Absorbing Cantor sets in dynamical systems: Fibonacci maps

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Abstract

In this paper we shall show that there exists a polynomial unimodal map $f: [0,1] \to [0,1]$ which is

- non-renormalizable (therefore for each x from a residual set, $\omega(x)$ is equal to an interval),
- for which $\omega(c)$ is a Cantor set and
- for which $\omega(x) = \omega(c)$ for Lebesgue almost all x.

So the topological and the metric attractor of such a map do not coincide. This gives the answer to a question posed by Milnor [Mil].

1 Introduction

One of the central themes in the theory of dynamical systems is the concept of attractors. However, there is no complete consensus about the 'correct' definition of this notion. In particular it is not clear whether an attractor should attract a topologically big set or a set which is large in a metric sense. So, if $f: M \to M$ is a dynamical system defined on a manifold M, then we could define a closed forward invariant set X to be a *topological* respectively a *metric attractor* if

1. its basin

$$B(X) = \{x \ ; \ \omega(x) \subset X\}$$

contains a residual subset of an open neighbourhood of X, respectively $B(X)$ has positive Lebesgue measure;

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2. there exists no closed forward invariant set X' which is strictly included in X for which $B(X)$ and $B(X')$ coincide up to a meager set respectively up to a set of measure zero.

Here $\omega(x)$ is the set of limit points of $f^n(x)$ as $n \to \infty$. Moreover, we say that A is a residual (resp. meager) set if it is the countable intersection (union) of open dense (closed nowhere dense) sets. For a discussion on these definitions, see [Mil]. If X is a periodic attractor, a hyperbolic attractor, a 'Feigenbaum attractor' (see for example [MS] and for the invertible case see [GST]), or one of the known strange attractors, see [BC], then X is both a metric and a topological attractor. Of course, there are some pathological cases: for example the horseshoe of a C^1 diffeomorphism can have positive Lebesgue measure and certainly is no topological attractor, see [Bow]. In this paper we present a non-pathological example for which the distinction does matter. More precisely, we want to show that there exists a smooth discrete dynamical system $f\colon M \to M$ where M is a smooth manifold with an 'absorbing Cantor set' X (this terminology comes from [GJ]). This means that X is a closed forward invariant minimal set $X \subset M$ with zero Lebesgue measure, such that its basin $B(X)$ has positive Lebesgue measure but its complement is a residual set. As far as we know this example is the first smooth dynamical system with such an 'absorbing Cantor set'.

In our case $M = [0,1]$ and f is a smooth unimodal interval map – this means f has one extremal point – and for simplicity we shall also assume that $f(0) = f(1) = 0$. A prototype of such map is

$$f(x) = \lambda \left[1 - |2x-1|^{\ell}\right]$$

where $\lambda > 0$ is chosen so that f maps the interval $[0,1]$ inside itself and f has the so-called Fibonacci-type dynamics. We shall define this in the next section.

There are many publications in which it was conjectured that a smooth map $f\colon [0,1] \to [0,1]$ cannot have an absorbing Cantor set. (We should note, however, that in 1992 Misha Lyubich and Folkert Tangerman made computer estimates suggesting that absorbing Cantor sets do exist for Fibonacci maps of the form $x \mapsto x^6 + c_1$.) Moreover, there are several results which prove that these sets cannot exist in particular cases, see [JS1], [LM] and in the general quadratic unimodal case [L1] when $\ell = 2$. We shall show that absorbing Cantor sets do exist when ℓ is a large real number.

Main Theorem
There exists ℓ_0 with the following property. Let f be a C^2 unimodal interval map with a critical point of order $\ell \geq \ell_0$ and with the Fibonacci combinatorics. Then f has an absorbing Cantor attractor X.

Here we say that c is a critical point of a C^2 map f if $Df(c) = 0$ and the *order of the critical point* is said to be ℓ if there exists a C^2 diffeomorphism ϕ between two neighbourhoods of c such that

$$f \circ \phi(x) = f(c) - |x - c|^{\ell}$$

for x close to c. It is easy to show that our methods also give examples of multimodal smooth interval maps for which each critical point is quadratic and which have an absorbing Cantor set: simply choose the map so that the return map near some critical point is a unimodal map of Fibonacci-type while the orbit of this critical point contains at least ℓ other critical points. However, it is not clear whether absorbing Cantor attractors also appear generically in one-parameter families:

Question *Does the space of smooth maps $f\colon [0,1] \to [0,1]$ with an absorbing Cantor set form a codimension-one subset of the space of all smooth interval maps?*

Of course, it follows from the Main Theorem that there exists on each smooth manifold a smooth mapping with an absorbing Cantor set. We conjecture that one can also construct invertible examples:

Conjecture *For each $n \geq 2$ dimensional smooth manifold M, there exists a diffeomorphism $f\colon M \to M$ which has an absorbing Cantor set.*

In the complex one-dimensional direction there are related results:

Theorem [NS2]
For each sufficiently large even integer ℓ there exists $c_1 \in \mathbb{R}$ such that the map $f(z) = z^\ell + c_1$ has the following properties:

- *the set $\omega(0)$ is a Cantor set with zero Lebesgue measure;*
- *the set of points $z \in \mathbb{C}$ for which $\omega(z)$ is contained in $\omega(0)$ has positive Lebesgue measure;*
- *the set of points whose forward iterates remain bounded has no interior.*

In particular, the Julia set of $z \mapsto z^\ell + c_1$ has positive Lebesgue measure. This map has the Fibonacci dynamics (to be defined in the next section).

1.1 Some comments on the Main Theorem and its proof

In fact, the attractor X from the Main Theorem is equal to $\omega(c)$ and this set has zero Lebesgue measure, see [Mar] and also [MS]. If the map f from the Main Theorem is a unimodal polynomial with a unique critical point in $\mathbb{C}$ (or if has negative Schwarzian derivative and f has no attracting fixed points) then $B(X)$ has full Lebesgue measure and its complement is a residual set. We should remark that a smooth map as above may have one or more periodic attractors, but that even then the attractor X has a basin which attracts a set of positive Lebesgue measure (and the critical point is density point of $B(X)$). This is not completely surprising because $\omega(c)$ is not accumulated by periodic attractors, see [MMS] and also [MS][Chapter IV].

In the theory of unimodal interval maps with negative Schwarzian derivative of f, i.e., with

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\frac{D^2 f(x)}{Df(x)} < 0$$

and for which the order of the critical point is finite, one has a well-known classification, see [Gu], [BL], [Ke] and also [MS].

1. f has a stable periodic orbit O which is both a topological and metric attractor;

2. f is infinitely renormalizable, i.e., there exists a nested sequence of intervals $I_n \ni c$ shrinking to c and a sequence of integers $q(n) \to \infty$ such that $I_n, \ldots, f^{q(n)-1}(I_n)$ are disjoint and $f^{q(n)}(I_n) \subset I_n$. In this case $\omega(c)$ is a Cantor set of zero Lebesgue measure which is both a topological and metric attractor;

3. f is not infinitely renormalizable. In this case there exists a cycle of intervals Z (a finite union of intervals) such that $B(Z)$ is dense and has full Lebesgue measure. The set Z is a topological attractor, but not necessarily a metric attractor: in principle, there could be a Cantor set $X \subset Z$ such that $B(X)$ has full Lebesgue measure (but is not dense).

From our theorem it follows that the possibility mentioned in the last case really does occur if ℓ is large. In the quadratic case, i.e. $\ell = 2$, the results of [L1] imply that Z is a metric attractor as well.

Any map with an absorbing Cantor set has no absolutely continuous invariant probability measure, because Lebesgue almost all points wander densely on the support of the measure by the Birkhoff Ergodic Theorem. If the Schwarzian derivative of f is negative and $\ell = 2$ then it is shown in [LM] that f has an absolutely continuous invariant probability measure by showing that the summability condition from [NS1] is satisfied. In particular, f has no absorbing Cantor set in this case. The methods of proof in [LM] are a mixture of real tools and tools from the theory of complex analysis and hyperbolic geometry. This result was generalized in [KN]: in that paper it was shown that the same results hold for $1 < \ell \leq 2 + \epsilon$ provided $\epsilon > 0$ is small. The tools in [KN] are entirely based on real estimates, and also no use is made of [NS1] (because the summability condition fails if $\ell > 2$).

As mentioned, our result implies that f has no absolutely continuous invariant probability measure for ℓ large. In fact, as Henk Bruin has shown in [Br], this already follows from Proposition 3.4.

We expect that the methods of this paper can be extended to show that for Fibonacci maps of 'bounded type' (a notion which we shall discuss in the section about the combinatorial properties of Fibonacci maps) with a rather flat critical point, the same result holds.

Let us now give an outline of the proof that $\omega(x)$ is equal to the Cantor set $\omega(c)$ for Lebesgue almost all x.

- First we will show that there exists a nested sequence of intervals $(u_n, \hat{u}_n)$ containing c and that the size of the annulus $A_n = (u_n, \hat{u}_n) \setminus (u_{n+1}, \hat{u}_{n+1})$ is very small compared to the size of $(u_{n+1}, \hat{u}_{n+1})$ if the order ℓ of the critical point is large.

- Next we let $I_n, \hat{I}_n$ be the components of A_n and show that some iterate f^{S_n} of f maps I_n diffeomorphically inside $\cup_{k\geq n-2}(I_k \cup \hat{I}_k)$ and that this map is not 'too' non-linear. Because of 1) this implies that 'most' points are mapped closer to c by this iterate.

- Finally, we combine 1), 2) and a kind of random walk argument to show that typical points are in the basin of $\omega(c)$.

2 Combinatorial properties of the Fibonacci map

In this section we shall define and state some properties of the Fibonacci map. It is well-known that maps with these properties exist, see [HK] or [LM] and also the sequel to this paper. In the companion paper [NS2] we shall construct such a map 'by hand'. Let $f\colon[0,1] \to [0,1]$ be a unimodal map with $f(0) = f(1) = 0$. For each $x \neq c$ there exists a 'symmetric' point $\hat{x} \neq x$ with $f(\hat{x}) = f(x)$. For $i \geq 0$ and $x \in [0,1]$, let $x_i = f^i(x)$ and choose $x_{-i} \in f^{-i}(x)$ so that the interval connecting this point to c contains no other points in the set $f^{-i}(x)$. Note that if c is not a periodic point there are always precisely two such points c_{-i} (which are symmetric with respect to each other). Let $S_0 = 1$ and define S_i inductively by

$$S_i = \min\{k \geq S_{i-1};\ c_{-k} \in (c_{-S_{i-1}}, \hat{c}_{-S_{i-1}})\}.$$

f is called a *Fibonacci* map if the sequence S_i coincides with the Fibonacci numbers: $S_0 = 1$, $S_1 = 2$ and $S_{k+1} = S_k + S_{k-1}$, i.e., the sequence $1, 2, 3, 5, 8, \ldots$. The proof of the following proposition can be found in [LM], [KN] and also in [NS2].

Let us denote by z_k the nearest point to c in the set $f^{-S_k}(c)$. It should be clear from the context whether z_k is to the left or right of c. Moreover, for $x \in [0,1]$ let us write

$$x^f = f(x)$$

(usually, x will be close to c and so x^f close to $c^f = f(c)$).

Proposition 2.1 *A Fibonacci map $f\colon[0,1] \to [0,1]$ satisfies the following properties.*

- *f is non-renormalizable;*

- *c_{S_k} and $c_{S_{k+2}}$ are on opposite sides of c.*

- *$c_{S_n} \in (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ and $c_i \notin (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ for each $0 < i < S_n$.*

- *$c_{-S_n} \in (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ and $c_{-i} \notin (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ for each $0 < i < S_n$.*

- *If T is the maximal interval adjacent to c such that $f^{S_k}_{|T}$ is monotone, then $f^{S_k}(T) = (c_{S_k}, c_{S_{k-2}})$.*

- *If $T_k \ni c^f$ is the largest interval on which $f^{S_k-1}_{|T_k}$ is monotone, then*

$$T_k = (z^f_{k-1}, t^f_k)$$

where $t^f_k > c^f$ and $f^{S_k-1}(T_k) = (c_{S_{k-2}}, c_{S_{k-4}})$ (note that t^f_k is not the f-image of some point t_k, so this notation is just to suggest that t^f_k is close to c^f).

- *$T_k, \dots, f^{S_k-1}(T_k)$ has intersection multiplicity 3 (this means that each point of $[0,1]$ is contained in at most 3 of these intervals).*

Proof: For the proof of this result we refer the reader to [KN] and [LM]. The statement about the disjointness can be found in [LM][Lemma 4.3]. □

From the fact that c_{-1} exists it follows that f has a orientation reversing fixed point q. Let us define inductively a sequence of points u_n as follows. Let $u_0 = q$ and let us define u_{n+1} to be nearest point to c with

$$u_{n+1} \in f^{-S_n}(u_n)$$

so that u_{n+1} is on the same side of c as $c_{S_{n+1}}$. In particular, $u_1 = \hat{u}_0 = \hat{q}$. Moreover, let $\tilde{u}_{k+1}$ be the point in $\{u_{k+1}, \hat{u}_{k+1}\}$ which is on the same side of c as u_k.

