Measures with infinite Lyapunov exponents for the periodic Lorentz gas

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December 12, 1994

Abstract

We study invariant measures for the Lorentz gas which are supported on the set of points with infinite Lyapunov exponents. We construct examples of such measures which are measures of maximal entropy and ones which are not.

Keywords: billiard, Lorentz gas, Lyapunov exponent, measure of maximal entropy.

1 Introduction

For a given dynamical system a measure of maximal entropy captures information about the "most chaotic" part of the dynamics. More precisely a measure of maximal entropy is a probability measure invariant under the dynamics whose metric entropy is equal to the topological entropy of the given system. For systems with finite topological entropy much is known about measures of maximal entropy. Axiom A diffeomorphisms have a unique measure of maximal entropy on each topologically transitive component and this measure is Markovian [11]. Recently in [9] it was shown that $C^{1+\alpha}$ diffeomorphisms of compact Riemannian manifolds have at most countably many ergodic measures of maximal entropy and these measures are Markovian. For systems with infinite topological entropy nothing in this direction has been known up to now, although the main tool used to derive the above results, Markov partitions, has been developed for some time [4].

In [6] it was shown that the billiard ball map for the periodic Lorentz gas has infinite topological entropy. In this article we study the set of points with infinite Lyapunov exponents. Using the cell structure developed in [4, 10] we construct an ergodic invariant probability measure with infinite topological entropy supported on this set. Since the topological entropy is infinite this is a measure of maximal entropy. From the construction it is clear that there many such measures can coexist on a single component of topological transitivity. We also construct an ergodic invariant probability measure with finite entropy which is supported on this set showing that infinite exponents do not necessarily lead to infinite entropy.

2 Cell structure and Lyapunov exponents

We study a periodic Lorentz gas on a plane. For simplicity, we assume that every fundamental domain of this gas contains a single round scatterer. Let the fundamental domain be a unit square and the scatterer be the circle of radius $r > 0$ centered at the origin. Thus we get a periodic array of circles of radius r centered at sites of the integral lattice. A pointlike particle moves freely at unit speed between the circles and reflects elastically off them. The circles are immovable and rigid.

We assume that $r < 1/2$, so that the moving particle is not trapped between four neighboring circles. Then the particle can move freely without collisions indefinitely; such Lorentz gases are said to have *no horizon*.

Since the gas is periodic we can consider a fundamental domain: the configuration space Q of this system is the unit torus $0 \leq q_1, q_2 < 1$ without the disc $q_1^2 + q_2^2 \leq r^2$ (mod 1). The phase space is $M = Q \times S^1$. This is a billiard system of Sinai type (with dispersing boundary). The billiard ball map T is defined on the surface

$$M_1 = \{(q,v) \in M : q \in \partial Q \text{ and } (v,n) \geq 0\}$$

where n is the inward unit normal vector to the boundary ∂Q of the domain Q. The map T is simply the first return map on the surface M_1, i.e. it sends the particle at a reflection point to its next reflection. We introduce the coordinates (s, φ) on M_1, where s is the arc length on the circle ∂Q and φ is the angle between the vector v and the above normal vector n to the circle, $0 \leq s < 2\pi r$ and $-\pi/2 \leq \varphi \leq \pi/2$. Since s is a cyclic coordinate, M_1 is a cylinder. The map T on M_1 preserves the measure $d\nu = c_\nu \cos\varphi ds\, d\varphi$, where $c_\nu = (2\pi r)^{-1}$ is the normalizing factor.

Sinai [13] was first to study the properties of the map T in detail. He proved the T is hyperbolic, i.e. has nonzero Lyapunov exponents a.e., and constructed stable and unstable fibers at a.e. point $x \in M_1$. He also developed a proof of ergodicity of T, which was later improved in [2]. In addition to ergodicity, the mixing and K-property of T was established in [13, 2], and its Bernoulli property was proved in [8]. Another proof of ergodicity, which worked for multidimensional Lorentz gas as well, was provided in [14]. Sinai [13] derived a formula for the Kolmogorov entropy of the map T, which was later reproved and studied in [7]. In particular, it was shown in [7] that the entropy $h(T)$ has the following asymptotics as $r \to 0$:

$$h(T) = 2\ln(1/r) + O(1).$$

Markov partitions for the map T were constructed in [4]. Those provide a symbolic representation of T by topological Markov chains with countable alphabet. Based on Markov partitions, it was later shown in [6] that the topological entropy of T is infinite. In particular, the natural ergodic measure ν on M_1 is not a measure of maximal entropy, since its entropy is finite. The statistical properties of the map T were studied in [5]: a stretched exponential bound on the decay of correlations was established and the central limit theorem along with its weak invariance principle was proved. Note that the periodic Lorentz gas with no horizon, apparently, displays a 'superdiffusive' behavior, as was conjectured and explained in [1].

We will use the 'cell structure' of the surface M_1 described in detail in [4, 10]. The map T has a countable number of domains of continuity, which accumulate at a finite number of singular points, at which the time of the first return is infinite. We will call such points *supersingular*. For example, four points on the circle ∂Q with coordinates $(0, \pm r)$ and $(\pm r, 0)$ and with $\varphi = \pm\pi/2$ are such supersingular points (they produce eight supersingular points in M_1). There might be more supersingular points for small radius r. The domains of continuity of the map T (we call them *cells*) form a fairly standard structure in the neighborhood of every supersingular point, independent of r (the structure is the same if the scatterers are not necessarily circles but smooth convex domains on the torus). The structure of cells is shown in Fig. 1. We denote cells A_n, $n \geq 2$, where n means that the first return time is about n on the cell A_n. Fig. 2 shows which points are included in the cell A_n. The sizes of the cells are shown in Fig. 1. Here $O(n^{-a})$ means a value between $c^{-1}n^{-a}$ and cn^{-a} for some $c > 1$. Since there are only a finite number of supersingular points, we can assume that the value of c is the same for all of them.

The inverse map T^{-1} also has a countable number of domains of continuity, which accumulate at the supersingular points. They have a symmetric form shown by dashed lines in Fig. 1. We denote them by A'_n, $n \geq 2$, and call 'inverse cells'. Clearly, any cell A_n is mapped by T onto an inverse cell A'_n with the same value of n but located in the neighborhood of another supersingular point. Fig. 3 shows how A_n is mapped onto A'_n under T.

It is shown in [4, 5] that unstable directions for the map T are continuous at every supersingular point and the limit $(ds^u, d\varphi^u)$ of the unstable directions is positive and finite: $0 < d\varphi^u/ds^u < \infty$. Likewise, the limit of the stable directions at every supersingular point is negative and finite: $0 > d\varphi^s/ds^s > -\infty$. Therefore, we have transversality of stable directions and increasing sides of the reverse cell A'_n and transversality of unstable directions in the neighborhood of supersingular point and decreasing (long) sides of the cells A_n. Thus it is clear from Fig. 3 that for all points $x \in A_n$ the one-step expansion in the unstable direction has a factor $O(n^{3/2})$, and one-step contraction in the stable direction has a factor $O(n^{3/2})$.

We will study points $x \in M_1$ such that $T^i x$ belongs to some cell A_{n_i} (near some supersingular point) with large $n_i = n_i(x) > 0$ for every $i \in \mathbb{Z}$. In other words, we will study points whose trajectories stay very close to supersingular points all the time. It is clear that the positive Lyapunov exponent of any such point is

$$\begin{aligned} \Lambda_+(x) &= \lim_{I\to\infty} \frac{3}{2I} \sum_{i=1}^{I} \ln n_i(x) + O(1) \\ &= \lim_{I\to\infty} \frac{3}{2I} \sum_{i=1}^{I} \ln n_{-i}(x) + O(1) \end{aligned} \tag{1}$$

In particular, if both limits in (1) are infinite, then $\Lambda_+(x) = \infty$. Similar formulas hold for the negative Lyapunov exponent, $\Lambda_-(x)$.

In virtue of (1), any point $x \in M_1$ such that $n_i(x) \to \infty$ as $|i| \to \infty$ has infinite Lyapunov exponents. Such points form a Cantor-like nonempty set concentrated in the vicinity of supersingular points. Obviously, this set does not support any finite invariant measure, because its every trajectory is attracted by supersingular points.

