

Period Doubling, Entropy, and Renormalization

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Abstract

We show that in any family of stunted sawtooth maps, the set of maps whose set of periods is the set of all powers of 2 has no interior point, *i.e.*, the combinatorial description of the boundary of chaos coincides with the topological description. We also show that, under mild assumptions, smooth multimodal maps whose set of periods is the set of all powers of 2 are infinitely renormalizable.

1 Introduction

The present work is motivated by the following folklore conjecture (see also [OT]):

Conjecture A. A real polynomial map of f with set of periods (of its periodic orbits)

$$P(f) = \{1, 2, 4, \dots, 2^i, \dots\} = \{2^n : n \in \mathbb{N}\}$$

can be approximated by polynomials maps with positive entropy and by polynomials maps with finitely many periodic orbits.

This conjecture is now established for quadratic polynomials (as a consequence of [Su] or [La]) and work is in progress toward generalization for higher degree polynomials [Hu]. The interest in such a conjecture comes from Theorems A and B below (see section 2.1) and the fact that *topological entropy* (conceived as an invariant of topological conjugacy [AKM]) is also one way to measure the complexity of the dynamics of a map (see section 2.1): one is trying to describe how maps with simple dynamics can be deformed to maps with complicated dynamics, or, as one says, chaotic maps. Tradition, as well as the availability in this framework of a greater set of techniques, have put some emphasis on the particular case of polynomial maps, as in Conjecture A. However, the problem of the transition to chaos is more generally interesting in the category of smooth maps, in particular smooth endomorphisms of the interval, for which we recall the following:

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Conjecture B. All endomorphisms of the interval $f \in C^k(I)$, $k \geq 1$, with $P(f) = \{2^n : n \in \mathbb{N}\}$ are on the boundary of chaos and on the boundary of the interior of the set of zero entropy.

We first show that any *stunted sawtooth map* (see section 2.2), whose set of periods is the set of all powers of 2, can be approximated by stunted sawtooth maps with positive entropy and by stunted sawtooth maps with only finitely many periodic orbits (see section 2.3). This result solves the symbolic dynamic version of the above conjectures in the sense that stunted sawtooth maps carry all possible *kneading data* (see section 2.2) of multimodal maps.

We also make a second step toward Conjecture A by proving that maps with $P(f)$ as above, which satisfy some smoothness conditions (and in particular polynomial maps), are *infinitely renormalizable* (see section 2.4).

Sections 3 and 4 contain proofs of the results formulated in section 2.

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2 Preliminary definitions and results

2.1 Topological entropy of one dimensional maps

A point x is a *periodic point* of period n of a map f if $f^i(x) \neq x$, $0 < i < n$, and $f^n(x) = x$. The orbit of x is then called a *periodic orbit* (of period n). If $n = 1$, x is called a *fixed point*. The *topological entropy* $h(f)$ of a continuous map f on a compact metric space X with metric d can be defined as follows [Bo].

Given $\epsilon > 0$, $n \in \mathbb{Z}_+$, we say a subset $S \subset X$ is (n, ϵ) -*separated* if

$$x, y \in S, x \neq y \Rightarrow \exists m : 0 \leq m < n \text{ such that } d(f^m(x), f^m(y)) > \epsilon.$$

Set $H(f, n, \epsilon)$ to be the maximal cardinality of (n, ϵ) -separated sets, then

$$h(f) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log H(f, n, \epsilon).$$

For maps on an interval, the following result gives a necessary and sufficient condition for the positivity of topological entropy.

Theorem A. ([BF], [Ml]) A continuous map of an interval to itself has positive topological entropy if and only if it has a periodic point whose period is not a power of 2.

Remark: The “if” part is from [BF], the “only if” part from [Ml].

From Theorem A and [BlH], one gets the following.

Theorem B. In the space $C^k(I)$, $k \geq 1$, of C^k endomorphisms of an interval I , if a map f is on the boundary of positive topological entropy then the set $P(f)$ of its periods is $\{2^n : n \in \mathbb{N}\}$. The same is true for f on the boundary of the interior of the set of maps with zero topological entropy.

Remark: Conjectures A and B are about the converse of Theorem B.

We next give another necessary and sufficient condition for the positivity of topological entropy, which will be an important tool for us. This requires some more terminology.

