

COMPLEX BOUNDS FOR CRITICAL CIRCLE MAPS

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ABSTRACT. We use the methods developed with M. Lyubich for proving complex bounds for real quadratics to extend E. De Faria's complex a priori bounds to all critical circle maps with an irrational rotation number. The contracting property for renormalizations of critical circle maps follows.

In the Appendix we give an application of the complex bounds for proving local connectivity of some Julia sets.

1. INTRODUCTION

The object of our consideration will be the family of analytic orientation-preserving self-homeomorphisms of the circle $f : \mathbf{T} \rightarrow \mathbf{T}$ with one critical point at 0. To fix our ideas we will assume that the critical point has order three, although our considerations are valid in the general case.

We will further refer to such maps as *critical circle maps*. A critical circle map has a well defined rotation number denoted further by $\rho(f)$. Examples of critical circle maps with any given rotation number are provided by the maps of the standard family $x \mapsto x + \theta - \frac{1}{2\pi} \sin(2\pi x)$.

We will be considering only mappings with irrational rotation number. For such mappings Lanford [La1, La2] has defined the infinite sequence of renormalizations.

De Faria in [dF1, dF2] has extended these renormalizations to the complex plane, and has developed the renormalization theory for critical circle maps, parallel to the Sullivan theory for unimodal maps of the interval. The key condition for applicability of this theory is the existence of certain geometric bounds on the renormalized maps, *the complex a priori bounds*.

Applying Sullivan's methods to circle mappings de Faria has shown the existence of complex a priori bounds for the class of mappings with $\rho(f) \in \text{dioph}^2$.

In this paper we use the method developed with M. Lyubich [LY] to show the existence of complex a priori bounds for all critical circle mappings with irrational ρ .

The method was originally used in [LY] to treat a special case of combinatorics of quadratic maps of an interval, *the essentially bounded combinatorics*. However, this particular case for the interval maps corresponds to the most general case for circle maps, which allows us to prove the following.

Theorem 1.1. *Let f be a critical circle map of Epstein class. Then f has complex a priori bounds.*

The theorem relies on the following key cubic estimate for the renormalizations of the map f :

$$|R^k f(z)| \geq c|z|^3, \quad (1.1)$$

with an absolute $c > 0$. The proof of the key estimate is outlined in §3.

The above result allows one to extend the renormalization theory developed by de Faria to all critical circle mappings (in preparation).

In the appendix we give another application of the complex a priori bounds, a new proof of the result of C. Petersen on local connectivity of Julia sets of some Siegel quadratics.

Acknowledgments. I would like to thank my academic advisor Mikhail Lyubich for a lot of helpful conversations and suggestions. I am grateful to Carsten Petersen for discussing some of his results with me. I also thank Scott Sutherland for his help with computer pictures. Finally most of this work was done at MSRI at Berkeley, and I would like to thank them for their hospitality.

2. PRELIMINARIES

2.1. General notations and terminology. We use $|J|$ for the length of an interval J , dist and diam for the Euclidean distance and diameter in \mathbb{C} . The notation $[a, b]$ stands for the (closed) interval with endpoints a and b without specifying their order.

Two sets X in Y in \mathbb{C} are called K -commensurable or simply commensurable if

$$K^{-1} \leq \text{diam } X / \text{diam } Y \leq K$$

with a constant $K > 0$ which may depend only on the specified combinatorial bounds. A configuration of points x_1, \dots, x_n is called K -bounded if any two intervals $[x_i, x_j]$, and $[x_k, x_l]$ are K -commensurable.

We say that an annulus A has a *definite modulus* if $\text{mod } A \geq \delta > 0$, where δ depends only on the specified bounds.

For a pair of intervals $I \subset J$ we say that I is contained *well inside* of J if for each component L of $J \setminus I$ we have $|L| \geq K|I|$ where the constant $K > 0$ depends only on the specified bounds.

Let f be an analytic map whose restriction to the circle T is a critical circle map. We reserve the notation $f^{-i}(z)$ for the i -th preimage of $z \in T$ under $f|_T$.

2.2. Hyperbolic disks. Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J \equiv \mathbb{C} \setminus (\mathbb{R} \setminus J)$ denote the plane slit along two rays. Let $\overline{\mathbb{C}_J}$ denote the completion of this domain in the path metric in \mathbb{C}_J (which means that we add to \mathbb{C}_J the banks of the slits).

By symmetry, J is a hyperbolic geodesic in \mathbb{C}_J . The *geodesic neighborhood* of J of radius r is the set of all points in \mathbb{C}_J whose hyperbolic distance to J is less than r . It is easy to see that such a neighborhood is the union of two \mathbb{R} -symmetric segments of Euclidean disks based on J and having angle $\theta = \theta(r)$ with \mathbb{R} . Such a hyperbolic disk will be denoted by $D_\theta(J)$ (see Figure 1). Note, in particular, that the Euclidean disk $D(J) \equiv D_{\pi/2}(J)$ can also be interpreted as a hyperbolic disk.

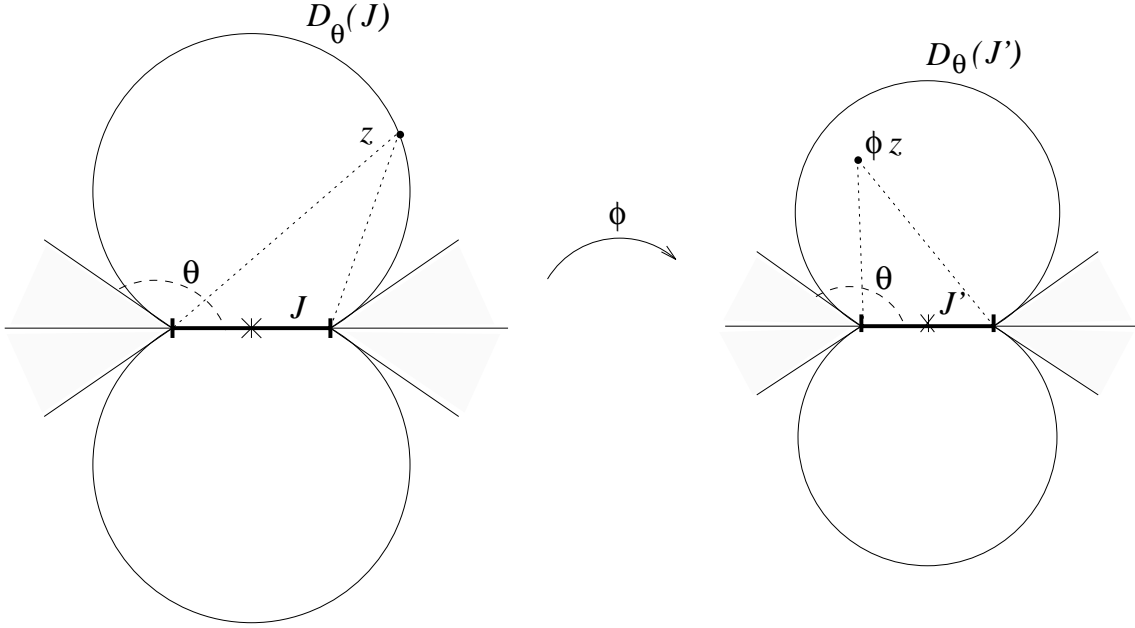


FIGURE 1

These hyperbolic neighborhoods were introduced into the subject by Sullivan [S]. They are a key tool for getting complex bounds due to the following version of the Schwarz Lemma:

Schwarz Lemma . *Let us consider two intervals $J' \subset J \subset \mathbb{R}$. Let $\phi : \mathbb{C}_J \rightarrow \mathbb{C}_{J'}$ be an analytic map such that $\phi(J) \subset J'$. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

Let $J = [a, b]$. For a point $z \in \bar{\mathbb{C}}_J$, the *angle between z and J* , $\widehat{(z, J)}$ is the least of the angles between the intervals $[a, z]$, $[b, z]$ and the corresponding rays $(a, -\infty]$, $[b, +\infty)$ of the real line, measured in the range $0 \leq \theta \leq \pi$.