However, there are points $x \in M_1$ for which the sequence $\{n_i(x)\}$ has infinity as a limit point (both as $i \to \infty$ and $i \to -\infty$) but that sequence is 'recurrent', i.e. for every $n \geq 2$ there are asymptotic frequencies

$$p_n^{\pm}(x) = \lim_{I\to\infty} \frac{\#\{i \in [1, I] : n_{\pm i}(x) = n\}}{I} \tag{2}$$

and $\sum_n p_n^+(x) = \sum_n p_n^-(x) = 1$. If the sequences $\{p_n^+(x)\}$ and $\{p_n^-(x)\}$ decay slowly enough, then the corresponding point x will have infinite Lyapunov exponent. It is clear that the following condition is sufficient for infinite Lyapunov exponents:

$$\sum_n p_n^{\pm}(x) \ln n = \infty \tag{3}$$

3 Measures with infinite Lyapunov exponents

Here we construct ergodic measures for the map T with finite and infinite Kolmogorov entropy, such that a.e. point $x \in M_1$ with respect to those measures has infinite Lyapunov exponents.

The construction starts with the following observation. There are positive constants $c > 1$ and $n_* \geq 1$ such that every cell A_n with $n > n_*$ intersects all inverse cells A'_m near the same supersingular point with $c\sqrt{n} \leq m \leq c^{-2}n^2$ so that both longer sides of A_n cross both longer sides of A'_m (as shown in Fig. 3). This observation is based on Fig. 1 and 3 and was made in [4].

Let $n_0, n_1, n_2, \ldots$ be a sequence of integers such that $n_i > n_*$ for all $i \geq 0$ and $c\sqrt{n_i} \leq n_{i+1} \leq c^{-2}n_i^2$ for all $i \geq 0$. Then there is a sequence of cells A_{n_i} such that $A_{n_i} \cap T^{-1}A_{n_{i+1}} \neq \emptyset$ for all $i \geq 0$, and the intersection

$$\cap_{i=0}^{\infty} T^{-i} A_{n_i}$$

is a monotone curve in the cell A_{n_0} which stretches from its top (short) side to its bottom (short) side.

The same is true for inverse cells: there is a sequence of inverse cells A'_{n_i} such that the intersection

$$\cap_{i=0}^{\infty} T^{i} A'_{n_i}$$

is a monotone curve stretching from the top short side of A'_{n_0} to its bottom short side.

Consequently, for any double-infinite sequence of integers $\{n_i\}$, $-\infty < i < \infty$, such that $n_i > n_*$ and $c\sqrt{n_i} \leq n_{i+1} \leq c^{-2}n_i^2$ for all $i \in \mathbb{Z}$ (note that this condition is symmetric: $c\sqrt{n_i} \leq n_{i-1} \leq c^{-2}n_i^2$) there is a sequence of cells $\{A_{n_i}\}$ such that the intersection

$$\cap_{i=-\infty}^{\infty} T^{-i} A_{n_i} \tag{4}$$

is a single point in the cell A_{n_0}.

We now fix an increasing sequence of integers, $N_0, N_1, N_2, \ldots$, such that $N_{i+1} = [c^{-4}N_i^2] + 1$ for all $i \geq 0$ and $N_0 \gg n_*$. It has the following two properties:

(i) for any double-infinite sequence $\{n_i\}$, $-\infty < i < \infty$ such that $n_i = N_{s(i)}$ with some $s(i) \in \mathbb{Z}^+$ and $|s(i) - s(i+1)| \leq 1$ for all $i \in \mathbb{Z}$. Note the intersection (4) is a single point in the cell A_{n_0};

(ii) for all $i \geq 0$

$$N_i \geq c^{-4(2^i-1)} N_0^{2^i}$$

so that if N_0 is large enough, the sequence $\{N_i\}$ grows at the following super-exponential rate:

$$N_i \geq 2^{2^i} \tag{5}$$

The set of double-infinite sequences $\{n_i\}$ described by the condition (i) above is a topological Markov chain with a countable number of states, which can be identified with $N_1, N_2, \ldots$. The allowed transitions from every state N_i are the ones to N_i itself and to the two neighboring states, N_{i-1} and N_{i+1}. The only exception is the first state, N_1, from which the transitions to itself and to N_2 are allowed.