Furthermore, let

$$y_n = f^{S_n}(c_{S_{n+2}}) \ , \quad y^f_n = f(y_n).$$

Proposition 2.2 *A Fibonacci map $f\colon [0,1] \to [0,1]$ satisfies the following properties.*

- *$f^{S_n}(u_{n+1}) = u_n$ and $f^{S_n}(u_n) = u_{n-2}$;*

- *in particular, f^{S_n} maps $(\tilde{u}_{n+1}, u_n)$ diffeomorphically onto (u_n, u_{n-2}) (note that this last interval contains c);*

- *the points u^f_n, $c^f_{S_n}$, $c^f_{S_n+S_{n+2}}$, y^f_n and z^f_n are ordered as in the picture below (we state the ordering near c^f rather than near c so that we do not need to be careful about on which side of c these points lie).*

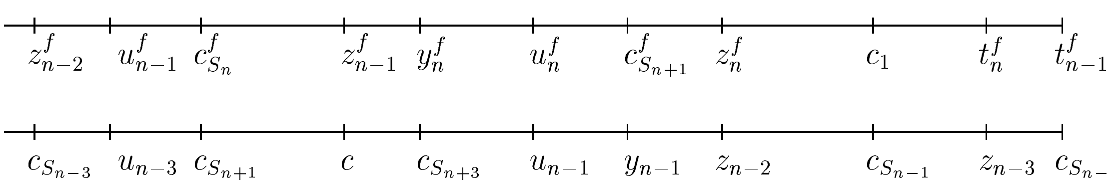

Figure 2.1: Points and their images under $f^{S_{n-1}-1}$.

Proof: The proof of these statements can be found in [KN]. It can be derived from Figure 2.2 below. □

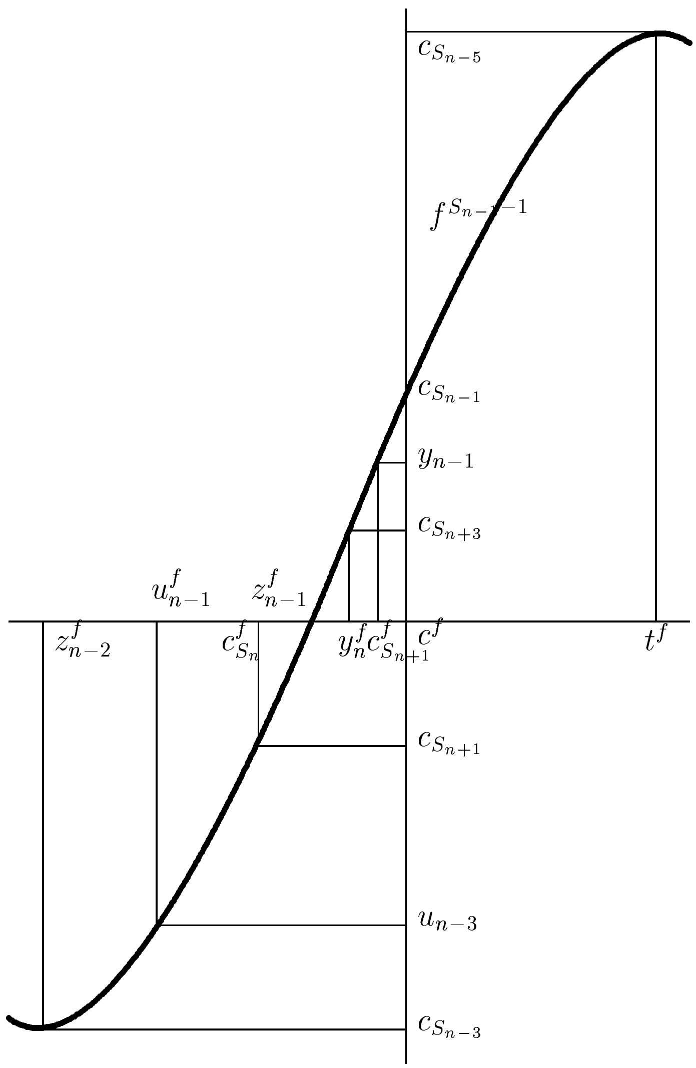

Figure 2.2.

If f has no wandering intervals (and this is the case under the present assumptions, see [MS][Chapter IV]), then $\omega(c)$ is a minimal Cantor set.

3 The estimates

In this section we shall estimate the rate of approach of the sequences $u^f_k, c^f_{S_k}, z^f_k$ to c^f. The basic tool is that of the distortion of cross-ratios.

Remark 3.1 *Since f is non-flat at $x = c$ we can assume (by applying a suitable C^2 coordinate change) that f is of the form*

$$f(x) = f(c) - |x - c|^\ell$$

near $x = c$. This will simplify some of the estimates somewhat.

Hence

$$\frac{|f(x) - f(c)|}{|x - c|} = M(x)$$

where $M(x)$ is a continuous function which is equal to $|x - c|^{\ell-1}$ near $x = c$. Moreover,

$$\frac{Df(x)}{\ell|f(x) - f(c)|/|x - c|} = 1$$

near $x = c$. We shall use these facts repeatedly.

3.1 The cross-ratio and the Koebe Principle

Let $j \subset t$ be intervals and let l, r be the components of $t \setminus j$. Then the cross-ratio of this pair of intervals is defined as

$$C(t, j) := \frac{|t|}{|l|}\frac{|j|}{|r|}.$$

Let f be a smooth function mapping t, l, j, r onto T, L, J, R diffeomorphically. Define

$$B(f, t, j) = \frac{|T|\,|J|}{|t|\,|j|}\,\frac{|l|\,|r|}{|L|\,|R|} = \frac{C(T, J)}{C(t, j)}.$$

It is well known that if $Sf = f'''/f' - 3(f''/f')^2/2 \le 0$ then $B(f, t, j) \ge 1$. In the next proposition it is stated that this ratio also cannot be decreased too much by a C^2 map f with non-flat critical points.

Proposition 3.1 *Let f be a C^2 map with non-flat critical points. Then there exists a function $o(\epsilon) > 0$ with $o(\epsilon) \to 0$ as $\epsilon \to 0$ such that for any intervals $j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. Let l, r be as above and let L, J, R, T be the images of l, j, r, t under f^n. Then*

$$B(f^n, t, j) = \frac{|T|}{|L|}\frac{|J|}{|R|}\frac{|l|}{|t|}\frac{|r|}{|j|} \ge \exp\left(-o(\epsilon) \cdot \sum_{i=0}^{n-1} |f^i(t)|\right)$$

where $\epsilon = \max_{i=0}^n |f^i(t)|$. (If $Sf < 0$ then $B(f^n, t, j) > 1$.)

Proof: See Theorem IV.2.1 in [MS]. □

From this it follows in particular that if $f^n|T$ is a diffeomorphism and j is reduced to the point x then

$$\frac{|Df(x)|}{|L|/|l|} \geq \exp\left(-o(\epsilon)\cdot\sum_{i=0}^{n-1}|f^i(t)|\right)|R|/|T|$$

and if l is reduced to the point y then

$$\frac{|Df(y)|}{|J|/|j|} \leq \exp\left(o(\epsilon)\cdot\sum_{i=0}^{n-1}|f^i(t)|\right)|T|/|R|.$$

We shall also need the following lemma. In fact, instead of this lemma one could use the Koebe Principle stated below (and the Koebe Principle can be derived from the next lemma).

Lemma 3.1 *Let f be a C^2 map with non-flat critical points. Then there exists a function $o(\epsilon) > 0$ with $o(\epsilon) \to 0$ as $\epsilon \to 0$ such that for any n and any interval t for which $f^n|t$ is a diffeomorphism one has the following property. Let $j_1, j_2 \subset t$ be two disjoint intervals and let l_i, r_i be the components of $t \setminus j_i$ where we assume $l_1 \subset l_2$ and $r_1 \supset r_2$. (So this is the case if j_1 lies to the left of j_2 and l_i is to the left of j_i for $i = 1, 2$.) Let L_i, J_i, R_i, T be the images of l_i, j_i, r_i, t under f^n. Then*

$$\frac{|j_1|}{|J_1|}\frac{|J_2|}{|j_2|} \geq \mathcal{O}\cdot\frac{|J_2\cup R_2||R_2|}{|J_1\cup R_1||R_1|} = \frac{|J_2\cup R_2||R_2|}{|J_1\cup J\cup J_2\cup R_2||J\cup J_2\cup R_2|}$$

and

$$\frac{|j_1|}{|J_1|}\frac{|J_2|}{|j_2|} \leq \mathcal{O}\cdot\frac{|L_2||L_2\cup J_2|}{|L_1||L_1\cup J_1|}.$$

where

$$\mathcal{O} = \exp\left(\pm o(\epsilon)\cdot\sum_{i=0}^{n-1}|f^i(t)|\right)$$

and $\epsilon = \max_{i=0,\dots,n-1}|f^i(t)|$. (If $Sf < 0$ then we can take $\mathcal{O} = 1$.)

Proof: Let j be the interval connecting j_1 and j_2. Multiplying the following two cross-ratio inequalities from the previous proposition, the result follows immediately.

$$\frac{|J|}{|J_1|}\frac{|J_1\cup J\cup J_2\cup R_2|}{|J_2\cup R_2|} \geq \mathcal{O}\frac{|j|}{|j_1|}\frac{|j_1\cup j\cup j_2\cup r_2|}{|j_2\cup r_2|}$$

and

$$\frac{|J_2|}{|J|}\frac{|J\cup J_2\cup R_2|}{|R_2|} \geq \mathcal{O}\frac{|j_2|}{|j|}\frac{|j\cup j_2\cup r_2|}{|r_2|}.$$

The second inequality follows similarly. □

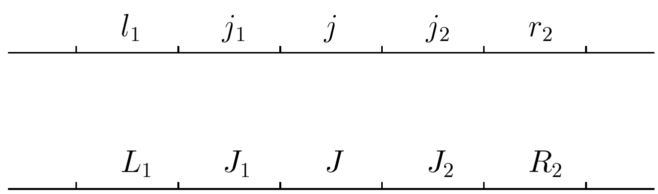

Figure 3.1: Intervals j_i, l_i, r_i and their images.

We should remark that if we take T_k to be the maximal interval containing c_1 on which f^{S_k-1} is a diffeomorphism, then from Proposition 2.1,

$$\sum_{i=0}^{S_k-1} |f^i(T_k)| \leq 3.$$

So this implies that we can apply the previous results immediately to $f^{S_k-1}|T_k$. In fact, $\max_{i=0}^{S_k-1} |f^i(T_k)| \to 0$ as $k \to \infty$:

Lemma 3.2 *For each $\epsilon > 0$ there exists $\delta > 0$ such that if $f^n(I)$ is not contained in an immediate basin of a periodic attractor and $|f^n(I)| \leq \delta$, then $\max_{i=0}^{n-1} |f^i(I)| \leq \epsilon$.*

Proof: If the lemma is not satisfied, then there exists a sequence of intervals I_i with $|I_i| \geq \epsilon$ and a sequence $n(i)$ with $|f^{n(i)}(I_i)| \to 0$ where $f^{n(i)}(I_i)$ is not completely contained in the immediate basin of some periodic attractor. By taking subsequences, there exists an interval I such that $\inf_{i\geq 0} |f^i(I)| = 0$ and such that I is not completely contained in the basin of a periodic attractor. This is impossible because f has no wandering intervals, see [MS][Chapter IV, Theorem A]. Indeed, by the Contraction Principle, see [MS][IV.5.1] if I is an interval with $\inf_{i\geq 0} |f^i(I)| = 0$ then either I is completely contained in the basin of a periodic attractor or a wandering interval. □

In fact, we shall also have to estimate the cross-ratio distortion of iterates of f which are not of the form f^{S_i}. For this we shall need the Koebe Principle and an estimate on the total size of orbits of some intervals. Let us say that an interval T contains a τ *-scaled neighbourhood* of an interval $J \subset T$ if each component of $T \setminus J$ has at least size $\tau|J|$.

Proposition 3.2 (Koebe Principle) *Let f be a C^2 map with non-flat critical points. Then there exists a function $o(\epsilon) > 0$ with $o(\epsilon) \to 0$ as $\epsilon \to 0$ such that for any intervals*

$j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. If $f^n(t)$ contains a τ-scaled neighbourhood of $f^n(j)$ then

$$\frac{|Df^n(x)|}{|Df^n(y)|} \le \left[\frac{1+\tau}{\tau}\right]^2 \exp\left(o(\epsilon)\cdot\sum_{i=0}^{n-1}|f^i(j)|\right) \tag{3.1}$$

for each $x, y \in J$ where $\epsilon = \max_{i=0}^{n}|f^i(t)|$.

Moreover, if $f^n(t)$ contains no periodic attractor then there exists t' with $j \subset t' \subset t$ for which

$$f^n(t') \text{ is a } \tau/2- \text{ scaled neigbourhood of } f^n(j)$$

and

$$|f^i(t')| \le K|f^i(j)|$$

for all $i = 0, 1, \ldots, n$. Here K depends on f, τ, ϵ and $\sum_{i=0}^{n-1}|f^i(j)|$.

Proof: This is a combined statement of Theorem IV.3.1 and Theorem IV.1.1 in [MS] and of Lemma 8.3 from [Str]. (Note that we do not assume that $\sum|f^i(t)|$ is bounded but merely that $\sum|f^i(j)|$ is bounded. □

In the next proposition we shall give a condition for orbits of intervals to have a finite total length. We shall need this proposition only in the case that the Schwarzian derivative of f is not negative to estimate the term $\mathcal{O}$ in Lemma 3.1.