We will use the following observation to control the expansion of the inverse branches.

Lemma 2.1. *Under the assumptions of the Schwarz Lemma, let us consider a point $z \in \mathbb{C}_J$ such that $\text{dist}(z, J) \geq |J|$ and $\widehat{(z, J)} \geq \epsilon$. Then*

$$\frac{\text{dist}(\phi z, J')}{|J'|} \leq C \frac{\text{dist}(z, J)}{|J|}$$

for some constant $C = C(\epsilon)$

Proof. Let us normalize the situation in this way: $J = J' = [0, 1]$. Notice that the smallest (closed) geodesic neighborhood $\text{cl } D_\theta(J)$ enclosing z satisfies:
 $\text{diam } D_\theta(J) \leq C(\epsilon) \text{dist}(z, J)$ (cf Fig. 1).

Indeed, if $\theta \geq \epsilon/2$ then $\text{diam } D_\theta(J) \leq C(\epsilon)$, which is fine since $\text{dist}(z, J) \geq 1$. Otherwise the intervals $[0, z]$ and $[1, z]$ cut out sectors of angle size at least $\epsilon/2$ on the circle

$\partial D_\theta(J)$. Hence the lengths of these intervals are commensurable with $\text{diam } D_\theta(J)$ (with a constant depending on ϵ). Also, by elementary trigonometry these lengths are at least $\sqrt{2} \text{dist}(z, J)$, provided that $\text{dist}(z, J) \geq |J|$.

By Schwarz Lemma, $\text{dist}(\phi z, J') \leq \text{diam}(D_\theta(J'))$, and the claim follows. \square

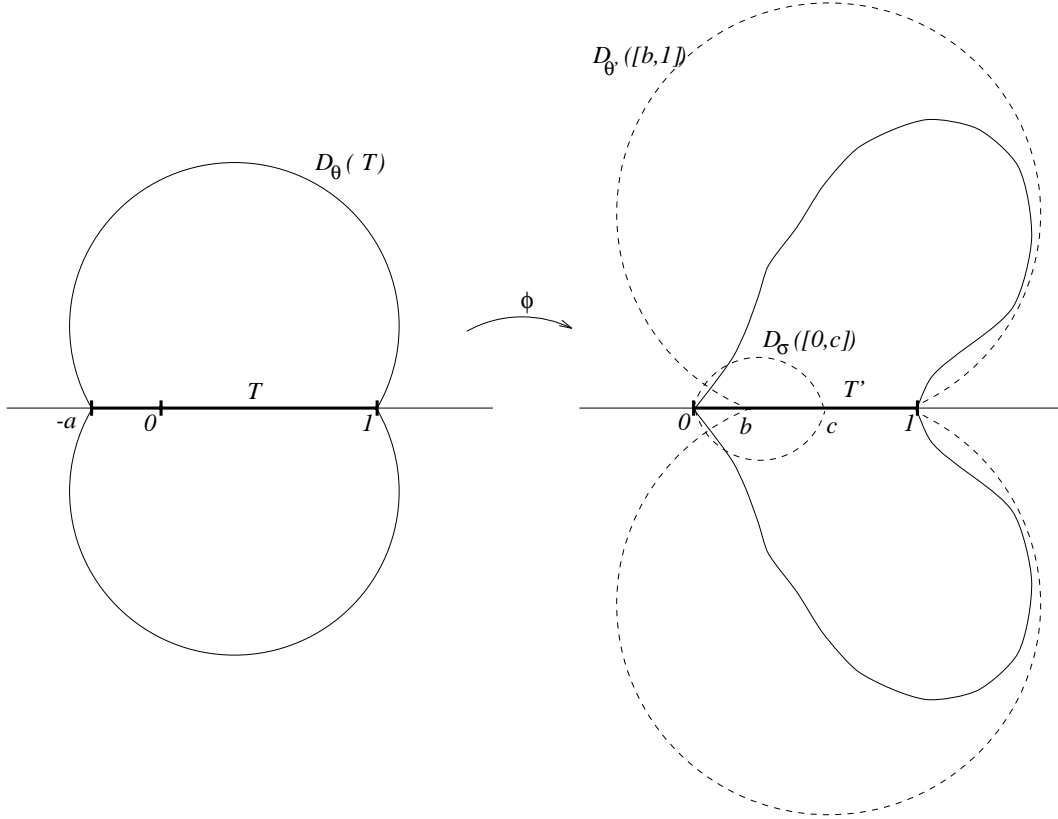


FIGURE 2

2.3. Cube root. In the next lemma we collect for future reference some elementary properties of the cube root map. Let $\phi(z)$ be the branch of the cube root mapping the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ into $\{z \mid |\arg(z)| < \pi/3\}$.

Lemma 2.2. *let $K > 1$, $\delta > 0$, $K^{-1} \leq a \leq K$, $T = [-a, 1]$, $T' = [0, 1]$. Then:*

- $\phi D_\theta(T) \subset D_{\theta'}(T')$, with θ' depending on θ and K only.
- Moreover, there exist $b, c \in [0, 1]$, such that $0, b, c, 1$ form a $C(K)$ -bounded configuration, and $\phi D_\theta(T) \subset D_{\theta'}([b, 1]) \cup D_\sigma([0, c])$ for $\sigma < \pi/2$ cf. Fig. 2.
- If $z' \in \phi D(T) \setminus D([-\delta, 1 + \delta])$, then

$$\widehat{(z', T')} > \epsilon(K, \delta) > 0 \text{ and } C(K, \delta)^{-1} < \text{dist}(z', T') < C(K, \delta).$$

2.4. Real Commuting Pairs and Renormalization. We present here a brief summary on renormalization of critical circle mappings. A more extensive exposition can be found for example in [dF2].

Let f be a critical circle mapping with an irrational rotation number $\rho(f)$. Let $\rho(f)$ have an infinite continued fraction expansion

$$\rho(f) = [r_0, r_1, r_2, \dots] = \frac{1}{r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \dots}}}.$$

We say that ρ is of *bounded type* if $\sup r_i < \infty$. This is equivalent to $\rho \in \text{dioph}^2$.

Denote by q_m the moments of closest returns of the critical point 0. Note that the numbers q_m appear as the denominators in the irreducible form of the m -th truncated continued fraction expansion of $\rho(f)$, $\frac{p_m}{q_m} = [r_0, \dots, r_{m-1}]$.

Let $I_m \equiv [0, f^{q_m}(0)]$. By Świątek- Herman real a priori bounds ([Sw, H]), the intervals I_m and I_{m+1} are K -commensurable, with a universal constant K provided m is large enough.

The dynamical first return map on the interval $I_m \cup I_{m+1}$ is f^{q_m} on I_{m+1} and $f^{q_{m+1}}$ on I_m . The consideration of pairs of maps

$$(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}})$$

leads to the following general definition due to Lanford and Rand (cf. [La1, La2, Ra1, Ra2]).

Definition 2.1. A (real) commuting pair $\zeta = (\eta, \xi)$ consists of two real orientation preserving smooth homeomorphisms $\eta : I_\eta \rightarrow \mathbb{R}$, $\xi : I_\xi \rightarrow \mathbb{R}$, where

- $I_\eta = [0, \xi(0)]$, $I_\xi = [\eta(0), 0]$;
- Both η and ξ have homeomorphic extensions to interval neighborhoods of their respective domains which commute, i.e. $\eta \circ \xi = \xi \circ \eta$ where both sides are defined;
- $\xi \circ \eta(0) \in I_\eta$;
- $\eta'(x) \neq 0 \neq \xi'(y)$, for all $x \in I_\eta \setminus \{0\}$, and all $y \in I_\xi \setminus \{0\}$.

