, it is enough to show that we can slightly perturb $\{H_t\}$ so that some H_t is strictly positive on A . But since a non-constant closed trajectory exists, there must be at least two rank 1 functions $\tilde{H}_{t_1}, \tilde{H}_{t_2}$ with distinct kernels: one can then apply the same kind of argument but using this time a function f which is equal to two bump functions covering the two connected components of $K_1 \cap N(A)$, where K_1 is the kernel of \tilde{H}_{t_1} . This will transform H_{t_1} into a function strictly positive over A while reducing slightly some positive values of H_{t_2} . □

Remark 4.6 In order to make the last step above work, we used the fact that H_t is not identically 0 for any t . This is permissible because the path γ was assumed to be a geodesic and hence, according to Definition 1.1, must be regular. However, it is not necessary to assume regularity here: one can use the same trick as above to make a regular path of the same length as the given one. To see this, choose t_0 so that H_{t_0} is not identically zero, and let $f : S \rightarrow [0, 1]$ be a smooth function such that f is 0 on the set of all fixed minima of the family $\{H_t\}$ and equal to 1 out of some neighbourhood of this set. Then, for all small ν and all t in some neighbourhood $(t_0 - 2\xi, t_0 + 2\xi)$ of t_0 , $\max(H_t - \nu f)$ is reached on the same set as $\max H_t$ and $\nu f < H_t$ everywhere. It now suffices to compose ϕ_t with Ψ_t generated by λf , where $\lambda : [0, 1] \rightarrow (-\varepsilon, \varepsilon)$ has vanishing integral, reaches its minimum on $(t_0 - \xi, t_0 + \xi)$, and its maximum on $[0, 1] - (t_0 - 2\xi, t_0 + 2\xi)$. □

Step 2. Construction of the scrubbing motion

By Step 1, $m = \inf_t \inf_A H_t$ is > 0 . Since p is a minimum of H_t , the linearized isotopy $L_t = d\phi_t$ at p of the Hamiltonian \tilde{H}_t always rotates in the same direction (clockwise, in fact). Therefore, our hypothesis implies that it rotates some ray by more than a full turn, and it follows that there exists a closed trajectory $\alpha : [0, 1] \rightarrow \mathbf{R}^2 = T_p S$ of $\lambda \tilde{H}_t$ for some $\lambda \in (0, 1)$.¹ We construct an optimal deformation of the path ϕ_t , $0 \leq t \leq 1$, which increases the minimum of each $H_t|_{D(4\delta)}$, by composing ϕ_t with a loop $\psi_t^{\delta, \rho}$ which moves the points near p round a small loop (this is our *scrubbing motion*).

For each t , and each sufficiently small δ, ρ , consider the symplectic diffeomorphism $\psi_t^{\delta, \rho}$ of $D(3\delta)$ whose restriction to $D(2\delta)$ is the translation by $\rho\alpha_0(t)$ where

$$\alpha_0(t) = \alpha(t) - c, \quad c = \alpha(0),$$

and which is smoothed to the identity on the annulus $D(3\delta) - D(2\delta)$.

We construct the $\psi_t^{\delta, \rho}$ so that they form a closed path, that is

$$\psi_1^{\delta, \rho} = \psi_0^{\delta, \rho} = \mathbf{1}.$$

Thus each point of $D(2\delta)$ describes a small loop during this Hamiltonian isotopy.

Let F_t be the non autonomous Hamiltonian which generates the isotopy $\{\psi_t^{\delta, \rho}\}$. Since $\psi_t^{\delta, \rho}(x) = x + \rho\alpha_0(t)$ on $D(\delta)$, the function F_t must have the form

$$F_t(x) = \rho J\alpha'(t) \cdot x + z(t), \quad \text{for } x \in D(\delta).$$

where $z(t) = F_t(0)$. We normalize F_t by setting $F_t = 0$ on the boundary of $D(3\delta)$.

Lemma 4.7 $\int_0^1 z(t)dt = \text{area } \rho\alpha_0$.