Proposition 3.3 *Let f be a C^2 map with non-flat critical points. Then for each $\tau, S > 0$ there exist constants $\delta, S' > 0$ such that the following holds. Let x be a recurrent point of f, let U be an interval neighbourhood around x of size $< \delta$ and $D \subset U$ be some disjoint union of intervals I_i. Let F be a map defined on $D = \cup I_i$ such that for each interval I_i there exists an integer $j(i)$ and an interval $T_i \supset I_i$ such that*

1. *$F|I_i = f^{j(i)}$ maps I_i onto some union of intervals I_i and $F(I_i)$ contains at least two of those intervals;*
2. *$f^{j(i)}|T_i$ is a diffeomorphism and $I_k \subset f^{j(i)}(I_i)$ implies $T_k \subset f^{j(i)}(T_i)$;*
3. *$f^{j(i)}(T_i)$ contains a τ-scaled neighbourhood of each interval $I_k \subset f^{j(i)}(I_i)$;*
4. *for each interval $I_j \subset F(I_i)$ one has $|I_j| \le (1 - \frac{1}{S}) \cdot |F(I_i)|$;*
5. *$\sum_{m=0}^{j(i)-1}|f^m(T_i)| \le S$.*

Then for each $n \in \mathbb{N}$ and each component J of the domain of F^n one has $F^n|J = f^j$ for some $j \in \mathbb{N}$ and there exists an interval $T' \supset J$ for which $F^n(T')$ contains a $\tau/2$-scaled neighbourhood of each element $I_k \subset F^n(J)$ and

$$\sum_{m=0}^{j-1}|f^m(T')| \le S'.$$

Proof: The idea of the proof of this proposition is essentially the same as that of [Str]. The proof of this proposition can be substantially simplified if f has negative Schwarzian derivative: in this case it is not necessary to choose δ small. (In fact we do not even need this lemma in that case.) However, in the general case, f could for example have a periodic interval (corresponding to basins of periodic attractors). This complicates matters to some extend.

Fix τ and K. Since f is C^2 each periodic point p of f of sufficiently large period k is repelling, see [MS][Theorem IV.B]. In particular, this holds for all periodic points which are in a δ neighbourhood of x, provided $\delta > 0$ is sufficiently small. For this reason we shall be able to apply Lemma 3.2.

Now let $\mathcal{I}$ be the partition of the domain of F of the intervals I_i and define inductively $\mathcal{I}_0 = \mathcal{I}$ and

$$\mathcal{I}_n = \mathcal{I}_0 \vee F^{-1}\mathcal{I}_0 \vee \ldots \vee F^{-n}\mathcal{I}_0.$$

So each element J of $\mathcal{I}_n$ is an interval which F^n maps diffeomorphically onto some interval I_i and each of these intervals is contained in U where $|U| < \delta$. Because of properties 2) and 3) there exists $j \in \mathbb{N}$ with $F^n|J = f^j$ and an interval $T \supset J$ which is mapped diffeomorphically onto a τ-scaled neighbourhood of $f^j(J) = I_k \in \mathcal{I}_0$. It suffices to show that there exists S' such that

$$\sum_{i=0}^{j} |f^i(J)| \le S' \quad \text{for each} \quad J \in \mathcal{I}_n. \tag{3.2}$$

Indeed, the components of the domain of F^n are elements from $\mathcal{I}_{n-1}$ and are mapped by F^{n-1} into an of elements of $\mathcal{I}_0$. However, because of property 5) the length of the remaining intervals up to the F^n-th iterate have uniformly bounded sum.

First we claim that there exists $\kappa < 1$ such that

$$\text{if } J \in \mathcal{I}_1 \text{ is contained in } I_i \in \mathcal{I}_0 \text{ then } |J| \le \kappa |I_i|. \tag{3.3}$$

This holds since $F(J)$ is equal to an interval $I_k \in \mathcal{I}_0$ while properties 3) and 4) imply that there exists an interval J' with $J \subset J' \subset I_i$ for which i) $F(J')$ is contained inside a $\tau/2$-scaled neighbourhood of $F(J) = I_k$ and ii) a definite proportion of $F(J')$ is outside $F(J) = I_k$. Moreover, because of 5) and the Koebe Principle there exists a universal constant $K_0 < \infty$ such that

$$\sup_{x,y \in J'} \frac{|DF(x)|}{|DF(y)|} \le K_0.$$

Combining this proves (3.3).

By using a 'telescope argument' we can improve this statement and show by induction that there exists $\kappa < 1$ such that for each $n \in \mathbb{N}$ there exists $\delta > 0$ such that if $|U| < \delta$, $J \in \mathcal{I}_n$ and J is contained in $I_i \in \mathcal{I}_0$ then

$$|J| \le \kappa^n |I_i|. \tag{3.4}$$

For $n = 0$ there is nothing to prove. So assume the statement holds for $n-1$ and consider $J \in \mathcal{I}_n$. If $F^n|J = f^j$ and $T \supset J$ so that $f^j|T$ is a diffeomorphism and $f^j(T)$ is a τ-scaled neighbourhood of $f^j(I) = I_k \in \mathcal{I}_0$ then

$$\sum_{i=0}^{j-1} |f^i(J)| \le n \cdot S \quad \text{and} \quad \max_{i=0,\dots,j-1} |f^i(T)| = o(|f^j(T)|), \tag{3.5}$$

where $o(t)$ is a function so that $o(t) \to 0$ if $t \downarrow 0$. Here we have used respectively property 5) and the previous Lemma 3.2. (We should note that $f^n(T) \subset U$ and so $|f^j(T)| = |F^n(J)| \le |U| \le \delta$.) Hence, by the Koebe Principle, there exists K_1 (which only depends on τ) such that for each n

$$\frac{|DF^n(x)|}{|DF^n(y)|} \le K_1 \tag{3.6}$$

for all $x, y \in J$ provided δ (and hence $F^n(T) \subset U$) is sufficiently small. (To get K_1 uniform we shrink δ for increasing n; by (3.5) and (3.1) this avoids the constants in the Koebe Principle to grow.) Now F^{n-1} maps each element of $\mathcal{I}_{n-1}$ diffeomorphically onto some element of $\mathcal{I}_0$ and each element of $\mathcal{I}_n$ onto an element of $\mathcal{I}_1$. From this, (3.6) and (3.3) it follows that each element J of $\mathcal{I}_n$ is a definite factor smaller than the element $I \in \mathcal{I}_{n-1}$ containing J. This proves (3.4).

Now of course (3.4) does not suffice because δ (and therefore the size of U) depends on n. Therefore, let us fix n_0 so large that

$$\kappa^{-n_0} \ge 4K_2 \quad \text{where} \quad K_2 = \left[\frac{1+\tau}{\tau}\right]^2$$

and write $G = F^{n_0}$. If J is an element of $\mathcal{I}_{kn_0}$ and $G^i(J) \supset J$ for some $0 \le i \le k$ then

$$|DG^i(x)| \ge 2 \quad \text{for all} \quad x \in J. \tag{3.7}$$

Indeed, we may assume that i is minimal and then $J, \dots, G^{(i-1)}(J)$ are disjoint. If $G^i = f^j$ then this gives that $J, \dots, f^j(J)$ have intersection multiplicity bounded by n_0. (This means that each point is contained in at most n_0 of these intervals.) Therefore, and since f^j maps some interval $T \supset J$ onto a τ-scaled neighbourhood of $f^j(J)$, it follows from the Koebe Principle that

$$|DG^i(x)| \;\ge\; \exp\left(-o(\epsilon) \cdot \sum_{i=0}^{j-1} |f^i(J)|\right) \frac{1}{K_2} \frac{|G^i(J)|}{|J|} \tag{3.8}$$

$$\;\ge\; \exp\left(-o(\epsilon) \cdot n_0\right) \frac{1}{K_2} \frac{|G^i(J)|}{|J|} \ge \frac{1}{2}\frac{1}{K_2}\kappa^{-n_0} \ge 2 \tag{3.9}$$

for each $x \in J$ provided $|f^j(J)| = |G^i(J) \le |U| \le \delta$ is sufficiently small. (This last inequality implies that $\epsilon = \max|f^i(J)|$ is small when δ is small.) Hence, if some interval returns then its size has increased by a uniform factor; as we shall now show this implies the total length of the intervals remains bounded. Indeed, consider again $J \in \mathcal{I}_{kn_0}$. Then

$$\sum_{i=0}^{k-1} |G^i(J)| \le \frac{1}{1-1/2} = 2. \tag{3.10}$$

This is because $G^{i_1}(J) \cap G^{i_2}(J) \neq \emptyset$ with $i_1 < i_2 \le k$ implies that $G^{i_1}(J) \subset G^{i_2}(J)$. Moreover, if $G^{i_1}(J), G^{i_2}(J) \subset G^{i_3}(J)$ and $i_1 < i_2 \le i_3 \le k$, then there exists $J' \supset G^{i_1}$ (which is an interval from a partition of the form $\mathcal{I}_{hn_0}$ with $h \in \{0, 1, \dots, k\}$) such that $G^{i_2 - i_1}(J') = G^{i_3}(J)$. Hence, by (3.7)

$$|G^{i_1}(J)| \le \frac{1}{2}|G^{i_2}(J)|.$$

Using this it follows that the total length of the interval $J, \dots, G^{k-1}(J)$ contained in one interval $G^{i_3}(J)$ is at most $\sum_{i\ge 0} 2^{-i} = 2$ times the length of $G^{i_3}(J)$. This implies (3.10). Now (3.10) gives that

$$\sum_{i=0}^{j-1} |f^i(J)| \le 2n_0$$

where $G^k = f^k$. So if $T \supset J$ is the interval which is mapped by f^j onto a $\tau/2$-scaled neighbourhood of $f^k(J)$, then

$$\sum_{i=0}^{j-1} |f^i(J)| \le S'$$

for some universal constant S'. Here have used the second part of the Koebe Principle. Thus we have proved (3.2). □

3.2 Two step bounds

For simplicity define

$$d_n = c_{S_n}$$

We shall use boldface letters to indicate the distance to the critical point (or value), so

$$\mathbf{d}_n = |d_n - c|, \quad \text{and} \quad \mathbf{d}_n^f = |d_n^f - c^f|.$$

This notation will also be used for the points we defined before, namely t_n^f is the critical point of the monotone branch of $f^{S_n - 1}$ near c^f lying on the other side of c^f than c (and therefore than z_n^f as well). The critical value corresponding to t_n^f is $c_{S_{n-4}} = f^{S_n - 1}(t_n^f)$.

$$z_n = c_{-S_n} \quad \text{and} \quad z_n^f = f(z_n)$$

where z_n could be either to the left or the right of c depending on the context. Moreover, remember that we defined

$$y_n = f^{S_n}(c_{S_{n+2}}) \quad \text{and} \quad y_n^f = f(y_n)$$

in Proposition 2.2. In the next lemmas the constant $\mathcal{O}$ from Proposition 3.1 will be written as $\mathcal{O}_n$, in order to indicate its dependence on S_n. Notice that $\mathcal{O}_n \to 1$ if $n \to \infty$ because of Lemma 3.2.

Lemma 3.3 *(See [KN]) Let* $\lambda_n^f = \mathbf{d}_{n-2}^f/\mathbf{d}_n^f$ *then* $\lambda_n^f > 3.85$ *and* $\ln(\mathbf{d}_{n-4}^f/\mathbf{d}_n^f) > 2.7$ *for sufficiently large* n.

Proof: Applying the cross-ratio inequalities we have

$$\begin{aligned} \frac{\mathbf{d}_n - \mathbf{y}_n}{\mathbf{d}_{n+2}^f}\frac{\mathbf{d}_{n-4}}{|t|} &= \frac{|J|}{|j|}\frac{|T|}{|t|} \geq \mathcal{O}_n \frac{|L|}{|l|}\frac{|R|}{|r|} \\ &\geq \mathcal{O}_n \frac{\mathbf{y}_n}{|\mathbf{z}_n^f - \mathbf{d}_{n+2}^f|}\frac{\mathbf{d}_{n-4} - \mathbf{d}_n}{|r|} \end{aligned}$$

where t, j, l, r are chosen as in the figure below.

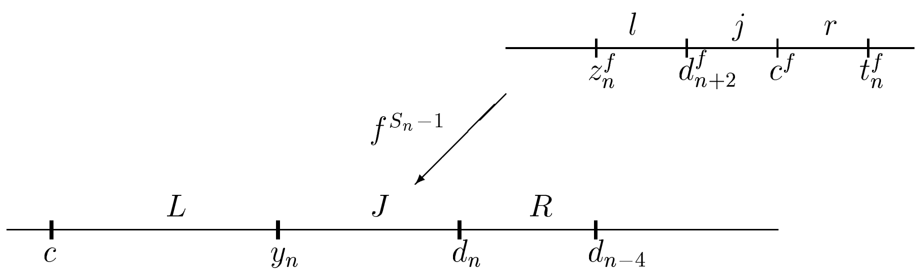

Figure 3.2.

