

Some Remarks on Periodic Billiard Orbits in Rational Polygons

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1. Introduction

A billiard ball, i.e. a point mass, moves inside a polygon Q with unit speed along a straight line until it reaches the boundary ∂Q of the polygon, then instantaneously changes direction according to the mirror law: “the angle of incidence is equal to the angle of reflection,” and continues along the new line (Fig. 1(a)). Despite the simplicity of this description there is much that is unknown about the existence and the description of periodic orbits in arbitrary polygons. On the other hand, quite a bit is known about a special class of polygons, namely, rational polygons. A polygon is called *rational* if the angle between each pair of sides is a rational multiple of π . The main theorem we will prove is

Theorem 1. *For rational polygons, periodic points of the billiard flow are dense in the phase space \mathcal{M} of the billiard flow.*

Theorem 1 is a strengthening of Masur’s theorem [M] who has shown that any rational polygon has “many” periodic billiard trajectories; more precisely, the set of directions of the periodic trajectories are dense in the set of velocity directions \mathbf{S}^1 . We will also prove some refinements of theorem 1: the “well distribution” of periodic orbits in the polygon and the residuality of the points $q \in Q$ with a dense set of periodic directions (precise statements of these results will be given in section 3).

The structure of the article is as follows. In section 2 we will give a brief description of billiards in polygons and some results related to theorem 1. In section 3 we will state the strengthenings of theorem 1, and the proofs will be given in section 4.

2. Description of billiards in polygons

The trace of a moving billiard ball is called a *billiard trajectory* or *orbit*. If a billiard trajectory hits a vertex of the polygon, then it is called *singular*. For convenience we will define such billiard trajectories by continuity from the left (with respect to a fixed orientation of the boundary), thus every trajectory is infinite and the singular trajectories are the discontinuities of the flow. It is convenient to consider the set of singular trajectory segments which start at a vertex and end at a vertex. Such segments are called *generalized diagonals* and the number of links is called the *length* of the generalized diagonal (Fig. 1(b)). We remark that the set of generalized diagonals is countable. If a billiard ball returns to its initial position and has the same velocity direction, then its orbit is called *periodic* (Fig. 1(c)).

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There is a very useful tool in the analysis of billiards in polygons: the *unfolding* of trajectories. Instead of reflecting the trajectory with respect to the side of the polygon it hits we reflect the polygon itself with respect to the same side. The two adjacent links of the trajectory then become part of a straight line (Fig. 2(a)). Continuing this procedure for ever unfolds the trajectory into a half-line through a forward *corridor* of polygons. The backward trajectory can be similarly unfolded. We enumerate the unfolded polygons starting with zero, while the length of an unfolding is the number of polygons it contains (Fig. 2(b)).

The set of pairs $\{(q, v)\}$, where $q \in Q$ is a position of the ball and $v \in \mathbf{S}^1$ is its velocity constitute the phase space \mathcal{M} of the billiard flow. The phase space \mathcal{M} can be thought of as a right prism foliated with a collection of “floors” $Q(\varphi) := Q \times \varphi$ with $\varphi \in \mathbf{S}^1$. Then the billiard flow in the phase space can be imagined as a straight line flow on the floors of the phase space. When the flow reaches the boundary it jumps from the point (q, φ) to the point (q, φ') on the floor $Q(\varphi')$, where φ and φ' are related by the mirror law (Fig. 3). In the case the polygon is rational, the flow is restricted to a finite number of floors. Using the identifications of the boundary one can glue these floors together to produce an orientable surface R_φ (see the survey article [Gu] for a full explanation).

For any $\varphi \in S^1$ for which R_φ contains no generalized diagonal the flow ϕ_t restricted to the invariant surface R_φ is *minimal*, i.e. the orbit of every point is dense [Gu]. Using Teichmüller theory Kerckhoff, Masur, and Smillie showed that for almost every $\varphi \in \mathbf{S}^1$ the flow $\phi_t|_{R_\varphi}$ is *uniquely ergodic*, i.e. the only ergodic invariant measure is the Lebesgue measure [KMS]. Using similar techniques Masur has shown that for a dense set of directions $\varphi \in S^1$ the flow ϕ_t restricted to R_φ has at least one periodic point [M]. However, his result gives no indication of the location of the periodic orbit on the invariant surface R_φ .