So let $f \in C^0(I)$ and p be a fixed point of f . A point x of I belongs to the *unstable manifold* $W^u(p, f)$ of p if, for every neighborhood V of p , $x \in f^n(V)$ for some positive integer n . It is easy to check that $W^u(p, f)$ is connected and invariant under f . A point $x \in I$ is a *homoclinic point* of f if there is a periodic point p of f of period n such that $x \neq p$, $x \in W^u(p, f^n)$ and $f^{mn}(x) = p$ for some $m \in \mathbb{N}$ [Bl].

Theorem C ([Bl]) A map $f \in C^0(I)$ has positive topological entropy if and only if it has a homoclinic point.

2.2 Multimodal and stunted sawtooth maps

Consider the continuous map $f : I \rightarrow I$ where $I = [c_0, c_{d+1}]$. For $d \geq 0$, assume there are points c_i , $0 < i < d + 1$ with $c_j < c_{j+1}$ for $0 \leq j \leq d$ such that f is monotone on each lap $[c_j, c_{j+1}]$, and not monotone on any segment of the form $[c_j, c_{j+2}]$. Such a map is then called *d-modal* or *multimodal* with *modality* d (one says *amodal* if $d = 0$, *unimodal* if $d = 1$, and then *bimodal*, and so on). The maximal interval $[a_i, b_i]$ containing c_i on which f is constant is called a *turning interval* and, more precisely, a *plateau* if $a_i < b_i$, and a *turning point* if $a_i = b_i$,

The *shape* of a d -modal map is the alternating sequence of $d + 1$ signs, starting with either $+$ or $-$ according as the map is increasing or decreasing on its initial lap. By the *kneading data* associated with a d -modal map f we will mean its shape together with the collection of signs

$$\text{sgn}(f^n(c_i) - c_j) \in \{-1, 0, 1\}$$

for $n > 0$ and $1 \leq i, j \leq d$. The i -ordered collection of signs with j fixed is the j^{th} *kneading sequence* of f , and the j -ordered collection of kneading sequences is the *kneading invariant* of f (for more on kneading theory, we refer to [MiT] and [BORT]). One might wish to first understand at the symbolic level some questions one formulates for polynomials or smooth maps. It is in fact more practical to consider continuous families of d -modal maps rich enough to exhibit all possible kneading data for d -modal maps, yet significantly easier to study than smooth maps. Such families exist, and we next recall the construction of one of them.

By the *sawtooth map* of shape $s_1 s_2 \cdots s_{d+1}$, $s_i \in \{+, -\}$, $s_{i+1} = -s_i$, $d \geq 1$, we mean the unique map $S_d : I \rightarrow I$ which is piecewise linear with slope $s_1(d + 1)$, $s_2(d + 1)$, \cdots , $s_{d+1}(d + 1)$ ($d + 1$ alternate values). This is a d -modal map with topological entropy $\log(d + 1)$, the largest possible value for d -modal maps.

Given any *critical value vector* $w = (w_1, w_2, \dots, w_d)$ satisfying

$$(w_j - w_{j+1}) \cdot s_{(j+1)} < 0, \quad w_j \in I, \quad j = 1, 2, \dots, d,$$

we obtain the *stunted sawtooth map* S_w from S_d by cutting off the tops and bottoms of the graph at heights $w_j, j = 1, 2, \dots, d$. The d -parameter family of stunted sawtooth maps S_w is complete in the following sense [MiT, DGMT].

Theorem D. To any d -modal map f there is a canonical d -modal stunted sawtooth map S_w which has exactly the same kneading data as f .

2.3 The first main result

Theorem 1. Suppose S_w is a stunted sawtooth map with

$$P(S_w) = \{2^i : i \in \mathbb{N}\}.$$

Then for any $\epsilon > 0$, there exist w' and w'' such that

$$|w' - w| < \epsilon, \quad |w'' - w| < \epsilon,$$

$h(S_{w'}) > 0$ and $S_{w''}$ has only finitely many periods.

Corollary 1. In the parameter space, the set $\{w : P(S_w) = \{2^n : n \in \mathbb{N}\}\}$ has no interior point, *i.e.*, the combinatorial and the topological descriptions of the boundary of chaos coincide.

