HAUSDORFF DIMENSION AND KLEINIAN GROUPS

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Abstract. Let G be a non-elementary, finitely generated Kleinian group, $\Lambda(G)$ its limit set and $\Omega(G) = \overline{\mathbb{C}} \backslash \Lambda(G)$ its set of discontinuity. Let $\delta(G)$ be the critical exponent for the Poincaré series and let Λ_c be the conical limit set of G. We prove that

(1) $\delta(G) = \dim(\Lambda_c)$.
(2) A simply connected component Ω is either a disk or $\dim(\partial\Omega) > 1$.
(3) $\Lambda(G)$ is either totally disconnected, a circle or has dimension > 1,
(4) G is geometrically infinite iff $\dim(\Lambda) = 2$.
(5) If $G_n \to G$ algebraically then $\dim(\Lambda) \leq \liminf \dim(\Lambda_n)$.
(6) The Minkowski dimension of Λ equals the Hausdorff dimension.
(7) If $\mathrm{area}(\Lambda) = 0$ then $\delta(G) = \dim(\Lambda(G))$.

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1. Statement of results

Consider a group G of Möbius transformations acting on the two sphere S^2. Such a group G also acts on the hyperbolic 3-ball $\mathbb{B}$ (with its boundary S^2 identified with the Riemann sphere). The limit set, $\Lambda(G)$, is the accumulation set (on S^2) of the orbit of the origin. We say the group is discrete if is discrete as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, (i.e., if the identity element is isolated). The ordinary set of G, $\Omega(G)$, is the subset of S^2 where G acts discontinuously, e.g., $\Omega(G)$ is the set of points z such that there exists a disk around z which hits itself only finitely often under the action of G. If G is discrete, then $\Omega(G) = S^2 \setminus \Lambda(G)$. G is called a Kleinian if it is discrete and $\Omega(G)$ is non-empty. The limit set $\Lambda(G)$ has either 0,1,2 or infinitely many points and G is called elementary if $\Lambda(G)$ is finite.

If we form the Poincaré series for a discrete G on $\mathbb{B}^3$,

$$\sum_G \exp(-s\rho(0, g(0))),$$

(ρ is the hyperbolic metric in $\mathbb{B}^3$) the series converges for all $s > s_0$ and the minimal such s_0 is called the *Poincaré exponent* of the group and is denoted $\delta(G)$. A point $x \in \Lambda(G)$ is called a *conical limit* point if there is a sequence of orbit points which converges to x inside a (Euclidean) non-tangential cone with vertex at x (such points are also called radial limit points or points of approximation). The set of such points is denoted $\Lambda_c(G)$. G is called *geometrically finite* if there is a finite sided fundamental polyhedron for the G action on $\mathbb{B}$. However, the only property we shall use is a result of Beardon and Maskit [7] that G is geometrically finite iff $\Lambda(G)$ is the union of $\Lambda_c(G)$, the rank 2 parabolic cusps and doubly cusped rank 1 parabolic fixed points of G. This makes it clear that $\dim(\Lambda_c) = \dim(\Lambda)$ and $\mathrm{area}(\Lambda) = 0$ in the geometrically finite case. (For our purposes we could take the Beardon-Maskit characterization to be the definition of geometrically finite.) By Selberg's lemma any finitely generated discrete group contains a finite index subgroup without torsion. This subgroup must have the same limit set as the original group, so for our purposes it will always be sufficient to assume G has no torsion.

In this paper "circle" will always refer to Euclidean circles or lines (e.g., circles on the sphere) as opposed to topological circles which we will always call closed Jordan curves. Similarly the terms "disk" or "ball" will always denote Euclidean balls.

The principle objective of this paper is to establish the following results.

Theorem 1.1. *If G is a non-elementary, discrete Möbius group on $\mathbb{B}$ then* $\delta(G) = \dim(\Lambda_c(G))$.

Theorem 1.2. *Suppose G is a finitely generated Kleinian group and Ω is a simply connected, invariant component of $\Omega(G)$. Then $\dim(\partial\Omega) = 1$ iff $\partial\Omega$ is a circle.*

Theorem 1.3. *If G is a finitely generated Kleinian group then its limit set is either totally disconnected, a circle or has Hausdorff dimension > 1.*

Theorem 1.4. *If G is a finitely generated Kleinian group which is geometrically infinite then $\dim(\Lambda(G)) = 2$.*

Theorem 1.5. *If $\{G_n\}$ is a sequence of N-generated Kleinian groups which converges algebraically to G then $\dim(\Lambda(G)) \leq \liminf_n \dim(\Lambda(G_n))$.*

Theorem 1.6. *If G is a finitely generated Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

Theorem 1.7. *Suppose that G is a non-elementary, finitely generated Kleinian group and that area$(\Lambda(G)) = 0$. Then $\delta(G) = \dim(\Lambda)$.*

These results are arranged roughly in order of dependence and the amount of required prerequisites from the theory of Kleinian groups and hyperbolic manifolds. Theorem 1.1 uses nothing but the definitions and a few simple properties of Möbius transformations. We have only been able to locate this result in the literature under the additional assumptions that G is geometrically finite or Fuchsian (e.g., see [53], [56]). The direction $\dim(\Lambda_c(G)) \leq \delta(G)$ is easy and well known and the opposite inequality is fairly simply (and known) when the Poincaré series diverges at the critical exponent. We should also note that Theorem 1.1 holds for a discrete group of Möbius transformations acting on the hyperbolic ball in any dimension as well as the rank 1 case in general.

It is known that the conical limit set corresponds to the geodesics starting at $0 \in \mathbb{B}$ which return to some compact subset of M infinitely often. Our proof shows that for any $\epsilon > 0$ there is a subset of $\Lambda_c(G)$ of dimension $\geq \delta(G) - \epsilon$ which corresponds to geodesics which never leave some some compact subset of M. Thus our proof shows that if M is a hyperbolic mainifold with finitely generated fundamental group then for any $x \in M$, the set of directions corresponding to geodesics rays starting at x which have compact closure has dimension $\delta(G)$.

Theorem 1.2 uses nothing about Kleinian groups except the definitions but does make use of some nontrivial results from the theory of conformal mappings and rectifiable sets. We shall actually give a long list of conditions which are equivalent to Ω not being a disk (see the end of Section 6. Among them:

(1) $\delta(G) > 1$.

(2) $\partial\Omega$ has tangents almost nowhere (with respect to harmonic measure).

(3) $\partial\Omega$ fails to have a tangent somewhere.

The elementary groups have to be excluded in Theorem 1.2 and Theorem 1.7 because a cyclic group consisting of parabolics has a one point limit set, but $\delta(G) = 1/2$.

Theorem 1.3 is essentially a corollary of Theorem 1.2 but requires the Ahlfors finiteness theorem and the Klein-Maskit combination theorems to reduce to the case considered in Theorem 1.2. The result was first formulated by Bowen in [19] in the case of quasi-Fuchsian groups with no parabolics. The convex, co-compact Kleinian case is proven in [53] and [20]. See also [55]. The general geometrically finite case is proven in [26]. In this paper we complete the discussion by including the geometrically infinite groups.

We should also note that Theorem 1.2 and Theorem 1.3 could be deduced from Theorem 1.4 and the known results for geometrically finite groups. However, our proofs of these results seem new even in the geometrically finite case, so we have included them in this paper.

Theorem 1.4 is the first place we have to invoke some nontrivial results about 3-manifolds. We need a result of Sullivan [57] that $\lambda_0 = \delta(G)(2-\delta(G))$ where λ_0 is the lowest eigenvalue for the Laplacian on $M = \mathbb{B}/G$. We also need exponential decay (in time) for the heat kernel of a manifold with first eigenvalue bounded away from zero.

Theorem 1.4 was previously known in special cases. Examples of groups with $\dim(\Lambda(G)) = 2$ were constructed by Sullivan in [54], and Canary [25] proved Theorem 1.4 holds if $M = \mathbb{B}/G$ is a "topologically tame" manifold and such that the thin parts have bounded type (in particular, if its injectivity radius is bounded away from zero). Our result shows these hypotheses are unnecessary.

Sullivan [56] and Tukia [60] independently showed that if G is a geometrically finite group then $\dim(\Lambda(G)) < 2$. Thus Theorem 1.4 implies that a finitely generated group is geometrically finite iff $\dim(\Lambda(G)) < 2$.

If $M = \mathbb{B}/G$, the convex core of M is defined to be $C(M) = C(\Lambda)/G$, where $C(\Lambda)$ is the convex hull of the limit set in $\mathbb{B}$. Burger and Canary prove in [22] that if G is a geoemtrically finite, $M = \mathbb{B}/G$ has infinite volume and $\delta(G) > 1$, then

$$2 - \frac{2\text{area}(\partial C(M))}{\text{vol}C(M)} \leq \dim(\Lambda(G)) \leq 2 - \frac{K}{\text{vol}(C_1(M))},$$

where $C_1(M)$ is a unit neighborhood of $C(M)$. Theorem 1.4 implies their result still holds in the geometrically infinite case (finitely generated groups are geometrically infinite iff $\text{vol}(C(M)) = \infty$; see Section 11).

To prove Theorem 1.5 we use our earlier results and an estimate of Canary which bounds the first eigenvalue of M in terms of the volume of the convex core $C(M)$. We also need the use the Margulis lemma. These results are only needed in the case when $\Lambda(G)$ has positive area; otherwise Theorem 1.5 follows immediately from earlier results. We will also see that the assumption that G and the $\{G_n\}$ are Kleinian is unnecessary; the result holds for any finitely generated Möbius groups.

Theorem 1.6 and Theorem 1.7 are both corollaries of a result relating $\delta(G)$ to Minkowski (or "box counting") dimension which we will state in Section 15. This result uses only Theorem 1.1, the Ahlfors finiteness theorem and some geometric arguments. Theorem 1.6 was proven in the geometrically finite case by Stratmann and Urbanski [52], and our proof is quite simple in this case. The geometrically infinite case follows immediately from Theorem 1.4, although there is also a proof which does not require Theorem 1.4.

The Ahlfors conjecture states that the limit set of a finitely generated discrete group of Möbius transformations is either the whole sphere or has zero area. We do not address the Ahlfors conjecture in this paper, but if it is true several of the arguments given here would simplify. As part of the proof of Theorem 1.4 we show that a finitely generated, geometrically infinite group with $\delta(G) < 2$ must have a limit set with positive area. Thus the Ahlfors conjecture implies $\delta(G) = 2$ for any geometrically infinite group. We do not know an argument for the converse direction, but both results are currently known to be true for topologically tame groups, [25].

In most of our results the case when G has no rank 1 parabolics is easier because then the action of G on $\Omega(G)$ has a compact fundamental polygon. When the parabolics introduce extra difficulties we usually give the proof first in the compact case and then go on to the general case. We will introduce the notion of "good" and "bad" horoballs. The good horoballs can be treated very similarly to the compact case but the bad horoballs require a little extra work (and can only occur in the geometrically infinite case). The rank 2 parabolics cause no difficulties since they do not correspond to cusps on $\Omega(G)/G$.

It might be worth noting that although the theory of hyperbolic 3-manifolds is intricate and highly developed, we we only need a few simple facts which we prove here. The only results we really use without proof are the Margulis lemma, Davies' Gaussian upper bound on the heat kernel and Sullivan's theorem relating the lowest eigenvalue for the Laplacian on $M = \mathbb{B}/G$ to the Poincaré exponent $\delta(G)$.

The remaining sections of this paper are organized as follows:

Section 2: We prove Theorem 1.1 and deduce from the proof that $\delta(G)$ is lower semi-continuous with respect to algebraic convergence.

Section 3: This is first of several sections devoted to Theorem 1.2 and Theorem 1.3. We define the "β"'s which measure the distance of a set to a line and we prove that a uniformly wiggley set (i.e., a set where the β's are bounded away from zero) has dimension larger than one.

Section 4: We recall that large Schwarzian for a conformal map implies large β's for the image domain. This implies that if G has no parabolics then $\Lambda(G)$ is either a circle or is uniformly wiggley. Thus we obtain the compact case of Theorem 1.2 as a corollary.

Section 5: We prove that if a simply connected invariant component is not a disk then the boundary has tangents almost nowhere. Here we need some known results about conformal mappings and Schwarzian derivatives. We also prove a result on the length of level lines that is needed in the next section.

Section 6: We prove Theorem 1.2 with parabolics.

Section 7: We show that if $\Omega(G)$ is a union of infinitely many disks, then $\dim(\Lambda(G)) > 1$. This is easy and previously known.

Section 8: We show that the limit sets of degenerate groups (groups where $\Omega(G)$ is connected and simply connected) are uniformly wiggley. This gives an alternate proof of Theorem 1.2 in this case.

Section 9: We prove Theorem 1.3 and deduce a variety of corollaries.

Section 10: We prove $\Lambda(G)$ is uniformly perfect and introduce the idea of "good" and "bad" horoballs and prove some simple facts about them.

Section 11: Here we gather together the facts about the convex hull of a hyperbolic 3-manifold that we will need in later sections.

Section 12: We prove Theorem 1.4.

Section 13: We prove Theorem 1.5.

Section 14: We introduce Minkowski and upper Minkowski dimension and relate it to the Besicovitch-Taylor index.

Section 15: We prove Theorem 1.6 and Theorem 1.7 by proving the following result: if G is finitely generated and $\Lambda(G)$ has zero area then $\delta(G)$ equals the upper Minkowski dimension of $\Lambda(G)$.

Section 16: We deduce some corollaries of our results in the special case when the groups belong to $\overline{T(S)}$, the closure of the Teichmüller space of a finite type surface S. For example, $\dim(\Lambda(G))$ is a lower semi-continuous function on $\overline{T(S)}$ and is continuous everywhere except at the geometrically finite cusps, where it is discontinuous.

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2. The conical limit set

First we recall the definition of Hausdorff dimension. Given an increasing function φ on $[0,\infty)$, we define

$$H^\delta_\varphi(E) = \inf\{\sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j), r_j \leq \delta\},$$

and

$$H_\varphi(E) = \lim_{\delta\to 0} H_\varphi^\delta(E).$$

This is the Hausdorff measure associated to φ. H_φ^∞ is called the Hausdorff content. It is not a measure, but has the same null sets as H^α. When $\varphi(t) = t^\alpha$ we denote the measure H_φ by H_α and we define

$$\dim(E) = \inf\{\alpha : H_\alpha(E) = 0\}.$$

For $\alpha = 1$ we sometimes denote H_1 by ℓ (for "length"). An upper bound for $\dim(E)$ can usually be produced by finding appropriate coverings of the set. We will be more interested in finding lower bounds. The usual idea is to construct a positive measure μ on E which satisfies $\mu(D(x,r)) \leq Cr^\alpha$. This implies $\dim(E) \geq \alpha$ since for any covering of E we have

$$\sum_j r_j^\alpha \geq C^{-1}\mu(D(x_j, r_j) \geq C^{-1}\mu(E) > 0.$$

Next we recall the definition of the critical exponent of Poincaré series of G,

$$\delta(G) = \inf\{\alpha : \sum_{g\in G}(1 - |g(0)|)^\alpha < \infty\}.$$

It is easy and well known that

$$\dim(\Lambda_c) \leq \delta(G),$$

so we will only prove the other direction. If the Poincaré series diverges at the critical exponent $\delta(G)$, then the Patterson measure ([47], [48], [49], [56]) lives on the conical limit set $\Lambda_c(G)$ and satisfies the estimate

$$\mu(B(x,r)) \leq Cr^{\delta(G))}.$$

So the theorem is known in this case and we must only treat the case where the Poincaré series converges at the critical exponent. In this case the Patterson measure is supported on $\Lambda(G) \setminus \Lambda_c(G)$. Our proof does not use Patterson measure or any property of divergence or convergence at the critical index. It does construct a measure on the conical limit set which mimics the behavior of the Patterson measure.

In this section we will prove,

Theorem 2.1. *Suppose G is a group of Möbius transformations with more than two limit points. Then*

$$\delta(G) \leq dim(\Lambda_c).$$

Proof. Let $\{z_n\}$ denote the orbit of 0 under G in the hyperbolic 3-ball, $\mathbb{B}$. Let $\delta = \delta(G)$ be the critical exponent for the Poincaré series of G and let $\epsilon > 0$. Choose a point $x \in \overline{\mathbb{C}} = \partial \mathbb{B}$ so that

$$\sum_{j:|z_j - x| < r} (1 - |z_j|)^{\delta - \epsilon} = \infty,$$

for every $r > 0$ (here $|z - x|$ denotes the spherical metric). We can do this by a simple compactness argument. Since G is non-elementary, x is not fixed by every element of G. Therefore we can choose elements $\{g_1, \dots, g_4\} \in G$ so that $x_i = g_i(x)$ are all distinct. Fix $r > 0$ to be so small that the balls B_i on $\overline{\mathbb{C}}$ (in the spherical metric) of radius $2r$ around the points $\{x_1, \dots x_4\}$ are pairwise disjoint.

Suppose M, N are large numbers (to be chosen below depending only on G and r). Let $A_n = \{z \in \mathbb{B} : 2^{-n-1} \leq 1 - |z| < 2^{-n}\}$. If it were true that

$$\sum_{j:z_j \in B_i \cap A_n} (1 - |z_j|)^{\delta - 2\epsilon} \leq M,$$

for all large enough n, then

$$\begin{aligned}\sum_{j:z_j\in B_i}(1-|z_j|)^{\delta-\epsilon} &\leq C\sum_n 2^{-n\epsilon}\sum_{j:z_j\in B_i\cap A_n}(1-|z_j|)^{\delta-2\epsilon}\\ &\leq CM\sum_n 2^{-n\epsilon}\\ &< \infty.\end{aligned}$$

This is a contradiction, so we must have

$$\sum_{j:z_j\in B_i\cap A_n}(1-|z_j|)^{\delta-2\epsilon}\geq M,$$

for infinitely many values of n. For each $i=1,2,3,4$, fix a value of n_i (depending on M and hence of r) for which this inequality holds.

Since the z_j's make up the orbit of a single point, they are uniformly separated in the hyperbolic metric of $\mathbb{B}$. Thus for any $A<\infty$ we may split the sequence into a finite number B of sequences (depending on A) each of which is separated by at least A in the hyperbolic metric. Therefore, to each point x_i we may associate a collection of points $\mathcal{G}_i(0)\subset\{z_j\}$ such that

$$\mathcal{G}_i(0)\subset B_i\cap A_{n_i},$$

$$z,w\in\mathcal{G}_i(0) \text{ implies } |z-w|\geq 3N2^{-n_i},$$

$$\sum_{j:z_j\in\mathcal{G}_i(0)}(1-|z_j|)^{\delta-2\epsilon}\geq M/B.$$

For each $z\in\mathcal{G}_i(0)$ let z^* denote its radial projection onto the sphere $\overline{\mathbb{C}}=\partial\mathbb{B}$. For $z\in\mathbb{B}$, let

$$B(z_j)=B(z_j^*,N(1-|z|)).$$

By hypothesis, the balls $B(z_j)$ are disjoint for all $z_j\in\cup_{i=1}^4\mathcal{G}_i(0)$.

