QUASI-PROJECTIONS IN TEICHMÜLLER SPACE

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November 23,1994

Many parallels have been drawn between geometric properties of the Teichmüller space of a Riemann surface, and those of complete, negatively curved spaces (see for example Bers [2], Kerckhoff [8], Masur [12], Wolpert [17]). This paper investigates one such parallel – the contracting properties of certain projections to geodesics.

We will use the Teichmüller metric throughout the paper (the other famous metric on Teichmüller space, the Weil-Petersson metric, is negatively curved although it is not complete – see Wolpert [16]. The Teichmüller metric is complete, but not negatively curved – see Masur [10].)

Every closed subset C of a metric space X determines a "closest-points" projection, defined as a map $\pi_C : X \to \mathcal{P}(C)$, where $\mathcal{P}(C)$ is the set of closed subsets of C. Namely,

$$\pi_C(x) = \{y \in C : d(x,y) = d(x,C)\}$$

where $d(x,C) = \inf_{y\in C} d(x,y)$. (Provided X is *proper*, i.e. R-balls are compact, $\pi_C(x)$ is never empty.) Suppose now that X is a simply-connected Riemannian manifold with non-positive sectional curvatures, and that C is a geodesic segment, ray or line. In this case $\pi_C(x)$ always consists of a single point. If the sectional curvatures are bounded above by a negative constant, then $||d\pi_C|| \to 0$ as $d(x,C)$ goes to ∞ (where $d\pi_C$ is the differential of π_C). In other words, π is contracting at large distances.

If $X = \mathcal{T}(S)$, the Teichmüller space of a surface S of finite type, this contraction property fails in general, but a coarse form of it can be shown to hold for a special class of geodesics. Call a geodesic (segment, ray, or line) L in $\mathcal{T}(S)$ *precompact* if its projection to the moduli space $\mathcal{M}(S)$ lies in a compact set. We say L is ϵ*-precompact* if $\epsilon > 0$ is a lower bound on the extremal length of any non-peripheral simple closed curve in S, for any conformal structure in L (see section 1 for definitions). Our main theorem is the following:

Theorem 3.1 *Given ϵ there are constants b_1, b_2 depending on $\epsilon, \chi(S)$ such that for any ϵ-precompact geodesic L, if $\sigma \in \mathcal{T}(S)$ is such that $d(\sigma, L) > b_1$ then, setting $r = d(\sigma, L) - b_1$,*

$$\mathrm{diam}(\pi_L(\mathcal{N}_r(\sigma))) \le b_2.$$

In particular, for some $b_0 = b_0(\epsilon, \chi(S))$ and all $\sigma \in \mathcal{T}(S)$ we have

$$\mathrm{diam}(\pi_L(\sigma)) \le b_0.$$

Here $\mathcal{N}_r$ denotes an r-neighborhood in the Teichmüller metric, and we have taken the liberty of denoting by $\pi_L(A)$ the union $\cup_{a\in A}\pi_L(a)$.

Precompact geodesics are actually rare: by Masur's ergodicity theorem [12], the infinite ones have zero measure in the geodesic flow. In Theorem 5.2 we show that Theorem 3.1 is sharp, in the sense that it fails for all non-precompact geodesics.

Theorem 3.1 is a consequence of a characterization of π_L in terms of an extremal-length ratio problem. Minimizing the distance of $\sigma \in \mathcal{T}(S)$ to a geodesic L is equivalent to solving the problem of finding

$$\inf_{\tau\in L} \sup_{\lambda} \frac{E_\tau(\lambda)}{E_\sigma(\lambda)}$$

where the supremum is over λ in $\mathcal{PMF}(S)$, the set of projectivized measured foliations of compact support in S. This is because the supremum of extremal length ratios is just $\exp 2d(\sigma,\tau)$. In section 2 we prove the following (stated a bit differently):

Theorem 2.1 *If $\sigma \in \mathcal{T}(S)$ and L is ϵ-precompact, the points $\tau \in L$ which appear in the solutions of the dual problem*

$$\sup_{\lambda} \inf_{\tau\in L} \frac{E_\tau(\lambda)}{E_\sigma(\lambda)}$$

are within a bounded distance of $\pi_L(\sigma)$.

In section 4 we give some consequences of the contraction theorem, which are directly analogous to well-known properties of hyperbolic space. Theorem 4.2 is a "stability" property for a class of quasi-geodesics: a quasi-geodesic whose endpoints are connected by an ϵ-precompact geodesic must remain in a bounded neighborhood of the geodesic. Theorem 4.3 bounds the projected images of regions in Teichmüller space which are analogous to horoballs in hyperbolic space. Theorem 4.4 is a lower bound for the translation distance of a pseudo-Anosov automorphism of $\mathcal{T}(S)$, in terms of distance from its axis; a similar estimate holds for arbitrary loxodromic isometries in $\mathbf{H}^n$.

§1. Preliminaries

We begin with a brief summary of results and notation. Let S be a surface of finite genus with finitely many punctures, and let $\mathcal{T}(S)$ be the Teichmüller space of conformal structures of finite type on S, where two structures are considered equivalent if there is a conformal isomorphism of one to the other, which is isotopic to the identity on S. By "finite type" we mean that each puncture has a neighborhood conformally equivalent to a punctured disk.

$\mathcal{T}(S)$ has a natural topology, and is homeomorphic to a finite-dimensional Euclidean space. The *Teichmüller distance* between two points $\sigma, \tau \in \mathcal{T}(S)$ is defined as

$$d(\sigma,\tau) = \tfrac{1}{2}\log K(\sigma,\tau),$$

where $K(\sigma,\tau) = K(\tau,\sigma)$ is the smallest possible quasi-conformal dilatation of a homeomorphism from (S,σ) to (S,τ) isotopic to the identity.