A *critical commuting pair* is a commuting pair (η, ξ) , which maps can be decomposed as $\eta = h_\eta \circ Q \circ H_\eta$, and $\xi = h_\xi \circ Q \circ H_\xi$, where $h_\eta, h_\xi, H_\eta, H_\xi$ are real analytic diffeomorphisms and $Q(x) = x^3$.

Given a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\tilde{\zeta}$ the pair $(\tilde{\eta}|_{\tilde{I}_\eta}, \tilde{\xi}|_{\tilde{I}_\xi})$ where tilde means rescaling by a linear factor $\lambda = \frac{1}{|I_\eta|}$.

For a critical circle mapping f one obtains a critical commuting pair from the pair of maps $(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}})$ as follows. Let \bar{f} be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m . Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ denote the translation $x \mapsto x + 1$. Let $\eta : \bar{I}_m \rightarrow \mathbb{R}$, $\xi : \bar{I}_{m+1} \rightarrow \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|_{\bar{I}_m}, \xi|_{\bar{I}_{m+1}})$ forms a critical commuting pair corresponding to $(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}})$. Henceforth, we shall abuse notation and use

$$(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}}) \tag{2.1}$$

to denote this commuting pair.

We see that return maps for critical circle homeomorphisms give rise to a sequence of critical commuting pairs (2.1). Conversely, regarding I_η as a circle (identifying $\xi(0)$ and 0), we can recover a smooth conjugacy class of critical circle mappings $f^\phi = \phi \circ f_\zeta \circ \phi^{-1}$, where $\phi : I_\eta \rightarrow I_\eta$ is a smooth orientation preserving homeomorphism with a cubic critical point at 0, and f_ζ is a circle homeomorphism defined by

$$f_\zeta(x) = \begin{cases} \xi \circ \eta(x), & \text{if } 0 \leq x \leq \eta^{-1}(0) \\ \eta(x), & \text{if } \eta^{-1}(0) \leq x \leq \xi(0) \end{cases} \quad (2.2)$$

Let $\rho(\zeta) = \rho(f_\zeta)$ be the *rotation number of the commuting pair* ζ . If $\rho(\zeta) = [r, r_1, r_2, \dots]$, one verifies that the mappings $\eta|_{[0, \eta^r(\xi(0))]}$ and $\eta^r \circ \xi|_{I_\xi}$ again form a commuting pair.

Definition 2.2. The *renormalization* of a real commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi}|_{\widetilde{I}_\xi}, \widetilde{\eta}|_{[0, \widetilde{\eta^r(\xi(0))}]}).$$

It is easy to see that renormalization acts as a Gauss map on rotation numbers, i.e. if $\rho(\zeta) = [r, r_1, r_2, \dots]$, $\rho(\mathcal{R}\zeta) = [r_1, r_2, \dots]$.

Finally note, that the renormalization of the real commuting pair (2.1) is the rescaled pair $(\widetilde{f^{q_{m+2}}}|_{\widetilde{I}_{m+1}}, \widetilde{f^{q_{m+1}}}|_{\widetilde{I}_{m+2}})$.

In this way for a given critical circle mapping with an irrational rotation number we obtain an infinite sequence of renormalizations $\{(\widetilde{f^{q_{i+1}}}|_{\widetilde{I}_i}, \widetilde{f^{q_i}}|_{\widetilde{I}_{i+1}})\}_{i=1}^\infty$.

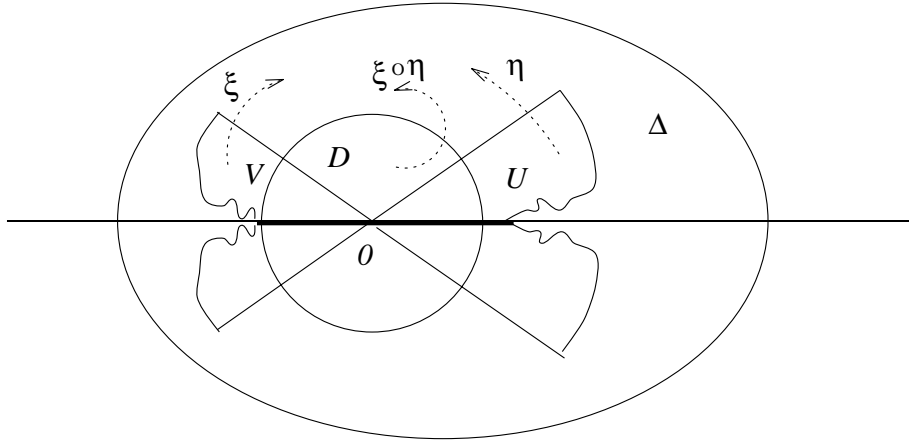


FIGURE 3

2.5. Holomorphic Commuting Pairs. Following [dF1, dF2] we say that a real commuting pair (η, ξ) extends to a *holomorphic commuting pair* (cf. Fig. 3) if there exist four \mathbb{R} -symmetric domains Δ, D, U, V , such that

- $\bar{D}, \bar{U}, \bar{V} \subset \Delta, \bar{U} \cap \bar{V} = \{0\}; U \setminus D, V \setminus D, D \setminus U$, and $D \setminus V$ are nonempty connected sets, $U \supset I_\eta, V \supset I_\xi$;

- mappings $\eta : U \rightarrow \Delta \cap \mathbb{C}_{\eta(J_U)}$ and $\xi : V \rightarrow \Delta \cap \mathbb{C}_{\xi(J_V)}$ are onto and univalent, where $J_U = U \cap \mathbb{R}$, $J_V = V \cap \mathbb{R}$;
- η and ξ have holomorphic extensions to D which commute, $\eta \circ \xi(z) = \xi \circ \eta(z) \forall z \in D$; $\eta \circ \xi : D \rightarrow \Delta \cap \mathbb{C}_{\eta \circ \xi(J_D)}$, where $J_D = D \cap \mathbb{R}$, is a three-fold branched covering with the only critical point at 0 .

Note that $J_D = (\eta^{-1}(0), \xi^{-1}(0))$.

Definition 2.3 (Complex Bounds). We say that a critical circle map f has *complex a priori bounds* if there exists M and $\mu > 0$, such that for all $m > M$ the real commuting pair (2.1) extends to a holomorphic commuting pair $(\Delta_m, D_m, U_m, V_m)$ with $\text{mod}(\Delta_m \setminus (D_m \cup U_m \cup V_m)) > \mu > 0$.

2.6. Epstein class. A map $g|I$ belongs to an *Epstein class*, if the restriction of g to I can be decomposed as $g \equiv h \circ Q$, where $Q : I \rightarrow J \equiv g(I)$ is a real cubic polynomial, and $h : J \rightarrow J$ is an orientation preserving diffeomorphism, which inverse h^{-1} extends to a univalent mapping $\mathbb{C}_{\tilde{J}} \rightarrow \mathbb{C}$, where $\tilde{J} \supset J$. A commuting pair (η, ξ) belongs to Epstein class if both maps do.

We will denote by \mathcal{E}_s the family of maps of Epstein class for which the length of each component of $\tilde{J} \setminus J$ is $s|J|$.

One immediately obtains:

Lemma 2.3. *An Epstein class \mathcal{E}_s is invariant under renormalization.*

We supply the space of maps of Epstein class with *Caratheodory topology* (see [McM1]).

The next Lemma follows from a standard normality argument.

Lemma 2.4. *For each $s > 0$ the space \mathcal{E}_s is compact.*

By real a priori bounds there exists a universal $s > 0$ such that $(f^{q_{m+1}}|I_m; f^{q_m}|I_{m+1}) \in \mathcal{E}_s$, for all sufficiently large m , provided f belongs to an Epstein class.

All maps in this paper are assumed to belong to some Epstein class \mathcal{E}_s . The bounds M and μ in the Definition 2.3 will depend only on the chosen value of s . Note that even for $\rho(f) \in \text{dioph}^2$ this improves the complex a priori bounds of E. de Faria which depend also on the combinatorics of the map.