Since the balls $\{B_1, \dots, B_4\}$ have disjoint doubles, any sufficiently small disk (depending on r) can intersect at most one of the balls. For any point $z = g(0)$ in the orbit of 0 consider the four balls $\{g(B_1), \dots, g(B_4)\}$. The preceding statement implies that if N is sufficiently large (depending only on r) then at most one of these balls can intersect $\mathbb{B} \setminus B(z)$. This determines our choice of N. Therefore, at least three of the balls are contained in $B(z)$. Without loss of generality, assume they are $g(B_1), g(B_2)$ and $g(B_3)$.

The Möbius transformation g has bounded distortion as a map from S^2 to itself except possibly at one point. More precisely,

Lemma 2.2. *Suppose $r > 0$. There is a $C < \infty$ (depending only on r) such that given any Möbius transformation g of $S^2 = \overline{\mathbb{C}}$ to itself we have*

$$C^{-1}(1 - |g(0)|) \leq |g'| \leq C(1 - |g(0)|),$$

except possibly on a disk D of radius r (both the derivatives and the disk are taken with respect to the spherical metric).

Proof. We may assume $g(0) \neq 0$ since otherwise the lemma is easy. Let z be the radial projection of $g(0)$ onto S^2 and choose R so big (depending only on r) so that

$$\omega(g(0), S^2 \setminus B(z, R(1 - |g(0)|), \mathbb{B}) \leq r^2.$$

Then $D = g^{-1}(B(z, R(1 - |g(0)|))$ is a disk of radius less than r and $|g'|$ is comparable to $1 - |g(0)|$ on its complement. $\square$

So for any $g \in G$, at least two of the disks (say B_1 and B_2) are bounded away from this point so we get

$$C^{-1}(1 - |g(0)|) \leq |g'| \leq C(1 - |g(0)|),$$

on B_1 and B_2 with constants depending only on r.

Note that if we choose n_i large enough (depending only on N) we may assume

$$\frac{1-|z|}{2N} \geq 1-|z_j| \geq \frac{1-|z|}{CN},$$

for some uniform C depending only on G and r.

Now for the orbit point $z = g(0)$ define $\mathcal{G}(z) = g(\mathcal{G}_1(0))$. Thus

$$\begin{aligned}
\sum_{z_j \in \mathcal{G}(z)} (1-|z_j|)^{\delta-2\epsilon} &\geq C^{-\delta+2\epsilon}(1-|z|)^{\delta-2\epsilon} \sum_{z_k \in \mathcal{G}_1(0)} (1-|z_k|)^{\delta-2\epsilon} \\
&\geq C^{-\delta+2\epsilon}(1-|z|)^{\delta-2\epsilon}\frac{M}{B} \\
&\geq C^{-2}\frac{M}{B}(1-|z|)^{\delta-2\epsilon} \\
&\geq (1-|z|)^{\delta-2\epsilon},
\end{aligned}$$

where the last line holds if M is large enough. Since C_1 depends only on r and B depends only on group G (more precisely it only depends on the injectivity radius of G at 0), this determines our choice of M.

We have now constructed a set of points $z_j = \mathcal{G}(z)$ which satisfy the following conditions:

$$z_j \in B(z^*, N(1-|z|)),$$

$$B(z_j^*, 2N(1-|z_j|)) \cap B(z_k^*, 2N(1-|z_k|)) = \emptyset \text{ for } j \neq k,$$

$$\sum_j (1-|z_j|)^{\delta-2\epsilon} \geq (1-|z|)^{\delta-2\epsilon},$$

$$\frac{1-|z|}{2N} \geq 1-|z_j| \geq \frac{1-|z|}{CN},$$

for some uniform C depending only on G and r (because the points in $G_1(0)$ do and the map g has uniformly bounded distortion on $G_1(0)$). It is now a standard argument to show that $\dim(\Lambda_c) \geq \delta - 2\epsilon$. Briefly it goes as follows.

Construct generations of points starting with $\mathcal{G}_0 = \{0\}$, and for each $z \in \mathcal{G}_n$, define points $\{z_j\}$ in $\mathcal{G}_{n+1}$ as above. To each point $z \in \mathcal{G} = \cup_n \mathcal{G}_n$, associate the disk $B_z = B(z^*, 2N(1-|z|))$. Then let

$$E_n = \cup_{z \in \mathcal{G}_n} B_z,$$

$$E = \cap_n E_n.$$

It is easy to see that $E \subset \Lambda_c(G)$.

Define a probability measure μ on E by setting $\mu(E_0) = 1$, and for $z \in \mathcal{G}_n$ with "parent" $z' \in \mathcal{G}_{n-1}$, set

$$\mu(B_z) = \frac{(1-|z|)^{\delta - 2\epsilon}}{\sum_{w \in \mathcal{G}(z')} (1-|w|)^{\delta - 2\epsilon}} \mu(B_{z'}).$$

It is easy to see by induction that

$$\mu(B_z) \leq (1-|z|)^{\delta-2\epsilon} \leq C \mathrm{diam}(B_z)^{\delta - 2\epsilon},$$

for each z in $\mathcal{G}$. We want to show this inequality is true for any disk D. Let D be any disk and let $D_0 = B_z$ be the lowest generation disk in our construction so that $D_0 \cap D \neq \emptyset$ but $D_0 \subset 2D$. Let D_1 be the parent of D_0. By the maximality of D_0 we have $D \subset 2D_1$. Since $2D_1$ is disjoint from any other balls of the same generation,

$$\begin{aligned} \mu(D) &\leq \mu(D_1) \\ &\leq C \mathrm{diam}(D_1)^{\delta - 2\epsilon} \\ &\leq C(NC)^{\delta-2\epsilon} \mathrm{diam}(D_0)^{\delta-2\epsilon} \\ &\leq 2C(NC)^{\delta-2\epsilon} \mathrm{diam}(D)^{\delta - 2\epsilon}. \end{aligned}$$

This is the desired inequality (the constant in front is larger, but is uniform over all disks; the power is the same).

Now we simply note that if $\{D_j\}$ is any covering of E by disks, then

$$0 < \mu(E) \geq \sum_j \mu(D_j) \leq C \sum \mathrm{diam}(D_j)^{\delta - 2\epsilon}.$$

By the definition of Hausdorff dimension, this implies

$$\dim(\Lambda_c(G)) \geq \dim(E) \geq \delta - 2\epsilon.$$

Since ϵ was arbitrary, we get Theorem 2.1. $\square$

The proof of Theorem 2.1 has the following corollaries.

Corollary 2.3. *Suppose $r > 0$ and let $C = C(r)$ and $N = N(r)$ be the constants in Lemma 2.2 and the proof above. Suppose G is a group of Möbius transformations on $\mathbb{B}$ and suppose there are three disjoint balls B_1, B_2, B_3 of (spherical) radius r and a collection of points $F \subset G(0) \cap A_n$ which satisfy*

$$z, w \in F \textit{ implies } |z - w| \geq N 2^{-n},$$

$$\sum_{z \in F \cap B_i} (1 - |z|)^\alpha \geq C^{-2}.$$

Then

$$\delta(G) \geq \alpha.$$

Once we have the conditions in the hypothesis, the proof of Theorem 2.1 proves the lemma. If G is a group then it satisfies these conditions for every $\alpha < \delta(G)$. If $\tilde{G}$ is another group which is very close to G (say G and $\tilde{G}$ have generators which are very close in $\mathrm{PSL}(2, \mathbb{C})$) then $\tilde{G}$ will also satisfy these conditions (since they only involve a finite number of elements in the group). Thus

Corollary 2.4. *Suppose G is a Möbius group generated by $\{g_1, \dots, g_n\}$. Given any $\delta_0 > 0$ there is an $\epsilon_0 > 0$ (depending only on δ and G) such that if $\tilde{G}$ is a group containing elements $\{\tilde{g}_1, \dots, \tilde{g}_n\}$ with $\|g_i - \tilde{g}_i\| < \epsilon$ (as elements of $PSL(2, \mathbb{C})$) then*

$$\delta(G') \geq \delta(G) - \delta_0.$$

Suppose $\{G_n\}$ is a sequence of m-generated Möbius groups each with a specific listing of its generators $G_n = \{g_{1n}, \dots g_{mn}\}$. We say that G_n converges *algebraically* to a Kleinian group G with generators $\{g_1, \dots, g_m\}$ if $g_{jn} \to g_j$ for each $1 \leq j \leq m$, as elements of $\mathrm{PSL}(2, \mathbb{C})$. See [38]. If we identify groups with points in $\mathrm{PSL}(2, \mathbb{C})^m$, this is just convergence in the product topology.

Corollary 2.5. *If $\{G_n\}$ is a sequence of discrete Möbius groups converging algebraically to G, then*

$$\liminf_{n \to \infty} \delta(G_n) \geq \delta(G).$$

This says that $\delta(G)$ is lower semi-continuous with respect to algebraic convergence. Strict inequality is possible even for sequences of Kleinian groups (e.g., one can choose a sequence $\{G_n\}$ of geometrically finite groups in $T(S)$ converging to a a geometrically finite cusp group G so that $(\delta(G_n) \to 2$ but $\delta(G) < 2)$.

We should also note that Corollary 2.5 is still true even if the groups involved are not discrete (assuming that we define $\delta(G)$ appropriately). If G is a finitely generated Möbius group which is not discrete, then its closure is a closed subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and hence a Lie subgroup. The only possible limit sets (i.e., accumulation sets on S^2 for the orbit of 0) are: zero, one or two points, a circle or the whole sphere. Let us set $\delta(G)$ to be $0, 1$ or 2 in these three cases respectively. In each case, it is easy to check that Corollary 2.5 still holds if we allow either the $\{G_n\}$ or G to be non-discrete groups.

3. Big β's imply big dimension

Although Kleinian groups are closely associated with 3-manifolds, our approach to Theorem 1.2 is completely two dimensional, exploiting connections between conformal mappings, the Schwarzian derivative and certain geometric square functions which first arose in the study of the Cauchy integral. A dyadic square Q in the plane is one of the form $Q = [2^{-n}j, 2^{-n}(j+1)] \times [2^{-n}k, 2^{-n}(k+1)]$. The side length of a square will be denoted $\ell(Q)$. For a positive number $\lambda > 0$, we let λQ denote the square concentric with Q but with side length $\lambda \ell(Q)$, e.g., $2Q$ is the "double" of Q.

Given a set E in the plane and a square Q we define $\beta(Q)$ and $\delta(Q)$ as

$$\beta(Q) = t^{-1} \inf_{L \in \mathcal{L}} \sup_{z \in E \cap 3Q} \operatorname{dist}(z, L),$$

where $\mathcal{L}$ is the set of all lines L intersecting Q and

$$\delta(Q) = \ell(Q)^{-1} \inf_{L \in \mathcal{L}} H(E \cap 3Q, L \cap 3Q),$$

where $H(E, F)$ denotes the Hausdorff distance between the sets,

$$H(E, F) = \max_{x \in E} \operatorname{dist}(x, F) + \max_{y \in F} \operatorname{dist}(y, E).$$

The second author proved in [36] that a planar set E lies on a rectifiable curve iff

$$\sum_Q \beta(Q)^2 \ell(Q) < \infty,$$

where the sum is over all dyadic squares in the plane. We shall show

Theorem 3.1. *Suppose E is a closed, connected set in the plane and $\beta_E(Q) \geq \beta_0 > 0$ for every square Q with $Q \cap E \neq \emptyset$ and $\ell(Q) \leq diam(E)$. Then $dim(E) \geq 1 + C\beta_0^2$, where C is an absolute constant.*

This is sharp (except for the choice of C) as can be seen by considering the standard iterative construction of a snowflake curve where each segment of length r is divided into four equal segments and the middle two are replaced by segments of length $r(\frac{1}{4} + \beta^2)$.

We will see later that if G has no parabolics then $\Lambda(G)$ has this property (i.e., is uniformly wiggley"). However, a general limit set need not have β's bounded away from zero everywhere (this fails near doubly cusped parabolic points), but they are large often enough to push the dimension above 1.

Proof of Theorem 3.1. Suppose E_0 is a connected compact set with large β's, Q is a square of side length $\ell(Q) = r = 2^{-N}$ with $\frac{1}{3}Q \cap E \neq \emptyset$ (here and later λQ denotes the square concentric with Q of side length $\lambda\ell(Q)$). Our first objective is to show that for small $\epsilon > 0$ we can find more than $1000\epsilon^{-1}$ subsquares of Q of sidelength $\epsilon\ell(Q)$ with disjoint doubles and such that each contains a point of E_0 in its middle third. We will then apply the same argument to each subsquare and use the resulting nested collection to squares to build a Frostman measure on E_0.

Define $E = (E_0 \cap Q) \cup \partial Q$. Note that E is connected. Fix some integer n so that $2^{-n} < r$ (possibly much smaller). Because E is connected it is easy to check that there are more than $\frac{1}{3}r2^n$ dyadic subsquares $\{Q_j\}$ of Q which lie in $\frac{2}{3}Q$ and such that $Q_j \cap K \neq \emptyset$ (e.g., consider concentric "annuli" between $\frac{1}{3}Q$ and $\frac{2}{3}Q$ made of squares of size $2^{-n}r$; there are $\frac{1}{3}r2^n$ such and each must intersect E).

From this we deduce

$$\sum_{Q:\ell(Q)=2^{-n}} \beta_E^2(Q)\ell(Q) \geq \frac{1}{3}\beta_0^2 r.$$

Thus for any integer $k \geq 1$ (recall $r = 2^{-N}$),

$$\sum_{n=N+1}^{N+k} \sum_{Q:\ell(Q)=2^{-n}} \beta_E^2(Q)\ell(Q) \geq \frac{1}{3}k\beta_0^2 r.$$

Suppose Γ_k is the shortest curve in the plane with the property that for each $z \in E$ we have $\mathrm{dist}(z, \Gamma_k) \leq 2^{-N-k}$. It is fairly easy to check that that for squares with $\ell(Q) \geq 10 \cdot 2^{-N-k}\beta_0^{-1}$, we have

$$\beta_{\Gamma_k}(Q) \geq \frac{1}{2}\beta_E(Q) \geq \frac{1}{2}\beta_0.$$

Therefore

$$\sum_{n=N+1}^{N+k} \sum_{Q:\ell(Q)=2^{-n}} \beta_{\Gamma_k}^2(Q)\ell(Q) \geq (k - 10\beta^{-1})\beta_0^2 r.$$

Choose $k > 20\beta_0^{-1}$. Then the term on the right is $\geq \frac{1}{6}k\beta_0^2 r$.

By the second author's characterization of rectifiable curves in [36], the length of Γ_k is at least

$$\ell(\Gamma_k) \geq C_0\beta_0^2 kr,$$

for some absolute constant C_0. We claim that this implies that there are more than $(C_0\beta_0^2 kr - 4r)C2^{N+k}$ boxes $\{Q_j\}$ of side length 2^{-N-k} such that $\frac{1}{3}Q_j \cap E \neq \emptyset$.

To prove this claim let $\{z_j\}$ be a collection of points on Γ_k so that $j \neq k$ implies $|z_j - z_k| \geq 2^{-N-k}$, but so that $\cup_j B(z_j, 2^{-N-k+2})$ covers E. Let $\mathcal{C}$ be the collection $\{\partial Q_k\}$ of all dyadic squares of size 2^{-N-k} contained in Q which contain some z_j. Let $\Gamma = \cup_{Q_j \in \mathcal{C}} \partial Q_j \cup \cup_j S_j$ be the union of the boundaries of these cubes, together with segments S_j which connect ∂Q_j with the point z_j. Then obviously

$$\ell(\Gamma) \leq 6 \cdot 2^{-N-k}|\mathcal{C}|.$$

Since Γ has the property that it passes within 2^{-N-k} of every point of E and since Γ_k was defined to be the shortest such curve, we must have

$$6 \cdot 2^{-N-k}|\mathcal{C}| \geq \ell(\Gamma) \geq \ell(\Gamma_k) \geq C_0\beta_0^2 k 2^{-N}.$$

and hence

$$|\mathcal{C}| \geq \frac{1}{6}C_0\beta_0^2 k 2^k.$$

So if k is large enough (e.g., $k > 60000\beta_0^{-2}C_0^{-1}$) Now set $\epsilon = 2^{-k}$. Then $|\mathcal{C}| \geq 10000\epsilon^{-1}$, and consist of disjoint dyadic squares of size $\epsilon\ell(Q)$ each of which contains a point of E. By throwing away 9/10's of the squares we can assume the remaining ones have disjoint triples, and this is what we wished to prove.

To finish the proof of the theorem, we build nested generations of squares using the construction above. The initial square Q_0 forms the first generation. The squares of size $\epsilon\ell(Q_0)$ constructed above form the first generation. In general, given a nth generation square containing a point of E_0 in its middle third, we construct $1000\epsilon^{-1}$ subsquares as above (with disjoint triples and containing a point of E_0 in their middle thirds), and put these into the $(n+1)$st generation.

We then define a measure μ by assigning each nth generation square equal mass (namely $(\epsilon/1000)^n$)). Since a nth generation square has size ϵ^n, this measure satisfies

$$\mu(Q) \leq C\ell(Q)^\alpha,$$

where

$$\alpha = \frac{\log\epsilon - \log 1000}{\log\epsilon} = 1 + \frac{\log 1000}{\log\frac{1}{\epsilon}} > 1,$$

and Q is an nth generation square. Since $\epsilon = 2^{-k}$ and $k \sim \beta_0^{-2}$ we get $\log\epsilon^{-1} \sim \beta_0^{-2}$. This gives the estimate in the theorem. It only remains to check that this inequality holds for all squares in the plane, but this is a standard argument. Thus

$$\sum \operatorname{diam}(D_j)^\alpha \geq \sum \mu(D_j) \geq \mu(E_0) > 0$$

for any covering $\{D_j\}$ of E_0, and therefore $\dim(E_0) \geq \alpha > 1$. $\square$

4. Large Schwarzian implies large β's

The Schwarzian derivative of a locally univalent function F is defined by

$$\begin{aligned} S(F)(z) &= [\frac{F''(z)}{F'(z)}]' - \frac{1}{2}[\frac{F''(z)}{F'(z)}]^2 \\ &= [\frac{F'''(z)}{F'(z)}] - \frac{3}{2}[\frac{F''(z)}{F'(z)}]^2. \end{aligned}$$

If we write $F' = e^\varphi$ then it can be rewritten as

$$S(F)(z) = \varphi'' - \frac{1}{2}(\varphi')^2.$$

Recall that $S(F) \equiv 0$ iff F is a Möbius transformation and that S satisfies the composition law

$$S(F \circ G) = S(F)(G')^2 + S(G).$$

In particular, if G is Möbius then

$$S(F \circ G) = S(F)(G')^2$$

$$S(G \circ F) = S(F).$$

In addition, given an $\epsilon > 0$, hyperbolic disk D and a compact neighborhood K of D, there is a $\delta > 0$ so that $|S(F)| \leq \delta$ on D implies F uniformly approximates a Möbius transformation on K to within ϵ.