3. Results

Throughout this section we will assume that Q is a rational polygon.

Theorem 1. *Periodic points of the billiard flow are dense in the phase space \mathcal{M} .*

We will also prove a slightly stronger theorem. For this purpose we need one more definition. A periodic orbit γ is called *ε -well distributed* on the table if for every convex set $A \subset Q$

$$\left| \frac{\text{length}(\gamma \cap A)}{\text{length}(\gamma)} - \mu_{\text{Leb}}(A) \right| < \varepsilon. \quad (1)$$

Here the length of a periodic orbit is its geometric length, not the number of links in its trajectory.

Theorem 2. *The set of ε -well distributed periodic points of the billiard flow is dense in the phase space \mathcal{M} for every $\varepsilon > 0$.*

Let V be the set of Q 's vertices and

$$G := \left\{ q \in Q \setminus V : \forall \varepsilon > 0, \text{ for a dense set of directions } \varphi, (q, \varphi) \text{ is an } \varepsilon\text{-well distributed periodic point} \right\}.$$

As another refinement of theorem 1 we have:

Theorem 3. G is residual.⁽¹⁾

The natural question arises: does $G = Q \setminus V$ for all rational polygons? Does $G = Q \setminus V \bmod(0)$ for all polygons?

Equality in the stronger sense holds for Coxeter chambers (polygons which tile the plane under their reflection group), almost integrable polygons (finite unions of Coxeter chambers) [Gu], regular polygons, and closely related polygons [V]. Next we consider

$$B := \left\{ q \in Q \setminus V : \text{there is no direction } \varphi \text{ for which } (q, \varphi) \text{ is periodic} \right\}.$$

We remind the reader that we have a standing assumption that Q is a rational polygon.

Theorem 4. *If Q is convex then B is contained in a finite union of segments. If Q is a triangle, then B is at most finite.*

Finally we turn to the question: for which angles φ is there at least one orbit which is not ε -dense? We show:

Theorem 5. *The set of directions for which there exists a non ε -dense orbit is an at most countable closed set for any $\varepsilon > 0$. There are examples when this set is not finite.*

4. Proofs of theorems

Proof of theorem 1:

First of all we fix a uniquely ergodic direction $\theta \in \mathbf{S}^1$. As mentioned above by avoiding a countable set of θ 's we can assume that R_θ contains no generalized diagonals. We claim that using the unique ergodicity we can choose N so large that for all $x \in R_\theta$ the first N links of x 's-forward orbit and the first N links of x 's-backward orbit are both $\varepsilon/2$ -dense in the surface R_θ .⁽²⁾ To see this note that the unique ergodicity of the flow implies that for every continuous function g the ergodic average $1/T \int_0^T g(\phi_t x) dt$ converges uniformly to $\int_{\mathcal{M}} g d\mu_{\text{Leb}}$ (see [W] for the proof which holds without change in this more general setting).

There are only a finite number of generalized diagonals of length less than or equal to $2N$. In the phase space \mathcal{M} these generalized diagonals lie on a finite number of floors $Q(\varphi_1), Q(\varphi_2), \dots, Q(\varphi_{k(N)})$. Let $\delta := \delta_N > 0$ be so small that for any θ' satisfying $|\theta - \theta'| < \delta$ we have $Q(\varphi_i) \cap R_{\theta'} = \emptyset$ for $i = 1, \dots, k(N)$ (Fig. 4). Therefore, if a generalized diagonal belongs to $R_{\theta'}$, then its length is greater than $2N$.

From Masur's theorem we know that there is a periodic point whose direction is arbitrarily close to θ . In particular, we can find a periodic point $(q_0, \varphi_0) \in \mathcal{M}$ satisfying $|\varphi_0 - \theta| < \delta$ and $< \varepsilon/(2 \cdot \text{const} \cdot N \cdot \text{diam } Q)$. Here, const is a constant depending on θ , which will be defined later. We consider the point (q_0, θ) and its forward unfolding of length N and its backward unfolding of length N . We claim that *either* the corridor

⁽¹⁾ A set is called *residual* if it contains a dense G_δ set, i.e. a countable intersection of open dense sets.