Proof: Let $\beta : [0, 1] \rightarrow S$ be a path from a point $\beta(0) \in \partial D(3\delta)$ to $\beta(1) = 0$. Then

$$\begin{aligned} z(t) &= F_t(\beta(1)) = \int_0^1 \langle dF_t, \dot{\beta}(s) \rangle ds \\ &= \int_0^1 \omega(X_t, \dot{\beta}(s)) ds, \end{aligned}$$

where $X_t = \dot{\psi}_t^{\delta, \rho}(\beta(s))$ and ω is the standard symplectic form $\omega(u, v) = (Ju) \cdot v$ on \mathbf{R}^2 . Thus $\int z(t)dt$ is the total flux through the arc β , that is, the total algebraic amount of surface area which crosses the fixed arc α during the whole isotopy. This flux is not only independent of the choice of β but may also be computed by taking any family of time dependent arcs β_t , provided that each β_t begins at $\beta(0)$ and ends at $\beta(1)$, and $\beta_0 = \beta_1$. (Here we are using the fact that we are working locally in S so that the integral of ω over the sphere formed by the images of the paths β_t is zero.) Take $\beta_t = \lambda_t \circ \tilde{\lambda}_t$ where $\tilde{\lambda}_t$ is the image of the fixed arc β by $\psi_t^{\delta, \rho}$ and λ_t is the straight segment in $D(\delta)$ from $\psi_t^{\delta, \rho}(0) = \rho\alpha_0(t)$ to $\{0\}$ oriented that way. The total flux is the sum of that through $\tilde{\lambda}_t$ and that through λ_t .

¹Here we use the parameter λ as conjugate value parameter instead of t . In dimension 2, this will lead to a simpler and more elegant theory, since there is then a canonical choice of the loop α . With time as conjugate parameter, one is forced to take a closed loop $\alpha : [0, 1] \rightarrow T_p S$ obtained by composing the closed trajectory $\tilde{\alpha} : [0, t'] \rightarrow T_p S$ with some more or less arbitrary slowing down map f as we described in the last section.

Since the former follows the flow of the isotopy, the flux crossing it is zero. To calculate the flux through the arcs λ_t we use the fact that these paths are entirely contained in the disc $D(\delta)$ on which $\phi_t^{\rho, \delta}$ is translation by $\rho\alpha_0(t)$ with constant Hamiltonian vector field $X_t = \rho\alpha_0'(t)$. The flux at time t passing through a moving arc λ_t is the difference between the infinitesimal flow which passes through λ_t as if λ_t were fixed, and the infinitesimal area swept out by λ_t . The latter contribution integrates over t to give the area enclosed by the loop $\rho\alpha_0$, while the former is:

$$\begin{aligned} \int_0^1 \int_0^1 \omega(\rho\alpha_0'(t), \frac{\partial}{\partial s} \lambda_t) ds dt &= \int_0^1 \int_0^1 \omega(\rho\alpha_0'(t), -\rho\alpha_0(t)) ds dt \\ &= - \int_0^1 \rho^2 \alpha_0'(t) \cdot J\alpha_0(t) dt = 2 \text{ area } \rho\alpha_0. \end{aligned}$$

□

Note Since the area of $\rho\alpha_0(t)$ is negative, this average value of $F_t(q)$, for any $q \in D(\delta)$, is equal to a negative constant. Of course the average value at some points in the annulus $D(3\delta) - D(\delta)$ must be positive since the Calabi invariant of the isotopy is 0.

Now consider the path $\psi_t^{\delta, \rho} \circ \phi_t$. It is generated by the Hamiltonian $K_t = F_t + H_t \circ (\psi_t^{\delta, \rho})^{-1}$. We write $H_t = \tilde{H}_t + R_t$ on $D(4\delta)$ for all t , where \tilde{H}_t is the 2-jet of H_t at p and $\frac{R_t(x)}{\|x\|^2} \rightarrow 0$ when $q \rightarrow 0$. Correspondingly, we set

$$\tilde{K}_t = F_t + \tilde{H}_t \circ (\psi_t^{\delta, \rho})^{-1}.$$

By construction $K_t = H_t$ outside $D(4\delta)$.

Step 3. Calculation of the minimum of \tilde{K}_t on the disc $D(\delta)$.

For $x \in D(\delta)$,

$$\begin{aligned} \tilde{K}_t(x) &= F_t(x) + \tilde{H}_t \circ (\psi_t^{\delta, \rho})^{-1}(x) \\ &= z(t) + \rho J\alpha_0'(t) \cdot x + \tilde{H}_t(x - \rho\alpha_0(t)) \end{aligned}$$

is a non-homogeneous polynomial of degree 2. We now show that its minimum is reached at a *critical point* lying *inside* $D(\delta)$ even when \tilde{H}_t has rank 1. The reason is that we chose α' so that $J\alpha'$ is parallel to the gradient of \tilde{H}_t , which, as we shall see, implies that the minimum of \tilde{K}_t may be computed as if the Hessians $d^2\tilde{H}_t : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ were invertible for all t .