We introduce some notation from the second author's paper [37]. For a domain Ω and a point $z_0 \in \Omega$ let $z_1 \in \partial\Omega$ be the closest point in the Euclidean metric and let

$\theta_0 = \arg(z_0 - z)$. Let $\tilde{z}_1 = z_1 + \delta(z_0, z_1)$ were $0 < \delta < 1$ is fixed. Define

$$L^\delta_{z_0} = \{z_1 + t\exp(i(\theta_0 - \frac{\pi}{2})) : t \in \mathbb{R}, |t| \leq \delta^{-1}|z_1 - z_0|\},$$

$$S^\delta_{z_0} = \{z : |z - z_1| \leq \delta^{-1}|z_1 - z_0|, |\theta_0 - \arg(z - \tilde{z}_1)| \leq \frac{\pi}{2}\}.$$

Then $L^\delta_{z_0}$ is a sort of tangent line to $\partial\Omega$ (as seen from z_0) and $S^\delta_{z_0}$ is a half-disk which approximates Ω near z_0. We say that Ω satisfies condition $M(\delta)$ at z_0 if either there exists $z \in L^\delta_{z_0}$ such that

$$\mathrm{dist}(z, \partial\Omega) \geq \delta \mathrm{dist}(z_0, \partial\Omega),$$

or

$$S^\delta_{z_0} \cap \partial\Omega \neq \emptyset.$$

Thus if $M(\delta)$ is satisfied at z_0 then the boundary of Ω "wiggles" to order δ near z_0. The following is Theorem 2 of [37]

Lemma 4.1. *Suppose Ω is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping. For every $\delta > 0$ there is a $\epsilon > 0$ so that if $s(z) = |S(\Phi)(z)|(1 - |z|)^2 \geq \epsilon$ then Ω satisfies $M(\delta)$ at $\Phi(z)$. Furthermore, $M(\delta)$ is satisfied at every point of Ω iff there are $C < \infty$ and $\epsilon > 0$ so that $s(z) > \epsilon$ on a C-dense set of the disk (with respect to the hyperbolic metric).*

The proof is a normal families argument. Since $\partial\Omega$ is connected it is easy to see that condition $M(\delta)$ implies that the square centered at z_0 of size $\delta^{-1}\mathrm{dist}(z_0, \partial\Omega)$ has $\beta_{\partial\Omega}(Q) \geq \delta^2$. Thus large Schwarzian implies large β's. Another version (more precise than we need in this paper) is given by

Lemma 4.2. [14] *Suppose E is compact, $\Omega = \overline{\mathbb{C}}\backslash E$ and $\Phi : \mathbb{D} \to \Omega$ the universal covering map. Suppose Φ is univalent on $D(w, \epsilon(1-|w|))$. Then*

$$|S(\Phi)(w)|(1-|w|^2)^2 \leq C\epsilon^{-2}\sum_{n=0}^{\infty}\delta_E(2^nQ)2^{-\mu n}$$

where $r = dist(\Phi(w), \partial\Omega)$ The number μ satisfies $0 < \mu < 1$ but can be taken as close to 1 as we wish. The constant C depends only on the choice of μ.

Using either of these estimates we can prove the following

Corollary 4.3. *Suppose Ω is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mappings. Suppose also that there is a $C < \infty$ and $\epsilon > 0$ so that every point of the disk is at most hyperbolic distance C from a point z where $|S(\Phi)(z)(1-|z|^2)^2| \geq \epsilon$. Then there is a $\beta = \beta(C, \epsilon) > 0$ so that $\beta_{\partial\Omega}(Q) \geq \beta > 0$ for every square Q such that $\ell(Q) \leq diam(Q)$ and $\frac{1}{3}Q \cap \partial\Omega \neq \emptyset$. In particular, $dim(\partial\Omega) > 1$ with estimates depending only on C and ϵ.*

Proof. We use the second lemma. Let $E = \partial\Omega$ and let Q be a square such that $\ell(Q) \leq \mathrm{diam}(Q)$ and $\frac{1}{3}Q \cap E \neq \emptyset$. Let $\beta_0 = e^{-10C}$. We may also suppose $\epsilon = 2^{-\mu N}$ for some integer N. If $\beta_E(Q) \geq \epsilon\beta_0$, there is nothing to do, so assume $\beta_E(Q) \leq \epsilon\beta_0$. By the Koebe 1/4 theorem, the hyperbolic metric on Ω is comparable to $\mathrm{dist}(z, E)^{-1}ds$, so by our hypothesis we can find a point $z \in \frac{1}{3}Q \cap \Omega$ with $\epsilon\beta_0\ell(Q) \leq \mathrm{dist}(z, E) \leq \epsilon\ell(Q)$, and such that if $w = \Phi^{-1}(z)$ then $|S(\Phi)(w)(1-|w|)^2| \geq \epsilon$.

Let Q' be the square centered at z with side length $2\mathrm{dist}(z, E)$. By the estimate above we know that one of the squares $Q', 2Q', \ldots 2^NQ'$ where $N \sim \log_2 \epsilon$ has $\delta_E \geq \epsilon$. Call this square Q''. Since $z \in \frac{1}{3}Q$ and $2^N\ell(Q) \leq \frac{1}{4}\ell(Q)$, Q'' must be a subsquare of

Q. Since $\ell(Q'') \geq \beta_0 \epsilon \ell(Q)$, we deduce

$$\delta_E(Q) \geq \beta_0 \epsilon \delta_E(Q'') \geq \beta_0 \epsilon^2 \equiv \beta.$$

This is what we wished to prove. The final statement is simply an application of Theorem 3.1. $\square$

Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann mapping and let $\hat{G} == \Phi \circ G \circ \Phi^{-1}$ denote the Fuchsian equivalent of G. We can now prove Theorem 1.2 in the case when $\hat{G}$ has no parabolics. In this case, the surface Ω/G is compact so that any point of $\mathbb{D}$ is within some bounded distance C on any orbit of $\hat{G}$. If Ω is not a disk then Φ is not Möbius, so $S(\Phi)$ does not vanish identically. Choose a point z so that $s(z) = |S(\Phi)(z)(1-|z|^2)^2| = \epsilon \neq 0$. Since $S(z)$ is constant on orbits of $\hat{G}$, we can apply Corollary 4.3 to deduce $\dim(\partial\Omega) > 1$.

5. Non-differentiability of the boundary

In this section we describe our "non-differentiability" result for invariant components. First we need some definitions which capture what we mean by "non-differentiable". A point $x \in \partial\Omega$ is called a (inner) tangent point of Ω if for any $\theta < \pi$, x is the vertex of a cone in Ω with angle θ, and this is not true for any $\theta > \pi$. The set of tangents is the same, up to a set of 1-dimensional measure zero, as the set $E \subset \partial\Omega$ of points which are vertices of some cone in $\Omega(G)$. If Ω is simply connected and $x \in \partial\Omega$, then x is called a twist point of Ω if

$$\limsup_{z \to x, z \in \Omega} \arg(x-z) = +\infty, \quad \liminf_{z \to x, z \in \Omega} \arg(x-z) = -\infty.$$

The harmonic measure ω for the domain Ω is the push forward of Lebesgue measure on the circle under a Riemann mapping onto Ω (the measure depends on the choice of Riemann mapping, but its null sets do not). A probability measure μ is said to be singular with respect to a measure ν if it gives full measure to a null set for ν.

Theorem 5.1. *Suppose G is a finitely generated Kleinian group with simply connected invariant component Ω. Then the following are all equivalent.*

(1) *Ω is not a disk.*

(2) *The set of inner tangent points of $\partial\Omega$ has zero 1-dimensional measure.*

(3) *Harmonic measure for Ω is singular to 1-dimensional measure.*

(4) *Almost every (with respect to harmonic measure) point of $\partial\Omega$ is a twist point.*

Conditions (2), (3) and (4) are equivalent for any simply connected domain and they each imply (1). Thus the main point is to show (1) implies any of (2), (3) or (4). Using known results on the behavior of harmonic measure (to be described below) this result implies,

Corollary 5.2. *Suppose G is finitely generated and $\Lambda(G)$ is not a circle. Then the harmonic measures for different components of $\Omega(G)$ are mutually singular.*

This last corollary is well known in many cases. For example, when G is a quasi-Fuchsian group it corresponds exactly to Mostow's theorem: given two finitely generated, first kind Fuchsian groups on the unit disk and a homeomorphism ϕ of the boundary which conjugates the actions on the circle, then either φ is Möbius or φ is singular (i.e., maps full measure to zero measure). A stronger version of Mostow's result is proven in [16] where it is shown that such a φ is either Möbius or maps a set of dimension < 1 to the complement of a set of dimension < 1.

If Ω is a component of $\Omega(G)$ then the stabilizer of Ω in G is also finitely generated (this is a consequence of the Ahlfors finiteness theorem [3], [5]). Thus if Ω is simply connected and $\Phi : \mathbb{D} \to \Omega_0$ is a Riemann mapping, $\hat{G} = \Phi \circ G \circ \Phi^{-1}$ is a finitely generated Fuchsian group of the first kind (i.e., its limit set is the whole circle) called the Fuchsian equivalent of G on Ω. We can actually prove Theorem 5.1 using the weaker hypothesis that the Fuchsian equivalent of G on Ω has a non-tangentially dense orbit. This happens if it is divergence type, i.e.,

$$\sum_{\gamma \in \hat{G}} (1 - |\gamma(0)|) = \infty.$$

This is the best one can expect since Astala and Zinsmeister [6] have shown that any convergence group (i.e., $\sum_{\gamma \in \hat{G}} (1 - |\gamma(0|) < \infty$) has a quasiconformal deformation to a Kleinian group whose limit set is a rectifiable curve (not a circle).

Suppose Ω is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping. By Plessner's theorem the unit circle $\mathbb{T}$ can be divided into two set E_0, E_1 such that almost every (Lebesgue measure) point of E_0, Φ' has a finite, non-zero non-tangential limit and almost everywhere on on E_1, Φ' is non-tangentially dense in the plane. Let ω denote the harmonic measure on Ω, i.e., $\omega(E) = |\Phi^{-1}|/2\pi$. By McMillan's theorem [44], almost every point of $\partial\Omega$ is either the vertex of a cone in Ω or is a twist point (see introduction) and these two sets correspond a.e. to E_0 and E_1 respectively via the map Φ. The set of the cone points has σ-finite 1-dimensional measure and on this set ω is mutually absolutely continuous with 1-dimensional Hausdorff measure. Makarov [41] proved that there is a subset F of the twist points which has zero 1-dimensional measure but the same harmonic measure as the full set of twist points. Thus harmonic measure on a simply connected domain always "lives" on a set of dimension 1, regardless of the Hausdorff dimension of the entire boundary. Pommerenke [50] proved that harmonic measure gives full measure to a set of σ-finite 1-dimensional measure and Wolff [62] has even extended this to arbitrary planar domains.

Given two disjoint simply connected domains Ω_1, Ω_2 with overlapping boundary $E = \partial\Omega_1 \cap \partial\Omega_2$, the two harmonic measures ω_1, ω_2 are mutually absolutely continuous on a subset $F \subset E$ iff almost every point (with respect to both measures) of F is the vertex of two cones, one in each of the two domains [13]. In this case, both harmonic measures are also mutually absolutely continuous with respect to 1-dimensional Hausdorff measure on F. Thus the measures ω_1 and ω_2 are mutually singular iff the set of "double cone points" has zero 1-dimensional measure. In particular, if almost every point (with respect to ω_1) of $\partial\Omega_1$ is a twist point then ω_1 is singular to 1-dimensional measure and singular to harmonic measure on any other disjoint domain. The converse is not true; there is a closed Jordan curve so that the harmonic measures for the two complementary domains are mutually singular but each is absolutely continuous with respect to 1-dimensional measure [11]. This is not possible for quasicircles however. A criterion for singularity of harmonic measures of two general disjoint domains is given in [12].

There is also a connection between rectifiability and the Schwarzian derivative.

Lemma 5.3. [15] *There is a $C < \infty$ such that if $\Phi : \mathbb{D} \to \Omega$ is univalent and $\partial\Omega$ is rectifiable (i.e., has finite 1-dimensional measure) then*

$$\iint_{\mathbb{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|^2)^3 dxdy < C\ell(\partial\Omega).$$

The same holds if Φ is only defined on a Lipschitz subdomain of $\mathbb{D}$. If Ω is a quasicircle and $dist(\Phi(0), \partial\Omega) \sim diam(\Omega)$ then the two sides are comparable with a constant depending only on the quasicircle bound.

This is Lemma 3.7 of [15] and is an easy computation involving Green's theorem and basic estimates for univalent mappings.

From the lemma we now deduce Theorem 5.1. Suppose Ω is a simply connected component of $\Omega(G)$. If Ω is not a circle, then $S(\Phi)$ is non-zero somewhere in the disk and therefore the invariant quantity $s(z) = |S(\Phi)(z)|(1-|z|^2)^2$ equals $\epsilon > 0$ on some orbit $\{z_n\}$ of the Fuchsian equivalent $\hat{G}$. Since $\hat{G}$ is a finitely generated Fuchsian group of the first kind, the orbit of any point is non-tangentially dense, i.e., for almost every point $x \in \mathbb{T}$ there is a sequence of points in the orbit approaching x through a Stolz cone with vertex x. Alternatively, if we associate to each orbit point z_n the interval I_n of length $1-|z_n|$ centered and $z_n/|z_n|$, then almost every point if the circle is in infinitely many of the intervals $\mathcal{F} = \{I_n\}$.

Now suppose Φ' has non-tangential limits on a set $E \subset \mathbb{T}$ of positive Lebesgue measure. Then by taking a union of small cones we could construct a "sawtooth" domain $W \subset \mathbb{D}$ with $|E \cap \partial W| \geq \frac{1}{2}|E|$ and such that $M^{-1} \leq |\Phi'| \leq M$ on W. Let $F = E \cap \partial W$. Because of the non-tangential density of the orbit we can use the Vitali covering lemma (e.g. page 109 of [61]) to obtain a disjoint covering of almost every point of F by intervals of the for $\{I_n\}$. In fact, by repeated applications of covering lemmas we can find infinitely many collections $\mathcal{F}_k = \{I_j^k\} \subset \mathcal{F}$ each of which is a disjoint covering of almost all of F, and so that no interval belongs to more than one collection.

For each orbit point z_n, let D_n denote the disk centered at z_n of radius $\frac{1}{2}(1-|z_n|)$.

By the Koebe 1/4 theorem

$$\begin{aligned}\iint_{D_n} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|^2)^3 dxdy &\geq C\epsilon^2 \iint_{D_n} |\Phi'(z)|(1-|z|)^{-1}dxdy \\ &\geq CM^{-1}\epsilon^2(1-|z_n|) \\ &\geq CM^{-1}\epsilon^2|I_n|.\end{aligned}$$

Therefore

$$\begin{aligned}\iint_{W} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|^2)^3 dxdy &\geq CM^{-1}\epsilon^2 \sum_{n:z_n\in W} |I_n| \\ &\geq CM^{-1}\epsilon^2 \sum_k \sum_{I_n\in\mathcal{F}_k} |I_n| \\ &= \infty.\end{aligned}$$

Therefore $\Phi(W)$ cannot have a finite length boundary by Lemma 5.3. However, $|\Phi'|$ is bounded by M on W, so $\Phi(W)$ must have a rectifiable boundary. This contradiction implies that Φ' cannot have non-tangential limits on any set of positive Lebesgue measure. Thus in terms of earlier remarks, almost every point of $\partial\Omega$ must be a twist point, the set of cone points has zero 1-dimensional measure and harmonic measure on Ω is singular to harmonic measure for any simply connected domain which is disjoint from Ω.

If we apply the proof not to the whole disk but to a "Carleson square" Q corresponding to an interval $I \subset \partial\mathbb{D}$,

$$Q = \{re^{i\theta} : e^{i\theta} \in I, 1-|I| < r < 1\},$$

then the proof shows that the length of $\Phi(rI)$ must tend to infinity. If we divide the circle into N equal intervals, we can apply the proof to each interval and then by my taking the minimum growth rate, get an estimate which is valid for all N intervals.

Since any interval of length $\geq 4\pi/N$ contains at least one interval from our collection, the growth rate also holds for such an interval. Thus

Lemma 5.4. *Suppose G is finitely generated with invariant component Ω which is not a disk. If Φ is a Riemann mapping onto Ω and $I \subset \partial\mathbb{D}$ then $\ell(\Phi(rI)) \to \infty$ as $r \to 1$ with estimates that only depend on G and the length of I.*

We finish this section with a remark concerning Makarov's law of the iterated logarithm from [41]. Not only did Makarov prove that the harmonic measure for a simply connected domain gives full mass to a set of dimension 1 (as discussed earlier), but also the following sharp converse. If we define

$$\varphi_C(t) = t\exp(-C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}),$$

then there is an absolute constant C_1 so that harmonic measure for any simply connected domain is absolutely continuous with respect to the Hausdorff measure for the gauge function φ_{C_1}. Conversely, there is a C_2 and examples of simply connected domains for which harmonic measure gives positive mass to a set of zero $H_{\varphi_{C_2}}$ measure. Such domains are called Makarov domains and have the worse possible behavior for harmonic measure. In [37] the second author proved that Ω is a Makarov domain if its boundary is "wiggley" on all scales (in a sense made precise in that paper). It is easy to check that Ω satisfies his condition if every point of the disk is within a bounded hyperbolic distance of a point where $s(z) = |S(\Phi)(z)(1-|z|^2)^2$ is bounded away from zero. We have already observed that if Ω is a component of a finitely generated group G such that the Fuchsian equivalent $\hat{G}$ contains only hyperbolics, then this condition is satisfied. Thus all such domains are Makarov domains. Presumably the same is true with parabolics, but we have not attempted to prove this.