We denote the collection of the above double-infinite sequences $\{n_i\}$ by Ω_1. In virtue of the property (i) every sequence $\omega = \{n_i\} \in \Omega_1$ corresponds to a point $x = x(\omega) \in M_1$ defined by the intersection (4). The set of points

$$\Omega_{1,M} = \{x(\omega) : \omega \in \Omega_1\}$$

is a closed Cantor-like subset of M_1 invariant under T, i.e. $T\Omega_{1,M} = T^{-1}\Omega_{1,M} = \Omega_{1,M}$.

Let μ_1 be a Markov measure on the symbolic space Ω_1 defined by the following transition probabilities: $\pi_{i+1,i} = 1/3$ for all $i \geq 1$, $\pi_{i-1,i} = 2/3$ for all $i \geq 2$ and $\pi_{1,1} = 2/3$.

(Here $\pi_{i,j}$ stands for the probability of transition from N_j to N_i). This Markov measure is ergodic and mixing, its stationary distribution is $p(N_i) = 1/2^i$ for $i \geq 1$.

The measure μ_1 projected from Ω_1 down to M generates an ergodic measure ν_1 for the map T, which is concentrated on $\Omega_{1,M}$. By the ergodic theorem, for ν_1-almost every point $x \in \Omega_{1,M}$ the asymptotic frequencies $p_n^{\pm}(x)$ defined by (2) exist and are equal to $p_n^{\pm}(x) = 1/2^i$ if $n = N_i$ for some $i \geq 1$ and zero otherwise. It is then a simple calculation based on (3) and (5) that the Lyapunov exponents are infinite a.e. in M with respect to the ergodic measure ν_1:

$$\sum_n p_n^{\pm}(x) \ln n = \sum_i p_{N_i}^{\pm} \ln N_i \geq \sum_i 2^{-i} \ln 2^{2^i} = \infty$$

The measure ν_1 constructed above has a finite entropy. Indeed, it is a Markov measure, and so its entropy, see, e.g., [11], is given by

$$h = -\sum_i p_i \sum_j \pi_{ij} \log \pi_{ij} \tag{6}$$

where π_{ij} are the transition probabilities and p_i is the stationary distribution. In fact, since for every state i only two transition probabilities π_{ij} are positive, as defined above, we have $h \leq \log 2$ for the measure μ_1.

The above construction of the Markov measure μ_1 can be modified so that its entropy will be infinite and Lyapunov exponents will be still infinite a.e. We outline the construction below.

We now consider all the double-infinite sequences $\{n_i\}$, $-\infty < i < \infty$, satisfying $n_i > n_*$ and $c\sqrt{n_i} \leq n_{i+1} \leq c^{-2} n_i$ for all $i \in \mathbb{Z}$, as defined above. We denote the set of these sequences by Ω_2. Obviously, Ω_2 is a topological Markov chain with a countable number of states which can be identified with $n_* + 1, n_* + 2, \ldots$. We will number these states by $1, 2, \ldots$ so that the ith state is identified with $n_* + i$. Every sequence $\omega = \{n_i\} \in \Omega_2$ corresponds to a point $x = x(\omega) \in M_1$ defined by the intersection (4). The set of points

$$\Omega_{2,M} = \{x(\omega) : \omega \in \Omega_2\}$$

is a closed Cantor-like subset of M_1 invariant under T, i.e. $T\Omega_{2,M} = T^{-1}\Omega_{2,M} = \Omega_{2,M}$.

We are going to find an ergodic and mixing Markov measure μ_2 on Ω_2 with transition probabilities π_{ij} and with stationary distribution p_i satisfying two conditions:

(iii) its entropy given by (6) is infinite;

(iv) one has

$$\sum_i p_i \ln(n_* + i) = \infty$$

so that, by the condition (3), the projection of the measure μ_2 on M_1 will have infinite Lyapunov exponents a.e.

The existence of Markov measures satisfying (iii) and (iv) is not based on the dynamics of the Lorentz gas or billiards, and we only sketch a proof. It is clear that (iv) is always satisfied if the probabilities p_i decay slowly enough, for example, if $p_i > \mathrm{const} \cdot (i \ln^2 i)^{-1}$. Next, we will take care of the condition (iii).