Remark: Let $f_i, i \in \{1, 2, 3\}$ be three d -modal maps of same shape with kneading invariants

$$K_i = \{K_{i,1}, K_{i,2}, \dots, K_{i,d}\}.$$

Assume that $P(f_2) = \{2^n : n \in \mathbb{N}\}$ and that, with the usual order on kneading sequences (see, *e.g.*, [BORT])

$$K_{1,j} < K_{2,j} < K_{3,j} \text{ if } s_j = +,$$

$$K_{1,j} > K_{2,j} > K_{3,j} \text{ if } s_j = -,$$

for $1 \leq j \leq d$. It then also follows from Theorem 1 that f_1 has only finitely many periodic orbits and that f_3 has positive topological entropy.

2.4 Renormalization

Let I be an interval. A map $f : I \rightarrow I$ is called *renormalizable* if there exists a proper subinterval J of I and an integer p such that

- (1) $f^i(J), i = 0, 1, \dots, p-1$, have no pairwise interior intersection,
- (2) $f^p(J) \subset J$.

Then $f^p|_J : J \rightarrow J$ is called a *renormalization* of f . A map $f : I \rightarrow I$ is *infinitely renormalizable* if there exist an infinite sequence $\{I_n\}_{n=1}^\infty$ of nested intervals and an infinite sequence $\{u(n)\}_{n=1}^\infty$ of integers such that $f^{u(n)}|_{I_n} : I_n \rightarrow I_n$ are renormalizations of f and the length of I_n tends to zero as $n \rightarrow \infty$.

Another purpose of this paper is to prove that maps f with $P(f) = \{2^i : i \in \mathbb{N}\}$, which satisfy some smooth conditions, are infinitely renormalizable. To this end, we shall prove the following abstract result for which we first recall some definitions.

Let $f \in C^0(I)$. A periodic point x of period n is *attracting* (resp. *one-sided attracting*) if there exists a neighborhood (resp. a one-sided neighborhood) U of x such that for any $y \in U$, $f^{nl}(y) \rightarrow x$ as $l \rightarrow \infty$. In both case we say the orbit of x is a *periodic attractor*. An open interval $J \subset I$ is called a *wandering interval* of f if

- 1) $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m, n, m \in \mathbb{N}$ and
- 2) $f^n(J)$ does not converge to a periodic orbit.

Theorem 2 Assume the multimodal map $f : I \rightarrow I$ with $P(f) = \{2^n : n \in \mathbb{N}\}$ has no wandering intervals, no plateaus, and no more than finitely many periodic attractors. Then f is infinitely renormalizable.

Let $f : I \rightarrow I$ be a map which is not constant on any open set. We say that f belongs to $\Gamma(2)$ if

- a) f is C^2 away from the turning points;
- b) For every $x_0 \in T_f$, there exists $\alpha > 1$, a neighborhood $U(x_0)$ of x_0 and a C^2 -diffeomorphism $\phi : U(x_0) \rightarrow (-1, 1)$ such that $\phi(x_0) = 0$ and

$$f(x) = f(x_0) \pm |\phi(x)|^\alpha, \quad \forall x \in U(x_0).$$

Theorem E. ($[Ma]$, $[MMS]$)

- (1) If $f \in \Gamma(2)$ then f has no wandering intervals.
- (2) Any $f \in \Gamma(2)$ has at most finitely many periodic attractors.

Combining this result with Theorem 2 yields:

Theorem 3. Any d -modal map $f \in \Gamma(2)$, with $P(f) = \{2^n : n \in \mathbb{Z}_+\}$ is infinitely renormalizable.

In particular, we have also:

Theorem 4 Any real polynomial map f with $P(f) = \{2^n : n \in \mathbb{N}\}$ is infinitely renormalizable.

Remark: Using results from [HS], the smoothness condition in (1) of Theorem E can be relaxed, which allows a proof of Theorem 3 with relaxed smoothness condition for unimodal maps.

Using more language from kneading theory, one could formulate a conjecture corresponding to Theorems 2 to 4 for general renormalizations (not just those at the boundary of chaos). Such a generalization completely escape the methods of the present paper.

3 Proof of Theorem 1

Lemma 1. Let $f : I \rightarrow I$ be a multimodal map with $P(f) = \{2^i : i \in \mathbb{Z}_+\}$, and let $\Omega(f)$ be the set of accumulation points of the periodic points of f . Then no point in $\Omega(f)$ is periodic, so that $\Omega(f)$ is not a finite set.