6. Proof of Theorem 1.2 with parabolics

We have already proven Theorem 1.2 when the Fuchsian equivalent $\hat{G}$ of G on Ω has no parabolic elements. In this section we will prove it assuming $\hat{G}$ does contain parabolics. This will also give a different proof for the co-compact case.

This section is not strictly necessary for proof of Theorem 1.2; we shall later give two more proofs for the case of degenerate groups and Theorem 1.2 can be deduced from this and the known results for geometrically finite groups. However, this argument seems to be necessary to prove $\delta(G) > 1$. In fact, we will show $\delta(G) > 1$ and use the fact that if Ω is G-invariant, then $\partial\Omega = \Lambda(G)$. Thus by Theorem 1.1 we deduce Theorem 1.2.

First we want to show that it is enough to estimate the Poincaré series along an orbit in $\Omega(G)$ (rather than along the orbit of $0 \in \mathbb{B}$. Recall that for $z \in \Omega(G)$ we set

$$d(z) = \operatorname{dist}(z, \partial\Omega(G)).$$

Lemma 6.1. *If G is a Kleinian group, 0 is the center of $\mathbb{B}$ and $z \in \Omega(G) \subset \partial\mathbb{B}$, then*

$$d(g(z)) \sim 1 - |g(0)|, \quad \textit{for all } g \in G$$

with constants that depend on z and G, but not on g.

Proof. Choose a ball $B \subset \Omega$ centered at z_0 so that $\operatorname{diam}(B) \leq \frac{1}{2}\operatorname{dist}(z_0, \partial\Omega$. Let $\omega_1 = \omega(0, B, \mathbb{B})$ be the harmonic measure of this ball in $\mathbb{B}$ with respect to the point zero and let ω_2 be the harmonic measure of $\frac{1}{2}B$ with respect to 0. Then by the conformal invariance of harmonic measure, for any $g \in G$, $g(0)$ is the unique point z so that

$$\omega(z, g(B), \mathbb{B}) = \omega_1,$$

$$\omega(z, g(\frac{1}{2}B), \mathbb{B}) = \omega_2.$$

By our choice of B and the Koebe 1/4 theorem, $g(\frac{1}{2}B) \subset \lambda g(B)$ for some $\lambda < 1$ independent of g. Therefore any $z \in \mathbb{B}$ which satisfies the two equalities above must satisfy

$$|g(z_0) - z| \leq C\text{diam}(g(B)),$$
$$1 - |z| \geq \frac{1}{C}\text{diam}(g(B)).$$

This proves the lemma. □

Note that this implies that the accumulation set of any point in $\Omega(G)$ is all of $\Lambda(G)$. In particular, if Ω is an invariant component of $\Omega(G)$ then $\partial\Omega = \Lambda(G)$.

Let $\hat{G}$ denote the Fuchsian equivalent of G and suppose 0 is a point where the Schwarzian derivative of the Riemann mapping $\mathbb{D} \to \Omega$ is large. Suppose $z_j = g(0) \in \mathbb{D}$ is an orbit point of zero and let $I_j \subset \partial\mathbb{D}$ be the interval on the boundary centered at $z_j/|z_j|$ with length $1 - |z_j|$. We let S_j denote the Carleson square with base I_j,

$$S_j = \{z : z/|z| \in I_j, 1 - |I_j| \leq |z| < 1\}.$$

Let

$$d_j = d(\Phi(z_j)) = \text{dist}(\Phi(z_j), \partial\Omega) \sim (1 - |z_j|)|\Phi'(z_j)|.$$

We will show

Lemma 6.2. *There is a $C < \infty$ (depending only on G) such that if $g \in \hat{G} \setminus \{Id\}$, and $z_0 = g(0)$ then there is a collection of orbit points $\{z_k\} = \{g_k(0)\} \subset T_0$ such that*

(1) $\sum_k d_k \geq 2d_0$,

(2) $(1 - |z_k|) \geq (1 - |z_0|)/C$.

(3) *the intervals $\{I_k\}$ are disjoint.*

Note that conditions (1) and (2) imply

$$\sum_k d_k^{1+\epsilon} \geq d_0^{1+\epsilon},$$

if ϵ is small enough (depending on C). Using condition (3), we can break the orbit of $0 \in \mathbb{D}$ into generations $\mathcal{G}_n$ so that

$$\sum_{z_k \in \mathcal{G}_n} d_k^{1+\epsilon} \geq \sum_{z_k \in \mathcal{G}_{n-1}} d_k^{1+\epsilon} \geq 1,$$

which implies

$$\sum_{z_k \in \hat{G}(0)} d_k^{1+\epsilon} = \infty.$$

By Lemma 6.1, this proves $\delta(G) > 1$.

Thus it suffices to prove Lemma 6.2 Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann map, normalized so that $S(\Phi)(0) \neq 0$. Conjugate g so that $\Phi(0) = \infty$ and $\operatorname{diam}(\partial\Omega) = 1$. If Ω/G is a surface with punctures, then we can find a G invariant collection of disjoint balls $\mathcal{B}_1 = \{B_j^1\}$ in Ω, each invariant under a parabolic element of G and so that $(\Omega \setminus \cup_j B_j^1)/G$ is compact (i.e., we are taking a neighborhood of each puncture on Ω/G and lifting it to Ω. Each B_j^1 is thus conjugate to one of a finite subcollection and each ball has a parabolic fixed point of G on its boundary. To each ball in $\mathcal{B}_1$ we associate smaller invariant balls $B_j^2 \subset B_j^1$ so that the hyperbolic distances between ∂B_j^1 and ∂B_2^j is 1. Also any point of Ω which is outside $\cup_j B_j^2$ is within a bounded hyperbolic distance of the orbit of ∞. Let C_1 denote this bound.

If Ω/G has no punctures, just replace the collections $\mathcal{B}_j$ by the empty set in the proof that follows. Note that in this case every point of Ω is within a bounded distance of the orbit of ∞.

For each $t > 0$ consider the level line of Green's function in the disk

$$\Gamma_t = \{z : |z| = 1 - e^{-t}\},$$

and for each orbit point $z_j \in \hat{G}(0)$ let

$$\Gamma_t^j = \Gamma_{t(1-|z_j|)} \cap S_j.$$

The first thing we want to see is that the image of Γ_t^j is very long if t is small enough (independent of j).

Lemma 6.3. *For any $M > 0$ there is a t_0 such that if $t \leq t_0$ then*

$$\ell(\Phi(\Gamma_y^j)) \geq M d_j = M \, dist(\Phi(z_j), \partial\Omega).$$

Proof. This is just Lemma 5.4 after rescaling z_j to be the origin. $\square$

We would like to take our orbits in Lemma 6.2 to be (in some sense) taken along the curve Γ_t^j. More precisely, we will break Γ_t^j into unit hyperbolic segments $\{\gamma_k\}$ and to each segment associate the closest orbit point z_k. If there were no parabolic points then each orbit point would be associated to a bounded number of segments, say N (depending on only G) and by the standard distortion theorems for conformal maps,

$$\ell(\Phi(\gamma_k)) \sim d(z_k).$$

Thus by throwing out repeats we would have a collection of points $z_k \in S_j$ with

$$\sum_k d_k \geq \frac{M}{N},$$

and

$$d_k \geq C d_j,$$

(where C depends on t and constants in certain distortion theorems for conformal maps). This proves the Lemma 6.2 when there are no parabolics.

If the surface Ω/G has punctures, then there may be points of Γ_t^j which are very far from the closest orbit of 0, and we need to replace such pieces by new curves which are closer to orbit points. The idea is that if Γ_t^j passes though the "bottom half" of a horoball B in the unit disk, then we can should replace $\Gamma_t^j \cap B$ by the arcs

$$\Gamma_B = \partial B \cap \{z : \frac{1}{2} t(1-|z_j|) \leq 1-|z| \leq t(1-|z_j|)\}.$$

These arcs have similar hyperbolic length to the the arc $\Gamma_t^j \cap B$, they are only slightly closer to the boundary of the disk, and they remain within a bounded hyperbolic distance of the orbit of 0. Most importantly, the Koebe 1/4-theorem implies that

$$\ell(\Phi(\Gamma_B)) \sim \ell(\Phi(\Gamma_t^j \cap B)).$$

Thus we have

Lemma 6.4. *For each z_j there is an arc $\tilde{\Gamma}_t^j$ (consisting of pieces of Γ_t^j and arcs of horoballs such that*

(1) $\tilde{\Gamma}_t^j \subset S_j \cap \{|z| \leq 1 - \frac{t}{2}(1-|z_j|)\}$.
(2) $\ell(\Phi(\tilde{\Gamma}_t^j)) \geq M d_j$.
(3) *every component of $\tilde{\Gamma}_t^j$ has hyperbolic length at least* 1.

We can now finish the proof of Lemma 6.2 just as in the case without parabolics described above. This finishes the lemma and hence completes the proof of Theorem 1.2.

Let us summarize the results from the last three sections. Suppose G is a finitely generated Kleinian group with a simply connected invariant component Ω. Recall that G is called an extended Fuchsian group if it is Fuchsian or has an index two Fuchsian subgroup. Then the following are equivalent:

(1) G is not an extended Fuchsian group.
(2) $\partial\Omega$ is not a circle.
(3) $\partial\Omega$ has infinite 1-dimensional measure.
(4) $\dim(\partial\Omega) > 1$.
(5) $\dim(\Lambda_c(G)) > 1$.
(6) $\delta(G) > 1$.
(7) $\partial\Omega$ fails to have a tangent somewhere.
(8) $\partial\Omega$ fails to have a inner tangent and all points except possibly the rank 1 parabolic fixed points.
(9) The set of inner tangent points of $\partial\Omega$ has zero 1-dimensional measure.
(10) Harmonic measure for Ω is singular to 1-dimensional measure (i.e., there is a subset of $\partial\Omega$ of full harmonic measure and 1-dimensional measure zero).
(11) Almost every (with respect to harmonic measure) point of $\partial\Omega$ is a twist point.
(12) Harmonic measures for distinct components of $\Omega(G)$ are mutually singular.

7. Groups with round components

Suppose G is a finitely generated Kleinian group and $\Omega(G)$ contains a component Ω which is a disk. Then either $\Omega(G)$ consists of exactly two components (both round disks) or has infinitely many components which are disks. In the first case, G is an extended Fuchsian group and the limit set is a circle. The second case is described by the following result.

Theorem 7.1. *Suppose G is a finitely generated Kleinian group and $\Omega(G)$ contains infinitely many components which are disks. Then $dim(\Lambda(G)) > 1$.*

This is slightly different than Larmen's result mentioned in the introduction, because we do not insist that every component be a disk (it could happen, for example, that there is another component of $\Omega(G)$ with accidental parabolics). This is known result; it is a special case of Theorem 1 of [26] which Canary and Taylor prove using a result of Furusawa [31]. It is also contained in results of Sullivan and of Patterson. Roughly speaking, such a group G must contain 2 Fuchsian subgroups G_1, G_2 whose limit circles are each contained in a fundamental domain for the other group. Furusawa's theorem then says that $\delta(G_1 * G_2) > \delta(G_1) = 1$. Since these groups are geometrically finite the same inequality holds for the dimension of the limit sets. An alternate approach is to note that the sum of the diameters of the orbit of $\Lambda(G_1)$ under G_2 diverges because G_2 is divergence type. Thus $\Lambda(G_1 * G_2)$ has infinite length. Sullivan's results then imply it has dimension > 1.

Some very interesting pictures of this type of limit set appear in [21]. Further results on such limit sets and the corresponding groups are given in [39].

We include a proof of Theorem 7.1 for completeness. It is a fairly standard computation involving Hausdorff measures. As usual, we pass to a finite index subgroup if necessary to remove any elliptic elements. Let D_1, D_2 be distinct round components (which are not just opposite sides of one circle) with stabilizers G_1, G_2. Since the orbit of D_1 under G_2 accumulates densely on ∂D_2 and vice versa, we may assume (by choosing new disks if necessary) that

$$\mathrm{dist}(D_1, D_2) \geq 1,$$

$$\operatorname{diam}(D_1) \leq 1/1000,$$

and the double of each disk in contained in a fundamental polygon of the other group (so the translates of the doubles are disjoint).

Fix values of $\delta > 0$ and $N < \infty$. Suppose we construct a Cantor set E by an iterative construction in which a disk D is replaced by at most N disks $\{D_j\}$ such that

(1) $D_j \subset 2D$, and $\{2D_j\}$ are disjoint.

(2) $\delta\operatorname{diam}(D) \leq \operatorname{diam}(D_j) \leq \operatorname{diam}(D)/100$.

(3) $\sum_j \operatorname{diam}(D_j) \geq 2\operatorname{diam}(D)$.

Then it is easy to see that $\dim(E) \geq \alpha(\delta, N) > 1$.

For D_1 we can choose δ, N and such disks $\{D_j^1\}$ by taking part of the orbit of D_2 under G_1 (using the fact that G_1 is divergence type). Similarly for D_2 and $\{D_j^2\}$. At a general step in the construction suppose we have a disk D' which is a "child" of D'' (i.e, $D' \subset 2D''$ and $\operatorname{diam}(D') \leq \operatorname{diam}(D'')/100$). Then D' corresponds to either D_1 or D_2 under the action of $G_1 * G_2$, so assume it is D_1. Then there is an element $g \in G_1 * G_2$ so that $g(D_1) = D'$ and $g(D_2) = D$. Its easy to check that g has bounded distortion on $2D_1$ (since it corresponds to the much smaller disk D'). Thus $\{g(D_j^1)\}$ satisfy the desired conditions with respect to D' and the construction may be continued (the constants may be different, but we have uniform bounds).

8. Degenerate limit sets have large β's

Suppose G is a finitely generated degenerate group, i.e., $\Omega(G) = \Omega$ has a unique component and this component is simply connected. If the Fuchsian equivalent $\hat{G} = \Phi \circ G \circ \Phi^{-1}$ has no parabolics, then $\Lambda(G)$ has "large β's" by the argument in Section 4.

We shall show this is true even if $\hat{G}$ contains parabolics. The point is that a rank 1 parabolic in a degenerate group cannot be doubly cusped, and this implies big β's.

Theorem 8.1. *If G is a degenerate group there is a $\beta_0 = \beta_0(G) > 0$ such that if $E = \Lambda(G)$, then $\beta_E(Q) > \beta > 0$ for every square Q such that $\ell(Q) \leq diam(Q)$ and $\frac{1}{3}Q \cap \partial\Omega \neq \emptyset$. In particular, $dim(\Lambda(G)) > 1$.*

Proof. As usual, by Selberg's lemma we may pass to a finite index subgroup without torsion (which clearly has the same limit set). Thus we assume G has no elliptics and let $R = \Omega/G = \mathbb{D}/\hat{G}$ denote the Riemann surface represented by G and $\hat{G}$. If $\hat{G}$ has no parabolics then Theorem 8.1 follows from Theorem 4.3 as noted in Section 4, so assume $\hat{g} \in \hat{G}$ is parabolic and let $g = \Phi \circ \hat{g} \circ \Phi^{-1}$ be the corresponding element of G. Note that g must also be parabolic and call its fixed point z_0.

Let B denote a horoball in $\mathbb{D}$ corresponding to $\hat{g}$. Let $\hat{\gamma}$ denote its boundary and h the hyperbolic translation length of $\hat{G}$ along $\hat{\gamma}$. Let $\hat{\gamma}_0$ denote a segment of hyperbolic length h on $\hat{\gamma}$. If we conjugate the group G so that the fixed point of g becomes ∞ then g becomes a translation. Then $\hat{\gamma}_0$ maps to a smooth arc and thus its easy to see that $\hat{\gamma}$ maps to a smooth quasicircle passing through infinity. In particular, its lies in the strip between two parallel lines. Thus before conjugating the fixed point to infinity, we see that $\gamma = \Phi(\hat{\gamma})$ is a quasicircle passing through z_0 and is contained between two circles tangent at z_0, say $B_1 \subset \Phi(B) \subset B_2$. Note that these two balls are invariant under g since they correspond to half-planes when z_0 is conjugated to ∞.

By taking B_1 smaller, if necessary, we may assume that B_3, the reflection of B_1 through the fixed point z_0, does not intersect B_2. Then B_3 is also invariant under g.

We wish to show that B_3 must contain points of $\Lambda(G)$ that are not too close to z_0. Change variables so that $z_0 = 0$, B_3 lies in the upper half plane and has diameter 1. Let $z = iy$ lie on the positive imaginary axis and note that $|z - g(z)| = O(|y|^2)$, as $y \to 0$. However, $\hat{g}$ moves every point of the disk outside B by at least some hyperbolic distance δ (depending on $\hat{g}$ and B) and therefore since $z \in B_3$ lies outside $B_2 \supset \Phi(B)$, g must move z by more than δ in the hyperbolic metric on Ω. Since Ω is simply connected the hyperbolic metric on Ω is comparable to $\mathrm{dist}(z, \partial\Omega)^{-1} ds$. Thus for $z = iy$ small we must have

$$\mathrm{dist}(z, \partial\Omega) \leq C\delta^{-1}|z - g(z)| \leq C\delta^{-1}y^2.$$

By considering the point $z = iy$ with $0 < y < \delta/C$ we see that $B_3 \cap \Lambda(G)$ must contain a point z_1 within $|y|^2$ of z. The orbit of z_1 under powers of g is a y^2-dense set on a circle of radius larger that $y/2$ tangent at the fixed point z_0. Since $\Lambda(G)$ is invariant under g it contains this sequence of points.

In fact, we have shown that a point $z \in B_3 \cap \Omega$ at distance r from z_0 can be at most distance $\sim r^2$ from $\partial\Omega = \Lambda(G)$. In particular, if Q is any square so that $Q \cap B_3$ is large (say contains a disk or radius $> \ell(Q)/100$) then $\beta_{\Lambda(G)}(Q) \geq \beta_0 > 0$ (and $\beta \geq 1/4$ for all sufficiently small square containing 0).

Now we can deduce that $\Lambda(G)$ has large β's everywhere. Suppose we have a square Q such that $\ell(Q) \leq \mathrm{diam}(Q)$ and $\frac{1}{3}Q \cap \partial\Omega \neq \emptyset$. If $\Omega \cap \frac{1}{10}Q$ contains no disk of size $\ell(Q)/100$, then $\beta_{\Lambda(G)}(Q) \geq 1/20$. So assume $\Omega \cap Q$ does contain a large disk and let $z_1 \in \frac{1}{10}Q$ be its center. If z_1 lies outside all the balls B_1 corresponding to parabolics in $\hat{G}$ described above, then $w_1 = \Phi^{-1}(z)$ lies only a bounded hyperbolic distance from an orbit where the Schwarzian derivative $|S(\Phi)|(z)(1 - |z|^2)^2$ is bounded away from zero. Therefore in this case $\beta_{\Lambda(G)}(Q) \geq \beta_0$ (where β_0 depends on G, but not on Q).