3. OUTLINE OF THE PROOF

3.1. Main Lemma. Fix n , and let $p = q_{n+1}$.

Let us consider the decomposition

$$f^p = \psi_n \circ f, \tag{3.1}$$

where ψ_n is a univalent map from a neighborhood of $f(I_n)$ onto $\mathbb{C}_{f^p(I_n)}$.

Lemma 3.1. *There exist a disc D_0 around 0 and universal constants C_1 and C_2 depending only on real a priori bounds, such that $\forall z \in \mathbb{C}_{f^p(I_n)} \cap D_0$ the following estimate holds:*

$$\frac{\text{dist}(\psi_n^{-1}z, f(I_n))}{|f(I_n)|} \leq C_1 \left(\frac{\text{dist}(z, I_n)}{|I_n|} \right) + C_2, \tag{3.2}$$

where ψ_n is the map from (3.1).

Thus the map $\widehat{\psi_n^{-1}}$ has at most linear growth.

Note that if $(z, I_n) > \epsilon > 0$, then the inequality (3.2) follows directly from Lemma 2.1 with the constants depending only on ϵ . Our strategy of proving Lemma 3.1 will be to monitor the inverse orbit of a point z together with the interval I_n until they satisfy this “good angle” condition.

Lemma 3.1 immediately yields the key cubic estimate:

Proposition 3.2 (Key estimate). *Let U be the neighborhood of \widetilde{I}_n which is mapped univalently by $\widetilde{f^p}$ onto $\mathbb{C}_{\widetilde{\mathbb{D}}_1(\widetilde{\mathbb{I}}_K)}$. There exist universal constants B and C , and a disc D_0 around 0 such that $\forall z \in U$ with $\widetilde{f^p}(z) \in \widetilde{D}_0$ and $|\widetilde{f^p}(z)| > B$, one has:*

$$|\widetilde{f^p}(z)| > C|z|^3.$$

The Theorem 1.1 now follows.

Remark 3.1. We remark that it follows from our estimate that the domains Δ_m, D_m, U_m , and V_m in the definition of complex bounds (2.3) can be chosen K -commensurable with I_m , with a universal constant K .

4. PROOF OF LEMMA 3.1

4.1. Let $g : U \rightarrow \mathbb{C}_{\mathbb{T}}$ be a map of Epstein class. Take an $x \in \mathbb{R}$, and $z \in \mathbb{C}_{\mathbb{T}}$. If we have a backward orbit of $x \equiv x_0, x_{-1}, \dots, x_{-l}$ of x which does not contain 0, the *corresponding* backward orbit $z \equiv z_0, z_{-1}, \dots, z_{-l}$ is obtained by applying the appropriate branches of the inverse functions: $z_{-i} = g_{x_{-i}}(z)$, where $g_{x_{-i}}(x) = x_{-i}$.

Let D_m denote the disc $D([f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)])$.

Let p be as above, and consider the inverse orbit:

$$J_0 \equiv f^p(I_n), J_{-1} \equiv f^{p-1}(I_n), \dots, J_{-(p-1)} \equiv f(I_n) \quad (4.1)$$

For a point $z \in \mathbb{C}_{I_n}$ consider the corresponding inverse orbit

$$z_0 \equiv z, z_{-1}, \dots, z_{-(p-1)} \quad (4.2)$$

We say that a point z_{-i} ϵ -jumps if $(z_{-i}, J_{-i}) > \epsilon$ for some fixed $\epsilon > 0$.

Let us call the moment $-i$ “good” if the interval J_{-i} is commensurable with J_0 . For example the first few returns of the orbit (4.1) to any of the I_m are good.

We would like to assert that the points of the orbit (4.2) either ϵ -jump at a good moment, or follow closely the corresponding intervals of the orbit (4.1).

The first step towards this assertion is the following

Lemma 4.1. *Let $J \equiv J_{-k}, J_{-k-q_{m+1}} \equiv J'$ be two consecutive returns of the backward orbit (4.1) to I_m , and let ζ and ζ' be the corresponding points of the orbit (4.2).*

Suppose $\zeta \in D_m$, then either $\zeta' \in D_m$, or $(\zeta', J') > \epsilon$, and $\text{dist}(\zeta', J') < C|I_m|$.

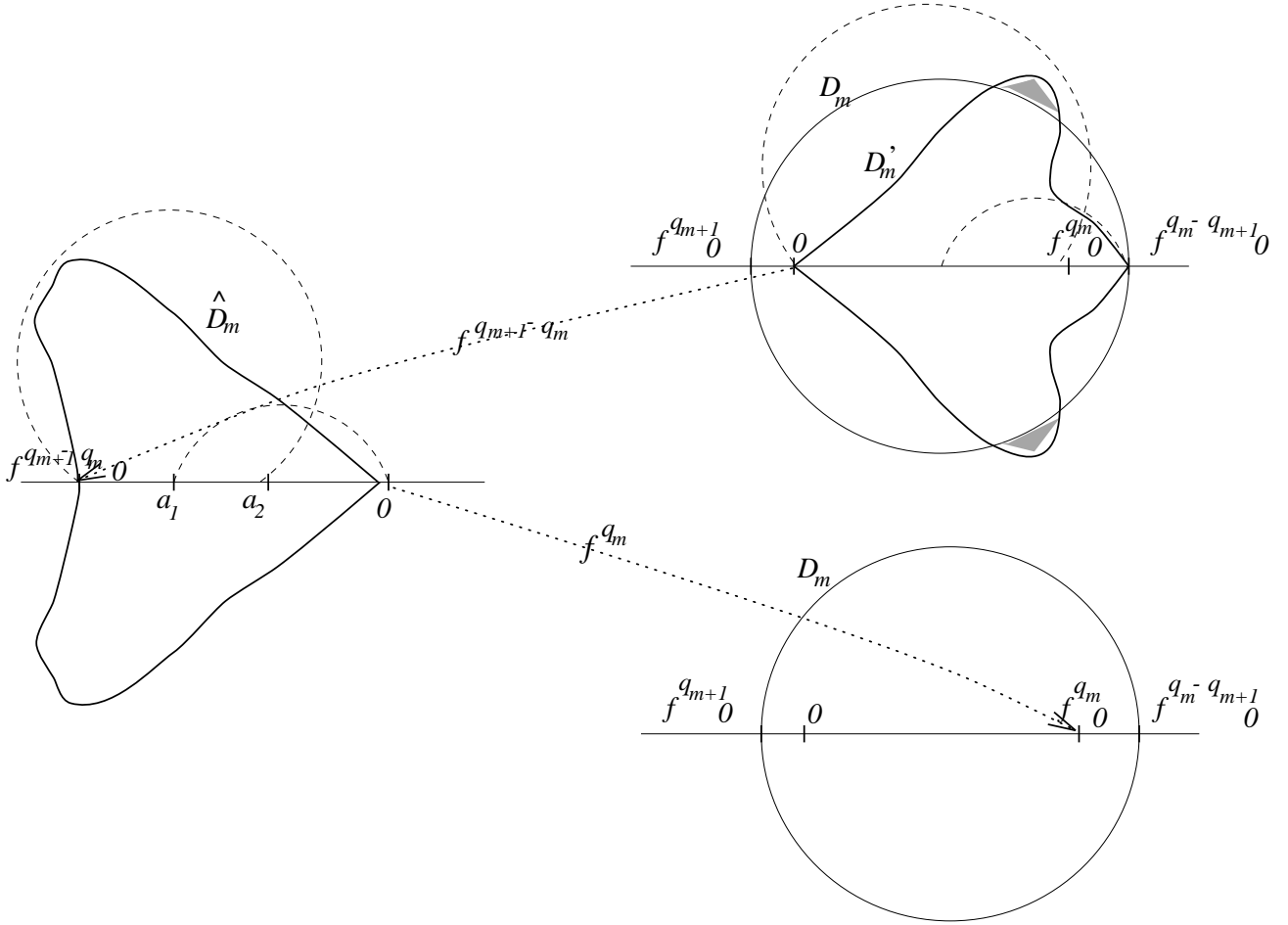


FIGURE 4

Proof. Let D'_m denote the pull back of D_m corresponding to the piece of backward orbit $J_{-k}, \dots, J_{-k-q_{m+1}}$, and let \hat{D}_m denote the pull back of D_m along the piece of the orbit $J_{-k} \rightarrow \dots \rightarrow J_{-k-q_m}$ (cf. Fig. 4). By Schwarz Lemma and by Lemma 2.2, there exist points $a_1, a_2 \in [f^{q_{m+1}-q_m}(0), 0]$, such that $f^{q_{m+1}-q_m}(0), a_1, a_2$, and 0 is a K -bounded configuration for some K independent on m , and angles θ and $\sigma < \pi/2$, also independent on m , such that $\hat{D}_m \subset D_\theta([f^{q_{m+1}-q_m}(0), a_1]) \cup D_\sigma([a_2, 0])$. Applying Schwarz Lemma, we obtain that $D'_m \subset D_m \cup D_\theta([0, f^{-q_{m+1}+q_m}(a_1)])$ and the claim immediately follows. \square