Proof: Let $p \in \Omega(f)$ be a periodic point of period 2^n . Denote $g = f^{2^n}$. Then p is a fixed point of g . Look at the map g near p . Because f hence g has isolated turning intervals, there are only three types of local behaviors for g .

1. g is monotone in a small neighborhood of p (if g is monotone reversing then g^2 is monotone preserving in a small neighborhood of p).
2. p is in the interior or at the end of one of the plateaus of g .
3. p is a turning point of g .

Look at three iterates of g or g^2 near p , one can easily see that neither g nor g^2 has any periodic points with higher period in a small neighborhood of p . This contradicts that p be an accumulation point of period-doubling periodic points of f . \square

Remark: Lemma 1 is false for continuous maps: examples with $\Omega(f)$ reduced to a point are easily provided.

Let $\Sigma = \{0, 1\}^{\mathbb{N}}$ and let σ stand for the *adding machine*, i.e., the map $\sigma : \Sigma \rightarrow \Sigma$ defined by $\sigma(x_i)_0^\infty = (y_i)_0^\infty$, where $y_i = 1 - x_i$ if $x_j = 1$ for all $j < i$ and $y_i = x_i$ otherwise. The following result is proved in [M2].

Theorem F. ([M2]) Let $f \in C^0(I)$ be a continuous map with $P(f) = \{2^n : n \in \mathbb{N}^+\}$. Suppose that K is an infinite closed invariant set of f which supports an ergodic f -invariant nonatomic probability measure. Then there exists a continuous map $h : K \rightarrow \Sigma$ such that

$$h \circ f = \sigma \circ h.$$

Furthermore $h^{-1}(s)$ contains at most two points for any point $s \in \Sigma$.

In the proof of Theorem F (see [M2]), K can be expressed as a disjoint union $K = K_0 \cup K_1$, where the supporting intervals of K_0 and K_1 have disjoint interiors and $f(K_i) = K_{1-i}$, where

$i = 0, 1$. Hence K_0 and K_1 are invariant under f^2 . K_0 and K_1 have the same bisections under f^2 and so on. Thus one can express K as the disjoint unions

$$K = \cup_{i=1}^{2^n} K_i^{(n)}$$

for $n \in \mathbb{N}$. Therefore at least one point of each fiber $h^{-1}(s)$, $s \in \Sigma$, is recurrent.

Lemma 2. Suppose that S_w is a stunted sawtooth map with set of periods

$$P(S_w) = \{2^i : i \in Z_+\}.$$

Let $\Omega(S_w)$ be the set of accumulation points of period-doubling periodic points of S_w and $\Sigma(S_w)$ be the set of the endpoints of the interval I and the closures of the turning intervals of S_w . Then the intersection of $\Omega(S_w)$ and $\Sigma(S_w)$ is not empty.

Proof: Suppose that the intersection of $\Omega(S_w)$ and $\Sigma(S_w)$ is empty and let $\epsilon > 0$ stand for the distance $d(\Omega(S_w), \Sigma(S_w))$. Let $K \subset \Omega(S_w)$ be as in Theorem F. Clearly $d(K, \Sigma(S_w)) > \epsilon$. Suppose that $x \in K$ is recurrent. There exists $l = 2^k, k \geq 1$, (by Theorem F) such that $|f^l(x) - x| < \frac{1}{8}\epsilon$. Again by Theorem F, one can assume that $f^l(x) > x$. Let V be the largest neighborhood of x on which f^l is monotone. The slope of f^l on V is d^l and clearly $V \supset (x - \epsilon/d^l, x + \epsilon/d^l)$.

Assume first that f^l is orientation preserving on U . Since $f^l(x) > x$ and $f^l(x - \frac{7}{8}\epsilon/d^l) = f^l(x) - \frac{7}{8}\epsilon < x + \frac{1}{8}\epsilon - \frac{7}{8}\epsilon = x - \frac{3}{4}\epsilon < x - \epsilon/d^l$, there exists a point $p \in (x - \frac{7}{8}\epsilon/d^l, x)$ such that $f^l(p) = p$. The unstable manifold $U(f^l, p)$ of f^l at p contains $(p, f^l(x) + \epsilon)$. We know that $f^l(f^l(x)) < f^l(x)$ by Theorem F. Let W be the largest neighborhood of $f^l(x)$ on which f^l is monotone. Then $W \supset (f^l(x) - \epsilon/d^l, f^l(x) + \epsilon/d^l)$. No matter whether f^l is preserving or reversing, $f^l(f^l(x) - \epsilon/d^l)$ (or $f^l(f^l(x) + \epsilon/d^l)$) is equal to $f^l(f^l(x)) - \epsilon$, which is less than $f^l(x) - \epsilon < x + \frac{1}{8}\epsilon - \epsilon = x - \frac{7}{8}\epsilon < p$. This implies that there exists $y \in (x, f^l(x)) \subset U(f^l, p)$ such that $y \neq p$ and $f^l(y) = p$, which is a homoclinic point, a contradiction. If f^l is orientation reversing on V , one proceeds similarly. \square