Otherwise, z_1 must lie inside a horoball B_1. Let g be the corresponding element of G with fixed point z_0. By the choice of B_1, B_3 described above and since $z_1 \in \frac{1}{10}Q$, $Q \cap B_3$ must contain a disk of size $\geq \frac{1}{100}\ell(Q)$. So by our remarks above, $\beta_{\Lambda(G)}(Q) \geq \beta_0$ (in fact it is quite large). This completes the proof of Theorem 8.1. $\square$

One can give yet another proof of $\dim(\Lambda(G)) > 1$ for degenerate groups with parabolics using the observations above, but without using Theorem 3.1. One notes that the horoball B_3 can be packed by an infinite number of ball accumulating to the fixed point, each containing points of the limit set and so that the sum of the radii diverges. This, together with the observation that conjugates of B_1 take up a fixed fraction of the area of $\Omega(G)$ in each such ball, allows one to define a nested sequence of balls such that the resulting Cantor set lies in the limit set and has dimension larger than one. The packing in question is roughly the one corresponding (via a Möbius transformation) to a unit packing of the upper half plane. The dimension of Cantor sets arising from this kind of packing of the disk were studied in [32].

9. Proof of Theorem 1.3 and some corollaries

Suppose G has a connected limit set. Then any component Ω of $\Omega(G)$ is simply connected. The subgroup fixing any component Ω of $\Omega(G)$ is a finitely generated Kleinian group G_Ω and Ω is an invariant component of its ordianry set. Thus by Theorem 1.2 either $\dim(\Lambda(G)) > 1$ or every component of $\Omega(G)$ is a disk. If the latter case holds then either $\Omega(G)$ has two components or infinitely many. If it has two then $\Lambda(G)$ is a circle. Otherwise $\Lambda(G) = \overline{\mathbb{C}} \setminus \cup_j D_j$ for some infinite collection of disjoint open disks. Thus $\dim(\Lambda(G)) > 1$ by Theorem 7.1. (Actually, Larman [40] proved that such a set has dimension > 1, regardless of whether the disks are associated to any group.) It follows from the Klein-Maskit combination theorems that either $\Lambda(G)$ is totally disconnected or $\Lambda(G)$ contains a component which is itself the limit set of finitely generated subgroup (see e.g. [2], [43]). Thus we get

Theorem 9.1. *If G is a finitely generated Kleinian group then its limit set is either totally disconnected, a circle or has Hausdorff dimension > 1.*

Our proof of Theorem 1.2 showed that if Ω is a simply connected component which is not a disk then $\delta(G) > 1$. Thus we obtain

Corollary 9.2. *If G is finitely generated and Ω is a simply connected invariant component of $\Omega(G)$ which is not a disk, then $\delta(G) = dim(\Lambda_c(G)) > 1$.*

It follows from the Klein-Maskit combination theorems that if G is finitely generated and $\Lambda(G)$ is totally disconnected then G is geometrically finite and hence $\delta(G) = \dim(\Lambda_c(G)) = \dim(\Lambda(G))$. Thus Theorem 9.1 implies

Corollary 9.3. *If G is a finitely generated Kleinian group then $dim(\Lambda(G)) > 1$ iff $\delta(G) > 1$.*

Recall that G is called degenerate if $\Omega(G)$ is connected and simply connected. As a special case of Theorem 1.2 we get

Corollary 9.4. *If G is degenerate then $dim(\Lambda(G)) > 1$ and $\delta(G) > 1$.*

Greenberg [34] showed that degenerate groups are geometrically infinite so Theorem 1.4 will imply that the limit set of a degenerate group has dimension 2, but we mention this weaker result here because it is a new result whose proof easy, whereas the stronger result requires much more. In [24] Canary shows that $\delta(G) = 2$ for any degenerate group.

A web group is one in which each component subgroup is quasi-Fuchsian (so the ordinary set is a union of quasidisks). By the decomposition theorem of Abikoff and Maskit [2] any finitely generated, geometrically infinite group contains either a degenerate group or a geometrically infinite web group (all component subgroups quasi-Fuchsian). In either case the limit set must have dimension > 1, so

Corollary 9.5. *If G is a finitely generated, geometrically infinite Kleinian group then $dim(\Lambda(G)) > 1$ and $\delta(G) > 1$.*

Later we will show that $\dim(\Lambda(G)) = 2$ for geometrically infinite groups, and $\delta(G) = 2$ if $\Lambda(G)$ has zero area, but this will be our best result for $\delta(G)$ in general.

10. Geometry of $\Omega(G)$

In this section we will prove some lemmas about the geometry of $\Lambda(G)$ and $\Omega(G)$ that we will use later. The two main facts we need are

(1) $\Lambda(G)$ is uniformly perfect.

(2) Show bad horoballs must be singly cusped and far apart.

For $z \in \Omega(G)$ define

$$d(z) = \mathrm{dist}(z, \partial\Omega(G)) = \mathrm{dist}(z, \Lambda),$$

where "distance" means spherical distance. Suppose Ω is a component of $\Omega(G)$ and let ρ denote the hyperbolic metric on Ω. Then a standard estimate says that

$$|d\rho(z)| \leq 2\frac{|dz|}{d(z)}.$$

Since an orbit is uniformly separated in the hyperbolic metric this implies its is also uniformly separated in the "$|dz|/d(z)$" metric, and hence each Whitney box contains only a bounded number of orbit points. Except near parabolic horoballs it is also true that Whitney cubes are near at least one orbit point. To prove this we need to show $\Lambda(G)$ is uniformly perfect.

A compact set K is called *uniformly perfect* if there exists $\epsilon > 0$ such that for any $x \in K$ and $r < \mathrm{diam}(K)$ there exists $y \in K$ such that

$$\epsilon r \leq |x - y| \leq r.$$

There are several well known equivalent formulations of this condition (e.g., [30], [33]). Suppose K is compact and Ω is its complement. Then the following are known to be equivalent:

(1) K is uniformly perfect.

(2) There is a positive lower bound for the length of the shortest closed hyperbolic geodesic in Ω.

(3) There is a constant $C < \infty$ so that
$$\frac{1}{C}\frac{|dz|}{d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}.$$

Lemma 10.1. *If G is a finitely generated, non-elementary Kleinian group then Λ is uniformly perfect. In particular, if ρ is the hyperbolic metric on a component Ω of $\Omega(G)$, then*
$$|d\rho(z)| \sim \frac{|dz|}{d(z)}.$$

Proof. From the discussion above it is enough to verify condition (2) for Ω: there exists ϵ so that every closed geodesic for ρ in Ω has hyperbolic length $\geq \epsilon$. By the Ahlfors finiteness theorem, $\Omega(G)/G = R_1 \cup, \dots, \cup R_N$ is a finite union of compact Riemann surfaces with a finite number of punctures, so there is a lower bound ϵ_0 for the length of the shortest closed geodesic (although if there are punctures there is no lower bound for homotopically non-trivial loops). Suppose there was a homotopically non-trivial closed loop γ in $\Omega(G)$ with length less than ϵ_0. Then Γ projects to a loop in a cusp region on $\Omega(G)/G$ which can be homotoped to a puncture. Thus γ can be homotoped in $\Omega(G)$ to curve with arbitrarily short hyperbolic length, which implies $\partial\Omega(G)$ must have an isolated boundary point. This is impossible if G is non-elementary. Thus $\Omega(G)$ has the desired property. $\square$

It is worth noting that the constants in the previous result depend on G and cannot be taken to depend, say, only on the number of generators. It has also been pointed out to us that this lemma (with almost the same proof) was earlier proven by Canary in [23].

Next we define what we mean by "good" and "bad" horoballs. Since G is finitely generated, the Ahlfors finiteness theorem [3] says that $\Omega(G)/G$ is a finite union of finite Riemann surfaces $R_1, \dots, R_N$, i.e., each is a compact surface with at most a finite number of punctures. Let $\{p_1, \dots, p_m\}$ be the punctures in $R = \cup_{i=1}^N r_i$, and for each p_i let B_i^* be a neighborhood of p_i which lifts to a Euclidean ball B_i in Ω which is invariant under some parabolic element of G (see Lemma 1 of [3]). Then $X = \cup_i R_j \setminus \cup_j B_j^*$ is compact, so as above, we can choose a finite set of points $E = \{z_1, \dots, z_P\} \subset \Omega(G)$ which project to an ϵ dense subset of X. It will also be convenient to assume (as we may) that each of the chosen horoballs B_j, is contained in a horoball in Ω of twice the diameter (so ∂B_j does not come too close to $\partial\Omega$ except near the parabolic fixed point.

Since there are only a finite number of surfaces in $\Omega(G)/G$, it is easy to see that there is an $\epsilon_0 > 0$ so that if γ is homotopically non-trivial loop on one of the surfaces R_i of length less than ϵ_0 then γ must be contained in one of the neighborhoods B_i^*. Thus any closed curve in $\Omega(G)$ of hyperbolic length $\leq \epsilon_0$ is either homotopically trivial or lies in an image of one of the balls B_i.

Suppose $B = g(B_i)$ for some $g \in G$. Suppose B is fixed by a parabolic element $h \in G$ with fixed point $p \in \partial B$. We say that B is doubly cusped it there is a another (disjoint) ball B_1 fixed by h.

Normalize, so that $\Lambda(G)$ has diameter 1. Given a $\eta > 0$ we say B is a "η-bad" horoball if

$$\sup_{z \in \partial B} d(z) \leq \eta \mathrm{diam}(B).$$

Otherwise we say B is "η-good". We will need the following simple facts about good and bad horoballs.

Lemma 10.2. *Suppose diam$(\Lambda) = 1$.*

(1) *There is a C_1 (depending only on η) so that for any η-good horoball B, and any $w \in \partial B$, there is a point $z \in G(E)$ such that*

$$C_1^{-1} d(w) \leq d(z) \leq C_1 d(w),$$

$$C_1^{-1} d(w) \leq |z - w| \leq C_1 d(w).$$

(2) *There is a η_2 (depending on G) so that if B is doubly cusped then B is η_2-good.*

(3) *If η is small enough (depending only on G), and if B is an η-bad horoball and $D \subset \Omega \cap \eta^{-1/2} B$ is a disk, then diam$(D) \leq \eta^{1/6}$diam(B). (the powers are not sharp).*

(4) *For any $\delta > 0$ there is a $\eta_3 > 0$ (depending only on δ) such that if B is a η_3-bad horoball then there is a disk $D \subset 3B$ such that diam$(D) \geq \frac{1}{3}$diam(B), and $D \setminus \Lambda$ contains no balls of radius $\geq \delta$diam(D).*

(5) *There is* $\eta_4 > 0$ *so that if* B_1, B_2 *are* η_4*-bad horoballs with* $diam(B_1) \leq diam(B_2)$ *then*

$$dist(B_1, B_2) \geq 100\, diam(B_1).$$

(6) *If* B_1, B_2 *are horoballs with*

$$diam(B_1) \leq diam(B_2) \leq 2 diam(B_1),$$

and $dist(B_1, B_2) \leq A diam(B_1)$*, then both* B_1 *and* B_2 *are* A^{-3}*-good (the power is not sharp).*

Proof. The first claim is easy, since if $w \in \partial B$ then it is within hyperbolic distance ϵ of a point in $G(E)$, and these points lie in the same or adjacent Whitney squares.

To prove (2) suppose $B = g(B_i)$. Then B is doubly cusped iff B_i is. Suppose they are and let $\hat{B}_i$ be the other ball corresponding to B_i and let $\hat{B} = g(\hat{B}_i)$. Choose η so that B_i is η-good. If B is η_2-bad with $\eta_2 << \eta$, its easy to see that g maps $\hat{B}_i$ to the exterior of a ball of diameter $\leq 2\mathrm{diam}(B)$. But this means that $\mathrm{diam}(\Lambda) \leq 2\mathrm{diam}(B) < 1$, a contradiction. Thus B must be η_2-good, as desired.

To prove (3), conjugate B by a linear map so it becomes the ball of radius $1/2$ centered at $i/2$ and it is fixed under an element of the form

$$h(z) = \frac{z}{1 + az}.$$

If B is η-bad then we must have $a \leq C\eta$. Therefore, a point $z \in \eta^{-1/2}B$, is displaced at most

$$|z - h(z)| = |\frac{az^2}{1 + az}| \leq C\eta |z|^2 \leq C\eta^{1/3} < \eta^{1/6}\epsilon_0/100,$$

if η is small enough. Suppose $z \in 3B \setminus 2B$ is the center of a disk $D \subset \Omega$ of radius $\eta^{1/6}$. Then D intersects $h(D)$ and connecting the centers gives a circular arc which projects to a loop on $\Omega(G)/G$ of length less than ϵ_0. Thus iterating D under powers of h gives a circle fixed by h and lying in Ω. Moreover, this circle projects to a curve in $\Omega(G)/G$ of length less than ϵ_0. Therefore it lies in a horoball. Hence side of this circle must lie in Ω. It cannot be the side containing B because this component contains points of Λ (the ones that are within distance η of $i \in \partial B$). Therefore the other side must be contained in Ω. This implies B is doubly cusped, hence η_2-good. This is a contradiction, so there are no such small disks.

Parts (4), (5), and (6) are all special cases of (3), so we are done. $\square$

11. The convex hull of the limit set

A discrete group of Möbius transformations is called geometrically finite if there is a finite sided fundamental polyhedron for its action on $\mathbb{B}$. For our purposes, however, the following characterization due to Beardon and Maskit [7] of geometrically finite groups is much more useful. Recall that a rank 1 parabolic fixed point p is called doubly cusped if there are two disjoint balls in $\Omega(G)$, tangent at p, and both invariant under the parabolic subgroup fixing p.

Proposition 11.1. *If G is a Kleinian group then G is geometrically finite iff $\Lambda(G)$ is the union of $\Lambda_c(G)$, the rank 2 parabolic fixed points and the doubly cusped rank 1 parabolic fixed points of G.*

Geometrical finiteness can also be characterized in terms of the convex hull of the limit set. If K is a compact set on $S^2 = \partial\mathbb{B}$ we will let $C(K) \subset \mathbb{B}$ denote its convex hull with respect to the hyperbolic metric on $\mathbb{B}$. If (as usual in this paper) G is a Kleinian group without torsion, we let $M = \mathbb{B}/G$ be the hyperbolic 3-manifold associated to G. Then $C(M) = C(\Lambda(G))/G \subset M$ is called the convex core of M.

Much of the interesting topology of M is associated to the topology of the convex core.

For $r > 0$ we define the radius r neighborhood of $C(M)$ as

$$C_r(M) = \{z \in \mathbb{B} : \mathrm{dist}(z, C(M)) < r\},$$

where distance is measure in the hyperbolic metric. Although we don't need it here, we should point out that Thurston has shown that G is geometrically finite iff $C_r(G)$ has finite volume for some (all) $r > 0$. We cannot take $r = 0$ because if G is any Fuchsian group, then $\Lambda(G)$ is contained in a circle, so the convex hull of $\Lambda(G)$ is contained in a hyperplane and hence has zero volume. Thus $C(M)$ has finite volume even for infinitely generated Fuchsian groups. However, for finitely generated groups this is not a problem.

Proposition 11.2. *If G is a finitely generated Kleinian and $C(M)$ has finite volume then G is geometrically finite.*

This is well known, but we will deduce it as a corollary of Lemma 11.5 later of this section. The equivalence of the many equivalent formulations of geometric finiteness is discussed in [18].

Our first goal is to see that $C(M)$ can be separated from $\Omega(G)$ by finite area surfaces. More precisely,

Lemma 11.3. *Suppose G is finitely generated Kleinian group (without torsion) and M is the corresponding hyperbolic 3-manifold. Let $\{\Omega_j\}_1^N$ be conjugacy classes of components of $\Omega(G)$ (i.e., the geometrically finite ends of M). For each Ω_j There is a surface $\{S_j\}$ in M so that the following holds.*

(1) *dist*$(S_j, C(M)) > 2$.

(2) *Each S_j has finite area.*

(3) *If $U_j = \{z \in M : \rho(z, S_j) < 1\}$ is a unit hyperbolic neighborhood of S_j then U_j has finite hyperbolic volume.*

(4) *The function $f(y) = (vol(B(y,1))^{-1/2}$ is integrable over U_j, i.e.,*

$$\int_{U_j} (vol(B(y,1))^{-1/2} dy < \infty.$$

(5) *The surfaces S_j separate $C(M)$ from the geometrically finite ends of M, i.e., there is a $\epsilon_0 > 0$ so that $w(z) \leq 1 - \epsilon_0$ on the component M_1 of $M \setminus \cup_j S_j$ containing $C(M)$.*

Proof: This consists of known facts, as described, for example, in [29]. In fact, we do not even need the sophisticated machinery developed there. For each component Ω_j we can simply take S_j to be the intersection of a fundamental polygon in $\mathbb{B}$ for G with the surface

$$\{z \in \mathbb{B} : 1 - |z| = \epsilon_0 \text{dist}(z/|z|, \Lambda(G)\}.$$

By Lemma 10.1 (which says $\Lambda(G)$ is uniformly perfect) hyperbolic area on this surface is boundedly equivalent to hyperbolic area on $\Omega(G)$ under the radial projection if ϵ_0 is small enough (depending only on the constants in Lemma 10.1; these depend on G, but may be taken absolute if $\Lambda(G)$ is connected). Thus (2) holds. Condition (3) follows because U_j is boundedly equivalent to $S_j \times [0,1]$. To prove (4) we note that the integral obvious converges on any compact piece of U_j and use the fact that the injectivity radius decreases exponentially to bound the integral in the cusps (also note that for $y \in U$, $\text{vol}(B(y,1)) \sim \text{inj}(y)$). Conditions (1) and (5) are easy to check if ϵ_0 is small enough. $\square$

We will also need the Margulis lemma (e.g., [8]). This says that there is an $\epsilon > 0$ so that if

$$M_{\text{thin}(\epsilon)} = \{x \in M : \text{inj}(x) \leq \epsilon\},$$

then every component of $M_{\text{thin}(\epsilon)}$ is one of three kinds:

(1) a torus cusp, i.e., a horoball in $\mathbb{B}$ modulo a rank 2 parabolic subgroup.