4.2. Saddle-node phenomenon. Let us note first that when q_{m+1}/q_m is large, the map $f^{q_{m+1}} : I_m \rightarrow I_m \cup I_{m+1}$ is a small perturbation of a map with a parabolic fixed point (see e.g. [H]). The next lemma is a direct consequence of real bounds.

Lemma 4.2. *Consider a family of maps $\{f_j^{q_{m_j+1}}|I_{m_j}\}$ with $f_j \in \mathcal{E}_s$ and $q_{m_j+1}/q_{m_j} \rightarrow \infty$. Then any limit point for this sequence in the Caratheodory topology has a parabolic fixed point.*

We would like to deal with the “dangerous” situation when the backward orbit (4.2) leaves D_m at a “bad” moment when J_{-i} is not commensurable with the original size. The next lemma takes care of this possibility.

Lemma 4.3. *Let us consider the map $f^{q_{m+1}}|_{I_m}$. Let $P_0, P_{-1}, \dots, P_{-k}$ be the consecutive returns of the backward orbit (4.1) to I_m , and denote by $\zeta_0, \dots, \zeta_{-k}$ the corresponding moments of the backward orbit of a point $\zeta_0 = z \in D_m$. If the ratio q_{m+1}/q_m is sufficiently big, then either $z' \equiv \zeta_{-k} \in D_m$, or $\widehat{(z', P_{-k})} > \epsilon$ and $\text{dist}(z', P_{-k}) \leq C|I_m|$*

Proof. To be definite, let us assume that the intervals P_{-i} lie on the left of 0. Without loss of generality we can assume that $z \in \mathbb{H}$. Let $\phi = f^{-q_{m+1}}$ be the branch of the inverse for which $\phi P_{-i} = P_{-(i+1)}$. As ϕ is orientation preserving on $(-\infty, f^{q_{m+1}+q_m}(0)]$, it maps the upper half-plane \mathbb{H} into itself: $\phi(\mathbb{H}) \subset \{z = re^{i\theta} | r > 0, \pi > \theta > 2\pi/3\}$.

By Lemma 4.2, if q_{m+1}/q_m is sufficiently large, the map ϕ has an attracting fixed point $\eta_\phi \in D(I_m) \subset D_m$ (which is a perturbation of a parabolic fixed point). By the Denjoy-Wolf Theorem, $\phi^n(\zeta) \xrightarrow{n \rightarrow \infty} \eta_\phi$ for any $\zeta \in \mathbb{H}$, uniformly on compact sets of \mathbb{H} . Thus for a given compact set $K \Subset \mathbb{H}$, there exists $N = N(K, \phi)$ such that $\phi^N(K) \subset D_m$. By a normality argument the choice of N is independent of a particular ϕ under consideration.

Suppose $\zeta_{-i} \notin D_m$. By Lemma 4.1 the set $K = (D_m \setminus \phi(D_m)) \cap \mathbb{H}$ is compactly contained in \mathbb{H} , and $\text{diam } K < C|I_m|$. For N as above we have $z' \in \bigcup_{i=0}^{N-1} \phi^i(K) \cup D_m$, and the lemma is proved. \square

4.3. The inductive step. The next lemma provides us with the inductive steps along the intervals I_m .

Lemma 4.4. *Let J be the last return of the backward orbit (4.1) to the interval I_m before the first return to I_{m+1} , and let J' and J'' be the first two returns of (4.1) to I_{m+1} . Let ζ, ζ', ζ'' be the corresponding moments in the backward orbit (4.2), $\zeta = f^{q_m}(\zeta')$, $\zeta' = f^{q_{m+2}}(\zeta'')$.*

Suppose $\zeta \in D_m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon$, or $\zeta'' \in D_{m+1}$.

Proof. Note that $J \subset [f^{q_m+q_{m+1}}(0), f^{q_m}(0)]$. By Lemma 2.2, $\zeta' \in D_\theta([f^{q_{m+1}-q_m}(0), 0])$ for some uniform constant θ . Denote by \tilde{J} , and \hat{J} the intervals of (4.1), such that $f^{q_{m+1}-q_m}(\tilde{J}) = J'$, $f^{q_m}(\hat{J}) = \tilde{J}$, and let $\tilde{\zeta}, \hat{\zeta}$ be the corresponding points in the orbit (4.2) (cf. Fig. 5). The interval $\tilde{J} \subset [f^{q_m}(0), f^{q_m-q_{m+1}}(0)]$, $\tilde{\zeta} \in D_\theta([f^{q_m}(0), f^{q_m-q_{m+1}}(0)])$. By Schwarz Lemma and Lemma 2.2 there are points $b_1, b_2 \in [0, f^{-q_{m+1}}(0)]$, such that $0, b_1, b_2$, and $f^{-q_{m+1}}(0)$ form a K -bounded configuration, and $\hat{\zeta} \in D_\sigma([0, b_1]) \cup D_\gamma([b_2, f^{-q_{m+1}}(0)])$ for uniform γ , and $\sigma < \pi/2$. The claim now follows by Schwarz Lemma. \square

4.4. Inductive argument. We start with a point $z \in D_1$. Consider the largest m such that D_m contains z . We will carry out induction in m . Let P_0, \dots, P_{-k} be the consecutive returns of the backward orbit (4.1) to the interval I_m until the first return to I_{m+1} , and denote by $z = \zeta_0, \dots, \zeta_{-k} = \zeta'$ the corresponding points of the orbit (4.2). By Lemma 4.1 and Lemma 4.3, ζ_{-i} either ϵ -jumps at a good moment when P_{-i} is commensurable with J_0 , and $\text{dist}(\zeta_{-i}, P_{-i}) \leq C|I_m|$, or $\zeta' \in D_m$.