Let α be a homotopy class of simple closed curves in S. Given $\sigma \in \mathcal{T}(S)$, one may define the *extremal length* of α in σ, as the inverse of the conformal modulus of the thickest embedded annulus homotopic to α in S:

$$E_\sigma(\alpha) = \inf_{A\sim\alpha} \frac{1}{Mod(A)}.$$

This number is positive provided α is non-peripheral – that is, not deformable into a puncture. We will make heavy use of Kerckhoff's theorem (see [7]), which states that

$$K(\sigma,\tau) = \sup_\alpha \frac{E_\tau(\alpha)}{E_\sigma(\alpha)}. \tag{1.1}$$

Kerckhoff in fact obtains this supremum as a maximum, by using a completion of the space of homotopy classes of non-peripheral simple closed curves known as the space of "measured foliations with compact support", $\mathcal{MF}(S)$ (see [3, 9]). Elements of $\mathcal{MF}(S)$ are equivalence classes of foliations of S with saddle singularities (negative index) in the interior, and index 1/2 singularities at the poles, equipped with transverse measures. The equivalence is via isotopy, and saddle-collapsing (Whitehead) moves. $\mathcal{MF}(S)$ carries a natural topology, and is homeomorphic to a Euclidean space of the same dimension as $\mathcal{T}(S)$. Multiplication of the measure by positive scalars gives a ray structure to $\mathcal{MF}(S)$, and the quotient of $\mathcal{MF}(S)$ (minus the empty foliation) by this multiplication is the projectivized space $\mathcal{PMF}(S)$, which is a sphere.

A homotopy class of simple closed curves, equipped with a real number giving the measure, is represented in $\mathcal{MF}(S)$ as a foliation whose non-singular leaves are in the homotopy class and fit together in a cylinder whose height is the measure. These special foliations are dense in $\mathcal{MF}(S)$, and Kerckhoff shows that the extremal length function extends to a function $E_\sigma(\lambda)$ which is continuous in both σ and λ, and scales quadratically:

$$E_\sigma(a\lambda) = a^2 E_\sigma(\lambda).$$

Moreover, the supremum in (1.1) is realized by a unique projective class in $\mathcal{PMF}(S)$.

Fixing $\sigma \in \mathcal{T}(S)$, let $QD(S,\sigma)$ denote the space of integrable holomorphic quadratic differentials on (S,σ) (see [4]). This space may be identified with $\mathcal{MF}(S)$ by associating to a quadratic differential Φ its horizontal foliation, which we call Φ_h. Hubbard-Masur [6] showed that this correspondence is a homeomorphism. From [7] we also have the equality

$$E_\sigma(\Phi_h) = ||\Phi|| \tag{1.2}$$

where $||\Phi|| = \int_S |\Phi|$. We note also that the vertical foliation Φ_v, which is the orthogonal foliation to Φ_h, is just $(-\Phi)_h$.

Geodesics in $\mathcal{T}(S)$ are determined by holomorphic quadratic differentials (and hence measured foliations) as follows: Fix $\sigma \in \mathcal{T}(S)$ and $\Phi \in QD(S,\sigma)$. The family

$\{\sigma_t\}_{t\geq 0}$ of conformal structures obtained by contracting along the leaves of Φ_h by e^t and expanding along Φ_v by e^{-t}, forms a geodesic ray in the Teichmüller metric. Geodesics exist and are unique between any two points in $\mathcal{T}(S)$. We also have

$$d(\sigma, \sigma_t) = t$$

and

$$E_{\sigma_t}(\Phi_h) = e^{-2t} E_\sigma(\Phi_h), \tag{1.3}$$

$$E_{\sigma_t}(\Phi_v) = e^{2t} E_\sigma(\Phi_v). \tag{1.4}$$

Our last group of definitions has to do with intersection number. The standard geometric (unoriented) intersection number between two simple closed homotopy classes may be extended to a continuous function on $\mathcal{MF}(S) \times \mathcal{MF}(S)$, homogenous in both arguments. One may best visualize this in the case when one of the arguments is a simple closed curve and the other is any measured foliation. The intersection number is the smallest possible transverse measure deposited on a homotopic representative of the simple closed curve.

Intersection numbers may be related to extremal lengths by the following basic lemma which appears for example in [15]:

Lemma 1.1. *If α, β are any two measured foliations on any Riemann surface X, then*

$$E_X(\alpha) E_X(\beta) \geq i(\alpha, \beta)^2.$$

We say a foliation λ is *complete* if for any other foliation μ, $i(\lambda, \mu) = 0$ implies that the underlying leaf structures of λ and μ are the same (up to isotopy and Whitehead moves).

Let L denote a geodesic segment, ray or line in $\mathcal{T}(S)$. We say that L is ϵ-*precompact*, for $\epsilon > 0$, if for all $\tau \in L$ and all non-peripheral simple closed curves γ in S, $E_\tau(\gamma) \geq \epsilon$. Equivalently, let $\mathcal{M}(S)$ denote the moduli space of S, which is the quotient of $\mathcal{T}(S)$ by the induced action of the group of homeomorphisms of S. A geodesic is precompact exactly when its image in $\mathcal{M}(S)$ is contained in a compact set.

We will have need of another observation, which appears in [10] and again in [14]:

Lemma 1.2. *If $\Phi_h \in \mathcal{MF}(S)$ determines an ϵ-precompact Teichmüller ray, then Φ_h is complete.*

§2. Two optimization problems

Let L be an ϵ-precompact geodesic segment, ray, or line and let $\overline{L}$ denote the complete geodesic line containing L. Choose an arbitrary origin and orientation on $\overline{L}$ and let the parameter t denote signed distance from the origin. Then L corresponds to an interval $[a, b]$, where $-\infty \leq a < b \leq \infty$ (if $a = -\infty$ we really mean $(-\infty, b]$, and similarly for $b = \infty$). Denote by $\overline{L}(t)$ the conformal structure on S given by the point parametrized by t; or by $L(t)$ if $t \in [a, b]$.