(2) a rank one cusp, i.e., a horoball in $\mathbb{B}$ modulo a rank 1 parabolic subgroup.

(3) a solid torus neighborhood of a closed geodesic, i.e., a r-neighborhood of a geodesic in $\mathbb{B}$ modulo powers of a loxodromic element fixing the geodesic.

Moreover, the components of $M_{\text{thin}(\epsilon)}$ (i.e., the horoballs in (1) and (2) and the neighborhoods in (3)) may be taken to be pairwise disjoint. Note that this implies that if $g \in G$ and B is a "thin component" (one of the thre types of regions described above), then $g(B)$ hits B iff $g(B) = B$ and g is in the subgroup associated to B above. Actually we will not need the full strength of the Margulis lemma. All we will use is that there is an $\epsilon > 0$ as described above, but this number may be allowed to depend on G.

We will need the following lemma in the proof of Theorem 1.5.

Lemma 11.4. *If G is a Kleinian group (not necessarily finitely generated) and $\Lambda(G)$ has positive area then $C(M)$ has infinite volume.*

Note that we need the hypothesis that G is Kleinian, since it is possible for G to be a discrete group with $\Lambda(G) = S^2$, but $C(M) = M$ to have finite volume (e.g., if G is co-compact).

Proof. We will prove this by showing there is a sequence $\{x_n\} \in C(M)$ with

$$\text{dist}(x_n, \partial C(M)) \to \infty,$$

and $\mathrm{inj}(x_n) > \epsilon$ for all n.

For $z \in \mathbb{B}$ define

$$w(z) = \max_{D \subset \Omega(G)} \omega(z, D, \mathbb{B}),$$

where the max is over all round disks in $\Omega(G)$. Then $C(\Lambda) = \{z : w(z) \leq 1/2\}$. Since w is a max of harmonic functions it is sub-harmonic but we will not need this. Instead of w we could simply deal with the distance function $dist(z, \partial C(M))$, which is essentially $-\log w$. It is also G invariant, so defines a function on M. It is easy to see that for any $R > 0$ there is an ϵ so that $w(x) < \epsilon$ implies $\mathrm{dist}(x, \partial C(M)) > R$.

Since $\Lambda(G)$ has positive area the Lebesgue density theorem gives us a point of density $z_0 \in \Lambda(G)$. Let γ be the hyperbolic geodesic connecting the origin to z_0 (i.e., a radius of $\mathbb{B}$) and consider points $x \in \gamma$ converging to z_0. Clearly $w(x) \to 0$, so $\mathrm{dist}(x, \partial C(M)) \to \infty$ as $x \to z_0$. Therefore we only have to show that the injectivity radius of x is $\geq \epsilon$ along some subsequence converging to z_0.

But if this is false then eventually γ must be in one of the three types of thin regions for all points close enough to the boundary. This implies z_0 is fixed by some element of G. Since such points form a countable subset of the limit set we may certainly assume z_0 is not one of them and we are done. $\square$

We will use the following result in the proof of Theorem 1.4 (but it is not essential; we will also sketch a proof which does not require it). We will use part of the proof in the proof of Theorem 1.6 and Theorem 1.7.

Lemma 11.5. *Suppose G is finitely generated and geometrically infinite. Then there is a sequence $\{x_n\} \in C(M)$ with $dist(x_n, \partial C(M)) \to \infty$ and $inj(x_n) > \epsilon$ for all n.*

Proof. The proof is similar to the last lemma but with a few extra technicalities because of the possibility of "bad" horoballs. We will carry out the proof in two cases

(1) Every rank 1 parabolic is doubly cusped.

(2) There is a rank 1 parabolic which in not doubly cusped.

Note that case (1) contains the case when G has no parabolic elements.

Proof for case (1) of Lemma 11.5*:* In this case, there is an $\eta > 0$ so that every horoball (if any exist) is η-good by Lemma 10.2. Let $z_0 \in \Lambda(G)$ be a point which is neither a parabolic fixed point nor in $\Lambda_c(G)$ (such a point exists by Proposition 11.1). Let γ be the hyperbolic geodesic from the origin in $\mathbb{B}$ to z_0. We will first show

$$\lim_{x \to z_0, x \in \gamma} w(x) = 0.$$

Suppose this is false, i.e., suppose there is an ϵ_0 such that $w(x) > \epsilon_0$ for all $x \in \gamma$. We will prove that z_0 is either in $\Lambda_c(G)$ or is a parabolic fixed point.

Let $\{\Omega_j\}_1^N$ be a maximal collection of non-conjugate components of $\Omega(G)$. In each we can find a finite sided fundamental polygon P_j (in Ω_j's hyperbolic geometry). Cover P_j by a finite number of Euclidean disks, including a horoball for each parabolic cusp. Let $\mathcal{D} = \{D_k\}$ denote the (finite) collection of disks obtained in this way. Note that there is a constant C_0 (depending only on our choice of $\mathcal{D}$) such that for any disk D in $\Omega(G)$ there is a D_k and a $g \in G$ so that $D \cap D_k \neq \emptyset$ and $\operatorname{diam}(g(D_k)) \geq C \operatorname{diam}(D)$. Thus there is a $C_1 > 0$ so that for any $x \in \mathbb{B}$ there is a D_k and $g \in G$ so that

$$\omega(x, g(D_k), \mathbb{B}) \geq C_1 w(x). \tag{11.1}$$

Therefore we may suppose we have $z_0 \in \Lambda(G)$ and there is an ϵ_0 such that for every

$x \in \gamma$ there exists $D_k \in \mathcal{D}$ and $g \in G$ such that

$$\omega(x, g(D_k), \mathbb{B}) \geq \epsilon_0.$$

Let $D(x) = g(D_k)$ be a choice of disk for each $x \in \gamma$.

If z_0 is not a parabolic fixed point then $D(x)$ must change infinitely often as $x \to z_0$. Since $D(x)$ is either compactly contained in Ω or a η-good horoball, the proof of Lemma 6.1 and Lemma 10.2 shows that there is a constant $C_3 > 0$ (depending only on our choice of $\mathcal{D}$ and ϵ_0) so that if

$$\omega(x, g(D_k), \mathbb{B}) \geq \epsilon_0,$$

for some $D_k \in \mathcal{C}$, then the orbit point $g(0)$ is contained in the cone with angle $\pi - C_3$ with vertex z_0. Thus in this case, there is a cone with vertex z_0 which contains infinitely many images of 0, i.e., $z_0 \in \Lambda_c(G)$.

Thus we have shown that $w(x) \to 0$ as $x \to z_0$, or equivalently $\text{dist}(x, \partial C(M)) \to \infty$. The proof now finishes just as in the previous lemma (if z_0 is not a fixed point of some element $g \in G$ then there is a sequence $\{x_n\} \to z_0$ in γ which lies outside the thin regions). This finishes the proof for case (1).

Proof for case (2) of Lemma 11.5*:* Now we consider the second case when there is a rank 1 cusp in $\Omega(G)$ which is not doubly cusped. Normalize so that this parabolic is $z \to z + 1$. Since ∞ is not doubly cusped, we may also assume that $\Omega(G) \cap \mathbb{H}$ contains no invariant horoball for this parabolic. The corresponding thin part in $\mathbb{B}$ will be

$$B_0 = \{(x, y, z) : z > C\},$$

for some C. If ∞ is singly cusped, normalize so the lower half plane is the largest invariant half plane in Ω. Consider the points $y^* = (0, y, C)$. Since these are on the boundary of the thin part the injectivity radii are all ϵ. Therefore we need only show $w(y^*) \to 0$ as $y \to \infty$.

Suppose not, say $w(y^*) \geq \nu > 0$ and let $D(y)$ be the largest disk seen from y^*. Suppose $D(y)$ is not the lower half plane. Then $\mathrm{diam}(D_y) \leq 1$ if y is large enough, because otherwise $D(y)$ would not be disjoint from its translate under the parabolic element $z \to z+1$. This would give a line in Ω and everything below this line would be in Ω, contradicting that the lower half plane is the largest half plane in Ω.

If $D(y)$ is constant for all large enough y then clearly the harmonic measure of $D(y)$ fromt eh point y^* tends to zero, which is what we want. Otherwise, $D(y)$ must change infinitely often. Assume this is the case.

If there is a sequence $y_n \to \infty$ such that $D(y_n)$ is η_n-bad with $\eta_n \to 0$, then we must have $\mathrm{diam}(D(y_n)) \to 0$. This holds because the estimates in the proof of Lemma 10.2 show that for bad horoballs the associated thin part in $\mathbb{B}$ has (Euclidean) diameter which is much greater than the diameter of the horoball in $\Omega(G)$. Since the thin regions in $\mathbb{B}$ are disjoint the thin region corresponding to $D(y_n)$ must have diameter $\leq C$ (by the normalization above). Thus $\mathrm{diam}(D(y_n)) \to 0$. Therefore we are done in this case.

Next suppose that there are no η-bad horoballs for some $\eta > 0$. Then we get a subsequence of disks $D(y_n)$ which are all images of the same disk in $\mathcal{D}$ and so that the diameters converge to some ν_0, $0 < \nu \geq \nu_0 \geq 1$. For two disks in this sequence consider the group element g_{nm} which maps $D(y_n)$ to $D(y_m)$. Also consider a linear conformal map $h_{nm}$from from $D(y_m)$ to $D(y_n)$. By compactness we can choose n, m and h_{nm} so that $g_{nm} \circ h_{nm}$ is arbitrarily close to the identity in $\mathrm{PSL}(2, \mathbb{C})$. Since h_{nm} preserves B_0, this contradicts the fact (the Margulis lemma) that $g_{nm}(B_0)$ can't intersect B_0. Thus the diameters of $D(y^*)$ must tend to zero. This completes the proof of Lemma 11.5. $\square$

Richard Canary has pointed out to us that Lemma 11.5 also follows easily from known results. His argument goes as follows: If $C(N)$ is non-compact, then (see [17]), there exists a sequence $\{\gamma_i\}$ of closed geodesics leaving every compact set. Pick ϵ less than the Margulis constant. Either, $length(\gamma_i) < \epsilon$ or some point on γ_i lies in the ϵ-thick part. In the first case, let C_i denote the component of the thin part containing γ_i, then ∂C_i (the boundary of C_i) intersects $C(N)$ (otherwise $C(N) \subset \partial C_i$ which implies that N is elementary), so pick x_i in the the intersection of ∂C_i and $C(N)$. In the second case pick $x_i \in \gamma_i \cap N_{thick}$. Then, $\{x_i\}$ leaves every compact set and every point in the sequence has injectivity radius at least ϵ.

Note that we obtain Proposition 11.2 as a corollary of Lemma 11.5 since the lemma gives us infinitely many disjoint balls which all have volume bounded uniformly from below. The proof also shows the following known fact (e.g., [43], it also follows from Proposition 11.1):

Corollary 11.6. *If G is a finitely generated, geometrically finite group, then every parabolic cusp of $\Omega(G)$ is doubly cusped (hence η-good for some η depending on G).*

We will use this in the proof of Theorem 1.6 and Theorem 1.7.

12. Geometrically infinite groups

In this section we will prove,

Theorem 12.1. *If G is a finitely generated and geometrically infinite then*

$$dim(\Lambda(G)) = 2.$$

If $\delta(G) = 2$ then this follows from Theorem 1.1. Therefore we may assume $\delta(G) < 2$. In this case Theorem 12.1 follows from

Theorem 12.2. *If G is a finitely generated, geometrically infinite group and $\delta(G) < 2$ then $\Lambda(G)$ has positive area.*

Note that the Ahlfors conjecture and Theorem 12.2 imply that $\delta(G) = 2$ for any finitely generated, geometrically infinite group G.

Proof of Theorem 12.2*:* Let $\Lambda = \Lambda(G)$ be the limit set of G. As usual, we assume G has no torsion. Let $\delta = \delta(G)$ be the critical index for the Poincaré series and λ_0 the base eigenvalue for the Laplacian on $M = \mathbb{B}/G$. By Corollary 9.5 we know that if G is geometrically infinite then $\delta(G) > 1$. Therefore by a result of Sullivan (Theorem 2.18 [57])

$$\lambda_0 = \delta(G)(2 - \delta(G)).$$

Let $C(\Lambda)$ be the convex hull of the limit set in $\mathbb{B}$ and $C(M) = C(\Lambda)/G \subset M$ be the convex core of M (see previous section). Let $S = \cup_j S_j$ and $U = \cup_j U_j$ be the surfaces and their unit neighborhoods given by Lemma 11.3. Let M_1 be the component of $M \setminus \cup_j S_j$. We will prove that Brownian motion is transient in M_1. More precisely, we will show that

Lemma 12.3. *Suppose M, S, M_1 are as above. Then there is a point $x_0 \in M_1$ such that a Brownian motion started at x_0 has a positive probability of never hitting S.*

First let us see why this implies Λ has positive area. Let $\tilde{x}$ be a point in the hyperbolic ball $\mathbb{B}$ which corresponds to the point x_0 in the previous paragraph and let $\tilde{M}_1$ be the lift of M_1 to the ball. Then a Brownian motion started at $\tilde{x}_0$ has a positive probability of hitting $S^2 = \partial\mathbb{B}$ without leaving $\tilde{M}_1$. Since $w < 1 - \epsilon_0$, such a path must first hit S^2 at a point of Λ. Since harmonic measure for $\mathbb{B}$ is mutually absolutely continuous with respect to area measure on S^2 this means that Λ has positive area. Therefore we need only prove Lemma 12.3.

Proof of Lemma 12.3*:* The lemma follows from known estimates on the heat kernel $K(x, y, t)$ for a Riemannian manifold.

Proposition 12.4 (Davies, Theorem 17 [27]**).** *If $\delta > 0$ then there is a constant $C = C(\delta)$ such that*

$$0 \leq K(x, y, t) \leq C \, vol(B(x, 1))^{-1/2} vol(B(y, 1))^{-1/2} e^{(\delta - \lambda_0)t} e^{-\rho(x,y)^2(4+\delta)^{-1}t^{-1}},$$

where $\rho(x, y)$ denotes the hyperbolic distance between x and y in M.

See also [28] and [35] and their references. To apply Davies' result we let $\delta = \lambda_0/2$. Choose $R > 0$ very large. By Lemma 11.5 we can choose a point $x \in C(M)$ where the injectivity radius $\mathrm{inj}(x) > \epsilon > 0$ is uniformly bounded below and

$$\mathrm{dist}(x, U) \geq \mathrm{dist}(x, \partial C(M)) \geq R.$$

The expected time a Brownian motion started at x spends in U is

$$\int_0^\infty [\int_U K(x, y, t) dy] dt.$$

By Davies' estimate this is bounded by

$$\mathrm{vol}(B(x,1))^{-1/2} \int_0^\infty e^{-\lambda_0 t/2} e^{-R^2/5t} dt \cdot \int_U \mathrm{vol}(B(y,1))^{-1/2} dy.$$

Using the fact that $\mathrm{inj}(x)$ is bounded below and part (4) of Lemma 11.3 this is bounded by

$$C \int_0^\infty e^{-\lambda_0 t/2} e^{-R^2/5t} dt.$$

This can be estimated by breaking the integral at $t = R$ to get

$$C \int_0^R e^{-\lambda_0 t/2} e^{-R/5} dt + C \int_R^\infty e^{-\lambda_0 t/2} dt \leq \frac{C}{\lambda_0}(e^{-\frac{1}{5}R} + e^{-\frac{\lambda_0}{2}R}).$$

Thus by choosing R sufficiently large we can make the the expected time a Brownian motion started at x spends in U is a small as we wish.

From this we wish to deduce that a Brownian motion started at x_0 never hits S with some positive probability. To do this we simply note that the expected time it takes a Brownian motion started at a point y of S to first leave U (i.e., to travel unit distance from S) is bounded away from zero independent of the starting point y. This is because the expected time to travel distance 1 in M is greater than or equal the expected time to travel this distance in the covering space $\mathbb{B}$, and this is bounded away from zero. Let $t_0 > 0$ be a lower bound for the expected time to travel distance 1.

Thus the expected time a Brownian motion started at x spends in U is at least the probability that it every hits S times the bound t_0. Now choose x_1 so that the expected time spent in U is less than $t_0/2$. Then the probability that a Brownian motion starting at x_1 ever hits S is $< 1/2$. This completes the proof of Theorem 12.1. $\square$

We do not actually need the full strength of Lemma 11.5. All we need is that there is a sequence $\{x_n\}$ with $\mathrm{inj}(x_n) \geq \epsilon$ and x_n leaving every compact set (rather than

$\text{dist}(x_n, \partial C(M)) \to \infty$). This is because we can cut U up into a compact piece and a finite number of small tips of cusps. If the volume of the small cusp pieces are small enough, then any uniform bound on the heat kernel shows that the expected time in these pieces is small. The time in the remaining compact part is small as long as the distance of x_n form the compact part is large. This is true as long as x_n tends to infinity in M.

Sketch of second proof of Lemma 12.3*:* It actually suffices to use a weaker estimate on the heat kernel and less information on the convex hull. All we need to know is that Brownian motion is transient on a complete, connected, infinite volume Riemannian manifold with lowest eigenvalue bounded away from zero. For example, the estimate

$$K(x, y, t) \leq C_1 e^{-C_2 t},$$

for some fixed x, y in some ball and $t > T_0$ would be sufficient for this.

We pay for the less precise estimate by a more involved construction on the manifold. Let S be the surfaces described by Lemma 11.3 which separate the convex core from the geometrically finite ends. Cut M along S and let M_1 be the component containing $C(M)$. Glue two copies of M_1 along S. We claim that the resulting manifold N (the double of M_1) has lowest eigenvalue bounded away from zero. If so then the heat kernel estimates apply to the new manifold N and we deduce that the expected time a Brownian motion spends in U (the unit radius neighborhood of the S) is finite. By the Borel-Cantelli lemma this says that there are points x in M from which the probability of ever hitting S is strictly less than 1 (in fact is as small as we wish). Thus there is a point $x \in N \setminus U$ from which the chance of ever hitting U is less than 1/2. But the two components of $N \setminus U$ are both exactly M_1. Thus Brownian motion in M_1 must have a positive probability of tending to infinity without ever hitting S.