Lemma 4.8 *There is a continuous path $p(t)$ in \mathbf{R}^2 on which \tilde{K}_t assumes its minimum over \mathbf{R}^2 . By choosing ρ sufficiently small, we may assume that $p(t) \in D(\delta)$ for all t . Further,*

$$\int_0^1 \min \tilde{K}_t = \int_0^1 (1 - \lambda)\lambda\rho^2 \tilde{H}_t(\alpha_0) dt > 0.$$

Proof: We prove the lemma in dimension 2, but it clearly holds in any dimension. As in §4.1, we will write

$$\tilde{H}_t(x) = \frac{1}{2}x \cdot B_t x,$$

for some matrix B_t . Then, the Hessian d^2H_t is the linear transformation given by the matrix B_t , and the closed trajectory α of the Hamiltonian flow of $\lambda\tilde{H}_t$ satisfies the equation

$$\alpha' = -\lambda JB_t \alpha.$$

Therefore

$$d\tilde{K}_t(x) = \rho J\alpha' + d\tilde{H}_t(x - \rho\alpha) = B_t(x - \rho\alpha_0 + \rho\lambda\alpha).$$

This is 0 when $x \in \rho\alpha_0 - \rho\lambda\alpha + \text{Ker}(\tilde{H}_t)$, and a smooth choice of critical points is given by

$$p(t) = \rho\alpha_0 - \rho\lambda\alpha = \rho(1 - \lambda)\alpha - \rho c.$$

It is clear that this is small if ρ is small, and that these critical points are absolute minima of \tilde{K}_t over \mathbf{R}^2 .

Observe that

$$\begin{aligned} \text{area } \alpha_0 &= \frac{1}{2} \int_0^1 J\alpha_0 \cdot \alpha'_0 dt \\ &= -\frac{1}{2}\lambda \int_0^1 \alpha_0 \cdot B_t\alpha_0 dt < 0. \end{aligned}$$

Therefore, by Lemma 4.7,

$$\begin{aligned} \int_0^1 \min \tilde{K}_t dt &= \int_0^1 \tilde{K}_t(p(t)) dt \\ &= \int_0^1 (z(t) + \rho^2 J\alpha'_0 \cdot (\alpha_0 - \lambda\alpha) + \frac{1}{2}\rho^2 \lambda^2 \alpha_0 \cdot B_t\alpha_0) dt \\ &= (-1 + 2(1 - \lambda) + \lambda)\rho^2 \lambda \int_0^1 \alpha_0 \cdot B_t\alpha_0 dt \\ &= (1 - \lambda)\rho^2 \int_0^1 \alpha_0 \cdot B_t\alpha_0 dt \\ &= (1 - \lambda)\lambda\rho^2 \int_0^1 \tilde{H}_t(\alpha_0) dt \end{aligned}$$

is strictly positive because α_0 is a non-constant trajectory of $\lambda\tilde{H}_t$. □

Step 4. The minimum of K_t .

In this step we show how to arrange that $\min K_t$ be strictly positive for all t . To begin we show that $\int_0^1 \min_{D(4\delta)} K_t$ is strictly positive.

Lemma 4.9 *If δ is sufficiently small, we may choose ρ so that*

$$\min_{D(4\delta)} K_t = \min_{D(\delta)} K_t \geq \min_{D(\delta)} \tilde{K}_t + \min_{D(2\delta)} R_t$$

with

$$\int_0^1 \left(\min_{D(\delta)} \tilde{K}_t + \min_{D(2\delta)} R_t \right) dt > 0.$$

Proof: Keeping δ fixed, and taking ρ sufficiently small with respect to $m = \min_t \max_A H_t$, we can insure that the minimum of $K_t|_{D(4\delta)}$ is reached inside $D(\delta)$. Now $H_t = \tilde{H}_t + R_t$ on $D(4\delta)$ where

$$\frac{R_t(x)}{\|x\|^2} \rightarrow 0 \text{ when } x \rightarrow 0.$$

Further $\tilde{K}_t = F_t + \tilde{H}_t \circ (\psi_t^{\delta, \rho})^{-1}$, where $(\psi_t^{\delta, \rho})^{-1}(D(\delta)) \subset D(2\delta)$. Thus, clearly,

$$\min_{D(\delta)} K_t \geq \min_{D(\delta)} \tilde{K}_t + \min_{D(2\delta)} R_t.$$