Using the non-flatness of c and the previous inequality (and $|t| > |r|$) we get

$$\begin{aligned} \frac{1}{\ell}\left(1 - \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-4}^f}\right) &\leq \frac{\mathbf{d}_{n-4} - \mathbf{d}_n}{\mathbf{d}_{n-4}} < \mathcal{O}_n \frac{\mathbf{d}_n - \mathbf{y}_n}{\mathbf{y}_n}\frac{|\mathbf{z}_n^f - \mathbf{d}_{n+2}^f|}{\mathbf{d}_{n+2}^f} \cdot 1 \\ &< \mathcal{O}_n \left(\frac{\mathbf{d}_n}{\mathbf{d}_{n+1}} - 1\right) \cdot \left(\frac{\mathbf{d}_{n+1}^f}{\mathbf{d}_{n+2}^f} - 1\right) \\ &< \mathcal{O}_n \frac{1}{\ell}\left(\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} - 1\right) \cdot \left(\frac{\mathbf{d}_{n+1}^f}{\mathbf{d}_{n+2}^f} - 1\right) \\ &\leq \mathcal{O}_n \frac{1}{\ell}\left(\sqrt{\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+2}^f}} - 1\right)^2 . \end{aligned}$$

Here we have used $0 < \ell a^{\ell-1} < (b^\ell - a^\ell)/(b-a) < \ell b^{\ell-1}$ and $(\sqrt{ab}-1)^2 \geq (a-1)(b-1)$. Finally we get

$$1 - (\lambda_n^f \lambda_{n-2}^f)^{-1} \leq \mathcal{O}_n(\sqrt{\lambda_{n+2}^f} - 1)^2$$

which yields the analogous inequality for $\lambda_\infty^f = \liminf \lambda_n^f$, and thus $\lambda_\infty^f > 3.85$ and $\liminf \ln(\mathbf{d}_{n-4}^f/\mathbf{d}_n^f) = 2\ln\lambda_\infty^f > 2.7$. One can obtain better estimates for large ℓ using

$\ell(\lambda-1)/\lambda \le \ln \lambda^f \le \ell(\lambda-1)$. □

Lemma 3.4 *Let $a \in (z_n, \hat{z}_n)$, $a^f = f(a)$, $b = f^{S_n}(a)$ and $b^f = f(b) = f^{S_n}(a^f)$. Then for n large enough*

$$|Df^{S_n}(a^f)| \le \frac{\mathbf{b}^f}{\mathbf{a}^f} \ln(\frac{\mathbf{d}^f_{n-4}}{\mathbf{b}^f}) \ln(\frac{\mathbf{d}^f_n}{\mathbf{b}^f}) \left(\frac{\mathbf{d}^f_{n-4}}{\mathbf{b}^f}\right)^{\frac{1}{\ell}}.$$

Proof: We use the cross-ratio for f^{S_n-1} with l is shrunk to a point $l = \{a^f\}$ and $j = (a^f, c^f)$ and $r = (c^f, t^f_n)$.

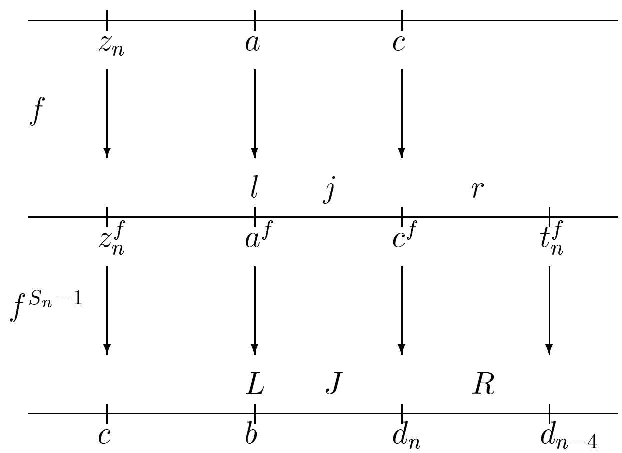

Figure 3.3.

In the cross-ratio inequality we can use $|r| < |l| + |j| + |r|$ and have

$$\begin{aligned}
|Df^{S_n}(a^f)| &= |Df(b)|\,|Df^{S_n-1}(a^f)| \\
&\le \mathcal{O}_n \ell \frac{\mathbf{b}^f}{\mathbf{b}} \frac{\mathbf{d}_n - \mathbf{b}}{\mathbf{a}^f} \frac{\mathbf{d}_{n-4} - \mathbf{b}}{\mathbf{d}_{n-4} - \mathbf{d}_n} \\
&= \mathcal{O}_n \frac{\mathbf{b}^f}{\mathbf{a}^f} \ell \cdot \frac{\mathbf{d}_n - \mathbf{b}}{\mathbf{d}_n} \ell \cdot \frac{\mathbf{d}_{n-4} - \mathbf{b}}{\mathbf{d}_{n-4}} \frac{1}{\ell \cdot \frac{\mathbf{d}_{n-4} - \mathbf{d}_n}{\mathbf{d}_n}} \frac{\mathbf{d}_{n-4}}{\mathbf{b}} \\
&\le \mathcal{O}_n \frac{\mathbf{b}^f}{\mathbf{a}^f} \frac{\ln \frac{\mathbf{d}^f_n}{\mathbf{b}^f} \ln \frac{\mathbf{d}^f_{n-4}}{\mathbf{b}^f}}{\ln \frac{\mathbf{d}^f_{n-4}}{\mathbf{d}^f_n}} \left(\frac{\mathbf{d}^f_{n-4}}{\mathbf{b}^f}\right)^{\frac{1}{\ell}},
\end{aligned}$$

which implies the statement as $\mathcal{O}_n \to 1$ and $\ln \frac{\mathbf{d}^f_{n-4}}{\mathbf{d}^f_n} > 2$. Here we have used Lemma 3.3 and the obvious inequalities $\frac{x-1}{x} \le \ln(x) \le x - 1$. □

Remark 3.2 *We shall use this lemma several times. In order to simplify the notation let us introduce $\varrho^f_n := \max\{\mathbf{d}^f_k/\mathbf{d}^f_{k+1}\ ;\ n - N_0 \le k < n\}$, where $N_0 \le 10$ may change* *from one lemma to another. Suppose that b in the previous lemma satisfies $\mathbf{d}_{n+i} \leq \mathbf{b}$ for some $i \leq 6$, then $\mathbf{d}^f_n/\mathbf{b}^f \leq (\varrho^f_{n+i})^i$, $\mathbf{d}^f_{n-4}/\mathbf{b}^f \leq (\varrho^f_{n+i})^{i+4}$ and*

$$|Df^{S_n}(a^f)| \leq \frac{\mathbf{b}^f}{\mathbf{a}^f} \cdot (4+i) \cdot \ln(\varrho^f_{n+i}) \cdot i \cdot \ln(\varrho^f_{n+i})(\varrho^f_{n+i})^{\frac{4+i}{\ell}} \ ,$$

where in fact ϱ^f could have been taken as $\max\{\mathbf{d}^f_k/\mathbf{d}^f_{k+1} \ ; \ n-i-4 \leq k < n\}$.

Lemma 3.5 *We have the following estimate*

$$|Df^{S_m}(d^f_{m+1})| \leq 160 \cdot \frac{\mathbf{d}^f_{m+2}}{\mathbf{d}^f_{m+1}} \ln^4(\varrho^f_{m+2}) \cdot (\varrho^f_{m+2})^{\frac{13}{\ell}} \ .$$

Proof: We decompose $Df^{S_m}(d^f_{m+1}) = Df^{S_{m-2}}(y^f_{m-1})Df^{S_{m-1}}(d^f_{m+1})$ and use previous lemma and remark to both factors. First we put in the lemma $n = m-2$, $a = y_{m-1}$, $b = c_{S_{m+2}}$ and $i = 4$ in the remark.

$$|Df^{S_{m-2}}(y^f_{m-1})| \leq \frac{\mathbf{d}^f_{m+2}}{\mathbf{y}^f_{m-1}} \cdot 32 \cdot \ln^2(\varrho^f_{m+2}) \cdot (\varrho^f_{m+2})^{\frac{8}{\ell}}$$

Then we put $n = m-1$, $a = d_{m+1}$, $b = y_{m-1}$ and as $d_m \in (y_{m-1}, \hat{y}_{m-1})$ we have $i = 1$.

$$|Df^{S_{m-1}}(d_{m+1})| \leq \frac{\mathbf{y}^f_{m-1}}{\mathbf{d}^f_{m+1}} \cdot 5 \cdot \ln^2(\varrho^f_m) \cdot (\varrho^f_m)^{\frac{5}{\ell}} \ .$$

The result follows taking ϱ^f depending on 9 consecutive k: $m-7 \leq k < m+2$. □

The next lemma prepares the last tool in this subsection. It describes the estimation (both ways) of $Df^{S_m}(c^f)$.

Lemma 3.6 *We have for large m*

$$\frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m+1}}(\varrho^f_m)^{-\frac{4}{\ell}} \leq |Df^{S_m}(c^f)| \leq 2 \cdot \frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m+2}} \cdot \ln(\varrho^f_{m+1}) \cdot (\varrho^f_{m+1})^{\frac{1}{\ell}}$$

Proof: We use the same trick as in Lemma 3.4.

$$|Df^{S_m}(c^f)| = \ell \, \frac{\mathbf{d}^f_m}{\mathbf{d}_m} \, |Df^{S_m-1}(c^f)| \ .$$

For one side we use the cross-ratio for f^{S_m-1} on $l = (z^f_m, c^f)$, $r = (c^f, t^f_m)$.

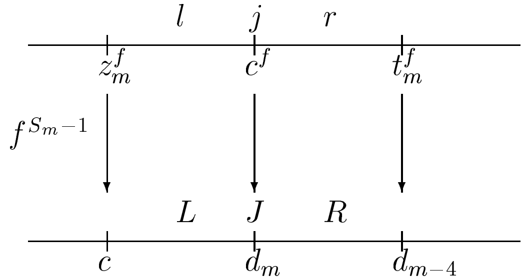

Figure 3.4.

Then we obtain

$$\begin{aligned} Df^{S_m}(c^f) &\geq \mathcal{O}_m \frac{\mathbf{d}^f_m}{\mathbf{d}_m} \ell \frac{\mathbf{d}_{m-4} - \mathbf{d}_m}{\mathbf{d}_{m-4}} \frac{\mathbf{d}_m}{\mathbf{z}^f_m} \\ &\geq \mathcal{O}_m \ln \frac{\mathbf{d}^f_{m-4}}{\mathbf{d}^f_m} \frac{\mathbf{d}_m}{\mathbf{d}_{m-4}} \frac{\mathbf{d}^f_m}{\mathbf{z}^f_m} \\ &> \mathcal{O}_m \frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m+1}} \left(\frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m-4}} \right)^{\frac{1}{\ell}} . \end{aligned}$$

For the other side we take $l = (z^f_m, d^f_{m+2})$, $j = (d^f_{m+2}, c^f)$ and $r = (c^f)$.

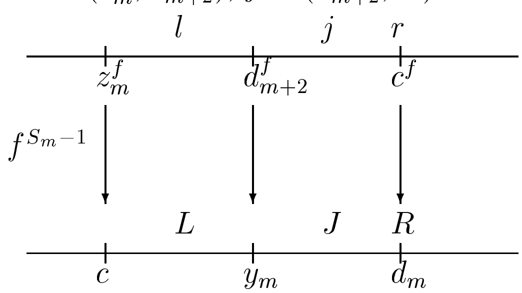

Figure 3.5.

We obtain, using $\mathbf{d}_{m+1} \leq \mathbf{y}_m$,

$$\begin{aligned} |Df^{S_m}(c^f)| &\leq \mathcal{O}_m \frac{\mathbf{d}^f_m}{\mathbf{d}_m} \ell \frac{\mathbf{d}_m - \mathbf{y}_m}{\mathbf{d}^f_{m+2}} \frac{\mathbf{d}_m}{\mathbf{z}^f_m} \frac{\mathbf{z}^f_m - \mathbf{d}^f_{m+2}}{\mathbf{y}_m} \\ &\leq \mathcal{O}_m \frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m+2}} \ell \frac{\mathbf{d}_m - \mathbf{d}_{m+1}}{\mathbf{d}_m} \frac{\mathbf{d}_m}{\mathbf{d}_{m+1}} \\ &\leq \mathcal{O}_m \frac{\mathbf{d}^f_m}{\mathbf{d}^f_{m+2}} \ln(\varrho^f_{m+1})(\varrho^f_{m+1})^{\frac{1}{\ell}} . \end{aligned}$$

And again ϱ^f could have been taken as $\max\{\mathbf{d}^f_k/\mathbf{d}^f_{k+1} \; ; \; m-4 \leq k \leq m\}$. □

3.3 The 1-step bounds

We can get a better upper estimate if we combine the previous calculations.