⁽²⁾ This fact can also be derived from the minimality of the direction θ .

of length N for the forward trajectory of (q_0, φ_0) or the corridor of length N for the backward trajectory of (q_0, φ_0) coincides with the same length corridor of (q_0, θ) .

Suppose this is not true. The forward corridors of (q_0, θ) and (q_0, φ_0) coincide for a while. Define $j_1 < N$ to be the length of the forward part of the corridors of (q_0, θ) and (q_0, φ_0) that coincide. The two corridors “branch” apart at polygon number $j_1 - 1$; let A be the common vertex of branching corridors. Similarly, let $j_2 < N$ be the length of the backward corridors and B the common vertex of the backward branching corridors. Then the straight line segment AB inside the corridor of (q_0, θ) is a generalized diagonal of length $j_1 + j_2 - 1 < 2N$. The direction of AB lies in the interval of directions $\{t : \theta \leq t \leq \varphi_0\}$ (we assume $\varphi_0 > \theta$) and thus its distance to θ is less than δ (Fig. 5). This contradicts the choice of δ .

Without loss of generality we will assume that both the periodic trajectory and the uniquely ergodic trajectory stay in the same forward corridor. The endpoints of these two trajectories lie on the boundary of the $(N-1)$ st polygon. Since $|\varphi_0 - \theta| < \varepsilon / (2 \cdot \text{const} \cdot N \cdot \text{diam}Q)$ it follows that if the const is small enough, then the distance between the endpoints of the two unfolded trajectories is less than $\varepsilon/2$ (Fig. 6). Furthermore, the first N links of the uniquely ergodic trajectory are $\varepsilon/2$ -dense in R_θ . These two facts combined show that the periodic trajectory is ε -dense in R_{φ_0} . This can be done for every uniquely ergodic direction θ . The set $\{R_\theta : \phi|_{R_\theta} \text{ is uniquely ergodic}\}$ is dense in the whole phase space \mathcal{M} . Since ε is arbitrary this completes the proof of theorem 1. ■

Proof of theorem 2:

We given an equivalent definition of ε -well distribution. We fix an embedding $Q \subset \mathbf{R}^2$. Let $A_{p,q,r,s}$ be the intersection of Q with the open ball with center $(p/q, r/s)$ and diameter $1/(\max(q, s))$. Enumerating gives rise to the countable basis $\{\tilde{A}_i\}$. A periodic orbit γ is ε -well distributed on the table if for each set \tilde{A}_i with $\text{diam}(\tilde{A}_i) > \varepsilon$

$$\left| \frac{\text{length}(\gamma \cap \tilde{A}_i)}{\text{length}(\gamma)} - \mu_{\text{Leb}}(\tilde{A}_i) \right| < \varepsilon. \tag{1}$$

For a fixed sufficiently small $\varepsilon > 0$ for a convex set A with sufficiently small diameter equation 1 automatically holds, and we can approximate any convex set A with large diameter by finite unions and intersections of the \tilde{A}_j . Using this reasoning one can show that for every $\delta > 0$ there is a $\varepsilon > 0$ so that any ε -well distributed point in the sense above is δ -well distributed in the sense of section 2 and $\varepsilon \rightarrow 0$ as $\delta \rightarrow 0$.

We need to modify several steps in the proof of theorem 1. In the proof of theorem 1 we choose N so large that the first N links of x 's forward and backward orbit are both $\varepsilon/2$ -dense. The proof given there actually shows the stronger result, both orbit segments are $\varepsilon/2$ -well distributed as well (the definition of well distributed is analogous to the one for periodic points introduced in section 3). We choose the integer N slightly larger so that the first N links are $\varepsilon/3$ -well distributed. We also choose N so large that for any $k \geq N$ if the first k links of the orbit of any point x are $2\varepsilon/3$ -well distributed then the first $k + 1$ links are ε -well distributed.