We have just seen that $\int \min_{D(\delta)} \tilde{K}_t$ has the form $c\rho^2$, where the constant c is independent of δ, ρ . On the other hand, $\int \min_{D(2\delta)} R_t = o(\delta^2)$ by the definition of R_t . Therefore, to prove the second part of the lemma, it suffices to show that we may choose $\rho = \rho(\delta)$ to be a linear function of δ . To check this, consider the dependency on δ of all parameters introduced so far. In Step 1 we introduced a fixed parameter ξ , and parameters ε, m . These have the form $\varepsilon = \text{const } \delta^2$, $m = \text{const } \delta^2$ since they both only depend on the value of the fixed function H_t (or of the fixed functions H_{t_1}, H_{t_2}) over $A(\delta)$. In Step 2, the functions F_t depend only on the parameter $\rho = \rho(\delta)$ which determines the size of the closed orbit. To insure that the scrubbing motion can be smoothed out to the identity on $D(3\delta) - D(2\delta)$, one may choose ρ such that $\max_t \rho \|\alpha(t)\| \leq \delta/6$, and to be sure that $\min_{D(4\delta)} K_t$ is reached on $D(\delta)$, it is enough to choose ρ so that the minimum over $D(4\delta)$ of the linear part of K_t be smaller than $m/3$, which means that $4\delta \max_t \|\rho\alpha'\| = 4\delta\rho \max_t \|\alpha'\| < m/3 = \text{const } \delta^2$. Thus $\rho(\delta)$ depends linearly on δ , as required. \square

We now use the technique of Proposition 2.1 again to deform the Hamiltonian $K_t, 0 \leq t \leq 1$, so that $\min_{D(4\delta)} K_t$ is strictly positive for all t . To do this, compose the isotopy with ψ_t generated by the Hamiltonian $F_t, 0 \leq t \leq 1$, defined by $F_t = \beta(t)f$ where $f : D(4\delta) \rightarrow [0, 1]$ is a S^1 -invariant bump function equal to 1 on $D(3\delta)$ and 0 near $\partial D(4\delta)$, and where $\beta : [0, 1] \rightarrow (m_0, m_1)$ has vanishing integral, with $m_0 = -\max_t \min_{D(4\delta)} K_t$ and $m_1 > -\min_t \min_{D(4\delta)} K_t$. As before, this composition has the same end points $\mathbb{1}, \phi_1$, it does not increase the Hofer length of the path. It is now generated by a Hamiltonian, still denoted by H_t , which is the same as before everywhere except on $D(4\delta)$ where each H_t is now strictly positive.

Step 5. Completion of the proof of Theorem 4.4.

Repeating the above process near each of the finite number of fixed minima of H_t , we deform H_t to a Hamiltonian K_t with

$$\max_x K_t(x) = \max_x H_t(x), \quad \min_{x \in N} K_t(x) > \min_{x \in N} H_t(x) = 0,$$

for all t , where N is some neighbourhood of all fixed minima. Then, of course, $\{K_t|_{S-N}\}_{t \in [0, 1]}$ has no fixed minimum, and Proposition 2.1 implies that we can perturb $\{K_t\}_{t \in [0, 1]}$ so that their maxima are the same as those of H_t , but with minima satisfying

$$\int_0^1 \min_S K_t > \int_0^1 \min_S H_t.$$

Thus $\mathcal{L}(\{K_t\}) < \mathcal{L}(\{H_t\})$. Further, we may clearly choose $\{K_t\}$ to be as close to $\{H_t\}$ as we want in the C^∞ -topology. Thus γ is not a local minimum of \mathcal{L} . \square

Finally, note that the proof of Theorem 4.4 shows:

Theorem 4.10 *Let $\{H_t\}_{t \in [0, 1]}$ be a Hamiltonian defined on any symplectic manifold M , and $\gamma = \{\phi_t\}, 0 \leq t \leq 1$ the corresponding isotopy. Assume that each fixed extremum of $\{H_t\}$ is isolated among the set of fixed extrema. If γ is a stable geodesic, there exist at least one fixed minimum p and one fixed maximum P at which the differential of the isotopy has no non constant closed trajectory in time less than 1.*

Proof: The proof of Theorem 4.4 in the 2-dimensional case applies directly. Actually, the hypothesis on dimension has been used only once, namely to deduce that each fixed extremum is isolated. The only other argument of the proof which should be treated in a slightly different way is the use of t -conjugate values instead of λ -conjugate values. In arbitrary dimensions, one cannot derive the existence of a closed trajectory of $\lambda\tilde{H}_t, 0 \leq t \leq 1$ from the existence of a closed trajectory of $\tilde{H}_t, 0 \leq t \leq t'$. Thus, as we indicated above, the loop α must be replaced by a closed loop $[0, 1] \rightarrow T_p M$ obtained by composing the closed trajectory $\tilde{\alpha} : [0, t'] \rightarrow T_p M$ with a slowing down map $f : [0, 1] \rightarrow [0, t']$. The rest of the proof is similar, although the proof of Lemma 4.8 in Step 3 must be adapted accordingly. \square

This theorem has the following obvious corollary:

Corollary 4.11 *Let M be a compact symplectic manifold, and let $\phi \in \text{Ham}(M)$ be generic in the sense that all its fixed points are isolated. Then, any stable geodesic $\phi_t, 0 \leq t \leq 1$, from the identity to ϕ must have at least two fixed points at which the linearised isotopy has no non-constant closed trajectory in time less than 1.*

5 Symplectomorphisms of S^2

This section is devoted to proving the following result.