For each $\alpha \in \mathcal{MF}(S)$ and $t \in \mathbf{R}$, let $E_t(\alpha)$ be short for the extremal length $E_{\overline{L(t)}}(\alpha)$. Fixing a point $\sigma \in \mathcal{T}(S)$, define also

$$R_{\sigma,t}(\alpha) = \frac{E_t(\alpha)}{E_\sigma(\alpha)},$$

which we will tend to abbreviate as $R_t(\alpha)$. Note that R_t is invariant under scaling, making it a function defined on projective classes in $\mathcal{PMF}(S)$.

For each t we have $d(\sigma, L(t)) = \frac{1}{2}\log\sup_\alpha R_t(\alpha)$, so finding $\pi_L(\sigma)$ reduces to minimizing this quantity over $[a,b]$.

Accordingly, define

$$\begin{aligned}\mathrm{Minmax}(\sigma, L) = \{(\alpha,t) \in \mathcal{MF}(S)\times[a,b] : R_t(\alpha) &= \inf_s \sup_\beta R_s(\beta) \\ &= \sup_\beta R_t(\beta)\}\end{aligned}$$

(where infima and suprema are over $[a,b]$ and $\mathcal{MF}(S)$, respectively). Denote by $A_{mM}(\sigma,L) \subset \mathcal{MF}(S)$ the projection of $\mathrm{Minmax}(\sigma,L)$ to its first factor, and let $T_{mM}(\sigma,L) \subset [a,b]$ denote the projection of $\mathrm{Minmax}(\sigma,L)$ to its second factor. The projection $\pi_L(\sigma)$ is exactly the set $\{L(t) : t \in T_{mM}(\sigma,L)\}$.

Now consider the dual problem,

$$\begin{aligned}\mathrm{Maxmin}(\sigma, L) = \{(\alpha,t) \in \mathcal{MF}(S)\times[a,b] : R_t(\alpha) &= \sup_\beta \inf_s R_s(\beta) \\ &= \inf_s R_s(\alpha)\}\end{aligned}$$

and similarly define T_{Mm} and A_{Mm}. Note that, in this problem, the first optimization, $\inf_s R_s(\beta)$, is independent of σ. The main idea of this paper is to replace the Minmax problem with the Maxmin problem. In particular,

Theorem 2.1. *Let L be an ϵ-precompact geodesic in $\mathcal{T}(S)$. Then for any $\sigma \in \mathcal{T}(S)$, $T_{mM}(\sigma,L)$ and $T_{Mm}(\sigma,L)$ have bounded diameters, and are a bounded distance apart. These bounds depend only on ϵ and $\chi(S)$.*

Proof. The first step is to show that $E_t(\alpha)$, as a function of t, is coarsely approximated by a strictly convex function. This in particular will show that the minima of $E_t(\alpha)$ occur on sets of bounded diameter.

Let Φ be the holomorphic quadratic differential on $\overline{L}(0)$ that determines the geodesic. The extremal lengths of the horizontal and vertical foliations of Φ vary exponentially with t according to (1.3) and (1.4). We also have $E_0(\Phi_h) = E_0(\Phi_v) = ||\Phi||$ by (1.2), and we may normalize so that $||\Phi|| = 1$.

We will need to know (see e.g. Gardiner-Masur [5]) that Φ_h and Φ_v *fill up* S, in the sense that every non-zero element of $\mathcal{MF}(S)$ has positive intersection with at least one of them. Thus we may define the strictly positive function

$$e_t(\alpha) = \tfrac{1}{2}\left(\frac{i(\alpha,\Phi_h)^2}{E_t(\Phi_h)} + \frac{i(\alpha,\Phi_v)^2}{E_t(\Phi_v)}\right)$$

which will serve as an approximation for $E_t(\alpha)$ (see lemma 2.2).

We can then introduce the following quantities, after fixing $\sigma \in \mathcal{T}(S)$.

- Let s_α denote the unique value of t where $e_t(\alpha)$ is minimal (possibly $s_\alpha = \infty$ or $-\infty$).
- Let $m(\alpha)$ be the set of t values where $E_t(\alpha)$ is minimal.
- Let $r_t(\alpha) = \frac{e_t(\alpha)}{E_\sigma(\alpha)}$. This function is an approximation for $R_t(\alpha)$, and takes its minimum at $t = s_\alpha$.
- Let $\widetilde{A}_{Mm}(\sigma, L) = \{[\alpha] \in \mathcal{PMF}(S) : r_{s_\alpha}(\alpha) = \max_{\beta \in \mathcal{MF}(S)} r_{s_\beta}(\beta)\}$.
- Let $\widetilde{T}_{Mm}(\sigma, L) = \{s_\alpha : [\alpha] \in \widetilde{A}_{Mm}(\sigma, L)\}$.

Note that $\widetilde{A}_{Mm}$ and $\widetilde{T}_{Mm}$ are defined similarly to A_{Mm} and T_{Mm}, respectively, with R_t replaced with r_t. These definitions can be made for the following reason: The function $r_t(\alpha)$ is invariant under scaling of α, so it may be defined for $[\alpha] \in \mathcal{PMF}(S)$. Since intersection number and extremal length are continuous functions, it follows that the minimal values $r_{s_\alpha}(\alpha)$ depend continously on $[\alpha]$. Therefore a maximum is realized, since $\mathcal{PMF}(S)$ is compact.

The proof will divide into the following basic steps:

(1) Show that $e_t(\alpha)$ and $r_t(\alpha)$ are approximations for $E_t(\alpha)$ and $R_t(\alpha)$, respectively. As a consequence, the set of minima $m(\alpha)$ is contained in a bounded neighborhood of s_α.
(2) Show that for a solution $\alpha \in A_{Mm}(\sigma, L)$ of the Maxmin problem, $R_{s_\alpha}(\alpha)$ and $r_{s_\alpha}(\alpha)$ are also good approximations to $\exp 2d(\sigma, L(s_\alpha))$.
(3) Conclude that $\operatorname{diam} \widetilde{T}_{Mm}$ is bounded, and that T_{Mm} and T_{mM} lie in a bounded neighborhood of $\widetilde{T}_{Mm}$.