Proof of Theorem 1: Since all turning points of S_w (if any) are eventually fixed by S_w , by Lemma 1 they are not in $\Omega(S_w)$. Let $\Lambda = E(S_w) \cap \Omega(S_w)$. For any $\epsilon > 0$, we push the concave plateaus up a little bit and the convex plateaus down a little bit, if one of their end points belongs to Λ , to get another stunted sawtooth map $S_{w'}$ with $|w - w'| < \epsilon$. If a periodic orbit of S_w has no point in the interior of any plateau of S_w then it is also a periodic orbit of $S_{w'}$. Thus $S_w \subset S_{w'}$ so that the arguments in the proof of Lemma 2 show that $S_{w'}$ has a homoclinic point. Thus $S_{w'}$ has positive topological entropy.

We next push all concave plateaus down a little bit and all convex plateaus up a little bit to get another stunted sawtooth map $S_{w''}$ with $|w - w''| < \epsilon$. Suppose that $P(S_{w''}) = \{2^n : n \in \mathbb{N}\}$. Then by the proof of Lemma 2, S_w has positive topological entropy, a contradiction. \square

4 Proof of Theorem 2

Lemma 3. Suppose $f \in C^0(I)$ has no attracting cycles and $P(f) = \{2^n : n \in \mathbb{N}\}$. Let K be as in Theorem F, and $K = K_0 \cup K_1$ with $f(K_i) = K_{1-i}, i = 0, 1$. Assume that $\sup\{x : x \in K_0\} < \inf\{x : x \in K_1\}$. Let $[K_i]$ denote the smallest closed interval containing $K_i, i = 0, 1$. Then for $i = 0, 1$ there exists a periodic point q of f with period $n = 1$ or 2 in the gap between $[K_0]$ and $[K_1]$ such that $U(f^2, q)$ contains $[K_i]$.

Proof: By Theorem F, there exists a fixed point p of f in the gap between $[K_0]$ and $[K_1]$. For $i = 1$, let $s = \inf\{x : x \in K_1\}$ and consider the map f^2 on the interval $[p, s]$. p is fixed by f^2 . Let q be the largest fixed point of f^2 in $[p, s]$. Because of Lemma 1, $q \neq s$ and then $f^2(y) > y$ for any $y \in (q, s)$. Clearly $f^2(K_1) \subset K_1$ and $f^2(s) > s$. Therefore s , hence $[K_1]$ by connectivity, is in $U(f^2, q)$. The case $i = 0$ is treated similarly. \square .

From Theorem F and Lemma 3, one has

Corollary 2. Suppose $f \in C^0(I)$ has no attracting cycles and $P(f) = \{2^n : n \in \mathbb{N}\}$. Let K be as in Theorem F, with $K = \bigcup_{i=1}^{2^n} K_i^{(n)}, n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists a periodic point p of period m , where $m = 2^n$ or 2^{n+1} , whose orbit is contained in the set $\bigcup_{i=1}^{2^n} [K_i^{(n)}] \setminus \bigcup_{i=1}^{2^{n+1}} [K_i^{(n+1)}]$ and such that $U(f^m, p)$ contains some $K_i^{(n+1)}$, where $1 \leq i \leq 2^{n+1}$.

Lemma 4 Assume a continuous map $f : I \rightarrow I$ with $P(f) = \{2^n : n \in \mathbb{Z}_+\}$ has no wandering interval, no plateau, only finitely many turning points and finitely many periodic attractors. Let K be as in Theorem F. Then the semi-conjugacy h in Theorem F is actually a conjugacy.