This proves the lemma, except for verifying that N has first eigenvalue bounded away from 0. We will not verify this in detail, but simply note that since M has constant negative curvature and lowest eigenvalue > 0, Buser's inequality (e.g. [24]) implies the Cheeger constant for M is bounded away from zero. From this one proves that the Cheeger constant for the manifold with boundary M_1 is non-zero, and from this that the Cheeger constant for the doubled manifold N is non-zero. Then Cheeger's estimate say the first eigenvalue for N is non-zero, as desired. $\square$

13. Lower semi-continuity of Hausdorff dimension

In this section we will prove,

Theorem 13.1. *If G is a finitely generated Möbius group and $\{G_n\}$ is a sequence of Möbius groups converging algebraically to G then*

$$\liminf_n dim(\Lambda(G_n)) \geq dim(\Lambda(G)).$$

In particular, if G is geometrically infinite then this result and Theorem 1.4 imply that $\lim_n \dim(\Lambda(G_n)) = 2$.

If G is not discrete then Theorem 13.1 follows from Theorem 1.1 and the remarks following Corollary 2.5. If all but finitely many of the $\{G_n\}$ are non-discrete then by passing to a subsequence we may assume $\Lambda(G_n)$ is always all of S^2, are all circles or are always two or less points. We may also assume that the limit sets $\Lambda_n = \Lambda(G_n)$ converge in the Hausdorff metric to a set Λ_∞. Moreover, because of the special form of the sets we must have

$$\dim(\Lambda_\infty) \leq \liminf_n \dim(\Lambda_n).$$

It is easy to check that the limit set of G is a subset of the limit of the sets Λ_n.

Therefore we may assume all the groups are discrete. If all but finitely many of the $\{G_n\}$ satisfy $\Lambda(G_n) = S^2$, there is nothing to do, so we may as well assume (by passing to a subsequence) that the $\{G_n\}$ are all Kleinian groups.

If $\Lambda(G)$ has zero area or $\delta(G) = 2$ then $\delta(G) = \dim(\Lambda(G))$ and this follows from Corollary 2.5. Therefore we need only consider the case when $\delta(G) < 2$ and $\Lambda(G)$ has positive area (possibly the whole sphere) If G where geometrically finite then either $\Lambda(G) = 0$ ([4]) or $\delta(G) = 2$. In either case we are finished, so we may assume that G is geometrically finite.

By passing to a subsequence we may assume

$$\lim_n \dim(\Lambda(G_n)),$$

exists and equals the liminf of the original sequence. If all but finitely many of the groups $\{G_n\}$ are geometrically infinite then

$$\liminf_n \dim(\Lambda(G_n)) = 2,$$

by Theorem 12.1 so there is nothing to do. Otherwise, by passing to another subsequence, we may assume all the G_n's are geometrically finite. We may also assume $\lim_n \delta(G_n) = \alpha$ exists and is strictly less than 2 (otherwise the result follows from the inequality $\dim(\Lambda(G_n)) \geq \delta(G_n)$). To finish the proof we need

Theorem 13.2. *If $\{G_n\}$ is a sequence of geometrically finite Kleinian groups which converges algebraically to a finitely generated, geometrically infinite discrete group G then $\delta(G_n) \to 2$.*

This result follows from two known results

Proposition 13.3 (Canary). *If G is a n-generated, geometrically finite group then*

$$\lambda_0 \leq \frac{A_n}{vol(C(M))},$$

where A_n is a constant that only depends on the number of generators of G.

Proof: This is essentially Theorem A of [24] except that there Canary proves

$$\lambda_0 \leq A \frac{\chi(\partial C(M))}{\mathrm{vol}(C(M))},$$

where A is an absolute constant and where χ denotes the Euler characteristic. However, the Euler characteristic of $\partial C(M)$ is the same as that of $\Omega(G)/G$, because there is always a homeomorphism between the two (e.g., see Epstein and Marden's paper [29]). By the Bers inequality (a quantitative version of the Ahlfors finiteness theorem) the area, and hence the Euler characteristic, of $\Omega(G)/G$ can be bounded in terms of n, the number of generators of the group G. Thus Canary's results says

$$\lambda_0 \leq \frac{A_n}{\mathrm{vol}(C(M))},$$

where A_n depends only on the number of generators. $\square$

Proposition 13.4. *Suppose $\{G_n\}$ is a sequence of n-generated, geometrically finite Kleinian groups such that*

$$\sup_n vol(C(M)) \leq M < \infty.$$

If the sequence $\{G_n\}$ converges algebraically to a finitely generated, geometrically infinite, discrete group G then $\Lambda(G)$ has zero area.

Proof: This is an easy case of a result obtained by E. Taylor in [58] and is probably well known. Here we will sketch a proof which follows an argument given by Jørgensen and Marden in [38].

Suppose $\Lambda(G)$ has positive area. We will derive a contradiction. Suppose $\{G_n\}$ is a sequence of N-generated Kleinian groups converging algebraically to a Kleinian group G. The Borel selection theorem says that the set of compact subsets of a compact metric space is itself compact with the Hausdorff metric

$$d(E, F) = \max_{z \in E} \operatorname{dist}(z, F) + \max_{w \in F} \operatorname{dist}(w, E),$$

so by passing to a subsequence (which we also denote $\{G_n\}$) we may assume the sets $\Lambda_n = \Lambda(G_n)$ converge in the Hausdorff metric to a compact set Λ_∞.

We say a sequence $\{G_n\}$ converges polyhedrally to a group H if H is discrete and for some $x_0 \in \mathbb{B}$, the fundamental polyhedra (the Dirichlet polyhedron)

$$P(G_n) = \{z \in \mathbb{B} : \rho(z, x_0) \leq \rho(z, g(x_0)) \text{ for all } g \in G_n\},$$

converge to $P(H)$ uniformly on compact subsets of $\mathbb{B}$. By Proposition 3.8 of [38] any algebraically converging subsequence has a polyhedrally convergent subsequence and the polyhedral limit contains the algebraic limit (but they need not be equal).

A third notion of convergence of groups is geometric convergence. Given a sequence of groups $\{G_n\}$ we define

$$\operatorname{Env}\{G_n\} = \{g \in \operatorname{PSL}(2, \mathbb{C}) : g = \lim_n g_n, g_n \in G_n\},$$

and we say $G_n \to H = \operatorname{Env}\{G_n\}$ geometrically if for every subsequence $\{G_{n_j}\}$, $\operatorname{Env}\{G_{n_j}\} = \operatorname{Env}\{G_n\}$. Proposition 3.10 of [38] says that G_n converges geometrically to H if and only if it converges polyhedrally to H.

Thus we may assume that we have groups $G \subset H$ such that

(1) $G_n \to G$ algebraically,

(2) $G_n \to H$ polyhedrally and geometrically,

(3) $\Lambda_n \to \Lambda_\infty$ in the Hausdorff metric.

We first claim that $\Lambda(H) \subset \Lambda_\infty$. If $\Lambda_\infty = S^2$ there is nothing to do, so we may assume Λ_∞ is not the whole sphere. In this case we follow the proof of Proposition 4.2 of [38]. Let $\Omega = S^2 \setminus \Lambda_\infty$ and suppose K, K' are compact sets such that

$$K \subset \mathrm{int}(K') \subset K' \subset \Omega.$$

Suppose $h \in H$. Because $\{G_n\}$ converges geometrically to H we can write $h = \lim_n g_n$ with $g_n \in G_n$.

We claim that $h(K) \subset \Omega$. If not then $h(K)$ intersects Λ_∞, so $\mathrm{int}(h(K')$ also hits Λ_∞. This implies that $\mathrm{int}(h(K')$ intersects Λ_n for all large enough n, say $n \geq N_1$. Therefore $\mathrm{int}(g_n(K'))$ hits Λ_m for all $m \geq N_1$, for all sufficiently large n, say $n \geq N_2$. So if $N_3 = \max(N_1, N_2)$, then $n \geq N_3$ implies

$$\mathrm{int}(g_n(K')) \cap \Lambda_n \neq \emptyset.$$

Therefore

$$\mathrm{int}(K') \cap g_n^{-1}(\Lambda_n) = \mathrm{int}(K') \cap \Lambda_n \neq \emptyset.$$

This is a contradiction, so we must have $h(K) \subset \Omega$. This implies $h(\Omega) \subset \Omega$. Since the same argument applies to h^{-1}, we see that $h(\Omega) = \Omega$, or equivalently, $h(\Lambda_\infty) = \Lambda_\infty$. Since Λ_∞ is a closed set which is invariant under the group H we must have $\Lambda(H) \subset \Lambda_\infty$ as desired (recall that the limit set is the smallest closed H-invariant set if H is non-elementary).

Since $\Lambda(H) \subset \Lambda_\infty$ the convex hull $C(\Lambda(H))$ of $\Lambda(H)$ in $\mathbb{B}$ is contained in the convex hull $C(\Lambda_\infty)$ of Λ_∞. The convex hulls $C(\Lambda_n)$ of the sets Λ_n converge, uniformly on compacta, to $C(\Lambda_\infty)$. Thus for any $R < \infty$,

$$\mathrm{vol}(C(\Lambda(H)) \cap P(H) \cap B(x_0, R)) \le \liminf_n \mathrm{vol}(C(\Lambda_n) \cap P(G_n) \cap B(x_0, R)) \le M.$$

Thus $C(\Lambda(H)) \cap P(H)$ has finite volume. Here we are using the convexity of the sets Λ_n; in general, sets of zero volume can converge in the Hausdorff metric to sets of infinite volume.

The volume of the quotient manifold is easily seen to be continuous under geometric convergence, so we deduce that $\mathbb{B}/H$ has infinite volume. If $\Lambda(H) = S^2$ then $\mathrm{vol}(M) = \mathrm{vol}(C(M)$ is finite, which is a contradiction. Thus $\Lambda(H) \ne S^2$.

If $\Lambda(H)$ is a proper subset of S^2 it still has positive area since it contains the set $\Lambda(G)$. Thus by Lemma 11.4 the convex core of H has infinite volume. This is another contradiction and so completes the proof of Proposition 13.4. $\square$

Now that we have the two propositions, we can can finish the proof of Theorem 13.2. Suppose G is a finitely generated, geometrically infinite discrete group and $\{G_n\}$ are geometrically finite groups converging to G algebraically. If $\Lambda(G)$ has zero area then $\delta(G) = \dim(\Lambda(G)) = 2$ and $\delta(G_n) \to 2$ by Corollary 2.5. Thus we may assume $\Lambda(G)$ has positive area. By Proposition 13.4 we must have

$$\mathrm{vol}(C(G_n)) \to \infty,$$

so by Proposition 13.3 we get

$$\lambda_0(G_n) \to 0.$$

Since G is geometrically infinite we have $\delta(G) > 1$ by Corollary 9.5 and so by Corollary 2.5 we have

$$\liminf_n \delta(G_n) \ge \delta(G) > 1.$$

Therefore we may as well assume $\delta(G_n) \geq 1$ for all n. By Sullivan's result [57] $\lambda_0 = \delta(2-\delta)$ if $\delta \geq 1$ so,

$$\delta(G_n) \to 2,$$

as $n \to \infty$. This completes the proof of Theorem 13.2. $\square$

14. Upper Minkowski dimension

In this section we introduce upper Minkowski dimension and prove a simple lemma which we will need in the next section.

Suppose K is a bounded set in $\mathbb{R}^d$ (or any metric space for that matter) and let $N(K,\epsilon)$ be the minimal number of ϵ balls needed to cover K. We define the *upper Minkowski dimension* as

$$\overline{\text{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon},$$

and the *lower Minkowski dimension*

$$\underline{\text{Mdim}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of K and is denoted $\text{Mdim}(K)$. The Minkowski dimension says the the number of balls of size ϵ needed to cover K grows like $\epsilon^{-\text{Mdim}(K)}$. It is clear that we always have

$$\dim(K) \leq \underline{\text{Mdim}}(K) \leq \overline{\text{Mdim}}(K),$$

and strict inequality is possible in both places. A simple example where the Minkowski dimension exists, but is strictly larger than the Hausdorff dimension, is given by the countable set $K = \{0\} \cup \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Here it is not to difficult to prove

$$N(K,\epsilon) \sim \frac{1}{\sqrt{\epsilon}},$$

so that $\mathrm{Mdim}(K) = 1/2$. However, since the set is countable, $\dim(K) = 0$. A much more sophisticated class of sets where the two dimensions disagree is given in [45].

Given a compact set K in the plane let $\Omega = \mathbb{R}^2 \setminus K$ be its complement. A Whitney decomposition of Ω is a collection of squares $\{Q_j\}$ which are disjoint, except along their boundaries, and such that

$$\frac{1}{10}\mathrm{dist}(Q_j, \partial\Omega) \le \ell(Q_j) \le 10\mathrm{dist}(Q_j, \partial\Omega).$$

The existence of Whitney decomposition for any open set is a standard fact in real analysis (e.g., [51]). The squares $\{Q_j\}$ may also be taken to be dyadic.

For any compact set K we can define an exponent of convergence similar to the exponent of convergence of a Poincaré series,

$$\kappa = \kappa(K) = \inf\{\alpha : \sum_j \ell(Q_j)^\alpha < \infty\},$$

where the sum is taken over all squares in Whitney decomposition of $\Omega = K^c$ which are within distance 1 of K (we have to drop the "far away" squares or the series might not converge). It is easy to check that this does not depend on the particular choice of Whitney decomposition. This number has been rediscovered many times in the literature, but seem to have been first used by Besicovitch and Taylor in [10]. See also Tricot's paper[59] where he shows that it agrees with the upper Minkowski dimension. We include a proof of this fact since it is central to our argument.

Lemma 14.1. *For any compact set K, $\kappa \le \overline{Mdim}(K)$. If K also has zero area then $\kappa = \overline{Mdim}(K)$.*

Proof. We start with the easy direction, $\kappa \le \overline{\mathrm{Mdim}}(K)$. Let $D = \overline{\mathrm{Mdim}}(K)$ and choose $\epsilon > 0$. Then by the definition of $\overline{\mathrm{Mdim}}(K)$ we have that

$$N(K, 2^{-n}) \leq C2^{D+\epsilon}.$$

Let $\mathcal{S}_n$ be a covering of K by fewer than $C2^{D+\epsilon}$ squares of size 2^{-n}. If Q is a Whitney square with $2^{-n-1} \leq \ell(Q) < 2^{-n}$, then choose a point $x \in K$ with $\mathrm{dist}(x, Q) \leq \ell(Q)$. Let $S(Q) \in \mathcal{S}_n$ be the square containing Q. Since $S(Q)$ and Q have comparable sizes and there distance apart is at most $\ell(Q)$, we easily see that each $S \in \mathcal{S}_n$ can only be associated to a uniformly bounded number of Q's in the Whitney decomposition, say A.

Let W_n be the number of Whitney squares with size $2^{-n-1} \leq \ell(Q) < 2^{-n}$. Then

$$\begin{aligned}
\sum_j \ell(Q_j)^{D+2\epsilon} &\leq \sum_{n=0}^{\infty} W_n 2^{-n(D+2\epsilon)} \\
&\leq \sum_{n=0}^{\infty} CN(K, 2^{-n}) 2^{-n(D+2\epsilon)} \\
&\leq C \sum_{n=0}^{\infty} 2^{n(D+\epsilon)} 2^{-n(D+2\epsilon)} \\
&\leq C \sum_{n=0}^{\infty} 2^{-n\epsilon} \\
&< \infty,
\end{aligned}$$

which proves $\kappa \leq D + 2\epsilon$. Taking $\epsilon \to 0$ gives $\kappa \leq D = \overline{\mathrm{Mdim}}(K)$.

Now we assume K has zero area and will prove $\kappa \geq D = \overline{\mathrm{Mdim}}(K)$. As above, let $\epsilon > 0$ and suppose $\{Q_j\}$ is a Whitney decomposition of $\Omega = K^c$. By the definition of $\overline{\mathrm{Mdim}}(K)$ we have

$$N(K, 2^{-n}) \geq 2^{n(D-\epsilon)},$$

for infinitely many n. Suppose n_0 is a value where this occurs and let $\mathcal{S} = \{S_k\}$ be a covering of K with dyadic squares of size 2^{-n_0}. Let for each $S_k \in \mathcal{S}$, $\mathcal{C}_k = \{Q_{jk}\}$

be the collection of Whitney squares which intersect S_k. If we assume the Q_j are dyadic, then every square hitting S_k is contained in S_k. Since the area of K is zero, this gives

$$2^{-2n_0} = \text{area}(S_k) = \text{area}(S_k \setminus K) = \text{area}(S_k \cap \Omega) = \sum_{\mathcal{C}_k} \ell(Q_{jk})^2.$$

Therefore,

$$\begin{aligned}
\sum_{\mathcal{C}_k} \ell(Q_{jk})^{D-2\epsilon} &= \sum_{\mathcal{C}_k} \ell(Q_{jk})^2 \ell(Q_{jk})^{-2+D-2\epsilon} \\
&\geq \sum_{\mathcal{C}_k} \ell(Q_{jk})^2 \ell(S_k)^{-2+D-2\epsilon} \\
&= \ell(S_k)^{-2+D-2\epsilon} \sum_{\mathcal{C}_k} \ell(Q_{jk})^2 \\
&= \ell(S_k)^{-2+D-2\epsilon} \ell(S_k)^2 \\
&= \ell(S_k)^{D-2\epsilon} \\
&= 2^{-n_0(D-2\epsilon)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_j \ell(Q_j)^{D-2\epsilon} &\geq \sum_k \sum_{\mathcal{C}_k} \ell(Q_{jk})^{D-2\epsilon} \\
&\geq \sum_k \ell(S_k)^{D-2\epsilon} \\
&\geq N(K, 2^{-n_0}) 2^{-n_0(D-2\epsilon)} \\
&\geq 2^{n_0(D-\epsilon)} 2^{-n_0(D-2\epsilon)} \\
&= 2^{n_0\epsilon}.
\end{aligned}$$

Taking $n_0 \to \infty$, we get $\sum_j \ell(Q_j)^{D-2\epsilon} = \infty$, and hence $\kappa \geq D - 2\epsilon$. Taking $\epsilon \to 0$ gives the desired result. $\square$

15. Minkowski dimension equals Hausdorff dimension

In this section we prove,

Theorem 15.1. *If G is a finitely generated Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

Finitely generated is necessary, because it is possible to construct infinitely generated Kleinian groups where the Minkowski dimension of $\Lambda(G)$ fails to exist.

Theorem 15.2. *If G is finitely generated Kleinian group and $area(\Lambda(G)) = 0$ then $\delta(G) = dim(\Lambda)$.*

Thus if the Ahlfors conjecture holds, we would have $\delta(G) = \dim(\Lambda(G))$ for all finitely generated groups. Theorem 15.1 and Theorem 15.2 follow from this more technical looking result.