Of these, step 2 contains the essential geometric idea of the paper, and brings us most of the way to connecting the Maxmin and the Minmax problems.

Begin with this estimate:

Lemma 2.2. *There is a fixed $c_0 = c_0(\epsilon, \chi(S))$ such that if L is ϵ-precompact then for any $\alpha \in \mathcal{MF}(S)$ and $t \in [a, b]$,*

$$e_t(\alpha) \leq E_t(\alpha) \leq c_0 e_t(\alpha).$$

Proof. The left side actually does not depend on the precompactness of L – it follows directly from lemma 1.1, applied to α and each of Φ_h and Φ_v.

The right side follows from a compactness argument. Let Φ^t denote the image of Φ in $L(t)$ under the Teichmuller map from $L(0)$, scaled so that $||\Phi^t|| = 1$. Then as measured foliations we have $\Phi^t_h = e^{-t}\Phi_h$ and $\Phi^t_v = e^t \Phi_v$. Thus,

$$e_t(\alpha) = \tfrac{1}{2}\left(i(\alpha, \Phi^t_h)^2 + i(\alpha, \Phi^t_v)^2\right).$$

The ratio $E_t(\alpha)/e_t(\alpha)$ is therefore a continuous positive function of the projectivized measured foliation $[\alpha] \in \mathcal{PMF}(S)$ and the pair $(L(t), \Phi^t)$ in the total space of Riemann surfaces equipped with holomorphic quadratic differentials. Fixing the Riemann surface $L(t)$, the space $\mathcal{PMF}(S) \times PQD(L(t))$ is compact; and the function is invariant under the action of the mapping class group. Further, the image of $L(t)$ in the moduli space is restricted to a compact set by our definition of precompactness. It follows that $E_t(\alpha)/e_t(\alpha)$ is bounded. □

Now since $e_t(\alpha)$ is a strictly positive sum of exponentials, a brief calculation using (1.3,1.4) shows that, excluding the case $s_\alpha = \pm\infty$,

$$\tfrac{1}{2} e_{s_\alpha}(\alpha) \exp 2|t - s_\alpha| \leq e_t(\alpha) \leq 2 e_{s_\alpha}(\alpha) \exp 2|t - s_\alpha| \tag{2.1}$$

for $t \in [a,b]$. It follows immediately from (2.1) and lemma 2.2 that $m(\alpha)$ is constrained by

$$m(\alpha) \subseteq [s_\alpha - \tfrac{1}{2}\log 2c_0, s_\alpha + \tfrac{1}{2}\log 2c_0]. \tag{2.2}$$

This concludes step 1 of the proof.

Define

$$I_t(\alpha,\beta) = \frac{i(\alpha,\beta)^2}{E_t(\alpha)E_t(\beta)}.$$

We begin the proof of step 2 with the following crucial lemma. It states that the minima s_α and s_β for two measured foliations α, β can occur far apart only if α and β are sufficiently different to have a definite intersection number, as measured by I_t:

Lemma 2.3. *There are constants D and c_1, depending only on ϵ and $\chi(S)$, such that for any $\alpha, \beta \in \mathcal{MF}(S)$,*

$$|s_\alpha - s_\beta| \geq D \quad \Longrightarrow \quad I_{s_\alpha}(\alpha,\beta) \geq c_1.$$

Remark. This lemma is similar to (and was motivated by) lemma 3.2 from [15], in which the role of L is played by a hyperbolic 3-manifold N homeomorphic to $S \times \mathbf{R}$, and the role of the minimum point s_α is played by the geodesic representative of α in N.

Proof. Suppose the lemma is false; then there exists a sequence of ϵ-precompact geodesics L_i parametrized by $[a_i, b_i]$, with foliations α_i, β_i such that $|s_{\alpha_i} - s_{\beta_i}| \to \infty$ while $I_{s_\alpha}(\alpha,\beta) \to 0$.

Adjusting if necessary by automorphisms of S, we may assume that the points $L_i(s_{\alpha_i})$ lie in a fixed compact subset of a fundamental domain of the action of the mapping class group on $\mathcal{T}(S)$. Further, the quantity I_t is invariant under scaling of the measures, and so is defined on the compact space $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. Thus we may take a convergent subsequence and obtain a limit example (L, α, β) in which $I_{s_\alpha}(\alpha,\beta) = 0$, and $|s_\beta - s_\alpha| = \infty$. Without loss of generality we assume $s_\beta = +\infty$, so that L must contain an ϵ-precompact ray $[s_\alpha, \infty)$.

However, by lemma 1.2, this implies that the foliation Φ_h is complete. Since $s_\beta = +\infty$, we have $i(\beta, \Phi_h) = 0$, so that β must be topologically equivalent to Φ_h. On the other hand, the fact that $s_\alpha \neq +\infty$ implies that $i(\alpha, \Phi_h) > 0$, and therefore that $i(\alpha,\beta) > 0$, a contradiction. □

As a consequence of this lemma we can show that two foliations with sufficiently distant minima cannot both give a large minimal value for R_t.

Corollary 2.4. *For any $\alpha, \beta \in \mathcal{MF}(S)$, if $|s_\alpha - s_\beta| > D$ then*

$$R_{s_\alpha}(\alpha) R_{s_\alpha}(\beta) \le 1/c_1$$

where D, c_1 are the constants of lemma 2.3.

Proof. Lemma 2.3 implies that $i(\alpha,\beta)^2 \ge c_1 E_{s_\alpha}(\alpha) E_{s_\alpha}(\beta)$. On the other hand, lemma 1.1 assures us that $E_\sigma(\alpha)E_\sigma(\beta) \ge i(\alpha,\beta)^2$. Combining these inequalities yields the desired conclusion. □

We can now complete step 2 of the proof:

Lemma 2.5. *There exists a constant $c_3(\epsilon, \chi(S))$ such that*

$$R_{s_\lambda}(\lambda) \le \exp 2d(\sigma, L(s_\lambda)) \le c_3 R_{s_\lambda}(\lambda)$$

for $[\lambda] \in \widetilde{A}_{Mm}(\sigma, L)$.