Proof: Suppose that h is not a conjugacy. Then by Theorem F there exists a point $s \in \sum$ such that $h^{-1}(s) = \{x, y\}, x \neq y$. We claim that $h^{-1}(\sigma^n(s))$ eventually contains a single point when n is large enough. Let I_n denote the supporting interval of $h^{-1}(\sigma^n(s)), n \geq 0$. Since there are only finitely many turning points and $I_n, n \geq 0$, are pairwise disjoint, there exists $m > 0$ such that I_n contains no turning points for any $n \geq m$. If the claim is false, then I_m is a wandering interval, a contradiction. Therefore one can assume that $f(x) = f(y)$, where $x, y \in K$ and $x \neq y$. We separate our considerations into three cases.

Case 1. Suppose that f is monotone preserving in some neighborhoods of x and y (the proof is similar when f is monotone reversing near x and y). By Corollary 2, there exists a periodic point p of f of period $n, n = 2^k, k > 0$, such that the unstable manifold $U(f^n, p)$ of f^n at p contains the interval $[x, y], p$ is not in $[x, y]$ and p is close enough to x or y . By continuity of f , when p is close enough to x (or y), $f(p)$ is very close to $f(x) = f(y)$. By the intermediate value theorem, there are at least two points $a, b \in (x, y)$ such that $f(a) = f(b) = f(p)$. Then $f^n(a) = f^n(b) = f^n(p) = p$. Hence a is a homoclinic point, a contradiction.

Case 2. Suppose that f is monotone preserving in a neighborhood of x and monotone reversing in a neighborhood of y (the proof is similar when the situation is reversed). Denote $x_t = f^t(x) = f^t(y)$, $t \in \mathbb{N}$. Let $\{c_i\}_{i \in \alpha}$ denote the set of turning points of f , where α is a finite set. Clearly there exists $t_0 > 0$ such that $(x_t, c_i) \cap K \neq \emptyset$ for any $t \geq t_0$, denoted by $K_{(t,i)}$. In the interval (x, y) we select v and w near x and y respectively with $f(v) = f(w)$. Because of the nonexistence of wandering intervals, the itineraries of $f(p) = f(q)$ and $f(v) = f(w)$ under f are eventually different. Let $v_t = f^t(v)$, $t \in \mathbb{N}$. Then there exists $t > t_0$ such that there is at least one $c_i \in (x_t, v_t)$. Clearly $K_{(t,i)} \subset (x_t, v_t)$ and $f^t((x, v)) = f^t((w, y)) \supset (x_t, v_t)$. From Theorem F and Corollary 2, there exists a periodic point p of period $n = 2^k$, $k > 0$, where p is not in $[x, y]$, such that p is very near x (or y), $f^j(p)$ is in the supporting interval $[K_{(t,i)}]$ and the unstable manifold $U(f^n, p) \supset [x, y]$. Clearly there exists a point $u \in (x, v)$ (or $u \in (w, y)$) such that $f^t(u) = f^j(p)$. Hence $u \in U(f^n, p)$, $u \neq p$ and $f^{nt}(u) = f^n(f^j(p)) = p$. So u is a homoclinic point, a contradiction.

Case 3. If at least one of x and y is a turning point, the proof is just a slight modification of that of Case 1 or Case 2. \square

Proof of Theorem 2: Let K be as in Theorem F. By Lemma 4, $f : K \rightarrow K$ is conjugate to $\sigma : \Sigma \rightarrow \Sigma$ by h . Separate the turning points of f into two parts. Let S_1 (resp. S_2) denote the turning points of f contained (resp. not contained) in K . Denote $K = \cup_{i=1}^{2^n} K_i^{(n)}$, $n \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $S_2 \cap \cup_i^{2^n} [K_i^{(n)}] = \emptyset$. It is sufficient to prove for any $x \in S_2$ there exists n_0 such that for any $n > n_0$, x is not in $\cup_i^{2^n} [K_i^{(n)}]$. Suppose the contrary. Then there exists an infinite sequence of nested intervals $[K_i^{(n)}]$ containing x . Hence $x \in \cap_{n \in \mathbb{N}} [K_{i(n)}^{(n)}]$. Since the end points of $\cap_{n \in \mathbb{N}} [K_{i(n)}^{(n)}]$ are in the same fiber of h , which has to be a point, x is in K , a contradiction. Now let $n > n_0$. Then f maps each $[K_i^{(n)}]$ onto another $[K_j^{(n)}]$. Therefore f is infinitely renormalizable. \square

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