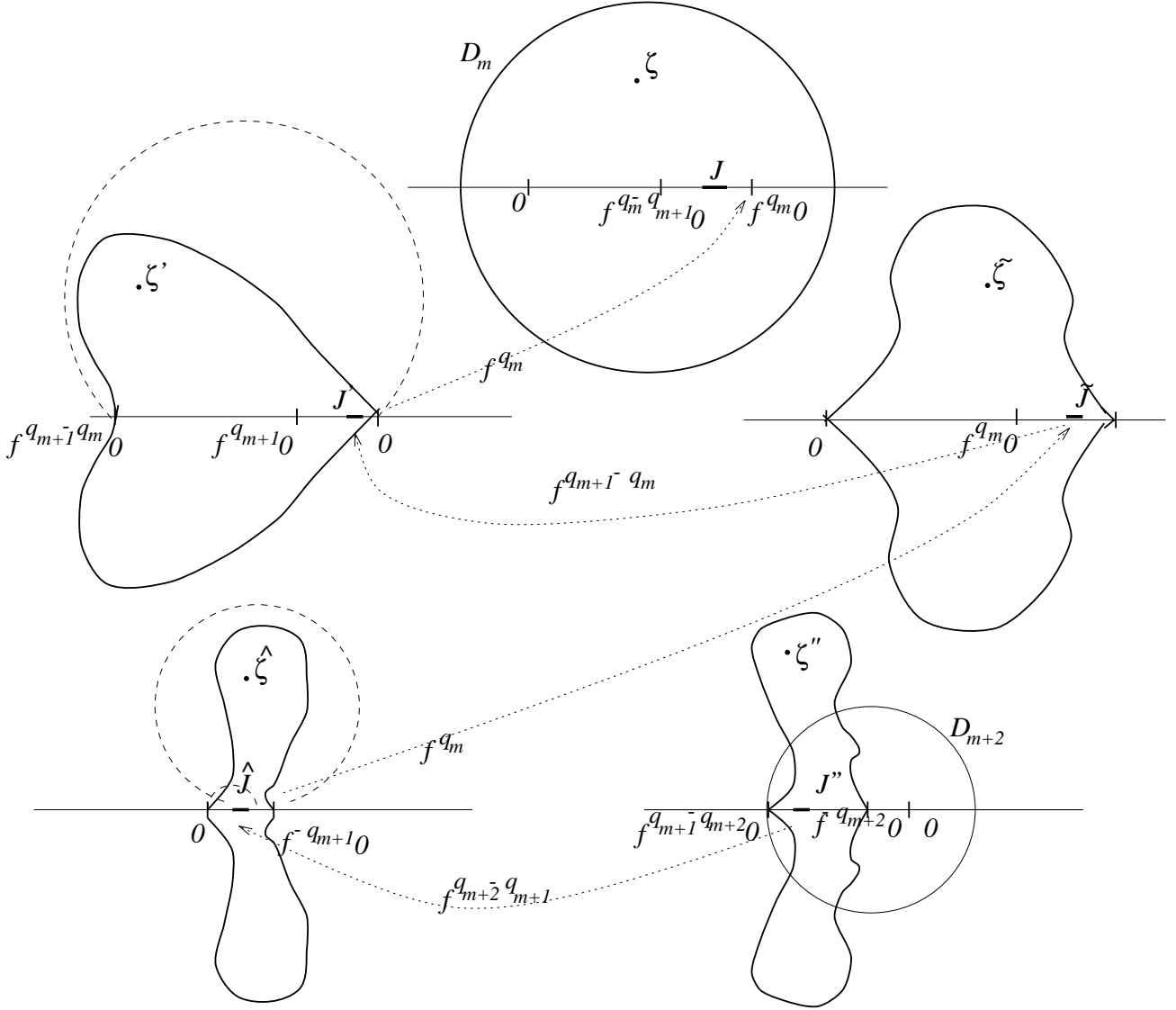


FIGURE 5

In the former case we are done by Lemma 2.1. In the latter case consider the point ζ'' which corresponds to the second return of the orbit (4.1) to I_{m+1} . By Lemma 4.4, either $(\zeta'', I_{m+1}) > \epsilon$, and $\text{dist}(\zeta'', I_{m+1}) \leq C|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

In the first case we are done again by Lemma 2.1. In the second case, the argument is completed by induction in m .

The following Remark is used in the Appendix.

Remark 4.1. By the above argument, in Lemma 3.1 and Proposition 3.2, we can let $D_0 = D_1$. We can choose instead $D_0 = D_\alpha([f^{q_2}(0), f^{q_1-q_2}(0)])$ for some $\alpha > \frac{\pi}{2}$, after an obvious change in the argument.

APPENDIX A. APPLICATION TO PROVING LOCAL CONNECTIVITY

A.1. Preliminaries. Let \mathbb{D} denote the unit disc, let $T = \partial\mathbb{D}$.

For a measurable set $S \subset \mathbb{C}$ let $\text{meas}(S)$ denote its planar Lebesgue measure.

Let $P_\theta(z) = e^{i2\pi\theta}z + z^2$. Denote by f_θ the Blaschke product

$$f_\theta(z) = e^{i2\pi\tau(\theta)} z^2 \frac{z-3}{1-3z}, \quad (\text{A.1})$$

where $\tau(\theta)$ is the unique real number for which $\rho(f_\theta) = \theta$. f_θ has a degree three critical point at 1, and we denote by W the component of $f_\theta^{-1}(\mathbb{D})$ not contained in \mathbb{D} . For a point $\zeta \in \mathbb{C}$, $f^i(\zeta) = 1$ we denote by $W(\zeta)$ the f^i -preimage of W attached to ζ , and we call ζ the *root point* of $W(\zeta)$.

The following theorem establishes the connection between f_θ and the quadratic polynomial P_θ .

Theorem A.1 (Douady, Ghys, Herman, Shishikura). *Let θ be of bounded type, and let Δ denote the Siegel disc of P_θ . Then there exists a quasi-conformal homeomorphism $\phi : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$, conformal on the immediate basin of infinity, such that $\phi(\mathbb{D}) = \Delta$, and $\phi \circ f_\theta = P_\theta \circ \phi$ on $\mathbb{C} \setminus \mathbb{D}$.*

Define $J_\theta = J(f_\theta) \setminus (\cup_{n \geq 0} f_\theta^{-n}(W) \cup \mathbb{D})$.

Note that for ϕ as in Theorem A.1, one has $J(P_\theta) = \phi(J_\theta)$. Petersen [P] proved the following result:

Theorem 1.2 (Petersen, [P]). *Let P_θ be as above. If θ is irrational of bounded type, then the Julia set $J(P_\theta)$ is locally connected and has Lebesgue measure zero.*

He actually gave a proof of another result which together with Theorem A.1 implies the above theorem.

Theorem 1.3 (Petersen, [P]). *For any irrational rotation number θ the set J_θ is locally connected and has Lebesgue measure zero.*¹

In what follows we present a new proof of Theorem 1.3, which bears a strong analogy to the proofs of local connectivity of the Julia sets for quadratic polynomials (see [HJ, McM2]).

We give a brief outline of the proof. First, following Petersen we construct a sequence of "puzzle-pieces". By the complex bounds they shrink down to the critical point 1. This yields the basis of connected neighborhoods around the critical point. We then present a "spreading around" argument to prove local connectivity at any other point of the set J_θ .

1.2. Complex bounds. Let the domain $G = \mathbb{C} \setminus \{0, \infty\}$. Consider the universal covering $\kappa : \mathbb{C} \rightarrow G$, $\kappa : z \mapsto e^{iz}$. Let $\bar{f} : \mathbb{C} \rightarrow \mathbb{C}$ denote the lift of the map f to this covering.

The map \bar{f} belongs to an Epstein class, and the cubic estimate (3.2) holds for this map.

Let $D \subset G$ be a domain such that $\text{cl} D \cap T \subset [f^{q_2}(1), f^{q_1-q_2}(1)]$. By remark 4.1 the domain D_0 in the Proposition 3.2 for the map \bar{f} can be chosen so that $\kappa(D_0) \supset D$.

¹In [P] Petersen proved the measure zero part of Theorem 1.3 for θ of bounded type only. The argument for the general case was suggested to us by M. Lyubich.