Proof. The left side of the inequality is immediate from (1.1).

Let $[\mu] \in \mathcal{PMF}(S)$ denote the unique projective class of measured foliations, such that

$$d(\sigma, L(s_\lambda)) = \tfrac{1}{2} \log R_{s_\lambda}(\mu),$$

and, in particular, $[\mu]$ maximizes R_{s_λ} over $\mathcal{PMF}(S)$. Thus our job is to find an upper bound for the ratio

$$Q = R_{s_\lambda}(\mu)/R_{s_\lambda}(\lambda).$$

The idea will be that, if Q is large, then by the exponential growth of e_t and the fact that the minimum of $r_t(\mu)$ is no larger than the minimum of $r_t(\lambda)$, we may conclude that $|s_\lambda - s_\mu|$ is large. But then corollary 2.4 implies that $R_{s_\lambda}(\lambda)$ and $R_{s_\lambda}(\mu)$ cannot both be large, which will yield a contradiction.

By virtue of the estimate (2.1) on e_t, we have

$$r_{s_\lambda}(\mu) \le 2e^{2|s_\lambda - s_\mu|} r_{s_\mu}(\mu).$$

Lemma 2.2 implies that $R_{s_\lambda}(\mu) \le c_0 r_{s_\lambda}(\mu)$, and by choice of λ we have $r_{s_\mu}(\mu) \le r_{s_\lambda}(\lambda) \le R_{s_\lambda}(\lambda)$. Therefore we conclude

$$Q \le 2c_0 e^{2|s_\lambda - s_\mu|},$$

or

$$|s_\lambda - s_\mu| \ge \tfrac{1}{2} \log Q/2c_0.$$

Now suppose $Q \ge 2c_0 \exp 2D$, with D the constant from lemma 2.3. Corollary 2.4 implies that

$$R_{s_\lambda}(\lambda) R_{s_\lambda}(\mu) \le \frac{1}{c_1}$$

and hence

$$Q \le (c_1 R^2_{s_\lambda}(\lambda))^{-1}.$$

To bound Q we need only note that there is a lower bound for $R_{s_\lambda}(\lambda)$: There is a constant ℓ_0 depending on $\chi(S)$ such that the shortest simple closed curve α on S has $E_\sigma(\alpha) \le \ell_0$. Since L is ϵ-precompact, $R_t(\alpha) \ge \epsilon/\ell_0$ for all $t \in [a,b]$. Tracing through our definitions, it follows that $r_{s_\alpha}(\alpha) \geq r_0 = \epsilon/\ell_0 c_0$, and hence that $R_{s_\lambda}(\lambda) \geq \epsilon/\ell_0 c_0$ as well. This gives the desired bound on Q. □

We proceed with the proof of step 3, and bound the diameter of $\widetilde{T}_{Mm}(\sigma, L)$. Suppose $\lambda, \mu \in \widetilde{A}_{Mm}(\sigma, L)$. If $|s_\lambda - s_\mu| \geq D$ then corollary 2.4 bounds $R_{s_\lambda}(\lambda) R_{s_\lambda}(\mu)$.

On the other hand, $R_{s_\lambda}(\lambda) \geq r_{s_\lambda}(\lambda)$, and by (2.1) $R_{s_\lambda}(\mu) \geq \frac{1}{2} r_{s_\mu}(\mu) \exp|s_\lambda - s_\mu|$. Since, as in the previous proof, $r_{s_\lambda}(\lambda) = r_{s_\mu}(\mu) \geq r_0$ for a fixed r_0, we get

$$\tfrac{1}{2} r_0^2 e^{2|s_\lambda - s_\mu|} \leq \frac{1}{c_1}$$

which bounds $|s_\lambda - s_\mu|$ by some c_4.

Similarly, if $\lambda \in \widetilde{A}_{Mm}(\sigma, L)$ and $\mu \in A_{Mm}(\sigma, L)$ then $\inf_t R_t(\mu)$ is maximal in $\mathcal{MF}(S)$ and in particular is at least r_0. Thus we may repeat the argument of the previous paragraph to obtain a bound on $|s_\lambda - s_\mu|$ in this case as well.

Using the bound (2.2) on $m(\alpha)$ in terms of s_α, and the fact that $T_{Mm}(\sigma, L) = \bigcup_{\alpha \in A_{Mm}(\sigma, L)} m(\alpha)$, we conclude that $T_{Mm}(\sigma, L)$ lies in a bounded neighborhood of $\widetilde{T}_{Mm}(\sigma, L)$, and thus its diameter is bounded, and half the theorem is proven.

To complete the proof, we must show that $T_{mM}(\sigma, L)$ is in a bounded neighborhood of $\widetilde{T}_{Mm}(\sigma, L)$.

Fix $t \in T_{mM}(\sigma, L)$, and let $[\mu] \in \mathcal{PML}(S)$ maximize R_t – so that

$$d(\sigma, L) = d(\sigma, L(t)) = \tfrac{1}{2} \log R_t(\mu).$$

Since $d(\sigma, L(s_\lambda)) \geq d(\sigma, L)$, and using lemma 2.5, we obtain

$$c_3 R_{s_\lambda}(\lambda) \geq R_t(\mu)$$

for any $\lambda \in \widetilde{A}_{Mm}(\sigma, L)$. But by choice of μ, $R_t(\mu) \geq R_t(\lambda)$. Finally, $R_t(\lambda) \geq r_t(\lambda) \geq \frac{1}{2} r_{s_\lambda}(\lambda) \exp 2|t - s_\lambda|$, and $R_{s_\lambda}(\lambda) \leq c_0 r_{s_\lambda}(\lambda)$. We conclude

$$c_0 c_3 \geq \tfrac{1}{2} \exp 2|t - s_\lambda|.$$

This bounds $|t - s_\lambda|$ by some c_5, and the proof of theorem 2.1 is complete. □

§3. The contraction theorem

Our main theorem follows almost directly from the results of the previous section.