Proposition 5.1 *There is a symplectomorphism ϕ of S^2 which is not the endpoint of any stable geodesic from the identity. A fortiori, there is no shortest path from the identity to ϕ .*

The proof uses properties of the Calabi invariant. Recall, from [1] for example, that if $(M, d\lambda)$ is an exact symplectic manifold, Cal is a homomorphism $\text{Ham}^c(M) \rightarrow \mathbf{R}$ defined by:

$$\text{Cal}(\phi) = \int_{M \times [0,1]} H_t \omega^n dt,$$

where H_t is any compactly supported Hamiltonian with time-1 map ϕ .¹ Thus

$$\text{Cal}(\phi) \leq \|\phi\|.$$

A crucial point is that H_t must be compactly supported. We will see below that if $\phi \in \text{Ham}(S^2)$ is the identity near both poles p_s, p_n , then the Calabi invariant of ϕ considered as an element of $\text{Ham}^c(S^2 - p_s)$ may be very different from the corresponding invariant calculated with respect to $\text{Ham}^c(S^2 - p_n)$. It is this fact which complicates the use of Calabi invariant on S^2 .

Before starting the construction, we prove the following easy lemma.

Lemma 5.2 *Let $\{\phi_t\}$ be any isotopy in \mathcal{P} with fixed minimum at p and fixed maximum at P , and let α be a path in M from p to P . Then $\mathcal{L}(\{\phi_t\})$ is the absolute value of the area swept out by α under the isotopy $\{\phi_t\}$.*

¹Although this definition does not appear to use the exactness of ω , this is needed to show that Cal is independent of the choice of the homotopy class of $\{\phi_t\}$. For general non-compact M , Cal is defined on the universal cover of $\text{Ham}^c(M)$.

Proof: There are several ways to see this. Here is a geometric argument. Let $H(x, t)$ be the Hamiltonian which generates $\{\phi_t\}$ and consider the surface S in its graph Γ_H made up of the characteristic lines starting at the points of α :

$$S = \{(\phi_t(\alpha(u)), H(\phi_t(\alpha(u)), t), t) : u, t \in [0, 1]\}.$$

Then the form $\Omega = \omega \oplus ds \wedge dt$ vanishes on S since it is a union of characteristic lines. Thus

$$\mathcal{L}(\{\phi_t\}) = \int_S ds \wedge dt = - \int_S \omega$$

is (up to sign) the area swept out by α under the isotopy. \square

On the 2-sphere S of radius 1 centered at the origin of \mathbf{R}^3 , take coordinates $\theta : S - \{p_s, p_n\} \rightarrow [0, 2\pi]$ and $z : S \rightarrow [-1, +1]$, where $\{p_s, p_n\}$ are the south and north poles, $\theta(x, y, z)$ is the positive angle of the point (x, y) with respect to the positive x -axis, and z is the height coordinate. The symplectic form is $d\theta \wedge dz$, with total area $A = 4\pi$. Thus the Hamiltonian flow of the function z is the positive rotation

$$(\theta, z) \mapsto (\theta + t, z).$$

We begin with the following proposition:

Proposition 5.3 *Let $h : \mathbf{R} \rightarrow \mathbf{R}$ be a smooth function either strictly convex everywhere or strictly concave everywhere, with $h'(\pm 1) \notin 2\pi\mathbf{Z}$, and ϕ the time 1 map of the Hamiltonian $H = h \circ z$ on S . Then the length of any stable geodesic $\psi_t, 0 \leq t \leq 1$, joining the identity to ϕ satisfies*

$$\mathcal{L}(\{\psi_t\}) \leq A$$

Proof: Let ψ_t be any stable geodesic from the identity to ϕ , generated by a Hamiltonian K_t . Let p, P be a fixed minimum and a fixed maximum of the family $\{K_t\}$, where by Corollary 4.11 the linearised Hamiltonian isotopy rotates no ray by more than a full turn. Then p, P belong to

$$\text{Fix}(\phi) = \{(\theta, z) \mid h'(z) \in 2\pi\mathbf{Z} \text{ or } z = \pm 1\}$$