Proposition 3.4 *For n and ℓ large enough the derivatives $Df^{S_n}(c^f)$ and the proportions $\mathbf{d}^f_n/\mathbf{d}^f_{n+1}$ are bounded from above and separated from below from 1 by constants independent of n and ℓ.*

Proof: Consider the following decomposition:

$$Df^{S_n}(c^f) = Df^{S_{n-2}}(d^f_{n-1}) \cdot Df^{S_{n-1}}(c^f) = Df^{S_{n-2}}(d^f_{n-1}) Df^{S_{n-3}}(d^f_{n-2}) \cdot Df^{S_{n-2}}(c^f).$$

By Lemma 3.5 used twice with $m = n-2$ and $m = n-3$ in the two first factors and by Lemma 3.6 used for $m = n-2$ in the third one we have

$$|Df^{S_n}(c^f)| \leq 160^2 \cdot 2 \cdot \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-1}^f}\frac{\mathbf{d}_{n-1}^f}{\mathbf{d}_{n-2}^f}\frac{\mathbf{d}_{n-2}^f}{\mathbf{d}_n^f} \cdot \ln^{(4+4+1)}(\varrho_n^f)(\varrho_n^f)^{\frac{13+13+1}{\ell}} = 51200 \cdot \ln^9(\varrho_n^f) \cdot (\varrho_n^f)^{\frac{27}{\ell}}.$$

This and the other part of Lemma 3.6 gives

$$\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} \leq 51200 \cdot \ln^9(\varrho_n^f) \cdot (\varrho_n^f)^{\frac{31}{\ell}} \ ,$$

where again ϱ_n^f depends on at most 10 consecutive quotients $\mathbf{d}_k^f/\mathbf{d}_{k+1}^f$, with $n - 10 \leq k < n$. This gives an upper bound of the growth of $\varrho_\infty^f = \limsup \varrho_n^f = \limsup \mathbf{d}_n^f/\mathbf{d}_{n+1}^f$ by

$$\varrho_\infty^f \leq 51200 \cdot \ln^9(\varrho_\infty^f) \cdot (\varrho_\infty^f)^{\frac{31}{\ell}} \ ,$$

(i.e. $\varrho_\infty^f < 10^{21}$ for large ℓ) and proves the upper bound part of the proposition. For the lower part one can use the estimates from Lemma 3.3, as from its proof it follows that

$$\frac{d_n^f}{d_{n+1}^f} \geq 1 + \frac{1 - e^{-2.7}}{\varrho_\infty^f - 1}.$$

□

Proposition 3.5 *There exists $K > 0$ independent of ℓ and n such that for ℓ and n large enough*

$$\mathbf{d}_n^f/\mathbf{u}_n^f > 1 + K.$$

Proof: We shall apply Proposition 3.1 together with Lemma 3.2 and the remark after the proof of Lemma 3.1. Consider $f^{S_{n-1}}$ and its interval of monotonicity $t = (z_{n-1}^f, t_{n-1}^f)$ around c^f. Let $l = (z_{n-1}^f, u_n^f)$, $j = (u_n^f, c^f)$ and $r = (c^f, t_{n-1}^f)$. Denote by T, L, J, R the images of t, l, j, r under $f^{S_{n-1}}$. Then

$$\begin{aligned} \frac{\mathbf{d}_n^f - \mathbf{u}_n^f}{\mathbf{u}_n^f} &\geq \frac{\mathbf{z}_{n-1}^f - \mathbf{u}_n^f}{\mathbf{u}_n^f} = \frac{|l|}{|j|} \geq \mathcal{O}_n \frac{|L|}{|J|}\frac{|R|}{|T|} = \mathcal{O}_n \frac{\mathbf{u}_{n-1}^f}{\mathbf{d}_{n-1}^f - \mathbf{u}_{n-1}^f}\frac{\mathbf{d}_{n-5}^f - \mathbf{d}_{n-1}^f}{\mathbf{d}_{n-5}^f} \\ &\geq \mathcal{O}_n \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-1}^f}\left(1 - \frac{\mathbf{d}_{n-1}^f}{\mathbf{d}_{n-5}^f}\right) \geq \mathcal{O}_n \frac{1 - \frac{1}{2.7}}{\varrho_n^f} \ , \end{aligned}$$

and this is bounded away from 0 uniformly in n. □

The main result of this section is the finally the following theorem.

Theorem 3.1

$$\frac{\mathbf{d}_n^f}{\mathbf{u}_n^f}\ ,\ \frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} \quad \textit{and} \quad \frac{\mathbf{u}_n^f}{\mathbf{u}_{n+1}^f}$$

are bounded and bounded away from one for all ℓ and n large enough. In particular, there are constants C_1, C_2 for which

$$\frac{C_1}{\ell} \leq \frac{|d_n - u_n|}{|u_n - c|}\ ,\ \frac{|\mathbf{d}_n - \mathbf{d}_{n+1}|}{|d_n - c|}\ ,\ \frac{|\mathbf{u}_n - \mathbf{u}_{n+1}|}{|u_n - c|} \leq \frac{C_2}{\ell}.$$

Proof: Follows from the previous two results and the fact that f has a critical point of order ℓ at c. □

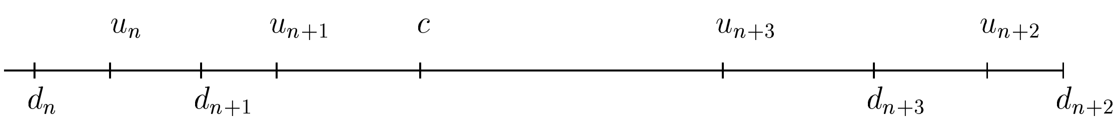

Figure 3.6: The points u_n and d_n. are on the same side of c. The points d_{n+2} and d_n are on opposite sides of c; z_{n-1} is between d_n and u_n.

4 The random walk argument

In this section we shall state and prove an abstract result about the evolution of typical points under a (nearly) Markov map with a kind of random walk structure. So let $(X, \mathcal{F}, m)$ be some space with probability measure m and σ-algebra $\mathcal{F}$. Let $\mathcal{A} = \{A_k : k = 0, 1, 2, \ldots\}$ denote a partition of X into $\mathcal{F}$-measurable sets, and let $F: X \to X$ be a $\mathcal{F}$-measurable transformation. We denote $\mathcal{A}_n = \bigvee_{k=0}^{n-1} F^{-k}\mathcal{A}$ and let $\mathcal{H}$ be the family of all measures of the form $F_*^n(m_{|A})$ with $A \in \mathcal{A}_{n+1}$ and $n \geq 0$. Now take $r, k_0 \in \mathbb{N}$ and define for $i \geq 1$,

$$a_i = m(A_{r+i-k_0-1}) \text{ and } \nu_i = \mu((F_{|A_r})^{-1}(A_{r+i-k_0-1}))/\mu(A_r)\ , \tag{4.1}$$

where μ is some measure from the class $\mathcal{H}$ defined above. Note that

$$\frac{a_{k+1}}{a_{j+1}} \frac{\nu_{j+1}}{\nu_{k+1}}$$

is equal to one if F preserves the measure μ.

In the remainder of this section the sequences of positive real numbers $(a_i)_{i\geq 0}$ and $(\nu_i)_{i\geq 1}$ will be assumed to have a particular exponential decay. Here a_i, ν_i are as above and a_0 will be a suitable constant corresponding to a constant which comes from 'Koebe space'. So we say that two such sequences satisfy the *scaling condition* with constants $\varrho_0, \varrho_1, \mathcal{O}_1, \mathcal{O}_2 > 0$ and $d \geq 0$, if

$$\varrho_0^{k-j} \leq \frac{\sum_{i=j}^{\infty} a_i}{\sum_{i=k}^{\infty} a_i} \leq \varrho_1^{k-j} \quad \text{for all } 0 \leq j \leq k \;, \tag{4.2}$$

$$\mathcal{O}_1 \cdot \frac{a_0}{\sum_{i=0}^{k} a_i} \cdot \frac{\nu_{\max}}{\sum_{i=0}^{k+1} a_i} \leq \frac{\nu_{k+1}}{a_{k+1}} \quad \text{for all } k \geq d, \tag{4.3}$$

$$\frac{\nu_{k+1}}{a_{k+1}} \leq \mathcal{O}_2 \cdot \frac{\nu_{\max}}{a_1} \quad \text{for all } k \geq 0, \tag{4.4}$$

where $\nu_{\max} = \max\{\nu_1, \ldots, \nu_{d+1}\}$. As we shall see in the lemma below, (4.2) means that the numbers a_i decay at a slow rate if $\varrho_i > 1$ are close to one. Moreover, this lemma implies that if $(\varrho_1 - 1)/(\varrho_0 - 1)$ is not too large then (4.3) is equivalent to the more symmetric expression

$$\mathcal{O}_1 \cdot \frac{\sum_{i=0}^{d} a_i}{\sum_{i=0}^{k} a_i} \cdot \frac{\sum_{i=0}^{d+1} a_i}{\sum_{i=0}^{k+1} a_i} \cdot \frac{a_{k+1}}{a_{\max}} \leq \frac{\nu_{k+1}}{\nu_{\max}}.$$

where $a_{\max} = \max\{a_1, \ldots, a_{d+1}\}$. The previous inequalities (4.2) and (4.3) combined show that the ratio of the 'mass' going from state A_r to state A_{r+i-k_0-1} compared to the mass going to one of the states $A_{r-k_0}, \ldots, A_{r-k_0+d}$ goes only down slowly with i. So - roughly speaking - a reasonably large set of points move to a state with much larger index. This suggests that m-typical points will move to states with larger and larger indices. This intuitive idea is formalized in the following theorem.

Theorem 4.1 *Suppose there is $k_0 \in \mathbb{N}$ such that $F(A_r) \subseteq \cup_{j=0}^{\infty} A_{r-k_0+j}$ for all $r \geq 2$. Then there exists for each $C > 1$, $\mathcal{O}_1, \mathcal{O}_2 > 0$, and $d \in \mathbb{N}$ a constant $\varrho \in (1,2)$ with the property that if the assumption stated below is satisfied, then there exists a set $D \in \mathcal{F}$ with $m(D) > 0$ such that for each $x \in D$ and each A_j the set*

$$\{k : F^k(x) \in A_j\}$$

has finite cardinality. The assumption is:

> *For any $1 < \varrho_0 < \varrho_1 < \varrho$ with $\frac{(\varrho_1 - 1)}{(\varrho_0 - 1)} \leq C$ there is $r_0 > 0$ such that for any $r \geq r_0$ and any $\mu \in \mathcal{H}$ with $\mu(A_r) > 0$ there exists $a_0 > 0$ such that the sequences $(a_i)_{i\geq 0}$ and $(\nu_i)_{i\geq 1}$ satisfy the* scaling condition *with constants $\varrho_0, \varrho_1, \mathcal{O}_1, \mathcal{O}_2$ and d, where a_i and ν_i $(i \geq 1)$ are defined as in (4.1).*

Before we turn to the proof of the theorem in the next three subsections, we state some simple properties of sequences satisfying the *scaling condition.* These properties give a better intuition for the meaning of this condition.

Lemma 4.1 *Let $(a_i)_{i\geq 0}$, $(\nu_i)_{i\geq 1}$ satisfy condition (4.2) and define $K_1 = (\varrho_1-1)\varrho_0/(\varrho_0-1)\varrho_1$. Then for any $0 \leq j \leq k$ we have*

$$1-\varrho_0^{-1} \quad \leq \quad \frac{a_j}{\sum_{i=j}^{\infty} a_i} \leq 1-\varrho_1^{-1} \tag{4.5}$$

$$\frac{1}{K_1}\varrho_0^{k-j} \quad \leq \quad \frac{a_j}{a_k} \leq K_1\varrho_1^{k-j} \ . \tag{4.6}$$

If also condition (4.4) is satisfied, then for $k \geq 1$,

$$\nu_k \leq \mathcal{O}_2 \nu_{\max} K_1 \varrho_0^{-(k-1)} \ . \tag{4.7}$$

Proof: The proof follows immediatly by calculation. For example (4.5):

$$\frac{a_j}{\sum_{i=j}^{\infty} a_i} = \frac{\sum_{i=j}^{\infty} a_i - \sum_{i=j+1}^{\infty} a_i}{\sum_{i=j}^{\infty} a_i} \leq 1-\varrho_1^{-1} \ .$$

□

4.1 The martingale argument

As before let $(X, \mathcal{F}, m)$ be some space with probability measure m and σ-algebra $\mathcal{F}$ and $\mathcal{A} = \{A_k : k = 0, 1, 2, \ldots\}$ a partition of X into $\mathcal{F}$-measurable sets. $F: X \to X$ is a $\mathcal{F}$-measurable transformation, $\mathcal{A}_n = \bigvee_{k=0}^{n-1} F^{-k}\mathcal{A}$, and $\mathcal{H} = \{F_*^n(m_{|A}) : A \in \mathcal{A}_n, n \geq 0\}$.

Observe that $\mathcal{A}$ is a Markov partition for F if and only if F^kA is an element of $\mathcal{A}$ for each $A \in \mathcal{A}_{k+1}$ and each $k \geq 0$. In order to make the following proposition most widely applicable we shall not assume that F is strictly Markov but formulate instead some restrictions on $\mathcal{H}$. Furthermore, even if F is topologically Markov, the nonlinearity of its branches still prevents F to be also measure theoretically Markov. Therefore we do not use in our proof a Markov-like model but instead a more flexible martingale construction. As a general reference to the theory of martingales we give [Sto].