We choose $\delta := \delta_{2N}$ and the periodic point (q_0, φ_0) so that it satisfies $|\varphi_0 - \theta| < \delta_{2N}$ and $< \varepsilon / (3 \cdot \text{const} \cdot$

$2N \cdot \text{diam}Q$). Then the periodic trajectory of (q_0, φ_0) shadows the $\varepsilon/3$ -well distributed uniquely ergodic trajectory for $2N$ links either forward or backward. Thus the first $2N$ links of this orbit are $2\varepsilon/3$ -well distributed and the first $2N + 1$ links are ε -well distributed. Suppose the number of links of the (q_0, φ_0) trajectory is M . If $M \leq 2N + 1$, then the trajectory is clearly ε -well distributed.

If $M > 2N + 1$ let the cyclically ordered set $\{L_1, L_2, \dots, L_M\}$ denote the links of the trajectory. We want to *partition* the trajectory into ε -well distributed trajectory segments of lengths between N and $2N$. From formula (1) it is clear that the concatenation of ε -well distributed trajectory segments is itself ε -well distributed. Therefore, the construction of such a partition will finish the proof.

To construct such a partition we start by *covering* the trajectory by trajectory segments, i.e. by ordered sets $\{L_i, \dots, L_{i+2N-1}\}$ of length $2N$ that are ε -well distributed and for which all its ordered subsets $\{L_j, \dots, L_{j+k-1}\} \subset \{L_i, \dots, L_{i+2N-1}\}$ of length $k \geq N$ are also ε -well distributed. Without loss of generality we suppose that $\{L_1, \dots, L_{2N}\}$ as well as all its consecutively ordered subsets of length $k \geq N$ are $2\varepsilon/3$ -well distributed. Now we apply this argument again to the point $\phi_{T(2N+1)}(q_0, \varphi_0)$ to conclude that either $\{L_2, \dots, L_{2N+1}\}$ or $\{L_{2N+2}, \dots, L_{4N+1}\}$ has this property. If the latter occurs, then the link L_{2N+1} is not covered. In this case we replace $\{L_1, \dots, L_{2N}\}$ by $\{L_1, \dots, L_{2N+1}\}$ which is ε -well distributed. Continuing this process inductively yields the desired covering (Fig. 7). To finish the proof we split the covering into a partition of ε -well distributed pieces of different lengths, but with all the lengths being at least N and at most $2N$. ■

Proof of theorem 3:

Fix a uniquely ergodic direction θ . Let

$$A_\varepsilon^\delta(\theta) := \left\{ q \in Q \setminus V : (q, \varphi) \text{ is an } \varepsilon\text{-well distributed periodic point for some } \varphi, |\varphi - \theta| < \delta \right\}.$$

In the proof of theorem 1 we showed that for all uniquely ergodic directions θ and for all $\varepsilon > 0$ there is a $\delta := \delta(\theta, \varepsilon) > 0$ such that $A_\varepsilon^\delta(\theta)$ is dense in Q . The set $A_\varepsilon^\delta(\theta)$ is also open because each periodic point is contained in an open strip, that is if $x = (q, \varphi)$ is periodic, then there is an open disc $D \subset Q \setminus V$ such that (q', φ) is periodic for all $q' \in D$ (see [GKT]). Choose a countable dense set of uniquely ergodic directions $\{\theta_i\} \subset \mathbf{S}^1$ and $\varepsilon_i > 0$ satisfying $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. Furthermore choose positive numbers $\delta_i \leq \delta(\theta_i, \varepsilon_i)$ such that $\lim_{i \rightarrow \infty} \delta_i = 0$. Then

$$\bigcap_{i \in \mathbf{Z}^+} A_{\varepsilon_i}^{\delta_i}(\theta_i) \subset G,$$

and thus G is residual. ■

Proof of theorem 4:

First we will show that the set B is contained in a finite number of line segments. To do this we will consider only billiard trajectories which hit some side of the polygon perpendicularly. Let φ_i be the direction perpendicular to the i th side of Q . As discussed in section 2, the invariant surface R_{φ_i} contains at most a

finite number of generalized diagonals. Any perpendicular trajectory which enters a vertex is automatically a generalized diagonal because its backward orbit also enters the same vertex. This means that only a finite number of points on the i th side have singular perpendicular trajectories. The number of such diagonals is clearly less than the number of floors of the invariant surface R_{φ_i} , multiplied by the number of vertices of Q . We remark that the number of R_{φ_i} 's floors is less than the greatest common denominator of all the inner angles of the polygon.