which is the union of a discrete set of parallels. By Lemma 5.2, $\mathcal{L}(\{\psi_t\})$ is equal to the area swept out by the curve $\psi_t(\alpha(s)), 0 \leq s \leq 1$, during the time interval $0 \leq t \leq 1$, where α is any path from p to P oriented accordingly. First assume that p, P do not belong to the same parallel. Call a path α from p to P **admissible** if it is locally the graph of a function $\theta(z)$: it is a smooth embedded curve everywhere transversal to the parallels, which can meet the poles p_n or p_s only at its end-points and only when $\{p_n, p_s\} \cap \{p, P\} \neq \emptyset$. Since h is strictly convex or concave, and $h'(\pm 1) \notin 2\pi\mathbf{Z}$, the map $d\phi$ at any $q \in \text{Fix}(\phi)$ has only the tangent space $T_q \text{Fix}(\phi)$ as eigenspace. Thus for any admissible curve α :

- a) α intersects $\phi(\alpha)$ transversally at interior points of α located on $\text{Fix}(\phi)$, and all these intersection points have same sign; and
- b) $\alpha(i) = \phi(\alpha(i)), i = 0, 1$, and the tangent vectors are transversal there.

Denote by $\sharp(p, P)$ the algebraic number of interior points of intersection, which is simply, up to a sign, the number of parallels in $\text{Fix}(\phi)$ lying strictly between p and P , thus independent of the choice of the admissible curve.

Lemma 5.4 $\sharp(p, P) = 0$.

Proof. Let α be an admissible curve from p to P . There is a Hamiltonian conjugation which sends K_t to a Hamiltonian \hat{K}_t on S such that $\hat{p}, \hat{P} = p_s, p_n$, and sends α to a meridian $\hat{\alpha}$. Then $\phi(\alpha)$ is sent to $\hat{\phi}(\hat{\alpha})$ which intersects $\hat{\alpha}$ at $\sharp(p, P)$ interior points of same sign. Note that the linearised isotopies $d\hat{\psi}_t$ at p_s, p_n rotate in the positive θ -direction because p_s, p_n are the minimum and maximum respectively, but no ray turns by more than a full turn. Further, the tangent vectors of $\hat{\alpha}$ and $\hat{\phi}(\hat{\alpha})$ at p_s and p_n are still transversal. Blow-up the sphere at p_s, p_n : the map $C = ([0, 2\pi]/\{0 = 2\pi\}) \times [-1, 1] \rightarrow S$ defined by the coordinates θ, z admits a unique lifting of the isotopy $\hat{\psi}_t(\hat{\alpha})$ such that $\hat{\psi}_t(\hat{\alpha}(0))$ is lifted to $(\theta, -1)$ where θ is the angle of the tangent vector of $\hat{\psi}_t(\hat{\alpha})$ at $\hat{p} = p_s$ (and similarly at p_n). Now lift again to the universal covering $\mathbf{R} \times [-1, 1] \rightarrow C$ to get an isotopy $\tau : [0, 1] \times [0, 1] \rightarrow \mathbf{R} \times [-1, 1]$ beginning with $\text{Im}(\tau(s, t = 0)) = \{0\} \times [-1, 1]$. Of course, the condition on the differential of $\hat{\psi}_t$ at \hat{p}, \hat{P} means that $\theta(\tau(i, t)), i = 0, 1$, are non-decreasing functions of t with values in $[0, 2\pi] \subset \mathbf{R}$. But the transversality of the tangent vectors of $\hat{\alpha}$ and $\hat{\phi}(\hat{\alpha})$ at the end points implies that these functions have values in $(0, 2\pi)$. Since the interior intersection points of $\tau(s, 1)$ with each of the liftings $\{2\pi k\} \times [-1, 1]$ of $\hat{\alpha}$ are all transversal and have same sign, the image of $\tau(s, 1)$ must lie inside $(0, 2\pi) \times [-1, 1]$, which means that $\sharp(p, P) = 0$. \square

It follows from the proof of this lemma that the area swept out by any curve joining p to P is at most A . By Lemma 5.2, this proves Proposition 5.3 when p, P do not belong to the same parallel.