Theorem 3.1. *Given ϵ there are constants b_1, b_2 depending on $\epsilon, \chi(S)$ such that for any ϵ-precompact geodesic L, if $\sigma \in \mathcal{T}(S)$ is such that $d(\sigma, L) > b_1$ then, setting $r = d(\sigma, L) - b_1$,*

$$\operatorname{diam}(\pi_L(\mathcal{N}_r(\sigma))) \leq b_2.$$

In particular, for some $b_0 = b_0(\epsilon, \chi(S))$ and all $\sigma \in \mathcal{T}(S)$ we have

$$\operatorname{diam}(\pi_L(\sigma)) \leq b_0.$$

Proof. Since $\pi_L(\sigma) = \{L(t) : t \in T_{mM}(\sigma, L)\}$, and t measures Teichmüller arclength along L, the bound on $\mathrm{diam}(\pi_L(\sigma))$ follows directly from theorem 2.1. (Note also that it is a logical consequence of the first diameter bound, with $b_0 = \max(b_2, 2b_1)$).

Let τ be such that $d(\sigma, \tau) \leq r$. Suppose $\lambda \in \widetilde{A}_{Mm}(\sigma, L)$ and $\mu \in \widetilde{A}_{Mm}(\tau, L)$. By the proof of theorem 2.1, it is sufficient to find a value for b_1 that allows us to bound $|s_\lambda - s_\mu|$ (note that s_λ and s_μ are indpendent of σ and τ).

Suppose $|s_\lambda - s_\mu| > D$, where D is the constant in lemma 2.3. Then corollary 2.4 again gives

$$R_{\sigma,s_\lambda}(\lambda) R_{\sigma,s_\lambda}(\mu) \leq \frac{1}{c_1}.$$

By choice of λ, and lemma 2.5,

$$R_{\sigma,s_\lambda}(\lambda) \geq c_3^{-1} \exp 2d(\sigma, L),$$

and similarly with σ and λ replaced by τ and μ. We note also that

$$E_\sigma(\mu) \leq e^{2r} E_\tau(\mu)$$

by choice of τ and by (1.1), and therefore

$$R_{\tau,s_\lambda}(\mu) \leq e^{2r} R_{\sigma,s_\lambda}(\mu).$$

By definition of s_μ and by lemma 2.2,

$$R_{\tau,s_\mu}(\mu) \leq c_0 R_{\tau,s_\lambda}(\mu).$$

Putting all this together, we obtain

$$\begin{aligned} d(\sigma, L) + d(\tau, L) &\leq \tfrac{1}{2} \log(c_3^2 R_{\sigma,s_\lambda}(\lambda) R_{\tau,s_\mu}(\mu)) \\ &\leq \tfrac{1}{2} \log(c_3^2 c_0 e^{2r} R_{\sigma,s_\lambda}(\lambda) R_{\sigma,s_\lambda}(\mu)) \\ &\leq r + C \end{aligned}$$

where $C = \frac{1}{2}\log(c_3^2 c_0/c_1)$. Now if we choose $b_1 > C/2$ and assume $d(\sigma, L) > b_1$, then since $r = d(\sigma, L) - b_1$ we have $d(\tau, L) \leq C - b_1 < b_1$. But, by the choice of τ and the triangle inequality, $d(\tau, L) \geq b_1$. This contradiction implies that $|s_\lambda - s_\mu| \leq D$. $\square$

§4. Applications

We shall need the following immediate consequence of the contraction theorem:

Corollary 4.1. *Let L be an ϵ-precompact geodesic in $\mathcal{T}(S)$ and let b_1, b_2 be the constants given in theorem* 3.1. *If $R > b_1$ and points $x, y \in \mathcal{T}(S)$ are connected by a path of length T which remains outside an R-neighborhood of L, then*

$$\mathrm{diam}(\pi_L(x) \cup \pi_L(y)) \leq b_2 \left(\frac{T}{R - b_1} + 1 \right). \tag{4.1}$$

Furthermore for any $x, y \in \mathcal{T}(S)$,

$$\mathrm{diam}(\pi_L(x) \cup \pi_L(y)) \leq d(x, y) + B(\epsilon). \tag{4.2}$$

Proof. Dividing the path from x to y into pieces of size at most $R - b_1$ and applying theorem 3.1 yields inequality (4.1). To obtain (4.2), let $R = 2b_1$ and consider the geodesic G connecting x to y. If G stays outside of an R-neighborhood of L then apply (4.1). If not, let x', y' be the closest points in G to x, y (respectively) such that $d(x', L) = d(y', L) = R$. The triangle inequality together with (4.1) yields

$$\mathrm{diam}(\pi_L(x) \cup \pi_L(y)) \le d(x, x') + 2R + d(x', y') + d(y, y') + 2b_2$$

which suffices. □

Define a (K, δ)-*quasi-geodesic* in $\mathcal{T}(S)$ as a path $\Gamma : [a, b] \to \mathcal{T}(S)$, parametrized by arclength, such that for any $s, t \in [a, b]$,

$$|s - t| \le K d(\Gamma(s), \Gamma(t)) + \delta.$$

Theorem 4.2. *Let Γ be a (K, δ)-quasi-geodesic path in $\mathcal{T}(S)$, whose endpoints are connected by an ϵ-precompact Teichmüller geodesic L. Then Γ remains in a $B(K, \delta, \epsilon)$-neighborhood of L.*

Proof. The idea of the argument is standard: any segment of Γ which is outside a sufficiently large neighborhood of L is, because of the contractive properties of π_L, much less efficient than the path obtained by going back to L and moving along the projection. Thus large excursions from L violate the quasi-geodesic property.