Define $\varphi : X \to \{0, 1, 2, \ldots\}$ by

$$\varphi(x) = n \text{ if } x \in A_n$$

and

$$\Delta\varphi := \varphi \circ F - \varphi \ .$$

Proposition 4.1 *Assume there are $r_0 \in \mathbb{N}$ and $M > 0$ such that for any $A \in \mathcal{A}_{k+1}$, $k \geq 0$, with $\varphi_{|F^kA} \geq r_0$ holds:*

$$\int_A (\Delta\varphi - 1) \circ F^k \, dm \quad \geq \quad 0 \quad \textit{and} \tag{4.8}$$

$$\int_A (\Delta\varphi)^2 \circ F^k \, dm \quad \leq \quad M \cdot m(A) \quad \textit{for all } n \geq 0 \ . \tag{4.9}$$

Then

$$\liminf_{n\to\infty} \frac{\varphi \circ F^n}{n} \geq 1 \quad m\text{-}a.s.,$$

and there exists a set $D \in \mathcal{F}$ *with* $m(D) > 0$ *such that for every* $x \in D$ *the trajectory* $x, Fx, F^2x, \ldots$ *visits each set* $A_k \in \mathcal{A}$ *only finitely often.*

Proof: Fix $s > r_0$ and denote by μ the normalized restriction of m to A_s. Let $\mathcal{F}_n$ be the σ-algebra generated by the partition $\mathcal{A}_{n+1}$. Then $\varphi \circ F^n$ is $\mathcal{F}_n$-measurable, i.e.,

$$E_\mu[\varphi \circ F^n | \mathcal{F}_n] = \varphi \circ F^n \ .$$

Define a stopping time $\tau : X \to \mathbb{N} \cup \{\infty\}$ by

$$\tau(x) = \begin{cases} \infty & \text{if } \varphi(F^n x) > r_0 \text{ for all } n \geq 0 \\ \min\{n \geq 0 \colon \varphi(F^n x) \leq r_0\} & \text{otherwise,} \end{cases}$$

and the random variables $(Z_n)_{n\geq 0}$ by

$$Z_n(x) = \begin{cases} \varphi(F^n x) & \text{if } \tau(x) > n \\ \varphi(F^{\tau(x)} x) & \text{if } \tau \leq n. \end{cases}$$

Then also the Z_n are $\mathcal{F}_n$-measurable. So for any $x \in X$ and $n \geq 0$ with $\tau(x) > n$,

$$\begin{aligned} & E_\mu[Z_{n+1}|\mathcal{F}_n](x) - Z_n(x) - 1 \\ = \ & E_\mu[(\Delta\varphi - 1) \circ F^n | \mathcal{F}_n](x) \\ = \ & \sum_{A \in \mathcal{A}_n} \chi_A(x) \cdot \frac{1}{m(A)} \int_A (\Delta\varphi - 1) \circ F^n \, d\mu \\ \geq \ & 0 \ , \end{aligned} \tag{4.10}$$

where we used (4.8) for the inequality. If $\tau(x) \leq n$, then $E_\mu[Z_{n+1}|\mathcal{F}_n](x) - Z_n(x) = 0$. Note that in both cases $E_\mu[Z_{n+1}|\mathcal{F}_n](x) \geq Z_n(x)$, i.e. $(Z_n, \mathcal{F}_n)_{n\geq 0}$ is a submartingale with respect to μ. Now define

$$W_n = Z_0 + \sum_{k=1}^{n} (E_\mu[Z_k|\mathcal{F}_{k-1}] - Z_{k-1}) \quad \text{ and } \quad M_n = Z_n - W_n$$

(this is, by the way, the Doob-decomposition of $(Z_n, \mathcal{F}_n)_{n\geq 0}$). Then $W_0 = Z_0 = s$ and $M_0 = 0$ μ-a.s., and $(M_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale:

$$M_{n+1} - M_n = Z_{n+1} - Z_n - W_{n+1} + W_n = Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n]$$

and therefore

$$E_\mu[M_{n+1}|\mathcal{F}_n] = E_\mu[M_n|\mathcal{F}_n] + E_\mu[Z_{n+1}|\mathcal{F}_n] - E_\mu[E_\mu[Z_{n+1}|\mathcal{F}_n]|\mathcal{F}_n] = M_n \ .$$

$(W_n, \mathcal{F}_{n-1})_{n\geq 1}$ is a predictable stochastic sequence with

$$W_{n+1} - W_n = E_\mu[Z_{n+1}|\mathcal{F}_n] - Z_n \geq \chi_{\{\tau > n\}}$$

because of (4.10). It follows that

$$W_n \geq n + s \quad \text{on } \{x \; ; \; \tau(x) \geq n\} \; . \tag{4.11}$$

Next note that on $\{\tau > n\}$ holds

$$\begin{aligned} & E_\mu[(M_{n+1} - M_n)^2 | \mathcal{F}_n] = E_\mu[(Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n])^2 | \mathcal{F}_n] \\ \leq \; & E_\mu[(Z_{n+1} - Z_n)^2 | \mathcal{F}_n] = E_\mu[(\Delta\varphi)^2 \circ F^n | \mathcal{F}_n] \\ \leq \; & M \; , \end{aligned}$$

where we used the fact that $E_\mu[(Z_{n+1} - Y)^2|\mathcal{F}_n]$ is minimized by $Y = E_\mu[Z_{n+1}|\mathcal{F}_n]$ for the first inequality and assumption (4.9) for the second one. On $\{\tau \leq n\}$ we have

$$M_{n+1} - M_n = Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n] = Z_{n+1} - Z_n = 0 \; .$$

Both estimates together yield $E_\mu[(M_{n+1} - M_n)^2] \leq M$, and we can apply Chow's version of the Hajek-Rényi inequality (see [Sto, Theorem 3.3.7]):

$$\mu \left\{ \max_{1 \leq i \leq n} \frac{|M_i|}{s - r_0 + i} \geq 1 \right\} \leq \sum_{i=1}^n \frac{E_\mu[(M_i - M_{i-1})^2]}{(s - r_0 + i)^2} \leq M \cdot \sum_{j > s - r_0} \frac{1}{j^2} < \frac{1}{2} \; , \tag{4.12}$$

if $s - r_0$ is large enough. Hence

$$\begin{aligned} \mu\{\tau < \infty\} &= \mu\left(\bigcup_{n \geq 1} \{\tau = n\} \right) \leq \mu\left(\bigcup_{n \geq 1} \{Z_n \leq r_0 \text{ and } W_n \geq n + s\} \right) \\ &\leq \mu\left(\bigcup_{n \geq 1} \{M_n \leq r_0 - s - n\} \right) \leq \mu\left(\bigcup_{n \geq 1} \{|M_n| \geq s - r_0 + n\} \right) \\ &= \sup_{n \geq 1} \mu \left\{ \max_{1 \leq i \leq n} \frac{|M_i|}{s - r_0 + i} \geq 1 \right\} \leq \frac{1}{2} \end{aligned}$$

for such s, i.e. $\mu\{\tau = \infty\} \geq \frac{1}{2}$.

Now a convergence theorem of Chow (see [Sto, Theorem 3.3.1]) asserts that

$$\lim_{n \to \infty} M_n/(s - r_0 + n) = 0 \quad \mu\text{-a.s.}$$

in view of the finiteness of the sum in (4.12). Hence, on $\{\tau = \infty\}$,

$$\liminf_{n \to \infty} \frac{\varphi \circ F^n}{n} = \liminf_{n \to \infty} \frac{Z_n}{n} = \liminf_{n \to \infty} \frac{W_n}{n} + \lim_{n \to \infty} \frac{M_n}{n} \geq 1 \quad \mu\text{-a.s.}$$

in view of (4.11). In particular, for each $x \in \{\tau = \infty\}$ the trajectory $x, Fx, F^2x, \ldots$ visits each element $A_k \in \mathcal{A}$ only finitely often. □

4.2 Some calculations

Let $(a_i)_{i\geq 0}$ and $(\nu_i)_{i\geq 1}$ be two sequences of positive real numbers which satisfy the *scaling condition* (4.2,4.3,4.4) with constants $\varrho_0, \varrho_1, \mathcal{O}_1, \mathcal{O}_2, d$. We assume additionally that $\sum_{i=0}^{\infty} a_i < \infty$ and $\sum_{i=1}^{\infty} \nu_i = 1$,

Proposition 4.2 *For any $E, \mathcal{O}_1, \mathcal{O}_2, C, d$ there is a $\varrho \in (1,2)$ such that if the numbers ϱ_0, ϱ_1 from above satisfy $1 < \varrho_0 < \varrho_1 < \varrho$ and $\frac{(\varrho_1-1)}{(\varrho_0-1)} \leq C$ then*

$$\sum_{j=1}^{\infty} j\nu_j > E \ , \tag{4.13}$$

and there is some constant $M > 0$ depending only on $\mathcal{O}_2, C, \varrho_0$ such that

$$\sum_{j=1}^{\infty} j^2\nu_j < M \ . \tag{4.14}$$

For the proof we need several lemmas.

Lemma 4.2 *Let $(q(j))$ be a positive increasing sequence. Then, if $n-1 \geq d+1$,*

$$\sum_{j=d+1}^{n-1} j \cdot \frac{q(j+1)-q(j)}{q(j)q(j+1)} \geq \sum_{j=d+1}^{n-1} \frac{q(n)-q(j)}{q(n)q(j)} \ .$$

If $q(\infty) := \lim_{n\to\infty} q(n)$ exists and if $\lim_{n\to\infty} n \cdot (q(\infty)-q(n)) = 0$, then the above inequality holds also for $n = \infty$.

Proof:

$$\begin{aligned}
\sum_{j=d+1}^{n-1} j \cdot \frac{q(j+1)-q(j)}{q(j)q(j+1)} &= -\sum_{j=d+1}^{n-1} j \cdot \left(\frac{1}{q(j+1)} - \frac{1}{q(j)}\right) \\
&= -\sum_{j=d+1}^{n-1} \frac{j}{q(j+1)} + \sum_{j=1}^{n-1} \frac{j}{q(j)} \\
&= \sum_{j=d+1}^{n-1} \frac{1}{q(j)}(j-(j-1)) + \frac{d}{q(d+1)} - \frac{n-1}{q(n)} \\
&= \sum_{j=d+1}^{n-1} \left(\frac{1}{q(j)} - \frac{1}{q(n)}\right) + d \cdot \left(\frac{1}{q(d+1)} - \frac{1}{q(n)}\right) \\
&\geq \sum_{j=d+1}^{n-1} \frac{q(n)-q(j)}{q(n)q(j)} \ .
\end{aligned}$$

As, under the additional assumption, $\lim_{n\to\infty} q(n) = q(\infty)$ and

$$\sum_{j=1}^{n-1} \frac{q(\infty)-q(n)}{q(n)q(j)} \leq \frac{n}{q(1)^2}(q(\infty)-q(n)) \to 0 \quad \text{as } n \to \infty \ ,$$

also the inequality for $n = \infty$ follows. □

Lemma 4.3 *Let* $1 < \varrho < 2^{1/d}$. *Then*

$$\sum_{k=d+1}^{\infty} \frac{1}{\varrho^k - 1} > \frac{1}{\ln \varrho} \ln \left(\frac{1}{2(d+1)(\varrho - 1)} \right) .$$

Proof: Fix $M > 0$ such that $\varrho^{M+1} - 1 > \varrho^M$. Then

$$\begin{aligned} \sum_{k=d+1}^{M} \frac{1}{\varrho^k - 1} &> \int_{d+1}^{M+1} \frac{1}{\exp(x \ln \varrho) - 1} dx \\ &= -(M-d) + \frac{1}{\ln \varrho} \left(\ln(\varrho^{M+1} - 1) - \ln(\varrho^{d+1} - 1) \right) \\ &\geq d + \frac{1}{\ln \varrho} \ln \left(\frac{1}{\varrho - 1} \cdot \frac{\varrho - 1}{\varrho^{d+1} - 1} \right) \\ &\geq \frac{1}{\ln \varrho} \ln \left(\frac{1}{2(d+1)(\varrho - 1)} \right) . \end{aligned}$$

Here we have used $1 < \varrho < 2^{1/d}$ in the last inequality. □

Lemma 4.4 *Let* (a_i), (ν_i) *and all constants be as in Proposition 4.2. Then*

$$\sum_{k=1}^{\infty} k\, \nu_k \geq \frac{\nu_{\max}}{2} \cdot \mathcal{O}_1 \cdot \frac{1}{K_1} \ln \left(\frac{1}{2(d+1)(\varrho_1 - 1)} \right)$$

provided $1 < \varrho_1 < 2^{1/d}$, *where* K_1 *was defined in Lemma 4.1.*

Proof: Let $q(j) = \sum_{i=0}^{j-1} a_i$. As the a_i decrease exponentially by (4.6), $q(\infty) = \lim_{j \to \infty} q(j)$ exists, and we can apply Lemma 4.2:

$$\begin{aligned} & \sum_{k=d+1}^{\infty} k \cdot \frac{q(k+1) - q(k)}{q(k) q(k+1)} \geq \sum_{k=d+1}^{\infty} \frac{q(\infty) - q(k)}{q(\infty) q(k)} \\ = & \frac{1}{q(\infty)} \sum_{k=d+1}^{\infty} \frac{\sum_{i=k}^{\infty} a_i}{\sum_{i=0}^{\infty} a_i - \sum_{i=k}^{\infty} a_i} = \frac{1}{q(\infty)} \sum_{k=d+1}^{\infty} \frac{1}{\sum_{i=0}^{\infty} a_i / \sum_{i=k}^{\infty} a_i - 1} \\ \geq & \frac{1}{q(\infty)} \sum_{k=d+1}^{\infty} \frac{1}{\varrho_1^k - 1} \geq \frac{1}{\sum_{i=0}^{\infty} a_i} \frac{1}{\ln \varrho_1} \ln \left(\frac{1}{2(d+1)(\varrho_1 - 1)} \right) . \end{aligned}$$

For the last two inequalities we have used (4.2) and Lemma 4.3. Observe next that by (4.3) we have

$$\frac{\nu_k}{\nu_{\max}} \geq \mathcal{O}_1 \cdot \frac{a_0}{\sum_{i=0}^{k-1} a_i} \cdot \frac{a_k}{\sum_{i=0}^{k} a_i} = \mathcal{O}_1 \cdot a_0 \cdot \frac{q(k+1) - q(k)}{q(k) q(k+1)} .$$

Combining this with the previous estimate we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} k\,\nu_k &= \nu_{\max} \sum_{k=d+1}^{\infty} k\,\frac{\nu_k}{\nu_{\max}} \\
&\geq \nu_{\max} \cdot \mathcal{O}_1 \cdot a_0 \cdot \sum_{k=d+1}^{\infty} k \cdot \frac{q(k+1)-q(k)}{q(k)q(k+1)} \\
&\geq \nu_{\max} \cdot \mathcal{O}_1 \cdot \frac{a_0}{\sum_{i=0}^{\infty} a_i} \cdot \frac{1}{\ln \varrho_1} \cdot \ln\left(\frac{1}{2(d+1)(\varrho_1 - 1)}\right) \\
&\geq \nu_{\max} \cdot \mathcal{O}_1 \cdot \frac{\varrho_0 - 1}{\varrho_0 \ln \varrho_1} \cdot \ln\left(\frac{1}{2(d+1)(\varrho_1 - 1)}\right) \\
&\geq \frac{\nu_{\max}}{2} \cdot \mathcal{O}_1 \cdot \frac{1}{K_1} \frac{\varrho_1 - 1}{\ln \varrho_1} \ln\left(\frac{1}{2(d+1)(\varrho_1 - 1)}\right) .
\end{aligned}$$