If p, P belong to the same parallel, there is no need to introduce $\sharp(p, P)$: take $\alpha = \phi(\alpha)$ a segment of the parallel to which p, P belong, and the above lifting argument shows that either $\theta(\tau(0, 1)) = \theta(\tau(1, 1)) = 0$ or $\theta(\tau(0, 1)) = \theta(\tau(1, 1)) = 2\pi$. In the first case, the area of τ is 0, and is A in the second one. The condition $h'(\pm 1) \notin 2\pi\mathbf{Z}$ is not necessary, but slightly simplifies the proof: without it, we would lose the transversality condition of tangent vectors of $\hat{\alpha}$ and $\hat{\phi}(\hat{\alpha})$ at the poles, and we would need to keep track of the signs to reach the same result. \square

Let $h : [-1, 1] \rightarrow \mathbf{R}$ be any smooth function with $h'(\pm 1) \notin 2\pi(\mathbf{Z} + \frac{1}{2}) = \{2\pi(k + \frac{1}{2}) \mid k \in \mathbf{Z}\}$. Set $\mathcal{Z} = \{\bar{z} \in (-1, 1) \mid h'(\bar{z}) \in 2\pi\mathbf{Z}\}$, and let us denote by $h_{\bar{z}} : [-1, 1] \rightarrow \mathbf{R}$ the map $h_{\bar{z}}(z) = h(\bar{z}) + \rho(z - \bar{z})$ where ρ equals $h'(\bar{z})$ if $\bar{z} \neq \pm 1$, and equals $2\pi k$ when $\bar{z} = \pm 1$ with k the unique integer that minimizes $|2\pi k - h'(\bar{z})|$. Thus $h_{\bar{z}}$ is the 1-jet of h at \bar{z} when $\bar{z} \neq \pm 1$, and is close to the 1-jet when $\bar{z} = \pm 1$. One should think of $h_{\bar{z}}$ as the correction term which is needed to make h compactly supported when considered as a function on $S^2 - \bar{x}$, where $\bar{x} \in h^{-1}(\bar{z})$. Of course, $h - h_{\bar{z}}$ does not quite have compact support in $S^2 - \bar{x}$, but its 1-jet at \bar{x} is zero which, as we shall see, means that we can use it to calculate the Calabi invariant about \bar{x} of a slight perturbation of its time-1 map. Finally, set

$$c(h) = -4\pi + \inf_{\bar{z} \in \mathcal{Z} \cup \{-1, 1\}} \left| \int_{-1}^1 (h - h_{\bar{z}}) dz \right|$$

if the right hand side is positive, and set $c(h) = 0$ otherwise.

The proof of the Proposition now boils down to the following:

Proposition 5.5 *Let $h : [-1, 1] \rightarrow \mathbf{R}$ be any smooth function with $h'(\pm 1) \notin 2\pi(\mathbf{Z} + \frac{1}{2})$, and let ϕ be the time 1 map of $H = h \circ z$. Then any stable geodesic $\psi_t, 0 \leq t \leq 1$, joining the identity to ϕ satisfies*

$$\mathcal{L}(\{\psi_t\}) \geq \frac{c(h)}{2}.$$

Proof. Let ψ_t be a stable geodesic from the identity to $\phi = \psi$ generated by K_t , $0 \leq t \leq 1$, with fixed minimum and fixed maximum p, P . We rescale K_t so that $K_t(p) = 0$ for all t . The point p belongs to $\text{Fix}(\phi) = z^{-1}(\mathcal{Z} \cup \{-1, 1\})$. We will calculate in two ways the Calabi invariant about p of a diffeomorphism $\tilde{\phi}$ which is very close to ϕ .

Suppose first that $z(p) \in \mathcal{Z}$. Let

$$G = (h - h_{\bar{z}}) \circ z$$

where $\bar{z} = z(p)$. Then p is a critical point of G , $G(p) = 0$ and, because the flows of $h_{\bar{z}}$ and h commute, G has time-1 map ϕ . Then let us denote by ϕ_t the flow generated by G . Now let β be a bump function with support very near the point p , and $\bar{\phi}_t$ be the isotopy generated by $\bar{G} = \beta G$. (Note that this has very small support.) Setting

$$\tilde{G} = ((1 - \beta)G) \circ \bar{\phi}_t$$

and denoting by $\tilde{\phi}_t$ its flow, one easily sees that:

$$\bar{\phi}_t \circ \tilde{\phi}_t = \phi_t \quad \text{or equivalently} \quad \bar{G} * \tilde{G} = G$$

where $*$ is defined in the proof of Proposition 2.1. Thus

$$\tilde{\phi} = \tilde{\phi}_1 = \bar{\phi}_1^{-1} \circ \phi$$

is very close to ϕ . Further, because $\tilde{G} = 0$ near p we may use it to calculate the Calabi invariant of $\tilde{\phi}$ about p , that is, the Calabi invariant of $\tilde{\phi}$ considered as an element of $\text{Ham}(S^2 - p)$. We find:

$$\begin{aligned} \text{Cal}_p(\tilde{\phi}) &= \int_t \int_S \tilde{G} \omega \\ &= \int_t \int_S G(\bar{\phi}_t(x)) \omega \quad + \varepsilon_1 \\ &= 2\pi \int_{-1}^1 (h - h_{\bar{z}}) dz \quad + \varepsilon_1 \end{aligned}$$

(In general, in what follows, there will be various small constants ε_i which can be made as small as we want by choosing appropriate bump functions.)