Theorem 15.3. *Suppose G is a finitely generated Kleinian group. If $area(\Lambda(G)) = 0$ then $\delta(G) = \overline{Mdim}(\Lambda(G))$.*

Proof of Theorem 15.1 *and Theorem* 15.2*:* To see how Theorem 15.3 implies Theorem 15.1 we consider two cases. First, if $\dim(\Lambda) = 2$ then

$$2 = \dim(\Lambda) \leq \underline{\mathrm{Mdim}}(\Lambda) \leq \overline{\mathrm{Mdim}}(\Lambda) \leq 2,$$

so all are equal to 2. On the other hand, if $\dim(\Lambda) < 2$, then Λ has zero area, so Theorem 15.3 applies and gives

$$\dim(\Lambda) \leq \underline{\mathrm{Mdim}}(\Lambda) \leq \overline{\mathrm{Mdim}}(\Lambda) = \delta(G) = \dim(\Lambda_c) \leq \dim(\Lambda),$$

so again, all these numbers are equal. Thus in both cases the Minkowski dimension exists and equals the Hausdorff dimension. The second case also proves Theorem 15.2.

$\square$

Our proof of Theorem 15.3 breaks up into three cases:

(1) $\Omega(G)/G$ is compact.

(2) $\Omega(G)/G$ has punctures and G is geometrically finite.

(3) $\Omega(G)/G$ has punctures and G is geometrically infinite (i.e., $\dim(\Lambda(G)) = 2$).

Note that the third case is not needed for Theorem 15.1, since the two dimensions must agree whenever the Hausdorff dimension is 2.

It is worth also noting that our proof uses Theorem 1.4, but this is not necessary. It is possible to prove Theorem 15.3 without using Theorem 1.4. This argument uses an intricate stopping time argument involving the "good" and "bad" horoballs and is completely "elementary" (e.g., two dimensional), but using Theorem 1.4 allows us to give a simpler proof. It may be possible that the implication could be reversed and Theorem 15.3 used to give an alternate proof of Theorem 1.4 but we have not done this (it would suffice to show that if G is geometrically infinite and $\delta(G) < 2$ then $\delta(G) < \overline{\mathrm{Mdim}}(\Lambda(G))$).

Since $\delta(G) = \dim(\Lambda_c) \leq \dim(\Lambda)$ and $\dim(\Lambda) \leq \overline{\mathrm{Mdim}}(\Lambda)$ we already know that

$$\delta(G) \leq \overline{\mathrm{Mdim}}(\Lambda),$$

but here is a simple proof of this fact which does not require Theorem 1.1. Chose a point $z_0 \in \Omega(G)$. By the Lemma 10.1 the critical exponent $\delta(G)$ for the Poincaré series of G is the same as the critical exponent for the series

$$\sum_{g \in G} d(g(z))^{\alpha}.$$

Because each Whitney square contains only a bounded number of images of z_0,

$$\sum_{g\in G} d(g(z_0))^\alpha \leq C \sum_j \ell(Q_j)^\alpha.$$

Hence the sum on the left converges whenever the sum on the right does. Thus

$$\delta(G) \leq \kappa(\Lambda) \leq \overline{\mathrm{Mdim}}(\lambda),$$

for any Kleinian group G.

We will now start the three cases of Theorem 15.3.

Proof of case (1) in Theorem 15.3*:* Now suppose G is finitely generated, $\Omega(G)/G = R_1 \cup \cdots \cup R_N$ is a union of compact surfaces and $\Lambda(G)$ has zero area. Let $\{Q_j\}$ be a Whitney decomposition of $\Omega(G)$. By the previous lemma there is an $\epsilon > 0$ so that every Q_j contains a hyperbolic ball of radius ϵ.

By the compactness of the surfaces $R_1, \dots, R_N$, we can choose a finite number of points $E = \{z_1, \dots, z_M\} \subset \Omega(G)$, so that E projects to an ϵ-dense set in $R_1 \cup \cdots \cup R_N$, i.e., every point of Ω is within distance ϵ of some point of $G(E) = \cup_{i=1}^M \cup_{g\in G} g(z_i)$. Thus each Whitney square Q_j contains at least one point of $G(E)$. Therefore

$$\sum_j \ell(Q_j)^\alpha \leq C \sum_{z\in G(E)} d(z)^\alpha.$$

By Lemma 6.1, the infinite series on the right hand side converges for $\alpha > \delta(G)$, hence so does the left hand side, i.e., $\kappa(\lambda) \leq \delta(G)$. Since we already proved the opposite inequality we have $\kappa(\Lambda) = \delta(G)$.

If $\Lambda(G)$ has zero area, then Lemma 14.1 implies

$$\overline{\mathrm{Mdim}}(\Lambda) = \kappa(\Lambda) = \delta(G),$$

as desired. $\square$

Proof of case (2) in Theorem 15.3*:* In this case we assume that G is geometrically finite, so by Corollary 11.6 every parabolic cusp of $\Omega(G)$ is double cusped. Thus by part (2) of Lemma 10.2 there is an $\eta > 0$ so that every horoball in $\Omega(G)$ is η-good.

Let $\epsilon > 0$ and choose n_0 so that

$$N(\Lambda, 2^{-n_0}) \geq 100 \cdot 2^{n_0(D-\epsilon/2)}.$$

Let $r = 2^{-n_0}$. Let $\mathcal{S} = \{S_k\}$ be a collection of $2^{n_0(D-\epsilon)}$ squares of size r so that the triples $3S_k$ are pairwise disjoint and $\frac{1}{3}S_k \cap \Lambda \neq \emptyset$ for each k.

For each η-good horoball with $\mathrm{diam}(B) \geq r/3$, let $\mathcal{G}_B$ be the collection of squares in $\mathcal{S}$ which so that $\frac{1}{3}S$ hits B. Let $\mathcal{G}$ be the union of all the $\mathcal{G}_B$. Note that the number of squares in $\mathcal{G}_B$ is at most $C\mathrm{diam}(B)/r$. Therefore if $R = \mathrm{diam}(B)$,

$$\begin{aligned}
\sum_{S \in \mathcal{G}_B} \ell(S)^{D-\epsilon} &\leq C\frac{R}{r} r^{D-\epsilon} \\
&= CRr^{D-1-\epsilon} \\
&\leq CRR^{D-1-\epsilon} \\
&= CR^{D-\epsilon}
\end{aligned}$$

By part (1) of Lemma 10.2 there is a orbit point $z \in G(E) \cap B$ such that

$$d(z) \sim \mathrm{diam}(B).$$

For this point,

$$d(z)^{D-\epsilon} \geq C \sum_{S \in \mathcal{G}_B} \ell(S)^{D-\epsilon}.$$

If more than half the squares in $\mathcal{S}$ belong to $\mathcal{G}$ then this argument shows

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \frac{1}{2} C 2^{n_0 \epsilon/2}.$$

If this happens for arbitrarily large n_0 then we have shown that $\delta(G) \geq D - \epsilon$, as desired.

Thus we may assume that fewer than half the elements of $\mathcal{S}$ are in $\mathcal{G}$. In this case, part (1) of Lemma 10.2 implies that each horoball B is associated to an orbit point z so that

$$\mathrm{diam}(B) \sim d(z).$$

Moreover, if we sum over all Whitney square hitting B then

$$\sum_{Q \cap B \neq \emptyset} \ell(Q)^\alpha \sim \mathrm{diam}(B)^\alpha \sim d(z)^\alpha.$$

Therefore the proof in this case is completed exactly as in the compact case. $\square$

Proof of case (3) of Theorem 15.3*:* We can consider the final case. Suppose that G is geometrically infinite (so $\dim(\Lambda(G)) = 2$ by Theorem 1.4) and assume $\mathrm{area}(\Lambda(G)) = 0$. By Lemma 14.1 $\kappa(\Lambda(G)) = \overline{\mathrm{Mdim}}(\Lambda(G)) = \dim(\Lambda(G)) = 2$. Let $\{B_j\}$ be a listing of the horoballs in $\Omega(G)$. Each horoball contains a Whitney square of comparable size so for any $\epsilon > 0$

$$\sum_j \mathrm{diam}(B_j)^{2-\epsilon/2} = \infty.$$

Thus if

$$\mathcal{B}_n = \{B_j : 2^{-n-1} \leq \mathrm{diam}(B_j) < 2^{-n}\},$$

and $N_n = \#\mathcal{B}_n$, we must have

$$N_n \geq 2^{n(2-\epsilon)},$$

for infinitely many values of n. Fix a value of n_0 where this holds and note that for at least half the balls B in $\mathcal{B}_n$ there is a second ball $B' \in \mathcal{B}_n$ such that

$$\mathrm{dist}(B, B') \leq 2^{n(1-\epsilon)} \leq \mathrm{diam}(B) 2^{-n\epsilon},$$

(otherwise we would have so many disjoint balls that we contradict the assumption that $\Lambda(G)$ has diameter 1). By part (6) of Lemma 10.2, this implies that both B and B' are $2^{-3n\epsilon}$-good horoballs. Let $\mathcal{G}_{n_0} \subset \mathcal{B}_{n_0}$ be the subcollection of $2^{-3n\epsilon}$-good horoballs. For any η-good horoball B let z be the point given in part (1) of Lemma 10.2 and let h be the primitive parabolic element fixing B_j. Then an easy calculation shows

$$\sum_{k\in\mathbb{Z}} d(h^k(z))^\alpha \geq C \mathrm{diam}(B)^\alpha \eta^{\alpha-1},$$

for any $0 < \alpha < 2$ and some C depending on G and α.

Thus if z_j is the good point in B_j given by (1) of Lemma 10.2,

$$\begin{aligned}
\sum_{z\in G(E)} d(z)^\alpha &\geq \sum_{B_j\in\mathcal{G}_{n_0}} \sum_{k\in\mathbb{Z}} d(h^k(z_j))^\alpha \\
&\geq C \sum_{B_j\in\mathcal{G}_{n_0}} \mathrm{diam}(B)^\alpha 2^{-3n_0\epsilon(\alpha-1)} \\
&\geq C 2^{n_0(2-\epsilon)} 2^{-n_0\alpha} 2^{-3n_0\epsilon(\alpha-1)} \\
&\geq C 2^{n_0[(2-\epsilon)-\alpha-3\epsilon(\alpha-1)]} \\
&\geq C 2^{n_0[(2-\alpha)-\epsilon-3\epsilon(\alpha-1)]}
\end{aligned}$$

If we first choose any $\alpha < 2$ and then choose ϵ sufficiently small, the exponent is positive. Thus $\delta(G) > \alpha$ for any $\alpha < 2$, i.e., $\delta(G) = 2$. This completes the proof of the third and final case of Theorem 15.3. $\square$

This finishes the proofs of Theorem 15.1 and Theorem 15.3.

16. Teichmüller spaces

In this section we shall consider $\dim(\Lambda(G))$ as a function on the closure of Teichmüller space $T(S)$ of a finite type hyperbolic surface S.

Given a finite type surface S (compact with a finite number of punctures (possible none)), the Teichmüller space $T(S)$ is the set of equivalence classes of quasiconformal mapping of S to itself. Each such is represented by a Beltrami differential μ which may be lifted to a Beltrami differential μ on the the upper halfplane, $\mathbb{H}$. Let Γ be a Fuchsian group acting on $\mathbb{H}$ such that $\mathbb{H}/\Gamma = S$. There is a quasiconformal mapping F of the plane which fixes $0, 1, \infty$ and such that $\overline{\partial} F/\partial F = \mu$ on $\mathbb{H}$ and so that F is conformal on the lower half plane. On the lower half-plane the Schwarzian derivative $S(F)$ satisfies

$$\|S(F)\| = \sup_z |\mathrm{Im}(z)|^2 |S(F)(z)| \geq 6 < \infty.$$

This realizes $T(S)$ as a bounded subset of a Banach space and gives a metric on $T(S)$. The closure of $T(S)$ with respect to this metric is denoted $\overline{T(S)}$ and the boundary by $\partial T(S)$. Points of $\overline{T(S)}$ may be identified with certain Kleinian groups which are isomorphic to Γ. Moreover, convergence in the Teichmüler metric above implies algebraic convergence of the groups. Recall that a group is called degenerate if $\Omega(G)$ has one component and this component is simply connected. Such groups must be geometrically infinite by a result of Greenberg [34]. $G \in \partial T(S)$ is called a cusp if there is a hyperbolic element in Γ which becomes parabolic in G. Bers showed $\partial T(S)$ consists entirely of degenerate groups and cusps and that degenerate groups form a dense G_δ set in $\partial T(S)$ in [9] (in fact, the cusps lie on a countable union of real codimension 2 surfaces). McMullen [46] proved there is a dense set of geometrically finite cusps in $\partial T(S)$.

Recall that Theorem 1.5 says that if $\{g_n\}$ converges algebraically to G then

$$\liminf_{n\to\infty} \dim(\Lambda(G_n))) \geq \dim(\Lambda(G)).$$

One special case where this holds is for $G \in \overline{T(S)}$, the closure of the Teichmüller space of a finite type hyperbolic Riemann surface S. Since $\Lambda(G)$ is at most 2 and is lower semi-continuous, it is continuous whenever it takes the value 2 (i.e., at the geometrically infinite groups). Since these points are dense on the boundary of Teichmüller space, this function must be discontinuous at the geometrically finite cusps on the boundary. Thus

Corollary 16.1. *Suppose S is a hyperbolic Riemann surface of finite type. Then $dim(\Lambda(G))$ is lower semi-continuous on $\overline{T(S)}$, and is continuous everywhere except at the geometrically finite cusps in $\partial T(S)$ where it is discontinuous.*

This also show that equality in Corollary 2.5 and Theorem 1.5 need not occur, because a geometrically finite cusp ($\delta(G) < 2$) can be approximated by by geometrically finite groups which are close to degenerate groups (so $\delta(G_n) \to 2$). The discontinuity at the geometrically finite cusped had been proved earlier by Taylor in [58]; he showed that for each geometrically finite cusp G there is a sequence $G_n \to G$ algebraically, but $G_n \to H$ geometrically where H is a geomtrically finite group containing G and

$$\dim(\Lambda(G_n)) \to \dim(\Lambda(H)) > \dim(\Lambda(G)).$$

If f is lower semi-continuous then $\{f \leq \alpha\}$ is closed. Thus,

Corollary 16.2. *The set $E_\alpha = \{G \in \overline{T(S)} : dim(\Lambda(G)) \leq \alpha\}$ is closed in $\overline{T(S)}$. The set $F_\alpha = \{G \in \partial T(S) : dim(\Lambda(G)) \leq \alpha < 2\}$ is a closed, nowhere dense subset of $\partial T(S)$.*

Since a lower semi-continuous functions takes a minimum on a compact set, Theorem 1.2 implies

Corollary 16.3. *dim*$(\Lambda(G))$ *takes a minimum value on* $\partial T(S)$ *and this minimum is strictly larger than* 1.

It is not clear where the minimum occurs. Canary has suggested it might occur at the cusp group corresponding to shrinking a minimum length geodesic on S to a parabolic, since this requires the "least" deformation of the Fuchian group (in some sense). Since $\dim(\Lambda(G))$ takes a minimum on $\partial T(S)$ which is > 1, any group in $\overline{T(S)}$ with small enough dimension must be quasi-Fuchsian. Thus

Corollary 16.4. *Suppose* $\{G_n\}$ *is a sequence of quasiconformal deformations of a Fuchsian group* G *(i.e.,* $\{G_n\}$ *is a sequence in* $T(S)$*,* $S = \mathbb{D}/G$*). If dim*$(\Lambda(G_n)) \to 1$*, then* $G_n \to G$*.*

Proof. This is immediate from Theorem 13.1 and the fact (from Theorem 1.2) that G is the only point in $\overline{T(S)}$ where $\dim(\Lambda) = 1$. □

Corollary 16.5. *Suppose* G *is a finitely generated and has a simply connected invariant component* Ω_0 *(possibly not unique). Let* $\Omega_0/G = S$*. Then for any* $\epsilon > 0$ *there is a* δ*, depending only on* S *and* ϵ*, such that dim*$(\Lambda(G)) < 1 + \delta$ *implies* G *is a* ϵ*-quasiconformal deformation of a Fuchsian group.*

Proof. If G is a quasi-Fuchsian group this follows from the previous result. Maskit [42] proved that a finitely generated Kleinian group with two invariant components is quasi-Fuchsian, so we may now assume G is a b-group (i.e., has exactly one simply connected, invariant component). If G is geometrically infinite then $\dim(\Lambda(G)) = 2$, which contradicts our assumption. Therefore, G must be geometrically finite. Abikoff

[1] proved that every geometrically finite b group covering S is on the boundary of the Teichmüller space $T(S)$, and so its dimension is bounded away from 1 by Corollary 16.3. $\square$

It is not true that the ϵ in Corollary 16.5 can be taken to depend only on the topological type of S (e.g., the number of generators of G). For example, given a surface with punctures S, it is possible to use the combination theorems to construct a b-group G with $\dim(\Lambda(G))$ as close to one as we wish and so that Ω_0/G is homeomorphic to (thought not conformally equivalent to) S.

Larman showed that there is an ϵ_0, such that if $\{D_j\}$ is a collection of three or more disjoint open disks then the dimension of $\overline{\mathbb{C}} \setminus \cup_j D_j$ is larger than $1+\epsilon_0$. A careful reading of Larman's paper [40] shows that his proof gives

Proposition 16.6. *There is an $\epsilon_0 > 0$ such that if $\{D_j\}$ is any collection of three or more disjoint open ϵ_0-quasidisks, then $dim(\mathbb{C} \setminus \cup_j D_j) > 1 + \epsilon_0$.*

Recall that a *web group* is a finitely generated Kleinian groups each of whose component subgroups is quasi-Fuchsian. Suppose G is web group. If $\Omega(G)$ has only two components then G is quasi-Fuchsian [43]). So suppose G has three or more components and let $\{G_1, \dots, G_n\}$ representatives of each conjugacy class of component subgroups. By the last corollary either one of these has limit set with dimension $> 1 + \delta$ or all are ϵ_0-quasicircle. In the latter case. Larman's theorem implies that $\Lambda(G)$ has dimension bigger than ϵ_0. In either case the dimension is bounded away from 1 by a number which only depends on the conformal structure of $\Omega(G)/G$. of generators for each component subgroup.

Corollary 16.7. *Suppose G is a finitely generated web group which is not quasi-Fuchsian. Then $dim(\Lambda) > 1 + \epsilon$ where ϵ depends only on the conformal types of the components of $\Omega(G)/G$.*

It is not true that the dimension of limit sets of proper web groups (i.e., not quasi-Fuchsian) is bounded uniformly away from 1. Canary, Minsky and Taylor have constructed examples of proper web groups (with a fixed number of generators) whose limits sets have dimension arbitarily close to 1 (personal communication).

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