For the last two inequalities we have used (4.5), $\varrho_1 < \varrho < 2$ and the definition of K_1 in Lemma 4.1. As $\ln(\varrho_1) \leq \varrho_1 - 1$, this proves the lemma. □

Lemma 4.5 *Let (a_i), (ν_i) and all constants be as in Proposition 4.2, and let K_1 be defined as in Lemma 4.1. Then for any $r > 0$ we have*

$$\sum_{k=1}^{\infty} k\,\nu_k > r(1 - \mathcal{O}_2 \nu_{\max} K_1\, r) .$$

Proof: The idea is to use the very rough estimation

$$\sum_{k=1}^{\infty} k\,\nu_k > \sum_{k=r}^{\infty} k\,\nu_k > r(1 - \sum_{k=1}^{r-1} \nu_k) .$$

Using (4.7),

$$\sum_{k=1}^{r-1} \nu_k \leq \mathcal{O}_2 \nu_{\max} K_1 \sum_{k=1}^{r-1} \varrho_0^{-(k-1)} \leq \mathcal{O}_2 \nu_{\max} K_1\, r$$

the lemma follows immediately. □

Proof of Proposition 4.2: Recall that $K_1 = (\varrho_1 - 1)\varrho_0/(\varrho_0 - 1)\varrho_1$. We have that $K_1 \leq C$. Fix $r = 2E$ and choose $\varrho \in (1,2)$ such that

$$\ln \frac{1}{2(d+1)(\varrho - 1)} = 64 \frac{\mathcal{O}_2}{\mathcal{O}_1} E^2 C .$$

Then $\ln \frac{1}{2(d+1)(\varrho_1 - 1)} > 64 \frac{\mathcal{O}_2}{\mathcal{O}_1} E^2 C$, and by Lemma 4.5 we have

$$\begin{aligned}
\sum_{k=1}^{\infty} k\,\nu_k &> 2E(1 - 2E\nu_{\max}\mathcal{O}_2 2C) \\
&> 2E\left(1 - \frac{\mathcal{O}_1}{16EC}\nu_{\max} \cdot \ln(\frac{1}{2(d+1)(\varrho_1 - 1)})\right) ,
\end{aligned}$$

which is bigger than E if $\nu_{\max} \ln \frac{1}{2(d+1)(\varrho_1-1)} < 8EC/\mathcal{O}_1$. Otherwise $\sum_{k=1}^{\infty} k\,\nu_k > E$ follows from Lemma 4.4. The existence of a uniform bound (4.14) for the second moments of (ν_k) follows from (4.7). □

4.3 The proof of Theorem 4.1

In this section we shall prove Theorem 4.1. Take $\mu \in \mathcal{H}$, i.e. fix some $A \in \mathcal{A}_{n+1}$ and consider the measure $\mu = F_*^n(m_{|A})$. Define ν to be the normalization of μ on A_r, i.e. $\nu(\cdot) = \mu(\cdot \cap A_r)/\mu(A_r)$. For $j \geq 1$ define

$$a_j = m(A_{r+j-k_0-1}) \text{ and } \nu_j = \nu((F_{|A_r})^{-1}(A_{r+j-k_0-1})) \ .$$

Let $E = k_0 + 2$. If the assumptions of Theorem 4.1 are satisfied, then we get from Proposition 4.2 that

$$\sum_{j=1}^{\infty} j\nu_j > k_0 + 2 \quad \text{ and } \quad \sum_{j=1}^{\infty} j^2\nu_j < M$$

where M does not depend on the particular measure μ. Hence

$$\sum_{j=1}^{\infty} j\nu(F_{|A_r})^{-1}(A_{r+j-k_0-1}) > k_0 + 2 \ .$$

Observe that $\Delta\varphi$ from Proposition 4.1 is equal to $j - k_0 - 1$ on $(F_{|A_r})^{-1}(A_{r+j-k_0-1})$. So

$$\begin{aligned} \int_{A_r} (\Delta\varphi - 1)\, d\nu &= \left[\sum_{j=1}^{\infty}(j - k_0 - 1)\nu(F_{|A_r})^{-1}(A_{r+j-k_0-1})\right] - 1 \\ &\geq \left[\sum_{j=1}^{\infty} j\nu(F_{|A_r})^{-1}(A_{r+j-k_0-1})\right] - k_0 - 2 \geq 0 \ , \end{aligned}$$

and similarly

$$\int_{A_r} (\Delta\varphi)^2\, d\nu < M \ .$$

Hence the assumptions of Proposition 4.1 are satisfied and this implies that the assertion of Theorem 4.1 holds.

5 Proof of the Main Theorem

In this section we shall complete the proof of the Main Theorem. So let f be a C^2 Fibonacci map with a critical point of order ℓ. First we should remark that the complement of the basin of $\omega(c)$ is a residual set. This can be seen as follows. From

Chapter IV of [MS] it follows that f has no wandering intervals (a wandering interval is an interval whose forward iterates are all disjoint and which is not in the basin of a periodic attractor). Moreover, f is not renormalizable and has positive topological entropy, see [HK]. It follows that f is semi-conjugate to a tent-map of the form

$$x \mapsto \lambda\,(1 - |2x - 1|)$$

and that the semi-conjugacy only collapses components of basins of periodic attractors. Clearly such components cannot be in the basin of the Cantor set $\omega(c)$. So it suffices to show that there exists a residual set of points x for which $\omega(x)$ (w.r.t. a tent-map) is equal to a cycle of intervals. This fact is well-known, see for example [Mil, page 189]. So the deepest part of the proof consists in showing that $B(\omega(c))$ has positive Lebesgue measure.

Let the points u_k, c_{S_k} and so on be defined as in Section 2 and choose as before $\tilde{u}_{k+1} \in \{u_{k+1}, \hat{u}_{k+1}\}$ so that it is on the same side of c as u_k. Define intervals $I_k = (u_k, \tilde{u}_{k+1})$ and $\hat{I}_k$ (the interval symmetric to I_k), and a map

$$F : \bigcup (I_k \cup \hat{I}_k) \to \bigcup (I_k \cup \hat{I}_k)$$

by

$$F|I_k = f^{S_k}\ .$$

Then for $k > 1$

$$F(I_k) = F(\hat{I}_k) = (u_{k-2}, u_k)\ .$$

Hence, if we let $A_k = I_k \cup \hat{I}_k$ $(k \geq 0)$, then $\mathcal{A} = \{A_k \colon k = 0, 1, 2, \ldots\}$ is a partition of $X = (u_0, \hat{u}_0)$, and F is Markov with respect to $\mathcal{A}$.

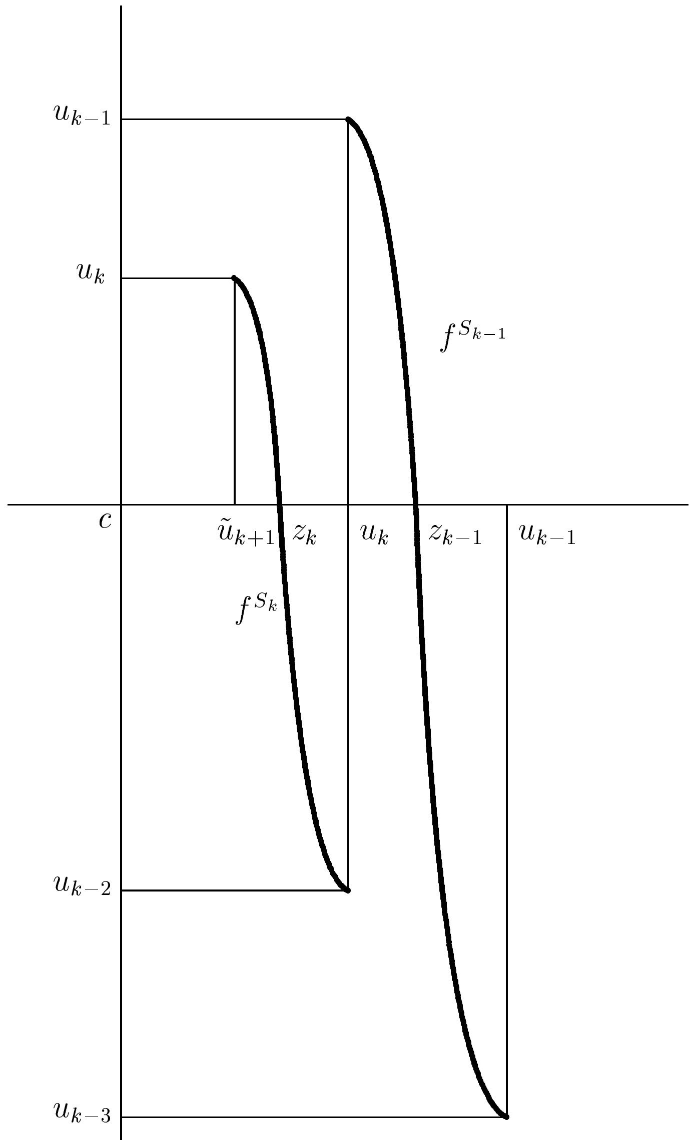

Figure 5.1.

In this section we shall show that F and $\mathcal{A}$ satisfy the assumptions of Theorem 4.1 with $d = 2$, $k_0 = 2$ and thus prove

Theorem 5.1 *For all sufficiently large ℓ holds: The set D of all points x for which the trajectory $(F^k x)_{k>0}$ visits each interval I_n and $\hat{I}_n$ at most finitely often, has positive Lebesgue measure.*

Let us first show that this result implies our Main Theorem, which states:

Theorem 5.2 *$\omega_f(c)$ is an absorbing Cantor set attractor for f provided ℓ is large enough.*

Proof: First we should remark that f is ergodic with respect to the (non-invariant) Lebesgue measure if its Schwarzian derivative is negative, see [BL]. Hence F is ergodic with respect to Lebesgue measure, and as $f^{-1}(D) = D$ and D has positive Lebesgue measure, D has full Lebesgue measure in this case. If f is a smooth Fibonacci map with a periodic attractor then, of course, f is not ergodic and $\omega(c)$ cannot be a 'global' attractor. But even in this case, the argument below will show that it attracts a set of points of positive Lebesgue measure. We should remark that $\omega(c)$ is not accumulated by periodic attractors, see [MMS] or [MS], so near $\omega(c)$ these periodic attractors are 'invisible'.

So consider a point $x \in X$ for which $(F^k x)_{k>0}$ visits each interval I_n and $\hat{I}_n$ at most finitely often, and denote by $t_1 < t_2 < t_3 < \dots$ the sequence of times for which $F^k x = f^{t_k} x$. We have to show that $\lim_{t\to\infty} \operatorname{dist}(f^t x, \omega_f(c)) = 0$. Along the subsequence t_k this holds as $\lim_{k\to\infty} f^{t_k} x = \lim_{k\to\infty} F^k x = c \in \omega_f(c)$. Consider now $t_k < t < t_{k+1}$ and suppose that $F^k x \in I_n$ (or $F^k x \in \hat{I}_n$). As f^{S_n} is monotone on I_n (and on $\hat{I}_n$) and as $f^{S_n}(I_n) = f^{S_n}(\hat{I}_n)$ is an interval contained in the union of the two central monotonicity intervals of $f^{S_{n-2}}$, the interval $V := f^{t-t_k}(I_n) = f^{t-t_k}(\hat{I}_n)$ is contained in the union of two adjacent monotonicity intervals of $f^{S_{n-2}+t_{k+1}-t}$. Furthermore, $f^t x \in V$, and as $c_{S_{n+1}} \in I_n$, V contains the point $f^{S_{n+1}+t-t_k}(c) \in \omega_f(c)$. Therefore $\operatorname{dist}(f^t x, \omega_f(c)) \le |V| \le 2\delta_{S_{n-2}+t_{k+1}-t} \le 2\delta_{S_{n-2}}$, where δ_k denotes the maximal length of a monotonicity interval of f^k, and $\lim_{k\to\infty} \delta_k = 0$ because f is non-renormalizable. □

Proof of Theorem 5.1: Let us show that we can apply Theorem 4.1 where we take $d = 2$, $k_0 = 2$, $X = (u_0, \hat{u}_0)$, $\mathcal{A}$ the partition from above and m the Lebesgue measure on X. So fix $r \in \mathbb{N}$ sufficiently large and consider $A_r = I_r \cup \hat{I}_r$. For $j > 0$ define

$$a_j := |A_{r+j-3}| = 2|I_{r+j-3}| = 2|u_{r+j-3} - \tilde{u}_{r+j-2}|$$

(observe that f is symmetric), and let

$$a_0 = \min(|c_{S_{r-2}} - u_{r-2}|, |c_{S_r} - u_r|) .$$

Note that a_1, a_2, a_3 is the size of A_{r-2}, A_{r-1}, A_r and that a_0 expresses 'Koebe space'. Now let $\mu \in \mathcal{H}$ be a measure of the form $\mu = F^n_*(m_{|A})$ where $A \in \mathcal{A}_{n+1}$ with $F^n(A) = A_r$. Denote by ν the normalization of μ and let for $j \ge 1$

$$\nu_j = \nu((F_{|A_r})^{-1}(I_{r+j-3})) .$$

We shall show that these numbers satisfy the *scaling condition* provided r and ℓ are large enough.