Now let us do the calculation using K_t . We will add the isotopy $\bar{\phi}_t^{-1}$ to ψ_t (we could tack it on at the end, that is do ψ_t a fraction faster, and then do $\bar{\phi}_t^{-1}$ quite quickly) to get an isotopy Ψ_t to $\tilde{\phi}$ generated by F_t . Note that $\Psi_t(p) = p$ for all t , and $\Psi_1 = \tilde{\phi} = \mathbb{1}$ near p .

Because p is a minimum of K_t , the rotation of $d\psi_t$ about p is always in the negative direction. Also, $\bar{\phi}_t$ is C^0 -small, and equals the identity outside the support of β and on the parallel $z = \bar{z}$ where $G = 0$. Therefore, it contributes a total of less than π to the twisting at p . Thus, the isotopy Ψ_t rotates S around p by an angle θ_p equal either to 0 or to -2π .

Let δ be a bump function supported in a little disc centered at p , and let $\bar{\Psi}_t$ be the isotopy generated by δF_t . As before, let $\tilde{\Psi}_t$ be the isotopy generated by $((1 - \delta)F_t) \circ \bar{\Psi}_t$. Then $\tilde{\phi} = \Psi_1 = \bar{\Psi}_1 \circ \tilde{\Psi}_1$. Since all three diffeomorphisms fix a neighbourhood of p , we can write:

$$\text{Cal}_p(\tilde{\phi}) = \text{Cal}_p(\bar{\Psi}_1) + \text{Cal}_p(\tilde{\Psi}_1).$$

Let us begin by computing the first term of the right hand side: $\bar{\Psi}_1 = \mathbb{1}$ except on a little annulus A centered at p , whose inner boundary is rotated through angle θ_p with respect to its outer boundary. The isotopy $\bar{\Psi}_t$ fixes the large disc outside A and moves a small disc near p . However, to calculate the Calabi invariant of $\bar{\Psi}_1$ about p , we must use an isotopy which fixes a neighbourhood of p . Thus this isotopy must rotate the large disc outside A through the non-negative angle $-\theta_p$, and therefore, viewed on the large disc outside A , centered at the point antipodal to p , this isotopy rotates the large disc through the non-positive angle θ_p . It follows easily that

$$\text{Cal}_p(\bar{\Psi}_1) = 4\pi\theta_p + \varepsilon_3.$$

Therefore

$$\text{Cal}_p(\tilde{\phi}) = \text{Cal}_p(\bar{\Psi}_1) + \text{Cal}_p(\tilde{\Psi}_1) = 4\pi\theta_p + c_p + \varepsilon_4$$

where $c_p = \int_t \int_S (K_t \omega)$.

Thus the two calculations give

$$\int_{-1}^1 (h - h_{\bar{z}}) \leq \frac{c_p}{2\pi} \leq 4\pi + \int_{-1}^1 (h - h_{\bar{z}})$$

which implies that $\frac{|c_p|}{2\pi} \geq c(h)$, and therefore that

$$\mathcal{L}(\{\psi_t\}) \geq \frac{|c_p|}{4\pi} \geq \frac{c(h)}{2}.$$

If p is one of the poles, the same argument applies if one takes $h_{\bar{z}} = h(\bar{z}) + 2\pi k(z - \bar{z})$ where $\bar{z} = z(p) = \pm 1$ and k is the integer which minimizes $|2\pi k - h'(\bar{z})|$. Indeed, the 2-jet of $G = (h - h(\bar{z}) - 2\pi k(z - \bar{z})) \circ z$ then generates a flow which rotates the tangent space $T_p S$ by less than π , and the same argument goes through. Here again, the hypothesis that $h'(\pm 1) \notin 2\pi(\mathbf{Z} + \frac{1}{2})$ is not necessary, but slightly simplifies the definition of $c(h)$. \square

Proof of Proposition 5.1

If $h : [-1, 1] \rightarrow \mathbf{R}$ is a strictly convex function with $h'(\pm 1) \notin \pi\mathbf{Z}$, and if the second derivative $h''(z)$ is large enough (for instance equal to a large constant), then $\frac{c(h)}{2} > A = 4\pi$, and there cannot exist a stable geodesic joining the identity to the time 1 map ϕ of $H = h \circ z$. \square

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