Fix $R = \max(Kb_2, 2b_1)$. Let $[s, t]$ be a maximal interval for which $\Gamma((s, t))$ is outside an R-neighborhood of L. Applying corollary 4.1, we have

$$\mathrm{diam}(\pi_L(\gamma(s)) \cup \pi_L(\gamma(t))) \le \frac{b_2}{2R}|s - t| + b_2.$$

The maximality of $[s, t]$ implies that $d(\Gamma(s), \pi_L(\Gamma(s))) = R$, and similarly for t. Thus

$$d(\Gamma(s), \Gamma(t)) \le 2R + \frac{b_2}{2R}|s - t| + b_2$$

On the other hand, the quasi-geodesic property of Γ gives a lower bound on the distance $d(\Gamma(s), \Gamma(t))$, and together we obtain (using the definition of R), a bound $|s - t| \le (4K + 2)R + 2\delta$. It follows that Γ cannot exit a $(2K + 2)R + \delta$ neighborhood of L. □

Let α be a measured foliation in $\mathcal{MF}(S)$. The set

$$Thin(\alpha, \delta) = \{\sigma \in \mathcal{T}(S) : E_\sigma(\alpha) \le \delta\}$$

is somewhat analogous to a horoball in hyperbolic space. As in hyperbolic space, it has infinite diameter, but its projection to a precompact geodesic is nevertheless bounded, bar one exceptional case:

Theorem 4.3. *Given ϵ, δ there is a constant B such that, if L is a complete ϵ-precompact geodesic and $\alpha \in \mathcal{MF}(S)$, then*

$$\mathrm{diam}(\pi_L(Thin(\alpha, \delta))) \le B + \mathrm{diam}(L \cap Thin(\alpha, \delta)).$$

Remarks: (1) A special case of this is when α is a non-peripheral simple closed curve, with weight 1. Then $L \cap Thin(\alpha, \delta)$ is uniformly bounded for all ϵ-precompact L, so we have a uniform bound on $\mathrm{diam}(\pi_L(Thin(\alpha, \delta)))$. (2) If a complete geodesic L is determined by a quadratic differential Φ, let us call $[\Phi_h]$ and $[\Phi_v]$ in $\mathcal{PMF}(S)$ the *endpoints at infinity* for L. Another special case occurs when $[\alpha]$ is equal in $\mathcal{PMF}(S)$ to an endpoint, and then both sides of the inequality are infinite.

Proof. In view of remark (2), let us assume that $[\alpha] \neq [\Phi_h], [\Phi_v]$ in $\mathcal{PMF}(S)$. Masur proved in [11] that, if L is ϵ-precompact, then its endpoints at infinity are *uniquely ergodic* – they support a unique projective class of transverse measures. Thus the assumption that $[\alpha]$ is not an endpoint implies that α is topologically distinct from both foliations, and therefore (by lemma 1.2) it intersects both of them non-trivially. It follows that $s_\alpha \neq \pm\infty$ and that $E_t(\alpha)$ has a positive infimum, E_0.

Fix $\delta_0 < E_0 c_1 r_0$, where c_1 and r_0 are the constants that appear in §2. We will show that $\pi_L(Thin(\alpha, \delta_0))$ lies in a bounded neighborhood of $L(s_\alpha)$.

By the results of section 2, it suffices to bound $|s_\lambda - s_\alpha|$ for $\lambda \in \widetilde{A}_{Mm}(\sigma, L)$, when $\sigma \in Thin(\alpha, \delta_0)$. Suppose $|s_\lambda - s_\alpha| \geq D$. Then by corollary 2.4 we get

$$R_{s_\alpha}(\alpha) R_{s_\alpha}(\lambda) \leq \frac{1}{c_1}.$$

On the other hand, $R_{s_\alpha}(\alpha) \geq E_0/\delta_0$ by assumption, and $R_{s_\alpha}(\lambda) \geq r_{s_\lambda}(\lambda) \geq r_0$ as in lemma 2.5, so we obtain a contradiction. It follows that $|s_\lambda - s_\alpha| < D$.

To prove the theorem for general $\delta > \delta_0$, we first observe that $Thin(\alpha, \delta)$ lies in a neighborhood of $Thin(\alpha, \delta_0)$ of radius $\frac{1}{2} \log \delta/\delta_0$ (since one can move along a Teichmüller geodesic defined by Φ such that $\Phi_h = \alpha$, and apply (1.3)).

Applying corollary 4.1 (inequality 4.2) we conclude immediately that the projection $\pi_L(Thin(\alpha, \delta))$ lies in a $(C + \frac{1}{2} \log \delta/\delta_0)$-neighborhood of $L(s_\alpha)$, for an appropriate $C(\epsilon)$. On the other hand, applying lemma 2.2 and estimate (2.1), we have $E_t(\alpha) \geq \frac{1}{2} \frac{E_0}{c_0} \exp 2|t - s_\alpha|$. Using the definition of E_0, we have

$$\mathrm{diam}(Thin(\alpha, \delta) \cap L) \leq \log \frac{\delta}{\delta_0} + \log 2 c_0 c_1 r_0.$$

This completes the proof of the theorem. □

Our last application is a geometric property of pseudo-Anosov homeomorphisms. Let $f : S \to S$ be pseudo-Anosov, and let f_* denote the induced action on $\mathcal{T}(S)$ (see [3] or [1]). Bers showed in [2] that the infimum $\inf_{\sigma \in \mathcal{T}(S)} d(\sigma, f_* \sigma)$ of translation distance is achieved in $\mathcal{T}(S)$ on a geodesic, namely the axis of f_*, on which f_* acts by translation. The contraction theorem can be used to give lower bounds on how fast the translation distance grows as we consider points far away from the axis (this question was suggested by Feng Luo).

Theorem 4.4. *Let $f : S \to S$ be a pseudo-Anosov homeomorphism, and $L \subset \mathcal{T}(S)$ the axis of f_*. There are $c_0, c_1 > 0$ such that, for all $x \in \mathcal{T}(S)$,*

$$d(x, f_* x) \geq c_0 d(x, L) - c_1.$$

Remark: it is easy to see that the rate of increase of $d(x, f_* x)$ is at most linear with $d(x, L)$, and so in an asymptotic sense the theorem gives maximal growth. However the constants c_0 and c_1 are completely non-constructive.