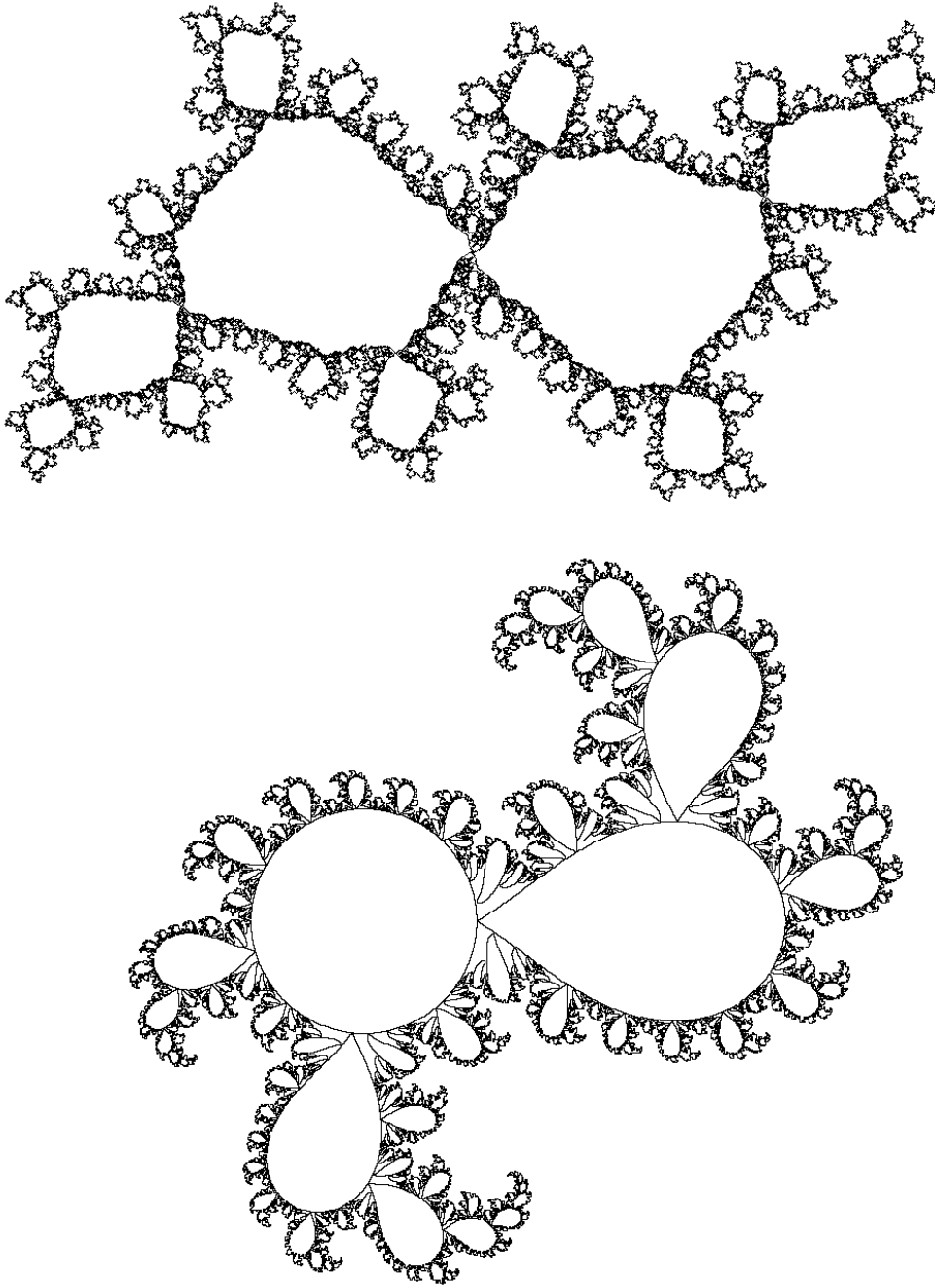


FIGURE 6. A quadratic Siegel Julia set (above) and the corresponding set J_θ

Using the bound on the derivative of the exponential map in the domain D_0 , we obtain the estimate (3.2) for the map f itself in the domain D , with constants depending only on the choice of the domain.

1.3. Construction of "puzzle-pieces". We construct a sequence of puzzle-pieces around the critical point following Petersen.

Let $\gamma'_0 \subset \partial W$, $\gamma_0 \subset \partial W$, and $f(\gamma'_0) = [f(1), 1]$, $f(\gamma_0) = T \setminus [f(1), 1]$. Let $\gamma'_1 \cap \gamma'_0 \neq \emptyset$, $\gamma_1 \cap \gamma_0 \neq \emptyset$, and $f(\gamma'_1) = \gamma'_0$, $f(\gamma_1) = \gamma_0$. Inductively define $\gamma'_i \cap \gamma'_{i-1} \neq \emptyset$, $\gamma_i \cap \gamma_{i-1} \neq \emptyset$, and $f(\gamma'_i) = \gamma'_{i-1}$, $f(\gamma_i) = \gamma_{i-1}$.

The curves γ'_i, γ_i converge to a repelling fixed point β of the map f .

Let $\Gamma' = \cup_i \gamma'_i \cup \beta$, $\Gamma = \cup_i \gamma_i \cup \beta$, and let $\hat{\Gamma} = f^{-1}(\Gamma')$, $\hat{\Gamma} \ni f^{-1}(1)$. Denote by R the external ray of external argument 0 landing at β and let R' be its preimage landing at the end point of $\hat{\Gamma}$. Finally, let E be an equipotential.

We let the puzzle-piece $P_0 \supset W$ be the closed domain cut out by the curves $R \cup \Gamma \cup [1, f^{-1}(1)] \cup \hat{\Gamma} \cup R'$ and E (cf. Fig. 7).

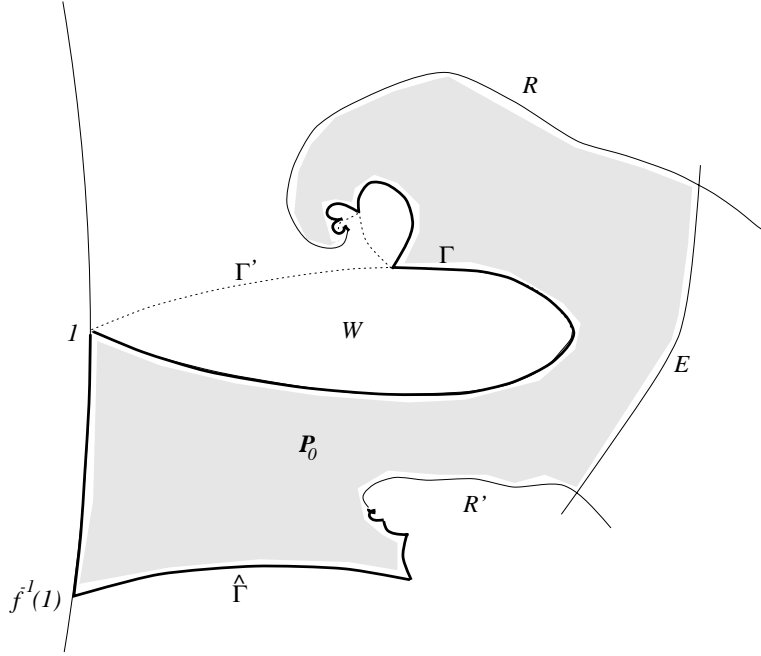


FIGURE 7

Let P_1 be the pullback of the domain P_0 corresponding to the inverse orbit $[1, f^{q_1}(1)] \subset P_0, [f^{-1}(1), f^{q_1-1}(1)], \dots, [f^{-q_1}(1), 1]$, and inductively, let P_n be the pullback of P_{n-1} corresponding to the orbit $[1, f^{q_n}(1)] \subset P_{n-1}, [f^{-1}(1), f^{q_n-1}(1)], \dots, [f^{-q_n}(1), 1]$.

Note that $P_n \cap T = [f^{-q_n}(1), 1]$, $f^{q_n}(P_n \cap \partial W) = [f^{q_n}(1), f^{-q_n-1}(1)]$, and $f^{q_n+q_{n-1}+q_{n-2}}(P_n \cap \partial W(f^{-q_n}(1))) = [1, f^{q_{n-1}+q_{n-2}}(1)]$.

By construction we immediately have the following

Proposition 1.4. *The intersection $P_n \cap J_\theta$ is connected.*

By Świątek-Herman real a priori bounds the intervals $[f^{-q_n}(1), 1]$ and $[1, f^{-q_{n-1}}(1)]$ are K -commensurable, with a universal constant K for sufficiently large n .