The perpendicular orbit through any other point on the i th side hits that side perpendicularly twice and thus is periodic (for details see [Bo][GSV]). For each point $q \in Q$ we consider the perpendiculars to the straight lines containing the sides of Q . Since Q is convex, for the nearest side of Q , the base point of this perpendicular will belong to an interior point of the side and not to its continuation. Consequently, any point q which is not covered by one of the finite number of generalized diagonals from the surfaces $\cup_i R_{\varphi_i}$ has a ‘‘perpendicular’’ periodic orbit passing through it.

We remark that using the ideas from the proofs of theorems 1-3 we can show a slightly stronger result. Namely, B is contained in a Cantor subset of these segments.

If Q is an acute or right triangle, then for each $q \in Q \setminus V$ the perpendicular to the lines containing the sides lie inside Q (if $q \in \partial Q$, then this perpendicular is just the point q). For obtuse triangles the same is true for at least two sides. Thus for rational triangles, for each $q \in Q \setminus V$ there are at least two distinct singular perpendicular billiard orbits passing through q . Each point $q \in Q \setminus V$ which is not covered by a perpendicular periodic trajectory must be covered by two distinct perpendicular generalized diagonals. However, any two generalized diagonals which intersect do so transversely or else they would coincide. If d_1, \dots, d_k denote the perpendicular generalized diagonals, then B is contained in $\cup_{i \neq j} (d_i \cap d_j)$, a finite set. ■

Question: For which polygons are all $q \in Q \setminus V$ at least double covered by perpendicular trajectories? Note that this does not hold for the regular hexagon although in this case it is easy to see that B is empty.

Proof of theorem 5:

Let

$$C_\varepsilon := \left\{ \varphi : \exists x \in R_\varphi \text{ such that the orbit of } x \text{ is not } \varepsilon\text{-dense} \right\}.$$

If R_φ contains no generalized diagonal, then the flow $\phi|_{R_\varphi}$ is minimal and thus every billiard orbit is dense. The set of generalized diagonals is at most countable and thus so is C_ε .

The set C_ε is clearly closed, for if $\varphi_i \in C_\varepsilon$ and $x_i \in R_{\varphi_i}$ is not an ε -well distributed point, then any weak limit point of the x_i is also not an ε -well distributed point.

An example where C_ε is countable is the following. We consider the L-shaped figure consisting of three identical squares as depicted in figure 8(a). In this figure a periodic orbit with 4 links which avoids the right hand square is shown. In figure 8(b), an unfolding of length 3 of the L-shaped figure along with a periodic orbit with 8 links is drawn. This periodic orbit also never enters the right hand square. In general, the

analogous unfoldings of length k contains a periodic orbit with $2k + 2$ links which avoids the right hand square. ■

Question: are there convex examples of this phenomenon?

5. Acknowledgements

MB is partially supported by NSF-DMS-9224667. GG is supported by the Alexander von Humboldt Stiftung. ST was supported during part of this research by the Deutsche Forschungsgemeinschaft.

6. References

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7. Figure captions

Fig. 1 (a) The mirror law, (b) a generalized diagonal, (c) a periodic orbit

Fig. 2 Unfolding a trajectory

Fig. 3 The mirror law in phase space

Fig. 4 The δ -neighborhood of R_θ does not contain a generalized diagonal of length less than $2N$

Fig. 5 Branching corridors

Fig. 6 $\varepsilon/2$ -shadowing of length N

Fig. 7 Covering the trajectory by well-distributed trajectory segments

Fig. 8 Periodic orbits in the L-shaped figure which do not enter the right hand square