Proof. Since the axis L is invariant by f_*, its image in the moduli space is a closed curve, so that L is precompact. Let b_1, b_2 be the constants given by theorem 3.1 for L.

For $x \in \mathcal{T}(S)$, let $R = d(x, L)$ and let $t = d(x, f_*(x))$. Let t_0 denote the translation distance of f_* on L, so that $t \geq t_0$, and $t = t_0$ only for $x \in L$.

For any $n > 0$ we can connect x to $f_*^n x$ by a chain of geodesics of length nt, which remains outside an $R - t/2$ neighborhood of L. By corollary 4.1 we have

$$\operatorname{diam}(\pi_L(x) \cup \pi_L(f_*^n x)) \leq b_2 \left(\frac{nt}{R - b_1 - t/2} + 1 \right).$$

(Note that we may assume that $R - b_1 - t/2 > 0$, because otherwise we would have $t \geq 2R - b_1$, and we would be done.) It follows that

$$d(x, f_*^n x) \leq 2R + b_2 \left(\frac{nt}{R - b_1 - t/2} + 1 \right).$$

On the other hand L is also the axis of f_*^n, so that $d(x, f_*^n x) \geq n t_0$. Letting n go to infinity, we conclude that

$$t_0 \leq b_2 \frac{t}{R - b_1 - t/2}.$$

Solving for t, we obtain the desired inequality. ☐

§5. The case of non-precompact geodesics

In this section we consider the sharpness of theorem 3.1. Let L be a geodesic which is not precompact. Then L has segments which enter, arbitrarily deeply, into the "thin parts" $Thin(\alpha, \delta)$ defined in the previous section.

For small δ, the geometry of $Thin(\alpha, \delta)$ is quite far from negatively curved. In [13] we showed that it is approximated by a product space in the following sense.

Let X_α denote $\mathbf{H}^2 \times \mathcal{T}(S \setminus \alpha)$, endowed with the *sup metric* $d_X = \max(d_{\mathbf{H}^2}, d_{\mathcal{T}(S\setminus\alpha)})$ of the metrics on the factors. There is a natural homeomorphism $\Pi : \mathcal{T}(S) \to X_\alpha$, defined using Fenchel-Nielsen coordinates, so that the coordinates in the hyperbolic plane $\mathbf{H}^2$ encode the length and twist parameters associated to α. In particular, the y coordinate in $\mathbf{H}^2$ (in the upper half-plane model) for $\Pi(\sigma)$ is $1/\ell_\sigma(\alpha)$ where ℓ_σ is hyperbolic length, and it follows easily that $\Pi(Thin(\alpha, \delta))$ is contained in $\{y > \ell_1\} \times \mathcal{T}(S \setminus \alpha)$ and contains $\{y > \ell_2\} \times \mathcal{T}(S \setminus \alpha)$, where ℓ_1, ℓ_2 are approximately inversely proportional to δ. The main theorem of [13] says that

Theorem 5.1. *(Theorem 6.1 of* [13]*) For sufficiently small δ, the map Π restricted to $Thin(\alpha, \delta)$ has bounded additive distortion; that is,*

$$|d_{\mathcal{T}(S)}(\sigma, \tau) - d_X(\Pi(\sigma), \Pi(\tau))| \leq c$$

for $\sigma, \tau \in Thin(\alpha, \delta)$, where c depends only on δ and the topological type of S.

This product geometry suffices to give a converse to theorem 3.1:

Theorem 5.2. *If a Teichmüller geodesic L is not precompact, then there is no choice of constants b_1, b_2 for which the contraction property of theorem* 3.1 *holds for* π_L.

Remark: This theorem does not rule out the diameter bound for images of single points which is stated in theorem 3.1. In fact it is not ruled out that $\pi_L(\sigma)$ is in all cases a single point; resolving this question seems to require finer techniques than we use here.

Proof. Using the product structure we can show that large segments of L can be replaced by quasi-geodesics that stray arbitrarily far from L. This will contradict theorem 4.2.

Since L is not precompact, for arbitrarily small $\delta > 0$ and large $T > 0$ there exists a segment in L of length T contained in $Thin(\alpha, \delta)$ for some α. (We are using here the fact that the Teichmüller metric on the moduli space is complete, or equivalently that the distance from the boundary of $Thin(\alpha, \delta)$ to $Thin(\alpha, \delta')$ goes to infinity as $\delta/\delta' \to \infty$ – in fact it is $\frac{1}{2}\log(\delta/\delta')$.)

Let σ_1, σ_2 be the endpoints of such a segment, and let $\Pi(\sigma_i) = q_i$ be their images in the product X_α, so that $|d(q_1, q_2) - T| < c$. Using the sup metric on X_α it is easy to construct two $(2,0)$-quasi-geodesics m, m' connecting q_1 to q_2, so that neither quasi-geodesic lies in a $(T-c)$-neighborhood of the other. (restricting to an $\mathbf{R}^2$ slice, construct an appropriate quadrilateral whose diagonal is the line $[q_1, q_2]$, and let m, m' be the two paths around the perimeter). Furthermore m, m' can be made to remain in $\Pi(Thin(\alpha, \delta'))$ for δ' slightly bigger than δ. Then applying Π^{-1}, we obtain two $(2,c)$-quasi-geodesics connecting σ_1 to σ_2, neither of which lies in a $(T-2c)$-neighborhood of the other. It follows that at least one of the quasi-geodesics cannot be in a bounded neighborhood of L, as T is chosen arbitrarily large. This violates the stability of quasi-geodesics proved in Theorem 4.2, and therefore π_L cannot have the contraction property. □

Remark: For a product space with the sup metric the stability of quasi-geodesics is actually violated in a stronger way than used above. See the last section of [13] for a brief discussion of this.

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