By the cubic estimate 3.2,

$$\text{diam } P_n \leq (C_1 \sqrt[3]{\frac{\text{diam } P_{n-1}}{|[f^{-q_n}(1), 1]|}} + C_2) \cdot |[f^{-q_n}(1), 1]|.$$

Hence, if $\frac{\text{diam } P_{n-1}}{|[1, f^{-q_{n-1}}(1)]|} > K_1$ for a large K_1 , then $\frac{\text{diam } P_n}{|[f^{-q_n}(1), 1]|} < \frac{1}{2} \cdot \frac{\text{diam } P_{n-1}}{|[1, f^{-q_{n-1}}(1)]|}$. It follows that for all sufficiently large n , the piece P_n is K_2 -commensurable with $[f^{-q_n}(1), 1]$ with a universal constant K_2 .

Together with Proposition 1.4 this implies

Proposition 1.5. *The set J_θ is locally connected at the critical point 1.*

Lemma 1.6. *P_n contains a Euclidean disc B with $\text{diam } B > K \text{diam } P_n$, for a universal constant $K > 0$.*

Proof. Note that by construction, $W(f^{-q_{n+2}}(1)) \subset P_n$. The claim now easily follows. \square

1.4. **“Spreading around” argument.** Choose any $z \in J_\theta$.

Assume first that there exists n such that $f^i(z) \notin P_k$ for any $i \geq 0$ and $k \geq n$. As f has an irrational rotation number on the circle this implies that the forward orbit $z_0 \equiv z, z_1 \equiv f(z), z_2 \equiv f(z_1), \dots$ does not accumulate on the circle, i.e. there exists $\epsilon > 0$, such that $z_i \in V_\epsilon \equiv \{\zeta, |\zeta| > 1 + \epsilon\}$.

As the set J_θ is locally connected at the critical point 1, there exist two external rays r_1 and r_2 landing at this point on different sides of W .

For a point ζ in the inverse orbit of the critical point let $r_1(\zeta), r_2(\zeta)$ be the preimages of the rays r_1 and r_2 , landing on ζ . Let L_ζ be the “limb” of the set J_θ , cut out by the rays $r_1(\zeta), r_2(\zeta)$.

Let a be an accumulation point of the sequence $\{z_k\}$. Choose a limb $L \equiv L_\zeta$ containing a , with $L \cap T = \emptyset$. Denote by k_n the moments when $z_{k_n} \in L$. Let $L_n \ni z$ be the pullback of L corresponding to the backward orbit $z_{k_n}, z_{k_n-1}, \dots, z_1, z_0 \equiv z$. We use the following general lemma to assert that $\text{diam}(L_n) \rightarrow 0$.

Lemma 1.7 ([L2], Prop. 1.10). *Let f be a rational map. Let $\{f_i^{-m}\}$ be a family of univalent branches of the inverse functions in a domain U . If $U \cap J(f) \neq \emptyset$, then for any V such that $\text{cl } V \subset U$,*

$$\text{diam}(f_i^{-m}V) \rightarrow 0.$$

This yields the desired nest of connected neighborhoods around z , and we are done.

Now let z_k be the first point in the orbit z_0, z_1, z_2, \dots contained in the piece P_n . Denote by

$$\Pi_0 \equiv P_n, \Pi_{-1}, \dots, \Pi_{-k} \tag{1.2}$$

the preimages of P_n corresponding to the inverse orbit z_k, z_{k-1}, \dots, z_0 .

Lemma 1.8. *The inverse orbit (1.2) hits the critical point 1 at most once.*

Proof. To be definite, assume that P_n is above the critical point 1. Note that if $\Pi_{-i} \cap T = \emptyset$ for some $i \leq q^{n+1}$, then the inverse orbit (1.2) never hits the critical point. Otherwise, denote by A and $B \equiv P_{n+1}$ the ‘‘above’’ and ‘‘below’’ $f^{q_{n+1}}$ -preimages of P_n , i.e. $f^{q_{n+1}}(A) = P_n$, $A \cap T \neq \emptyset$, and A is above 1, and similarly for B . Notice that $A \cap T = [f^{-q_{n+1}-q_n}(1), 1] \subset [f^{-q_n}(1), 1]$. Let $L_1 = P_n \cap \partial W$, and $L_2 = A \cap W$, then

$$\begin{aligned} f^{q_{n-1}+q_n+q_{n+1}}(L_1) &= f^{q_{n+1}}([f^{q_{n-1}+q_n}(1), 1]) = [f^{q_{n-1}+q_n+q_{n+1}}(1), f^{q_{n+1}}(1)] \\ &\supset [f^{q_{n-1}+q_n}(1), f^{q_{n-1}+q_n+q_{n+1}}(1)] = f^{q_{n-1}+q_n}([1, f^{q_{n+1}}(1)]) \\ &= f^{q_{n-1}+q_n+q_{n+1}}(L_2). \end{aligned}$$

Thus $L_1 \supset L_2$, and as two different preimages of W cannot cross, it follows that $A \subset P_n$. Hence $\Pi_{-q_{n+1}} \neq A$.

Finally, denote by $B' \cap T \neq \emptyset$ the f^{q_n} preimage of B , $f^{q_n}(B') = B$. $B' \cap T = [f^{-q_n}(1), f^{-q_n-q_{n+1}}(1)] \subset [f^{-q_n}(1), 1]$. Let $L_2 = P_n \cap W(f^{-q_n}(1))$, and $L_3 = B' \cap W(f^{-q_n}(1))$, then

$$\begin{aligned} f^{q_n+q_{n-1}+q_{n-2}}(L_2) &= [1, f^{q_{n-1}+q_{n-2}}(1)] \\ &\supset [f^{q_{n-1}+q_{n-2}-q_n-q_{n+1}}(1), f^{q_{n-1}+q_{n-2}}(1)] \\ &= f^{q_n+q_{n-1}+q_{n-2}}(L_3). \end{aligned}$$

Therefore, $L_2 \supset L_3$, and $B' \subset P_n$.

Thus, $\Pi_{-q_{n+1}-q_n} \cap T = \emptyset$ and the claim follows. \square

Now let $-k \leq -i \leq 0$ be the first moment such that $\Pi_{-i} \cap T = \emptyset$. By real bounds there exists an annulus A around $\Pi_{-(i-1)}$ with $\text{mod } A > K$ for a universal K , such that the critical value $f(1)$ is outside of A . Therefore, there exists an annulus with modulus K around Π_{-i} , such that the whole postcritical set of f is outside of it.

By Lemma 1.8 and Koebe theorem, the inverse branch $f^{-k} : P_n \rightarrow \Pi_{-k} \equiv P_n(z) \ni z$ has bounded distortion on the piece P_n . As $P_n(z)$ cannot contain a disc of definite radius, and by Lemma 1.6, $\text{diam}(P_n(z)) \rightarrow 0$, which yields the desired nest of connected neighborhoods around z .

1.5. Proof of measure zero statement. The following proof was suggested by M. Lyubich. First consider the set of points $J_1 \subset J_\theta$, $J_1 \equiv \{\zeta \in J_\theta \mid \exists n, f^i(\zeta) \notin P_k, \forall i \geq 0, k \geq n\}$. The set J_1 has zero Lebesgue measure by a theorem of M. Lyubich ([L1]).

For a piece P_n denote by P'_n the symmetric piece with respect to the circle T . Let $Q_n \equiv P_n \cup P'_n$. Note, that $\text{int } P'_n \cap J_\theta = \emptyset$. Consider now a point $z \in J_\theta \setminus J_1$. Let k be the first moment when the orbit $z_0 \equiv z, z_1, z_2, \dots$ enters the piece P_n . By the same argument as above, there is a piece $Q_n(z)$ around z , which is a bounded distortion pull-back of Q_n . The diameters of sets $Q_n(z)$ tend to zero.

Thus we obtain a sequence of balls $B_n \supset Q_n(z)$ around z , such that by Lemma 1.6,

$$\frac{\text{meas}(B_n \setminus J_\theta)}{\text{meas}(B_n)} > \delta > 0.$$

It follows that z is not a point of Lebesgue density of set J_θ , and this completes the proof of Theorem 1.3.

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