Because of the estimates from Theorem 3.1, it follows that there exist constants $C_1, C_2 \in (0, \infty)$ such that for large ℓ and large j,

$$1 + \frac{C_1}{\ell} \le \frac{|u_j - c|}{|\tilde{u}_{j+1} - c|} \le 1 + \frac{C_2}{\ell} .$$

It follows easily that for $k \geq j \geq 1$,

$$(1+\frac{C_1}{\ell})^{k-j} \leq \frac{|u_{r+j-3}-c|}{|u_{r+k-3}-c|} = \frac{\sum_{i=j}^{\infty} a_i}{\sum_{i=k}^{\infty} a_i} \leq (1+\frac{C_2}{\ell})^{k-j}$$

provided ℓ and r are sufficiently large. If $j = 0$ then, again because of Theorem 3.1

$$(1+\frac{C_1}{\ell})^{k} \leq \frac{|c_{S_{r-2}}-c|}{|u_{r+k-3}-c|} = \frac{\sum_{i=0}^{\infty} a_i}{\sum_{i=k}^{\infty} a_i} \leq (1+\frac{C_2}{\ell})^{k} .$$

(possibly with a different constant C_1). This gives condition (4.2).

Let us now show that condition (4.3) is satisfied with $d = 2$. We need to estimate

$$\frac{a_{j+1}}{\nu_{j+1}}\frac{\nu_{k+1}}{a_{k+1}} = \frac{2|I_{r+j-2}|}{\nu((F_{|A_r})^{-1}(A_{r+j-2}))}\frac{\nu((F_{|A_r})^{-1}(A_{r+k-2}))}{2|I_{r+k-2}|} \tag{5.1}$$

from below for $j = 0, 1, 2$, where we assume $k \geq j$. As $A_r = I_r \cup \hat{I}_r$, it suffices to estimate this expression with A_r replaced by I_r and also with A_r replaced by $\hat{I}_r$. Because of the symmetry of F, both cases can be treated in the same way, and we consider without loss of generality only the case with I_r. So we have to estimate

$$\frac{|I_{r+j-2}|}{\nu((F_{|I_r})^{-1}(A_{r+j-2}))}\frac{\nu((F_{|I_r})^{-1}(A_{r+k-2}))}{|I_{r+k-2}|} . \tag{5.2}$$

Let $\mathcal{I}$ be the partition into sets I_i and $\hat{I}_i$, and recall that F maps I_r (and also $\hat{I}_r$) diffeomorphically onto

$$\cup_{i=r}^{\infty}(I_i \cup \hat{I}_i) \cup \tilde{I}_{r-1} \cup I_{r-2} .$$

Denote by $\tilde{I}_k$ that one of the intervals I_k and $\hat{I}_k$ that is on the same side of c as I_{r-2}. Then, in case that $j = 0$ or $j = 1$, we have $(F_{|I_r})^{-1}(A_{r+j-2}) = (F_{|I_r})^{-1}(\tilde{I}_{r+j-2})$ and, as $A_{r+k-2} \supset \tilde{I}_{r+k-2}$, we must find a lower bound for

$$\frac{|\tilde{I}_{r+j-2}|}{\nu((F_{|I_r})^{-1}(\tilde{I}_{r+j-2}))}\frac{\nu((F_{|I_r})^{-1}(\tilde{I}_{r+k-2}))}{|\tilde{I}_{r+k-2}|} . \tag{5.3}$$

Moreover, $I_r \subset (z_{r-1}, c) \subset (c_{S_r}, c)$, and (z_{r-1}, c) is mapped by F diffeomorphically onto $(c_{S_{r-2}}, c_{S_r})$. As the partition

$$\mathcal{I}_{n+1} = \mathcal{I} \vee F^{-1}(\mathcal{I}) \vee \ldots \vee F^{-n}(\mathcal{I})$$

refines the partition $\mathcal{A}_{n+1}$, the set $A \in \mathcal{A}_{n+1}$ is a finite union of intervals $H \in \mathcal{I}_{n+1}$ with $F^n(H) = I_r$ or $F^n(H) = \hat{I}_r$. Fix such an interval H with $F^n(H) = I_r$. It follows from Proposition 2.1 that F satisfies the following extension properties:

- $F^n_{|H}$ is of the form f^s (in fact, f^s is a composition of maps of the form f^{S_i}) and therefore $F \circ F^n = f^{S_r+s}$;

- there exists an interval $T \supset H$ which is mapped by f^{S_r+s} diffeomorphically onto $(c_{S_{r-2}}, c_{S_r})$.

Hence if B is a subset of I_r, then for the measure $\mu = F^n_*(m_{|A})$,

$$\mu(B) = \sum_{H\in\mathcal{I}_{n+1};F^n(H)=I_r} |(F^n{}_{|H})^{-1}(B)|.$$

In particular,

$$\begin{aligned} &\frac{|\tilde{I}_{r+j-2}|}{\nu((F_{|I_r})^{-1}(\tilde{I}_{r+j-2}))}\frac{\nu((F_{|I_r})^{-1}(\tilde{I}_{r+k-2}))}{|\tilde{I}_{r+k-2}|} \\ = \ &\frac{\sum_{\{H\in\mathcal{I}_{n+1};F^n(H)=I_r\}} |(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+k-2})|}{\sum_{\{H\in\mathcal{I}_{n+1};F^n(H)=I_r\}} |(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+j-2})|}\frac{|\tilde{I}_{r+j-2}|}{|\tilde{I}_{r+k-2}|}\end{aligned}$$

which means that this last expression can be estimated from below as the infimum over all $H \in \mathcal{I}_{n+1}$ with $F^n(H) = I_r$ of the expression

$$\frac{|(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+k-2})|}{|(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+j-2})|}\frac{|\tilde{I}_{r+j-2}|}{|\tilde{I}_{r+k-2}|}$$

As we noted before, for each $H \in \mathcal{I}_{n+1}$ with $F^n(H) = I_r$, there exists an interval $T \supset H$ such that $F \circ F^n$ maps T diffeomorphically onto $(c_{S_{r-2}}, c_{S_r})$. Now F satisfies the assumptions of Proposition 3.3:

- the first assumption holds for obvious reasons;
- assumptions 2), 3) and 4) of this Proposition follow from the above extension properties and from the bounds from Theorem 3.1 (where τ is a constant which is independent of ℓ);
- in assumption 5) the constant K can be taken as the intersection multiplicity 3 from Proposition 2.1.

Hence if we take an interval $T' \supset H$ such that each component of $F \circ F^n(T' \setminus J)$ has exactly half the size of the corresponding component of $F \circ F^n(T \setminus J)$ then it follows from Proposition 3.3 that

$$\sum_{m=0}^{j-1} |f^m(T')| \le K'$$

for some universal number K' (provided we take r sufficiently large and therefore the set $\cup_{j\ge 0}A_{r-k_0+j}$ sufficiently small). Hence by Proposition 3.1 the cross-ratio distortion of $f^m|T'$ is bounded. Hence Lemma 3.1 implies that

$$\frac{|(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+k-2})|}{|(F^n{}_{|H})^{-1}\circ(F_{|I_r})^{-1}(\tilde{I}_{r+j-2})|}\frac{|\tilde{I}_{r+j-2}|}{|\tilde{I}_{r+k-2}|}$$

is at least

$$\mathcal{O}_1 \cdot \frac{|c_{S_{r-2}} - \tilde{u}_{r+j-2}||c_{S_{r-2}} - \tilde{u}_{r+j-1}|}{|c_{S_{r-2}} - \tilde{u}_{r+k-2}||c_{S_{r-2}} - \tilde{u}_{r+k-1}|}$$

if r is large enough (where $\mathcal{O}_1$ is a universal constant).

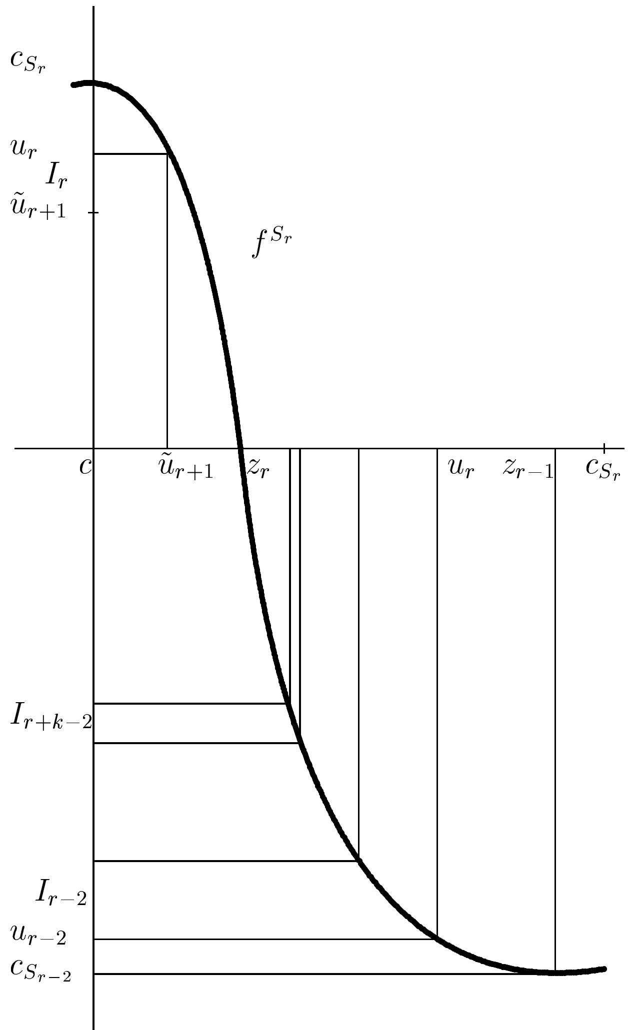

Figure 5.2.

By the choice of a_0 this is bounded from below by

$$\mathcal{O}_1 \cdot \frac{a_0}{\sum_{i=0}^{k} a_i} \cdot \frac{a_1}{\sum_{i=0}^{k+1} a_i} ,$$

and this is also a lower bound for (5.3) and hence in case $j = 0$ or 1 also for (5.2).

If $j = 2$, then both parts of $(F_{|I_r})^{-1}(A_{r+j-2}) = (F_{|I_r})^{-1}(I_{r+j-2}) \cup (F_{|I_r})^{-1}(\hat{I}_{r+j-2})$ are nonempty. Admitting an additional factor of $\frac{1}{2}$ for the lower bound it suffices to estimate (5.2) with A_{r+j-2} first replaced by $I_{r+j-2} = I_r$ and then by $\hat{I}_{r+j-2} = \hat{I}_r$. For

$\tilde{I}_{r+j-2}$ (that one on the same side of c as I_{r-2}), the same estimate as above works. For $\hat{\tilde{I}}_{r+j-2}$ (that one on the other side of c) we estimate (5.2) from below by

$$\frac{|I_r|}{\nu((F_{|I_r})^{-1}(I_r))} \frac{\nu((F_{|I_r})^{-1}(\hat{\tilde{I}}_{r+k-2}))}{|\hat{\tilde{I}}_{r+k-2}|} , \tag{5.4}$$

and along the same lines as above we find the lower estimate

$$\mathcal{O}_1 \cdot \frac{|c_{S_r} - u_r||c_{S_r} - u_{r+1}|}{|c_{S_r} - \hat{\tilde{u}}_{r+k-2}||c_{S_r} - \hat{\tilde{u}}_{r+k-1}|} .$$

By the choice of a_0 this is bounded from below by

$$\mathcal{O}_1 \cdot \frac{a_0}{\sum_{i=0}^{k} a_i} \cdot \frac{a_3}{\sum_{i=0}^{k+1} a_i} ,$$

and this is also a lower bound for (5.2) in case $j = 2$. This bounds (5.1) from below and hence proves (4.3).

We turn to the proof of (4.4).

$$\frac{\nu_{k+1}}{a_{k+1}} = \frac{\nu_1}{a_1} \frac{|I_{r-2}|}{\nu((F_{|A_r})^{-1}(I_{r-2}))} \frac{\nu((F_{|A_r})^{-1}(I_{r+k-2}))}{|I_{r+k-2}|} + \frac{\nu_3}{a_3} \frac{|\hat{I}_r|}{\nu((F_{|A_r})^{-1}(\hat{I}_r))} \frac{\nu((F_{|A_r})^{-1}(\hat{I}_{r+k-2}))}{|\hat{I}_{r+k-2}|} ,$$

and as above it suffices to estimate this expression with A_r replaced by I_r. Using the second inequality in Lemma 3.1 a rough upperbound for the first summand is given by

$$\mathcal{O}_2 \frac{\nu_1}{a_1} \cdot \left| \frac{c_{S_{r-2}} - c_{S_r}}{c_{S_r} - c} \right|^2 \leq \frac{\nu_1}{a_1} \cdot \left(1 + \frac{\sum_{i=0}^{\infty} a_i}{\sum_{i=3}^{\infty} a_i} \right)^2 \leq \frac{\nu_1}{a_1} \cdot (1 + \varrho_1^3)^2 < (27)^2 \frac{\nu_1}{a_1} ,$$

and the second one similarly by

$$\mathcal{O}_2 \frac{\nu_3}{a_3} \cdot \left| \frac{c_{S_{r-2}} - c_{S_r}}{c_{S_{r-2}} - c} \right|^2 \leq K_1 \varrho_1^2 \frac{\nu_3}{a_1} \cdot 2^2 \leq 16 K_1 \frac{\nu_3}{a_1} .$$

This yields (4.4).

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