Let a probability distribution, $||p_i||$, be given. We will show how to find a transition matrix, $\Pi = ||\pi_{ij}||$, preserving the distribution $||p_i||$, so that for every state $i \geq 1$ all the positive transition probabilities $\pi_{ij} > 0$ satisfy

$$c\sqrt{n_* + i} \leq n_* + j \leq c^{-2}(n_* + i)^2 \tag{7}$$

in which case the Markov measure defined by $||p_i||$ and $||\pi_{ij}||$ will be concentrated on Ω_2. For any $k \geq 1$ denote

$$q_k = p_k - p_{k+1} + p_{k+2} - p_{k+3} + \cdots$$

We assume that the probabilities p_i decrease monotonically, $p_1 > p_2 > \cdots$, and that

$$\frac{1}{2} p_k \leq q_k \leq \frac{3}{4} p_k \tag{8}$$

for every $k \geq 1$. To assure this, it is enough to assume that

$$\frac{1}{3} \leq \frac{p_{k+2} - p_{k+1}}{p_{k+1} - p_k} \leq 1 \tag{9}$$

for all $k \geq 1$, which simply means that $\{p_k\}$ decays without abrupt drops.

Now, for any $k \geq 1$ define a matrix of transition probabilities, $\Pi(k) = ||\pi_{ij}(k)||$, by $\pi_{ii} = 1$ for all $1 \leq i < k$, $\pi_{kk} = q_k/p_k$, $\pi_{i,i+1} = q_{i+1}/p_i$ for all $i \geq k$ and $\pi_{i,i-1} = q_i/p_i$ for all $i \geq k+1$. It is easy to check that all $\Pi(k)$ preserve the same distribution $||p_i||$. Now, let $1 = k_1 < k_2 < k_3 < \cdots$ be a sequence of numbers such that $k_m = [\bar{c}m]$ for some sufficiently large $\bar{c} > 2$. We then define the matrix of transition probabilities $\Pi = ||\pi_{ij}||$ by

$$\Pi = \Pi(k_1) \cdot \Pi(k_2) \cdot \Pi(k_3) \cdots$$

(here every entry π_{ij} requires only a finite number of multiplications). If $\bar{c}$ and n_* are large enough, this matrix clearly satisfies (7). Now, due to (8) all the positive entries of the matrices $\Pi(k)$, $k \geq 1$, are not smaller than $1/4$. In that case, the entropy of the conditional distribution,

$$h(i) = -\sum_j \pi_{ij} \log \pi_{ij}$$

increases to infinity as $i \to \infty$. Moreover, it is bounded below by some increasing sequence $h(i) \geq H_i$, $H_i \to \infty$ as $i \to \infty$, independent on the stationary distribution $||p_i||$. (The sequence H_i depends only on the value of $\bar{c}$ which is only determined by c and n_* in (7). Finally, we can find a probability distribution $||p_i||$ satisfying (9) and such that $\sum_i p_i H_i = \infty$. (Obviously, such a distribution always exists, whatever the increasing sequence $\{H_i\}$). Hence the condition (iii) is provided.

The Markov measure μ_2 satisfying the conditions (iii) and (iv) projected from Ω_2 down to M_1 induces a measure, ν_2, concentrated on $\Omega_{2,M}$. A.e. point with respect to ν_2 has infinite Lyapunov exponents. This measure has infinite entropy, and so it is a measure of maximal entropy. Note that for every mixing subshift of finite type with finite entropy [12] the measure of maximal entropy is a Markov one, and its transition probabilities are positive for all (topologically) allowed transitions between states. The measure of maximal entropy that we have constructed here is also a Markov one, but some of its transition probabilities are zero even for topologically allowed transitions. This means that we did not use all of the available topological richness of the chain Ω_2 defined above. The transitions that we have used were enough to make the entropy of the Markov measure infinite.

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Figure captions.

1. Fig. 1. A supersingular point S and a cell A_n near it.
2. Fig. 2. The outgoing vectors form the cell A_n.
3. Fig. 3. The image of A_n under the map T is a reverse cell, A'_n.